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A baseline model of industry evolution*

Sidney G. Winter[1], **Yuri M. Kaniovski**[2], **and Giovanni Dosi**[3]

[1] The Wharton School, University of Pennsylvania, 2000 Steinberg Hall-Dietrich Hall, Philadelphia, PA 19104.6370, USA (e-mail: winter@wharton.upenn.edu)
[2] School of Economics, Free University of Bozen · Bolzano, Via Sernesi 1, 39100 Bolzano, Italy (e-mail: Yuriy.Kaniovskyi@unibz.it)
[3] The Sant'Anna School of Advanced Studies, Via Carducci 40, 56127 Pisa, Italy (e-mail: gdosi@sssup.it)

Abstract. The paper analyses some general dynamic properties of industries characterized by heterogeneous firms and continuing stochastic entry.

After a brief critical assessment of some significant drawbacks of recent contributions to modeling of stochastic industrial dynamics, we propose a novel analytical apparatus able to derive some generic properties of the underlying competition process combining persistent technological heterogeneity, differential growth of individual firms and turnover. The basic model, we suggest, is indeed applicable with proper modifications to a large class of evolutionary processes, well beyond industrial dynamics.

Key words: Evolution – Competition – Stochastic entry – Industrial dynamics – Evolutionary games

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1 Introduction

This paper analyses the properties and outcomes of competitive dynamics in industries characterized by heterogeneous firms and continuing stochastic entry. In that setting, aggregate economic variables – such as prices, quantities and indirectly

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Correspondence to: G. Dosi

distributive shares –, and the structural characteristics of the industry – such as size distributions – are interpreted as stemming from an explicitly dynamic process combining persistent technological heterogeneity, differential growth of individual firms and turnover.

The empirical phenomena on industrial dynamics addressed by our model have been examined in several streams of scholarly literature[1].

There is, *first*, a substantial body of descriptive work based on longitudinal data sets with large numbers of firms and establishments: cf., in particular, the U.S. Census Bureau's Longitudinal Research Database (Audretsch, 1997; Dunne et al., 1988; Jensen and McGuckin, 1997), and a broadly similar data set developed at Statistics Canada (Baldwin, 1995). A great number of specific questions have been answered by these explorations. What stands out impressionistically, however, is the diversity of firms and the sense of continuing, highly dynamic, disequilibrium. The extent of turnover at the low end of the size distribution is particularly notable. For example, Dunne et al. (1988) summarizing the general picture have remarked among other things that "entry and exit rates at a point in time are ... highly correlated across industries so that industries with higher than average entry rates tend also to have higher than average exit rates" (p. 496); and that "... the market share of each entering cohort generally declines as the cohort ages. This occurs because high exit rates, particularly when the cohort is young, overwhelm any increase in the relative size of the surviving cohort members" (p. 513). Similar properties are emphasized also by Geroski (1995), who adds among other "stylized results" from the available evidence that a) "entry seems to be slow to react to high profits" (p. 427); b) "entry rates are hard to explain using conventional measures of profitability and entry barriers" (p. 430), and c) "prices are not usually used by incumbents to block entry" (p. 430).

Second, a growing evidence from the economics of innovation suggests widespread heterogeneity in the technological capabilities of different firms. (For discussions cf. Dosi, 1988; Freeman, 1994). This evidence intuitively matches that on quite diverse revealed performances by firm and by plant (Jensen and McGuckin, 1997; Doms et al., 1995).

Third, in the literature on "industry life cycles" (for overviews, cf. Klepper, 1997; Afuah and Utterback, 1997), the principal focus is the unfolding pattern of industrial evolution over time. Industries and/or product markets are viewed as entities that have historical starting points, that often have broadly similar patterns of development and ultimately disappear. Levels of entry and exit, degrees of concentration and other phenomena are shown to vary systematically within the historical time-frame of industry development. Moreover, this longitudinal evidence suggests that often (but not always) industrial evolution is punctuated by relatively sudden "shakeouts" which tend to shape the structure of the industry thereafter.

A *fourth* relevant literature is that of the "population ecology of organizations" (Hannan and Freeman, 1989; Carroll and Hannan, 1995; Carroll, 1997). Empirical work in the field is centrally concerned with explaining the variation over time

[1] For comprehensive overviews, see the Special Issues of the *International Journal of Industrial Organization*, No. 4, 1995 and of *Industrial and Corporate Change*, No. 1, 1997.

in the number of organizations undertaking a particular type of activity and their survival probabilities.

In a nutshell, the model that follows is meant to explore some generic properties of processes of industrial evolution whereby – in accordance with empirical "stylized facts" – (i) firms are heterogeneous in their technological capabilities; (ii) entry as well as exit occurs throughout the history of the industry; (iii) one observes wide variations in the rates of growth of firms, both cross-sectionally and over time; (iv) turbulence (in terms of market shares and, ultimately, of the identity of incumbent firms) is an equally persistent phenomenon; (v) a more or less prolonged "transitory" phase associated with the birth of an industry is often followed by a significantly different "mature" structure, possibly via a rather sudden endogenously generated shakeout. Moreover, (vi) such dynamic processes generally display skewed distributions of firms throughout (see discussions in Ijiri and Simon, 1974; Sutton, 1998; Dosi et al., 1995; among many others); and (vii) supply shocks do bear effect on aggregate prices, which in turn influence the opportunities of survival and growth of each firm.[2]

The model is a "baseline" one in two different senses.

In terms of microeconomic foundations, a number of important issues are resolved here by quite simple assumptions. This partly reflects the fact that the paper is in the evolutionary economics tradition, which generally abjures certain kinds of behavioral complexity (cf., Nelson and Winter, 1982). For example, imputation to individual actors of high levels of foresight and knowledge of system structure is avoided when simpler alternatives are adequate to explain aggregate phenomena, and there exists no direct empirical support for the more complex assumptions. This approach stands in sharp contrast to more mainstream economic models of competition among heterogeneous actors which accept full *ex ante* rationality of the individual actors as a fundamental modeling constraint. In our view, added rationality is added complication, and the model presented here provides a baseline that will permit an assessment of the incremental explanatory gain from such complications. In any case, as we shall show below, the major qualitative properties of the model hold also e.g. when one allows more sophisticated responses of potential entrants to perceived incentives.

Second, we do expect, however, that some of the simple assumptions will require elaboration and modification in future work if the model is to be brought into reasonable correspondence with reality: for some explorations in this spirit, cf. Winter et al. (2000). Hence, the model is a baseline not merely in the sense of a standard for comparison, but also as a starting point for further work. We anticipate that many of the results developed here will have at least heuristic value, if not direct application, in such future work.

In particular, in the following we study the properties of that special case of evolutionary dynamics whereby technological heterogeneity is bound from the start to some fixed menu of efficiency levels. One may conclude that this model is as "evolutionary" in its spirit as "evolutionary games". An obvious extension,

[2] For the purposes of this work, it is not crucial to know whether all price/quantity fluctuations are due to supply shocks (for an appraisal of the problem, see e.g., Judd and Trehan, 1995). It is enough that, at least, *part of them are*, as indeed plausible.

straight in the evolutionary spirit, is to allow for "open-ended" dynamics whereby both entrants and incumbents continuously learn and discover along the way novel techniques.[3] This is indeed what we have begun to do in the mentioned companion paper (Winter et al., 2000; see also Bottazzi et al., 2001) where one studies industry evolution driven by an expanding set of technological opportunities which entrants progressively tap.

Indeed, we conjecture, the modeling framework presented here might bear fruitful applications well beyond industrial dynamics to a few domains – including some of those currently addressed by evolutionary games – whereby populations of heterogeneous agents search, adapt and compete in partly unknown environments.

After introducing the spirit of the model with reference to a critical discussion the existing literature (Sect. 2), we present its basic structure (Sect. 3) and consider some important generalizations which can be treated with the same technique and do not affect the main qualitative conclusions (Sect. 4). Next, we develop in Section 5 two dynamic settings, namely a first one which analyses the dynamics of productive capacity associated with different efficiency levels, and a second one which, conversely, follows the fate of all individual firms appearing throughout the whole dynamic path.

Our model entails a stochastic system driven by the persistent random arrival of new firms, on the one hand, and on a systematic selection process linking investments (and ultimately survival) to realized profitabilities, on the other. Some properties of this system are analyzed in Sections 6 and 7, with respect to its "laws of motion" and to the time-averages of aggregate statistics such as the productive capacities and the numbers of firms in business associated with different efficiency levels.

These analytical results are followed in Section 8 by a computer simulation of the model, showing among other things the dynamics in the number, size and age of firms. The Appendix provides mathematical proofs of the main formal propositions of this work.

2 A preliminary view

The idea of the competitive process held here in its essence dates back quite a long time: indeed, it is quite germane to the view of competition retained by classical economists, and, later, in diverse fashions, by Marshall and Schumpeter. Just think for example of the classical view of prices and profits as attracted to their "normal" levels by inflows/outflows of investment, or think of the famous Marshallian metaphor of industries as "forests" with young, mature and dying trees. However, the static bias of a lot of contemporary work has also meant the neglect of these early intuitions. This applies to a good extent to traditional industrial economics too. For example, while it is true that in the "Structure-Conduct-Performance" (SCP) paradigm entry and entry barriers play a prominent role, it is equally true that the

[3] A model in this perspective, albeit explored only with simulation techniques is in Dosi et al. (1995).

analysis, if not entirely static is at least ahistorical: dates and sequences of events have no visible importance.[4]

Only in recent years, a few formal models have tried to address some of the dynamic "stylized facts" recalled above. So, for example, Jovanovic (1982) tries to model industry dynamics whereby heterogeneous agents are initially uncertain about their own efficiencies. As time goes on, the latter are estimated via stochastically independent observations. That work looks for a perfect foresight equilibrium whereby a generic firm adjusts its optimal production plan (including its optimal exit time) given its revealed performances at any t. Hence, the model is meant to yield an (equilibrium) path of "noisy selection" whereby above-average firms expand and below-average ones shrink and eventually die. In a similar spirit, Hopenhayn (1992) attempts to model firm-specific productivity shocks which follow a Markov process.

The main interpretative focus of the foregoing two formalizations is the "stylized fact", mentioned above, concerning the variability of growth rates and survival probabilities (empirically observed to be often dependent also on the size and age of firms themselves).[5] The model of Ericson and Pakes (1995) treats the same phenomena within the setting of a class of Markov-perfect Nash equilibria (in principle able to account also for persistent *turbulence*, i.e. "stylized fact (iv)", above). Firms are supposed to evolve according to their relative efficiency or "state of success", itself dependent upon exogenous drifts in competitive opportunities and endogenous (resource-expensive) stochastic changes in the 'pecking-order' of both incumbents and entrants.

Although these attempts (and others in a similar spirit) to rationalize some aspects of the observed dynamics of industries ought to be considered welcome developments, they do, however, display, in our view, some noticeable drawbacks, to a large extent due to the self-inflicted burden of rationalizing industrial dynamics as equilibrium paths of some kind microfounded upon sophisticated forward-looking agents. All this carries serious problems of both economic plausibility and, together, of mathematical coherence.

Take the model in Jovanovic (1982) (a similar argument applies to the model by Hopenhayn, 1992). There is a continuum of firms which are heterogeneous in that each is an independent realization of a representative firm. Their outputs – notwithstanding the fact that firms are verbally assumed to be infinitesimal – are i.i.d. nonnegative random variables. Consequently, the total output of the industry, as a sum of infinitely many i.i.d. nonnegative shocks, must be infinite with probability one at every time instant. Thus, the claim of this work of a finite deterministic equilibrium output, necessary for the model to hold, remains puzzling.[6]

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[4] For example, first moves advantages do not appear among the basic determinants of industry structure in the definitive account of the SCP paradigm, namely Scherer and Ross (1990); this applies even more so to the treatment of competition in standard textbooks on industrial organization: see, for example, Carlton and Perloff (1999).

[5] Other works addressing similar issues, include Lambson (1991), Jovanovic and MacDonald (1994), and Klepper (1996) which, in quite different veins, model some interaction between sunk costs and time-dependent learning also as the cause of 'shakeouts' (i.e. 'stylized facts (v)' above).

[6] In Kaniovski (2000) one identifies the implicit route to get a finite total supply in this setting, i.e. taking weighted normalizations of individual outputs rather than simple sums. The underlying idea is

The demands on individual rationality and the corresponding requirements on *ex ante* coordination among agents are even more stringent in Ericson and Pakes (1995). In short, a necessary condition for the model to apply is the collective identity of expectations about both drifts in competitive opportunities and investment plans. Unfortunately, this approach does not guarantee the optimality of individual trajectories. In its essence, the problem is as follows.

The collective dynamics used in the incumbent's optimization problem keeps a record of the number of firms occupying each feasible efficiency level. As such, it is not concerned with what happens to a firm upon exiting the industry. On the other hand, the law of motion of a firm must explicitly specify how the firm quits the industry. Ericson and Pakes (1995) states that the optimal path of the industry, a rational expectations equilibrium, obtains by aggregating the individual trajectories derived by solving the incumbent's optimization problem. However, as an aggregate of individual motions, the optimal path must specify what happens to any one firm quitting the industry. As discussed in much greater detail in Kaniovski (2001), it turns out that the model does not guarantee the consistency between "optimal" individual trajectories and the industry aggregate, which should, of course, in equilibrium, sum upon the former. It is a problem rather common amongst models purportedly nested upon rational expectation agents, but it is particularly demanding in set-ups whereby agents are also allowed to leave the modeled environment. (In fact, as one shows in Kaniovski, 2001, all "dead entities" are implicitly assumed to keep evolving alike living ones.)

Here, we take a quite different route and focus upon the properties of competitive processes where no *ex ante* consistency is assumed among individual expectations and behaviours. Hence markets *ex post* select among heterogeneous agents, whose stochastic entry to the market resembles much more the messy process described in the empirical literature.[7] In turn, the selection process is itself dependent on the aggregation of individual decisions, hence closing the feedback loop linking microeconomic heterogeneity, the dynamic path of the industry and individual growth/survival opportunities. The methodology that we shall follow in the modeling exercise below is somewhat similar in spirit to that advocated by the "bounds approach" pioneered by Sutton (1998), notwithstanding rather different building blocks. Both perspectives share the search for "a small number of mechanisms that appear to operate in a systematic way across the general run of industries [and whose] operation induces a number of bounds on the set of outcomes that we expect to observe in empirical data" (Sutton, 1998, p. 5). Hence, for example, Sutton establishes the lower bound of the skewedness of size distributions simply

germane to the attempts by e.g. Judd (1985) or Feldman and Gilles (1985) to formally validate the measurability of a total supply stemming from a continuum of independent individual outputs. So, as Judd (1985) puts it "the law of large numbers with a continuum of random variables is not inconsistent with basic mathematics" (p. 24.), However, as one argues in Kaniovski (2000), such a use of the law of large numbers in Jovanovic (1982) turns out to be rather at odds with economics. Moreover, it appears that Jovanovic (1982) interchangingly uses two notions for the output of a firm. One is a solution of the optimization problem of a price-taker. (Who, due to unspecified reason, operates with a generic inverse demand curve, rather then with a horizontal one.) The other is a measure of the index by which the firm is marked.

[7] See, for example, Aldrich and Fiol (1994) and Dosi and Lovallo (1997).

by isolating the effects of purely statistical independence effects in the growth and entry process. Here, we follow a similar strategy and try to identify some generic properties of a stochastic process of entry together with a systematic mechanism of selection operating via the impact of profit margins upon investment opportunities and ultimately survival probabilities. However, unlike Sutton we do not constrain the set of outcomes to those which fulfill some rationality criterion (i.e. entrants must cover their costs and they are rational enough in this entry decisions for this to happen), and some non-arbitrage principle (saying more or less that all profit opportunities are exploited). On the contrary, we shall explore the characteristics of the dynamic profile of industries wherein these criteria may be systematically violated.

3 The basic model

Consider an industry evolving in discrete time $t = 0, 1, \ldots$. At time $t = 0$ there are no firms ready to manufacture, but a random number of firms is drawn which will start manufacturing at $t = 1$. At time $t \geq 1$ the industry consists of n_t firms which are involved in manufacturing and new firms which enter at t and will be involved in manufacturing from $t + 1$ onward. Uniformly for the whole industry we have:

m: variable (marginal $\equiv$ average) costs per unit of output, $m > 0$, which, for simplicity, we assume for the time being identical across firms;

v: price per unit of physical capital, $v > 0$;

d: the "physical" depreciation rate, $0 < d \leq 1$.

The competitiveness of a firm represented in the industry is determined by (the inverse of) its capital per unit of output. Let us designate it by a_i for the i-th firm. The variable a_i takes a finite number of values $A_1 < A_2 < \ldots < A_k,\ k \geq 1$. A particular value is randomly assigned to a firm when it enters the industry.

The productive capacity (which is fully utilized) of the i-th firm is $Q_t^i = K_i(t)/a_i$, where $K_i(t)$ stands for the capital of the i-th firm at time t. The total productive capacity of the industry involved in manufacturing at time t is

$$Q_t = \sum_{i=1}^{n_t} Q_t^i.$$

(We set that the sum where the lower index exceeds the upper one equals to zero.) There is a decreasing continuous inverse demand function $p = h(q)$, mapping $[0, \infty)$ in $[0, h(0)]$ such that $h(0) < \infty$ and $h(q) \to 0$ as $q \to \infty$. As usual, p stands for the price and q for the demand. Thus, the market clearing price at time t is given as $h(Q_t)$. The gross profit per unit of output at t is $h(Q_t) - m$. Hence, the gross investment per unit of output at t reads $\lambda \max[h(Q_t) - m, 0]$, where $0 < \lambda \leq 1$. The constant λ captures the share of the gross profit which, in our "partial disequilibrium" model, does not leak out as the interest payments and shareholders' dividends. It can be considered as a measure for the propensity to invest. Then the total gross investment per unit of capital for the i-th firm at time t is $I(Q_t)/a_i$, where for $x \geq 0$

$$I(x) = \frac{\lambda}{v} \max[h(x) - m, 0] \geq 0.$$

We shall allow multiple entrants into the industry. Capitals of new entrants take values from the interval $[b, c]$, $0 < b < c < \infty$. A particular value is randomly assigned to a firm when it enters the industry. We postulate that the initial capitals are independent realizations of a random variable θ distributed over $[b, c]$. That is, the distribution of initial physical capital endowments do not depend upon the would-be competitiveness of entrants.

At each time t a random number of entrants is allowed for each level of capital per unit of output. Again, the distribution of the number of entrants does not depend upon their competitiveness (see however the extensions of the model below). It is given by an independent realization of a random variable γ taking the values $0, 1, \ldots, l$, where l is a positive integer ($P\{\gamma = l\} > 0$). The number of entrants at time t that have the j-th level of capital per unit of output is given by the j-th coordinate Γ_j^t of a k-dimensional random vector Γ^t. The vectors Γ^t are observations of Γ independent in t. Each coordinate of Γ is an independent realization of γ. The initial capitals of new entrants at time t with the j-th level of capital per unit of output are given by Γ_j^t independent realizations $\theta_{j,i}^t$, $1 \le i \le \Gamma_j^t$, of θ if $\Gamma_j^t > 0$ and are equal to 0 if $\Gamma_j^t = 0$.

Set Θ^t to be a k-dimensional vector such that

$$\Theta_j^t = \sum_{i=1}^{\Gamma_j^t} \theta_{j,i}^t.$$

Thus, Θ_j^t represents the total inflow of capital at t of firms with A_j as capital per unit of output. The random variables Γ_j^t and $\theta_{p,i}^s$ are independent for all possible combinations of indexes. Two remarks are in order. First, note that the random variables Γ and θ can be given a straightforward economic interpretation in terms of technological opportunities to potential entrants, scale of entry and entry barriers. (Indeed computer-simulated versions of the model allow experiments with different distributions intuitively corresponding to empirically diverse "entry regimes".) Second, for simplicity, we treat here the stochastic entry process as entirely exogenous – in particular entry does not depend on past or present industry profitability. The point of this assumption is not the affirmative claim that *all* entry is independent of profitability, but that *some* entry is (possibly more so in the vicinity of that notional equilibrium where on average firms earn zero net profits). Many models of rational entry under uncertainty, would have that feature, too. The principal qualitative result of our analysis *are not affected* by the addition of a layer of profit-dependent entry, though the quantitative results certainly are. Moreover, one can easily endogenize the entry process, and in fact in the next section we shall describe a modification of the model where the distribution of the number of entrants depends on the current profit margins.

To complete the description of this competitive environment we need some death mechanism. A firm is dead at time t and does not participate in the production process from $t+1$ onward if its capital at t is less than ϵb, $\epsilon \in (0, 1]$. A setup without mortality can be thought of as a limit case of this threshold when $\epsilon = 0$.

Given this setting, all random elements are defined on a probability space $\{\Omega, \mathcal{F}, P\}$.

4 Generalizations of the model

Let us consider some generalizations of the basic model presented above, which can be essentially studied by the same analytic means, without affecting the major qualitative conclusions.

1. On purpose, in the foregoing basic setting we have kept the behavioral assumptions to a minimum. In that vein, we have ruled out also any feedback from profitability to investment rules. However the qualitative properties of the model would not be affected if one allowed the rate of investment to fall when the price gets close to variable costs per unit of output. In that case one would just set gross investment per unit of output at t as

$$\lambda \max\left[h(Q_t) - m, 0\right] \quad \text{if} \;\; h(Q_t) - m > \delta$$

and

$$(\lambda - \eta) \max\left[h(Q_t) - m, 0\right] \quad \text{if} \;\; h(Q_t) - m \leq \delta,$$

where $\lambda \in (0, 1], \quad \eta \in [0, \lambda)$. Here δ gives a threshold of profitability when the investment policy changes.

2. The death levels may be different for firms with different efficiencies, say ϵ_i for all firms with A_i as capital per unit of output. At the same time the death criterion could be dependent on the total productive capacity at time t. That is, the i-th firm is dead at time t and does not participate in the evolution of the market from t onward if its productive capacity Q_t^i at t is less than ϵQ_t, where $\epsilon \in (0, 1)$ is some threshold value.

3. One can allow also variable costs to vary across firms. Assume that there are $n > 1$ possible levels of variable costs per unit of output $m_j, \; j = 1, 2, \ldots, n$. Assuming as above multiple entrants, one may postulate that at time $t \geq 0$ the number of newcoming firms which have A_i as output/capital ratio and m_j as variable costs is given by $\Gamma^t_{n(i-1)+j}$, that is the $n(i-1) + j$-th coordinate of Γ^t. Here $\Gamma^t, \; t \geq 0$, are independent realizations of a $n \times k$ dimensional vector Γ whose coordinates are independent realizations of the random variable γ defined in Section 3. Initial capital endowments of these firms are given by $\Gamma_{n(i-1)+j}$ independent realizations of the random variable θ. Moreover, these initial physical capitals for different time instants are independent and they do not depend either upon the number of newcoming firms.

4. Initial capitals of newcoming firms may be specific for a class of firms with a given capital/output ratio and given variable costs. Thus, instead of a single random variable θ, we may consider a collection of them, $\theta_{i,j}, \; 1 \leq i \leq k, \; 1 \leq j \leq n$. (Each of these variables is assumed to have a bounded support.) Similarly, the number of entrants can be specific for a class of firms with given capital/output ratio and variable costs.

5. One can make the distribution of the number of newcomers path dependent. In particular, it may depend upon how close are the current price and the marginal

costs of the firm which is deciding whether to enter or not. In fact, let $\phi_j(\cdot)$ be a decreasing function mapping $[0,\infty)$ to $(0,1]$, $j = 1,2,\ldots,n$. For example, $\phi_j(x) = \Phi_j \exp(-\phi_j x)$, where Φ_j and ϕ_j are positive constants, $\Phi_j \in (0,1]$. Given some distribution $p_s^{(i,j)}$, $s = 1,2,\ldots,l(i,j)$, the random variable $\gamma_{i,j}^t(Q_t)$ governing the number of firms with the output/capital ratio A_i and the variable costs m_j that enter the market at t can be as follows

$$\gamma_{i,j}^t(x) = \begin{cases} 0 & \text{with probability } \psi_j(x), \\ s & \text{with probability } p_s^{(i,j)}[1-\psi_j(x)], \end{cases}$$

where $\psi_j(x) = \phi_j(\max[h(x) - m_j, 0])$. By $l^{(i,j)}$ we denote the maximum feasible number of entrants with A_i as the capital per unit of output and m_j as variable costs. The random variables $\gamma_{i,j}^t(x_j)$ for any deterministic x_j are assumed to be stochastically independent in each of the indexes.

With this specification, $\phi_j(0) = 1$ implies that, when production is not profitable for firms with variable costs m_j, they will not enter the industry. Alternatively, $\phi_j(0) < 1$ means that, even when production is not profitable for them, some firms whose costs are m_j will be entering the industry. Thus, by assigning different values to $\phi_j(0)$, one can comply either with the "rational" assumption of no entry when the profit is nonpositive, or with a more realistic and empirically founded view that firms may well keep entering even though the instantaneous profitability is not positive.

In the following we shall study the long run behavior of the simplest variant of this industry, beginning with a formal description of its evolution.

5 Two dynamical settings

Set $Q_1^{A_j} = A_j^{-1}\Theta_j^0$ for the total productive capacity of those firms having A_j as capital per unit of output which are involved in manufacturing in the first time period, $j = 1,2,\ldots,k$. These firms perform the first cycle of production and new firms come in the industry. As the result, by the end of the first time period the productive capacity $q_1^{A_j}$ of firms having A_j as capital per unit of output reads

$$q_1^{A_j} = Q_1^{A_j}[1 - d + I(Q_1)A_j^{-1}] + V_1^j, \quad j = 1,2,\ldots,k,$$

where

$$Q_1 = \sum_{j=1}^{k} Q_1^{A_j}, \quad V_1^j = A_j^{-1}\Theta_j^1.$$

Conceptually, V_1^j is the total inflow of productive capacity of firms having A_j as capital per unit of output during the first time period. Q_1 stands for the total productive capacity involved in manufacturing during the first time period. Not all of the firms which manufactured during the first production cycle remain in the industry during the second time period. Those that have shrunk below the minimum threshold have to leave. Denote by $\mathcal{E}_1^j \geq 0$ the total outflow of productive capacity at time 1 of firms having A_j as capital per unit of output. Then the total productive capacity $Q_2^{A_j}$ of firms which are ready to produce during the second time period reads

$$Q_2^{A_j} = q_1^{A_j} - \mathcal{E}_1^j = Q_1^{A_j}[1 - d + I(Q_1)A_j^{-1}] + V_1^j - \mathcal{E}_1^j.$$

In the same way we get

$$Q_{t+1}^{A_j} = Q_t^{A_j}[1 - d + I(Q_t)A_j^{-1}] + V_t^j - \mathcal{E}_t^j, \quad t \geq 1. \tag{1}$$

Here $Q_t^{A_j}$ stands for the total productive capacity of those firms having A_j as capital per unit of output which manufacture during t-th production cycle. V_t^j denotes the total inflow of productive capacity of firms having A_j as capital per unit of output at time t, that is, $V_t^j = A_j^{-1}\Theta_j^t$, and $\mathcal{E}_t^j$ stands for the total outflow of productive capacity of such firms at time t due to mortality. By Q_t we denote the total productive capacity of firms involved in manufacturing at time t, that is,

$$Q_t = \sum_{j=1}^{k} Q_t^{A_j}, \ \ t \geq 1.$$

Taking into account (1), we see that this value evolves as

$$Q_{t+1} = Q_t(1 - d) + I(Q_t)\sum_{j=1}^{k} A_j^{-1}Q_t^{A_j} + V_t - \mathcal{E}_t, \quad t \geq 1, \tag{2}$$

where V_t represents the total inflow of productive capacity at time t, that is,

$$V_t = \sum_{j=1}^{k} V_t^j,$$

and $\mathcal{E}_t$ stands for the total outflow of productive capacity at time t due to mortality,

$$\mathcal{E}_t = \sum_{j=1}^{k} \mathcal{E}_t^j.$$

The random process given by (1) and (2) is not as such a Markov process. (However, it turns out to be one, if there is no death rule, and, hence, firms may shrink indefinitely but do not exit the industry.)

Note also that this setting does not account for the fate of any individual firm. Let us consider an alternative, explicitly microfounded, representation.

Since only the entry process of the model is stochastic, the state of the industry at any time t is determined given the detailed history of entry and output through $t - 1$ and the stochastic events of t. Further, the output history of the system to any t can be computed recursively on the basis of prior output history and current stochastic entry. Although only finitely many firm output levels are relevant up to any particular t, a full realization of the process involves an infinite number of firm histories. At any time, the part of the output history that has not happened yet is represented by an infinite list of zeroes; zeroes may also appear in the firm-specific output history because the corresponding firm has died. It is convenient for the representation to make room for every possible firm that could come into being; this means that zeroes also appear in the feasible output realizations at any given time because less than the maximum possible number (l) of firms may have entered in some previous periods.

With this picture in mind, let us introduce an infinite dimensional space R^{∞} of vectors with denumerably many coordinates. It is instructive to split the evolution of the industry in the dynamics of age cohorts. (Note that this is not necessary. Formally, the story remains the same regardless of whether one talks about cohorts or not. However, conceptually, the approach becomes much more transparent by using the language of cohorts.) By a cohort one means all firms that enter the industry at the same period. Then R^{∞} splits in infinitely many identical blocks, kl-dimensional real vector spaces R_i^{kl}, corresponding to age cohorts. (Recall that k denotes the total number of possible levels of capital/output ratios, while l stands for the maximum number of entrants for each of these levels. Consequently, one may observe up to kl firms in an age cohort.) Alternatively one may say that R^{∞} is the Cartesian product of $R_i^{kl},\ i \geq 1$. In formal terms,

$$R^{\infty} = \Pi_{i=1}^{\infty} R_i^{kl}.$$

Thus, for every $\mathbf{q} \in R^{\infty}$

$$\mathbf{q} = \Pi_{i=1}^{\infty} \mathbf{q}^i,$$

with $\mathbf{q}^i \in R_i^{kl}$. That is, the infinite output history $\mathbf{q}$ (which is the history of productive capacities as well, because firms fully utilize them) may be regarded as partitioned into vectors $\mathbf{q}^i$ of dimension kl, each of which may be be thought of as output levels (productive capacities) of a specific age cohort, i. (Here, as noted above, we "make room" in the notation for the outputs of firms that may not exist in a particular realization because less than the maximum possible number of entrants appeared in that cohort.) The notational convention adopted is that firms are numbered within types and within cohorts. Thus, for example, the firms of the third cohort, that is, those that have produced twice, are numbered from $2kl + 1$ to $3kl$. In a realization of the process, the deterministic part of the output change from period to period can be represented as follows.

As a first step in characterizing the evolution of the industry in terms of age cohorts, we define the transformations $\mathbf{D}^i(\cdot),\ i \geq 1$, on R^{∞}. In fact, the mapping $\mathbf{D}^i(\cdot)$ determines one-step changes of productive capacities of firms involved in the i-th age cohort. Consequently, $\mathbf{D}^i(\cdot)$ transforms productive capacities involved in R_i^{kl} into productive capacities accommodated by R_{i+1}^{kl}. According to our production dynamics, in terms of coordinates, $\mathbf{D}^i(\cdot)$ is given by the following relations:

$$D^i_{(j-1)l+p}(\mathbf{q}) = q^i_{(j-1)l+p}[1 - d + I\left(\sum_{i=1}^{\infty}\sum_{s=1}^{kl} q_s^i\right) A_j^{-1}]\chi_{L^i_{(j-1)l+p}(\mathbf{q})}.$$

Here $D_s^i(\cdot)$ stands for the s-th coordinate of $\mathbf{D}^i(\cdot)$, $\mathbf{q}$ is a state of the industry described by productive capacities of firms involved, $1 \le j \le k$, $\ 1 \le p \le l$, and $i \ge 1$. That is, i denotes the age of a cohort, j accounts for the level of capital costs, A_j, specific to the firm, and p is the order number assigned to this firm in the pool of all alive entities in the i-th age cohort whose capital/output ration is A_j. We restrict ourselves to vectors $\mathbf{q}$ with nonnegative coordinates and set $I(\infty) = 0$ for the case when the iterated sum involved in the above expression is infinite. Moreover, $L^i_{(j-1)l+p}(\mathbf{q})$ stands for the relation

$$q^i_{(j-1)l+p}[1 - d + I\left(\sum_{i=1}^{\infty}\sum_{s=1}^{kl} q_s^i\right) A_j^{-1}]A_j \ge \epsilon b.$$

It represents the condition under which a firm whose productive capacity at time i is $q^i_{(j-1)l+p}$ survives to period $i+1$, given that at i the productive capacities involved in manufacturing are described by the vector $\mathbf{q} \in R^\infty$.

$\chi_{L^i_{(j-1)l+p}(\mathbf{q})}$ is the indicator function of the relation $L^i_{(j-1)l+p}(\mathbf{q})$. We set that for a relation $\mathcal{A}$

$$\chi_{\mathcal{A}} = \begin{cases} 1, & \text{if } \mathcal{A} \text{ is true,} \\ 0, & \text{otherwise.} \end{cases}$$

The indicator function involved in the definition of $D^i_{(j-1)l+p}(\cdot)$ serves for the following purpose. Consider at time (age) $i \ge 1$ an alive firm having A_j as capital per unit of output. Let its productive capacity be $q^i_{(j-1)l+p} > 0$ (since it is alive). The question is whether it will be participating in the next production cycle or not. According to our mortality rule, it depends upon whether its capital at the end of the current production period is not less than or falls below the death threshold, ϵb. The investment rule adopted in the model gives

$$q^i_{(j-1)l+p}[1 - d + I\left(\sum_{i=1}^{\infty}\sum_{s=1}^{kl} q_s^i\right) A_j^{-1}]$$

for its production capacity at the end of the current production cycle, or, in capital terms,

$$q^i_{(j-1)l+p}[1 - d + I\left(\sum_{i=1}^{\infty}\sum_{s=1}^{kl} q_s^i\right) A_j^{-1}]A_j.$$

Hence, the firm survives and continues its production if this expression for capital is not less than ϵb. Otherwise, the firm dies and never returns to business.

We set $\mathbf{D}(\cdot)$ for the deterministic mapping R^∞ in R^∞ characterizing the deterministic evolution of productive capacities of all manufacturing firms of the industry. In terms of coordinates, this transformation is given by the following relations: $D_s(\mathbf{q}) = 0$ for $s = 1, 2, \dots, kl$, $D_{kli+(j-1)l+p}(\mathbf{q}) = D^i_{(j-1)l+p}(\mathbf{q})$ for $i \ge 1$, $1 \le j \le k$, $1 \le p \le l$. Here $D_m(\mathbf{q})$ denotes the m-th coordinate of $\mathbf{D}(\mathbf{q})$. Note that the first kl coordinates equal zero to accommodate newcomers, which currently do not produce: thus their productive capacities do not change and, consequently, they are not a subject to the dynamics captured by $\mathbf{D}(\cdot)$. The inflow process is given in the following way.

Define infinite dimensional random vectors $\mathbf{Y}^t,\ t \geq 0$, independent in t. Set

$$Y^t_{(j-1)l+p} = \theta^t_{j,p} A_j^{-1} \ \text{ for } \ p = 1, 2, \ldots, \Gamma^t_j,$$

$$Y^t_{(j-1)l+\Gamma^t_j+i} = 0 \ \text{ for } \ i = 1, 2, \ldots, l - \Gamma^t_j$$

if $\Gamma^t_j > 0$ and

$$Y^t_{(j-1)l+p} = 0 \ \text{ for } \ p = 1, 2, \ldots, l$$

if $\Gamma^t_j = 0$, also

$$Y^t_s = 0 \ \text{ for } \ s > kl.$$

Here $j = 1, 2, \ldots, k$ (note that in this case, as well as when defining $\mathbf{D}(\cdot)$, coordinates are numbered sequentially rather than in terms of cohorts. The coordinates of the state variable of the industry, $\mathbf{q}$, are numbered in terms of cohorts). The vector $\mathbf{Y}^t$ characterizes the inflow of productive capacity into the industry at time t. Now, the evolution of productive capacities of all firms involved in the economy can be given as follows

$$\mathbf{q}(t+1) = \mathbf{D}(\mathbf{q}(t)) + \mathbf{Y}^t,\ t \geq 1, \ \ \mathbf{q}(1) = \mathbf{Y}^0. \tag{3}$$

Since $\mathbf{Y}^t$ are independent in t, this expression defines a Markov process on R^∞. Also, since the deterministic operator $\mathbf{D}(\cdot)$ as well as the distribution of $\mathbf{Y}^t$ do not depend on time, the process is homogeneous in time. Conceptually, this phase space is formed by productive capacities of all firms which stay alive. More precisely, if $q^i_{(j-1)l+p}(t) > 0$ for some $p = 1, 2, \ldots, l$ and $t > i \geq 1$, then a firm with A_j as capital per unit of output which came to the industry at $t - i$ has been alive until t, that is, has manufactured $i - 1$ times, and continue to manufacture during the t-th time period.

Having outlined the specific features of this process of industrial change, let us proceed to the analysis of some of its properties.

6 Entry, mortality and long run balance relations

Start from the statement that, for any j, in a finite time with probability one there will be born at least one firm with A_j as capital per unit of output.

Lemma 1. *For each $j = 1, 2, \ldots, k$ in a finite random time τ_j with probability one there appears a firm with A_j as capital per unit of output.*

The proof of this lemma is given in the Appendix.
Now let us show that $Q_t,\ t \geq 1$, are bounded with certainty.

Lemma 2. *With certainty* $Q_t \leq Q_*$ *for* $t \geq 1$, *where* Q_* *is a constant depending upon parameters of the model.*

Proof. Dropping the nonpositive term, $-\mathcal{E}_t$, in (2), we get

$$Q_{t+1} \leq Q_t(1-d) + I(Q_t)\sum_{j=1}^{k} A_j^{-1} Q_t^{A_j} + V_t,\ t \geq 1. \tag{4}$$

Since $A_j^{-1} < A_1^{-1},\ j = 2, 3, \dots, k$, we conclude that

$$\sum_{j=1}^{k} A_j^{-1} Q_t^{A_j} \leq A_1^{-1} \sum_{j=1}^{k} Q_t^{A_j} = A_1^{-1} Q_t$$

and

$$V_t \leq lc \sum_{j=1}^{k} A_j^{-1} < lckA_1^{-1}.$$

Substituting these two estimates in (4), one has

$$Q_{t+1} < Q_t[1 - d + I(Q_t)A_1^{-1}] + lckA_1^{-1}. \tag{5}$$

Since $I(x) \to 0$ as $x \to \infty$, there is a finite output level $\bar{Q}$ such that

$$Q[I(Q)A_1^{-1} - d] + lckA_1^{-1} < 0 \quad \text{for} \ \ Q \geq \bar{Q}.$$

If $Q_t \geq \bar{Q}$ for some $t \geq 1$, then $Q_{t+1} < Q_t$ by (5). Thus, at $\bar{Q}$ and above this level, the sequence $Q_t,\ t \geq 1$, becomes decreasing. Otherwise, if $Q_t < \bar{Q}$ for some $t \geq 1$, by (5) we get

$$Q_{t+1} < Q_t[1 - d + I(0)A_1^{-1}] + lckA_1^{-1} <$$

$$\bar{Q}[1 - d + I(0)A_1^{-1}] + lckA_1^{-1} = \hat{Q}.$$

(Take into account that $I(x) < I(0)$ for all $x > 0$.) When $\hat{Q} < \bar{Q}$, this inequality implies that $Q_t < \bar{Q}$ for $t \geq 1$. Otherwise, when $\hat{Q} \geq \bar{Q}$, it means that $Q_t < \hat{Q}$ for $t \geq 1$. (Indeed, upon upcrossing the level $\bar{Q}$, this sequence is bounded by $\hat{Q}$. But at $\bar{Q}$ and above this level, it is a decreasing sequence.) Summing up, one may conclude that there is a finite deterministic upper bound Q_* of $Q_t,\ t \geq 1$. Moreover, $Q_* < \max(\bar{Q}, \hat{Q})$.

The lemma is proved.

Let us now show that, if there is a death threshold, then none of the firms can survive for an infinitely long time. The argument given below is instructive demonstrating how the selection mechanism in question works.

Theorem 1. *If* $\epsilon > 0$ *and* $P\{\gamma = 0\} = p_0 < 1$, *then each firm dies in a finite random time with probability one.*

The theorem is proved in the Appendix. The intuition is the following. For simplicity let $\epsilon < 1$ and $p_0 = 0$. (If $\epsilon = 1$ and/or $p_0 > 0$, we need a more sophisticated argument.)

Each firm enters with a physical capital which cannot be below b. If it dies, at the moment when this happens its capital must be less than ϵb. That is, a firm living a finite time must shrink during its life at least $1/\epsilon$ times. The intuitive proof (by contradiction) of the theorem starts by the observation that at least some firms, including some of the most efficient ones do die. Consequently, taking into account that the most efficient firms have the highest investment rate, a notional firm that would live infinitely long would shrink at least $1/\epsilon$ times during the life time of a most efficient one, if the latter dies in a finite time. Hence, to prove that no firm can live infinitely long, it is enough to show that the capital of every alive firm is bounded from above by a constant and that there is a long enough (depending upon the constant) or, better, an infinite chain of the most efficient firms coming and dying one after another.

The capital of an alive firm is bounded from above by the total physical capital of the industry which in turn is bounded with certainty by Lemma 2. Moreover, the capital of an alive firm is bounded from below by the death threshold, ϵb. Hence, the total number of alive firms is bounded from above with certainty by $Q_* A_k/\epsilon b$. Consequently, starting from a finite random time τ every newcoming firm dies in a finite time. Since $p_0 = 0$, at least one the most efficient firm comes every time instant. Thus, starting from τ, there is an infinite chain of the most efficient firms coming and dying one after another. This chain will push below the death threshold the physical capital of any candidate for living infinitely long.

The theorem is proved.

From (1) we get that

$$Q_{t+1}^{A_j} - Q_t^{A_j} = Q_t^{A_j}[I(Q_t)A_j^{-1} - d] + V_t^j - \mathcal{E}_t^j.$$

This implies for $n \geq 1$

$$Q_{n+1}^{A_j} - Q_1^{A_j} = \sum_{t=1}^{n} \left\{ Q_t^{A_j}[I(Q_t)A_j^{-1} - d] + V_t^j - \mathcal{E}_t^j \right\}. \tag{6}$$

Since $Q_t^{A_j}$, $t \geq 1$, are bounded with certainty, then by (6) we conclude that (with certainty)

$$\frac{1}{n}\sum_{t=1}^{n} \left\{ Q_t^{A_j}[I(Q_t)A_j^{-1} - d] + V_t^j - \mathcal{E}_t^j \right\} \to 0 \tag{7}$$

as $n \to \infty$. Due to the strong law of large numbers, with probability one

$$\frac{1}{n}\sum_{t=1}^{n} V_t^j \to A_j^{-1} E\gamma E\theta \tag{8}$$

as $n \to \infty$. Here by $E\gamma$ and $E\theta$ we denote the mean values of γ and θ correspondingly. By (7) and (8) we conclude the following.

Lemma 3. *For $j = 1, 2, \dots, k$ with probability one*

$$\frac{1}{n}\sum_{t=1}^{n}\left\{Q_t^{A_j}[d - I(Q_t)A_j^{-1}] + \mathcal{E}_t^j\right\} \to aA_j^{-1} \tag{9}$$

as $n \to \infty$. Here $a = E\gamma E\theta$, $j = 1, 2, \dots, k$.

Summing up relations (9), we get that with probability one

$$\frac{1}{n}\sum_{t=1}^{n}\left[dQ_t - I(Q_t)\sum_{j=1}^{k} Q_t^{A_j}A_j^{-1} + \mathcal{E}_t\right] \to a\sum_{j=1}^{k} A_j^{-1} \tag{10}$$

as $n \to \infty$. Relations (9) and (10) represent the most general long-run balance equations for the productive capacities involved in the market. They imply that

$$Q_t^{A_j}[d - I(Q_t)A_j^{-1}] + \mathcal{E}_t^j - aA_j^{-1},\ j = 1, 2, \dots, k,$$

and

$$dQ_t - I(Q_t)\sum_{j=1}^{k} Q_t^{A_j}A_j^{-1} + \sum_{j=1}^{k}\mathcal{E}_t^j - a\sum_{j=1}^{k} A_j^{-1}$$

fluctuate through time in such a way that on average positive deviations of these values from zero are compensated by their negative deviations.

However, the results given by Lemma 3 do not say anything about the limit behavior of time averages of $Q_t^{A_j}$ or Q_t. To study this issue, let us turn to the ergodic properties of process (3).

7 Ergodic properties of the industry

Is there any role for "history" in determining the long-run fate of our admittedly very simple industry? In order to answer the question, let us check its possible ergodic properties.

Define $\mathcal{B}^\infty$ the minimal σ-field in R^∞ generated by sets of the following form

$$A = \Pi_{j=1}^{\infty} A^j, \tag{11}$$

where A^j denotes a set from the σ-field of Borel sets $\mathcal{B}_j^{kl}$ in R_j^{kl}. For every such set A, one step transition probability of process (3) reads

$$p^1(\mathbf{q}, A) = P\{\mathbf{D}(\mathbf{q}) + \mathbf{Y} \in A\} = P\{\mathbf{Y}^* \in A^1\}\chi_{\mathbf{D}(\mathbf{q})\in\Pi_{i=2}^{\infty}A^i}. \tag{12}$$

Here $\mathbf{Y}^*$ stands for the kl-dimensional vector whose coordinates coincide with first kl coordinates of a generic vector $\mathbf{Y}$ having the same distribution as $\mathbf{Y}^t$, $t \geq 0$. Also, $\chi_{\mathbf{D}(\mathbf{q})\in\Pi_{i=2}^{\infty}A^i}$ is an indicator function. As explained above, it takes on value 1 or 0 depending whether $\mathbf{D}(\mathbf{q}) \in \Pi_{i=2}^{\infty}A^i$ or $\mathbf{D}(\mathbf{q}) \notin \Pi_{i=2}^{\infty}A^i$. The total productive capacity is bounded with certainty. Consequently, process (3) belongs with probability one to

$$L = \left\{ \mathbf{q} \in R^{\infty} : \quad \sum_{i=1}^{\infty} \sum_{j=1}^{kl} q_j^i \leq Q_*, \ q_j^i \geq 0 \right\}.$$

To study the ergodic properties of process (3), we need the following condition, due to Doeblin (see Doob, 1953, p. 192): there is a finite positive measure $\phi(\cdot)$ with $\phi(L) > 0$ and a number $\delta \in (0, 1)$ such that for all $\mathbf{q} \in L$

$$p^1(\mathbf{q}, A) \leq 1 - \delta \ \text{ if } \ \phi(A) \leq \delta.$$

For a set A as in (11) let $\psi(A) = P\{Y^* \in A^1\}$. Then, by (12), we conclude that $p^1(\mathbf{q}, A) \leq \psi(A)$. Hence, restricting ourselves to $\delta \leq 1/2$, we see that if $\psi(A) \leq \delta$, then $p^1(\mathbf{q}, A) \leq \delta \leq 1 - \delta$. Thus, Doeblin's condition holds for $\phi(\cdot) = \psi(\cdot)$ and all positive $\delta \leq 1/2$. (This argument is copied from Example 3 given in Doob, 1953, p. 193.)

Now, by Theorem 5.7 from Doob (1953), p. 214, we obtain that

$$\pi(\mathbf{q}, A) = \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} p^t(\mathbf{q}, A) \tag{13}$$

defines for each $\mathbf{q} \in L$ a stationary absolute distribution. Here $p^t(\mathbf{x}, \cdot)$ stands for the transition probability in t steps, that is,

$$p^t(\mathbf{q}, A) = \int_L p^{t-1}(\mathbf{y}, A) dp^1(\mathbf{q}, \mathbf{y}), \ t \geq 2.$$

The stationary distribution $\pi(\mathbf{q}, \cdot)$ turns out to be the same, that is $\pi_{\pounds}(\cdot)$, for all $\mathbf{q}$ belonging to the same ergodic set $\pounds$ (Doob, 1953, p. 210). Since $\phi(L) = 1$ and $\delta = 1/2$, there can be at most two ergodic sets (see Doob, 1953, p. 207). Because $p^1(x, \pounds) = 1$ for any $x \in \pounds$, it follows that the component of $\pounds$ belonging to R_1^{kl} must contain the support of $\mathbf{Y}^*$. As any ergodic set is unique up to a subset whose ϕ measure is zero, we conclude that *there is a single ergodic set*. Consequently, *the corresponding measure,* $\pi(\cdot)$, *is unique* as well. It has the following generic property

$$\int_L p^1(\mathbf{x}, A) d\pi(\mathbf{x}) = \pi(A). \tag{14}$$

In general, it is not possible to find from this relation an explicit expression for $\pi(\cdot)$. But, due to the specific structure of the operator $\mathbf{D}(\cdot)$, it allows to conclude that *the distribution* $\pi(\cdot)$ *nests in a finite dimensional space when* $\epsilon > 0$.

Lemma 4. *If* $\epsilon > 0$, *then there is a finite* $N < Q_* A_k / \epsilon b$ *such that, for every* A *as in (11),* $\pi(A) = 0$ *if* $A^i \neq \{\mathbf{0}^i\}$ *for some* $i > N$. *In other words, the support of* $\pi(\cdot)$ *belongs to* $\Pi_{i=1}^N R_i^{kl}$. *Here* $\mathbf{0}^i$ *denotes the zero vector of* R_i^{kl}.

Proof. Let $\mathbf{q}^*$ be a vector whose distribution is $\pi(\cdot)$. That is $P\{\mathbf{q}^* \in A\} = \pi(A)$ for every A as in (11). Equations (3) and (14) imply that the vectors $\mathbf{D}(\mathbf{q}^*) + \mathbf{Y}$ and

$\mathbf{q}^*$ have the same distribution. This observation allows to make two conclusions concerning $\mathbf{q}^*$.

First, with certainty, nonzero coordinates of $\mathbf{q}^*$ may not fall below $\epsilon b A_k^{-1}$. Indeed, the first k of them do not fall below bA_1^{-1} as initial productive capacities of firms whose capital ratio equals A_1. In general, the coordinates whose numbers are from $(j-1)l$ to jl do not fall below bA_j^{-1}. As $A_j^{-1} > A_k^{-1}$ for $j \neq k$, we conclude that the first kl nonzero coordinates of $\mathbf{q}^*$ do not fall below bA_k^{-1}. The rest do not fall below $\epsilon b A_k^{-1}$. Again, the productive capacity of a manufacturing firm whose capital ratio is A_j is bounded from below by $\epsilon b A_j^{-1}$. As $A_j^{-1} > A_k^{-1}$ for $j \neq k$, we see that $\epsilon b A_j^{-1} > \epsilon b A_k^{-1}$.

Second, with certainty, $\sum_{i=1}^{\infty} q_i^* < Q_*$. (This inequality obtains by applying to $\mathbf{D}(\mathbf{q}^*) + \mathbf{Y}$ an argument similar to the one used in Lemma 2.)

These two properties lead us to conclude that, with probability one, the number of nonzero coordinates of $\mathbf{q}^*$ falls below $Q_* A_k/\epsilon b$. Note that a nonzero element of $\mathbf{D}(\mathbf{q}^*) + \mathbf{Y}$ in a cohort block, say R_{i+1}^{kl}, by definition of $\mathbf{D}(\cdot)$, requires a nonzero element of $\mathbf{q}^*$ at the same position in R_i^{kl}. Consequently, the maximum number of cohort blocks with at least one nonzero element falls below $Q_* A_k/\epsilon b$ and these have to be subsequent cohort blocks each with a single nonzero element (at the same position within a cohort). In other words, $\pi(\cdot)$ nests with certainty in $\Pi_{i=1}^{N} R_i^{kl}$ for some $N < Q_* A_k/\epsilon b$.

The lemma is proved.

If there are distinct subsequent cyclically moving subsets, C_i, $i \geq 1$, of the ergodic set in question, then $p^1(\mathbf{q}, C_{i+1}) = 1$ for every $\mathbf{q} \in C_i$ (see Doob, 1953, p. 210.). Consequently, the component of C_{i+1} belonging to R_1^{kl} must contain a copy of the support of $\mathbf{Y}^*$. In other words, C_i and C_{i+1} coincide up to a subset whose ϕ measure is zero. In sum, *the Markov process in question does not have cyclically moving subsets.*

Since $\pi(\cdot)$ does not have cyclically moving subsets, drawing from Case (f) in Doob (1953, p. 214), we conclude that the limit in (13) exists as an ordinary (rather than Cesàro) limit. That is, taking into account that the stationary distribution is unique,

$$\lim_{t\to\infty} p^t(\mathbf{q}, A) = \pi(A) \text{ for each } \mathbf{q} \in L.$$

Let $\rho(\cdot)$ be a function measurable with respect to $\mathcal{B}^\infty$ and integrable with respect to $\pi(\cdot)$. By the strong law of large numbers (see Doob, 1953, p. 220), as $n \to \infty$,

$$\frac{1}{n}\sum_{t=0}^{n} \rho(\mathbf{q}(t)) \to \int_L \rho(\mathbf{x}) d\pi(\mathbf{x})$$

with probability one. This result shows that all sensible time averages of process (3) converge with probability one to deterministic limits. Thus, in particular, the following statement holds.

Theorem 2. *With probability one*

$$\frac{1}{n}\sum_{t=1}^{n} Q_t^{A_j} \to \int_L \sum_{i=1}^{\infty}\sum_{p=1}^{l} x^i_{(j-1)l+p}\,d\pi(\mathbf{x}), \tag{15}$$

$$\frac{1}{n}\sum_{t=1}^{n} Q_t^{A_j} I(Q_t) \to \int_L I\left(\sum_{i=1}^{\infty}\sum_{n=1}^{kl} x^i_n\right)\sum_{s=1}^{\infty}\sum_{p=1}^{l} x^s_{(j-1)l+p}\,d\pi(\mathbf{x}), \tag{16}$$

and, if $\epsilon > 0$,

$$\frac{1}{n}\sum_{t=1}^{n} \nu^j(t) \to \int_L \sum_{i=1}^{\infty}\sum_{p=1}^{l} \chi_{x^i_{(j-1)l+p}A_j \geq \epsilon b}\,d\pi(\mathbf{x}) \tag{17}$$

as $n \to \infty$. *Here* $j = 1, 2, \ldots, k$. *By* $\nu^j(t)$ *we denote the number of firms having* A_j *as capital per unit of output which are manufacturing at time* t.

Proof. The sums involved in (15)–(17) are measurable with respect to $\mathcal{B}^\infty$ nonnegative functions. Indeed, while for (15) and (17) this is straightforward, concerning (16) we have to take into account that $I(\cdot)$ is a continuous function by hypothesis. By definition of L, the sum involved in (15) is uniformly bounded from above by Q_*. Taking into account the continuity of $I(\cdot)$, we get that the expression in the right hand side of (16) is bounded from above by

$$Q_* \max_{x\in[0,Q_*]} I(x) < \infty.$$

Note that the minimal productive capacity of a firm with A_j as capital per unit of output which is manufacturing is $\epsilon b A_j^{-1}$. Since the total productive capacity of such firms does not exceed Q_* for any time instant, we conclude that the iterated sum in (17) is bounded from above by $A_j Q_*/\epsilon b$. Thus, the functions involved in the right hand sides of (15)–(17) are measurable and uniformly bounded. Consequently, they are integrable with respect to $\pi(\cdot)$. Applying the strong law of large numbers quoted above, we obtain the statement of the theorem.

The theorem is proved.

Relations (15)–(17) entail the following conceptual interpretations. First, the time average of the total productive capacity of firms that are in business and have A_j as capital per unit of output converges to a limit which is a deterministic function. Since

$$\max\left[h(Q_t) - m, 0\right] Q_t^{A_j} = \frac{v}{\lambda} I(Q_t) Q_t^{A_j}$$

represents the gross total profit of all firms having A_j as capital per unit of output at time t, the second relation says that the time average of this value converges to a deterministic limit. The third relation means that the average number of firms with A_j as capital per unit of output that are in business converges to a deterministic limit as well.

Remark. Arguing as above, we can derive expressions for the limits of the time averages of market shares of all firms whose capital ratio is A_j.

From (9), (15) and (16) we get the following result.

Corollary. *With probability one*

$$\frac{1}{n}\sum_{t=0}^{n}\mathcal{E}_t^j, \quad j = 1, 2, \dots, k,$$

converge as $n \to \infty$ and the corresponding limits, e^j, are deterministic satisfying the following relations

$$e^j = aA_j^{-1} - dq^j + \frac{\lambda}{v} r^j A_j^{-1}, \quad j = 1, 2, \dots, k,$$

where q^j and r^j denote the values in the right hand sides of (15) and (16).

Thus, this corollary gives expressions for the limit average total outflows of productive capacities of firms via the other ergodic characteristics of the industry and the parameters of the model.

The foregoing properties shed light on the possible path-dependence of the dynamics. Indeed, Doeblin's condition implies that events occuring at t and $t+n$ are getting more and more independent as $n \to \infty$. This ergodic property means as such lack of path-dependence. Moreover, the limits for the time averages for productive capacities, outflows of the latters, numbers of firms in business, etc. do not depend on the initial state. (A caveat is however in order: such ergodicity properties are likely to crucially depend on the lack of endogenous technical progress in our "baseline model" and are likely to be lost whenever one allows for cumulative forms of learning: see also the remarks in Winter et al., 2000.)

In order to explore some more detailed properties of the dynamics of industry structure, let us turn to numerical simulations of the model.

8 Some results of computer simulations

As an illustration of some qualitative properties, let us consider on a set of computer simulations of the model which we performed[8] (note that we have tried several runs, under different parametrizations and the properties discussed below appear to generally apply: more rigorous test for robustness are forthcoming). The runs presented below use the following parametrization: $k = 4$; $A_i = \{1, 2, 3, 4\}$; $m = 1$, $v = 1$, $d = 0.3$, $\lambda = 0.6$, $\epsilon = 0.5$. The demand function is $h(x) = 4.1667\exp(-0.5x)$ and the capitals of newcoming firms were uniformly distributed over $[0.02, 0.04]$. We set $l = 3$ and the number of newcoming firms is such that, in tune with the evidence, "bad" firms are more likely to arrive than "good" ones. The arrival probabilities are the following:

[8] The simulation used a program from the Laboratory for Simulation Development (LSD), a package providing a user-friendly environment for implementation of simulation models, developed by M. Valente (see Valente, 1997) at the International Institute for Applied Systems Analysis (IIASA) and subsequently at Ålborg university. It is publicly available via Internet, at *http://www.business.auc.dk/~mv/Lsd/lsd.html*.

	p_0	p_1	p_2	p_3
A_1	0.9	0.06	0.03	0.01
A_2	0.5	0.25	0.15	0.1
A_3	0.1	0.15	0.25	0.5
A_4	0.01	0.03	0.06	0.9

Figure 1 shows the dynamic of the price for $1 \leq t \leq 50$: it tends stabilize rather quickly around its long term average level, which indeed is below that value at which everybody shrinks (note that this is a property of the *price average*, while the actual price keeps fluctuating around this level).

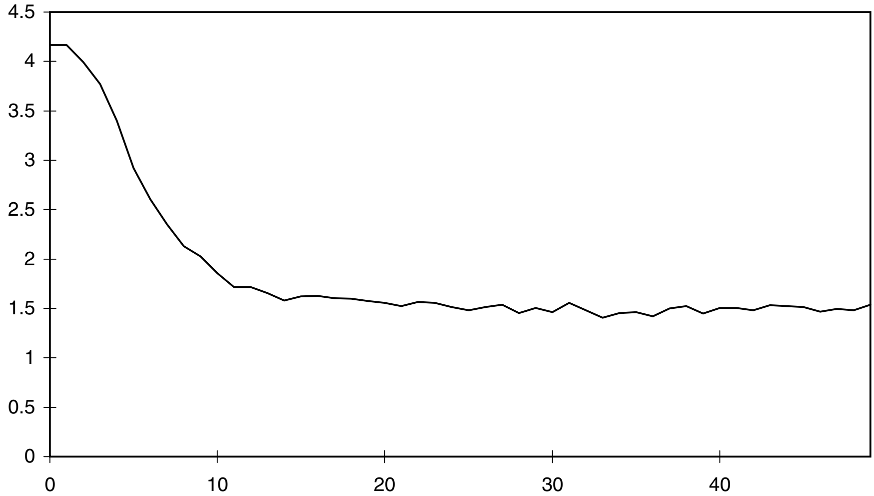

Fig. 1. The price $h(Q_t)$ for $1 \leq t \leq 50$

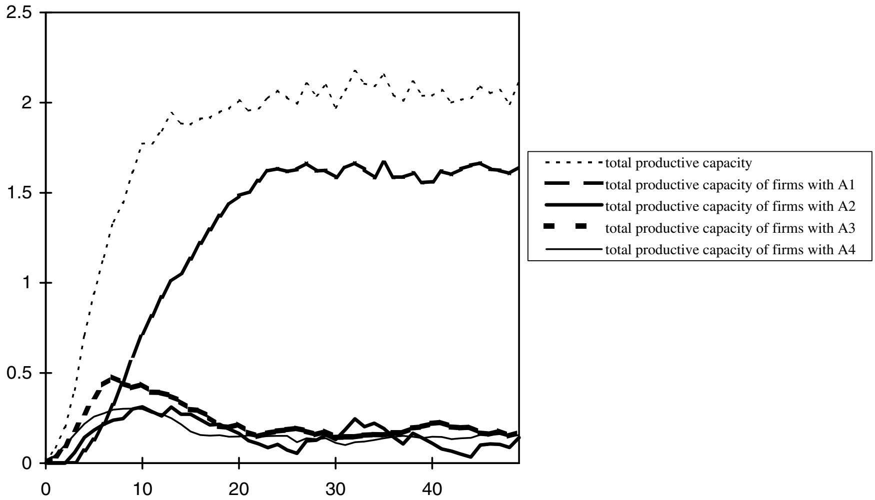

Fig. 2. Productive capacities for $1 \leq t \leq 50$

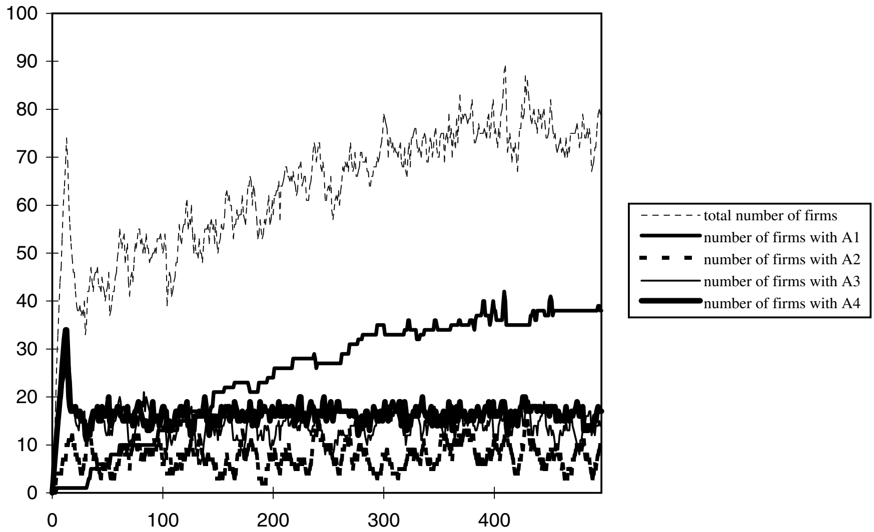

Fig. 3. Numbers of firms in business for $1 \leq t \leq 500$

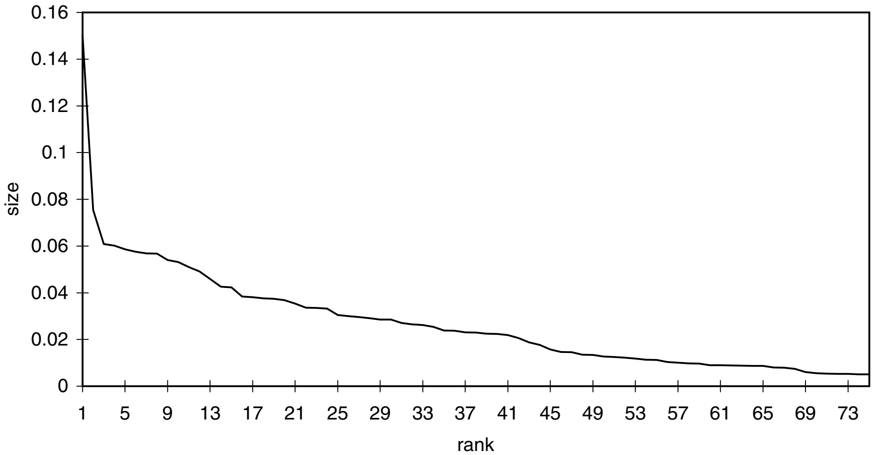

Fig. 4. The size distribution of firms at $t = 500$

Underlying that apparent stability of aggregate supply, one observes however a more messy micro structure with a fringe of productive capacity provided by the least efficient firms (Fig. 2), which keep entering and quickly die. Note also that while the total capacity stabilizes rather fast, this is not so for the total number of firms (cf. Fig. 3). That is to say, for a rather long "transitional dynamic" the "carrying capacity" of the market is not saturated and the relative aggregate stability of supply and price is supported by a net inflow of firms. In a sense, during the transitional dynamic, market selection operates less effectively allowing relatively long survival time also for less efficient firms and persistent "early mover advantages".

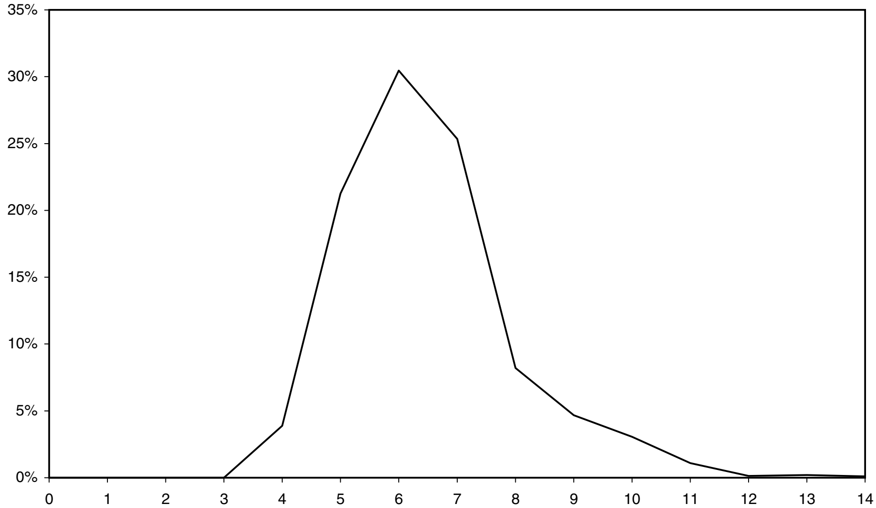

Fig. 5. The life time distribution of 98% of firms dead before $t = 500$ (for sake of visual clarity, the figure does not include 2% of firms surviving the longest time, i.e. around 100 periods)

Note also what looks like a "shakeout" at some point in the history of the industry (cf. Fig. 3) corresponding to the time when highest productivity firms start getting a hold on the market: the number of firms dramatically falls (purely due to an increase in the mortality rates) and non-best practice firms are relegated thereafter to a rather marginal, but persistent fringe of the industry.

Skewed size distributions appear from early on and remain thereafter (Fig. 4), notwithstanding the very simplified assumptions of our model (including the absence of learning by incumbents)[9]. Note also that despite size as such does not provide any specific advantage, one observes the early emergence of one or few (transient) market leaders (Fig. 4). When both production capacity and number of firms approach their stationary regimes, the competitive pressure prevents all firms (including the most efficient ones) from expanding indefinitely. Relatedly, the ensuing life time distribution, that is the number of production cycles firms perform before dying (cf. Fig. 5), assures that even the most efficient firms are bound to disappear in the long term. Indeed, the picture is very similar to the Marshallian view of the "forest", mentioned earlier, with a persistent turnover of trees (with, of course, the marginal fringe having the highest turnover rates).

Note also that the foregoing qualitative properties hold across different parametrizations of the model, although long-term averages and the length of the "transitional dynamics" depend of course on the parametrization itself.

[9] We are facing here a phenomenon similar to the well known Pareto law (see, for example, Ijiri and Simon, 1974), which in one of its formulations states that for a set of firms ranked according to their size, the size s and the rank r of a firm are related as $sr^{\beta} = A$, where β and A are positive constants.

9 Conclusions

In this paper we have developed a formal analytical apparatus able to treat the dynamics of industrial evolution and derive some generic properties of the underlying competition process. The continuing entry flow produces a continuing turnover in the firm population of the sort observed in real data. The size distribution of firms emerges as a derivative consequence of the combination of heterogeneity and turnover. Although "snapshots" of the distribution at different points in time might be similar (after the industry has approached its long term dynamic path), the firms occupying particular places in the distribution are generally different. This does not only apply to the lower end of the distribution, but to the whole universe of firms: in fact, we proved, under quite general conditions, that all firms are bound to die in a finite random time with probability one.

Moreover, the long-run proportions of firms of different efficiency levels reflect the interplay of selection forces and entry rates – indeed, in a fashion roughly analogous to the analysis of gene frequencies provided by the Hardy – Weinberg laws of population biology[10].

The view of the outcomes of the competition process, in a sense, is a formal vindication of the intuition of classical economists that conditions of entry and (heterogeneous) techniques of production determine some sort of "centers of gravity" around which actual prices, quantities and profitabilities persistently fluctuate. In fact, on the ground of the foregoing model, one is able to establish the limit properties of those time averages.

As mentioned earlier, the model is suitable to several extensions. An obvious one is the account of an endogenous process of arrival of new techniques and, hence, new productivity levels (see Winter et al., 2000). However, even in its "baseline" version presented here the model is able to account, *together*, for the "stylized facts" recalled at the beginning concerning turbulence, "life cycle" properties of industry evolution, skewed size distributions and persistent fluctuations in output and prices. The fundamental drivers of the process generating those phenomena are shown to rest upon an ever-lasting inflow of technologically heterogeneous firms and market selection.

Finally, note also that hardly any assumption of the foregoing formal apparatus binds its applications to the domain of industrial dynamics. On the contrary, *even as it stands*, it appears particularly suitable to a large ensemble of dynamic processes – including several of those formalized by evolutionary game theories. Even when assuming that the domain of notational exploration is finite and given from the start, the explicit account of random occurrences in heterogeneous populations subject to systematic selection pressures yields quite rich structures of the emerging evolutionary processes.

[10] The Hardy – Weinberg laws provide a quantitative statement of the fact that "deleterious" genes are continuously eliminated from the population by natural selection forces, but are replenished by mutation (see Wilson and Bossert, 1971).

Appendix

Lemma 1. *For each $j = 1, 2, \dots, k$ in a finite random time τ_j with probability one there appears a firm with A_j as capital per unit of output.*

Proof. Note that, since the variable γ is not deterministic, then $P\{\gamma = 0\} = p_0 < 1$. Also,

$$\{\tau_j = \infty\} = \bigcap_{n \geq 0} \{\tau_j > n\}.$$

Hence

$$P\{\tau_j = \infty\} = P\left\{\bigcap_{n \geq 0} \{\tau_j > n\}\right\}.$$

Since $\{\tau_j > n\} \supseteq \{\tau_j > n+1\}$, we have

$$P\left\{\bigcap_{n \geq 0} \{\tau_j > n\}\right\} = \lim_{n \to \infty} P\{\tau_j > n\}.$$

But

$$P\{\tau_j > n\} = p_0^{kn} \to 0 \quad \text{as} \;\; n \to \infty.$$

This completes the proof.

Theorem 1. *If $\epsilon > 0$ and $P\{\gamma = 0\} = p_0 < 1$, then each firm dies in a finite random time with probability one.*

Proof. The death threshold implies that if a firm lives infinitely long, then its capital does not drop below ϵb. Since the total productive capacity of the industry is bounded with certainty, we conclude that starting from a finite random time τ with probability one every newcoming firm dies in a finite time. Indeed, otherwise we would have infinitely many firms living infinitely long. This, by boundness from below of their physical capitals, would imply that the total productive capacity goes to infinity.

At time $t \geq 1$ consider two firms: one with capital c_t and capital per unit of output A_i and the other with capital c'_t and capital per unit of output $A_j \leq A_i$. Then

$$\frac{c_{t+1}}{c'_{t+1}} = \frac{c_t[1 - d + I(Q_t)A_i^{-1}]\chi_{c_t[1-d+I(Q_t)A_i^{-1}] \geq \epsilon b}}{c'_t[1 - d + I(Q_t)A_j^{-1}]\chi_{c'_t[1-d+I(Q_t)A_j^{-1}] \geq \epsilon b}}. \tag{a1}$$

Assume that there is a firm living infinitely long with positive probability. Set c_t for its capital at t and A_i for its capital per unit of output. Then

$$P\{c_t \geq \epsilon b, \; t \geq \tau'\} = \delta > 0, \tag{a2}$$

where τ' stands for the time instant when it comes to the industry.

Let us first $p_0 = 0$. Consider a time instant $t \geq \tau$. There is a most efficient firm coming at t. Set c'_t for its capital. Since we are in the time domain where every entrant dies in a finite time, also this most efficient firm dies at a finite time instant $t' > t$ with probability one. By (a1) we get that

$$c_{t'} \leq c_t \frac{c'_{t'}}{c'_t} \leq c_t \frac{\epsilon b}{b} \leq \epsilon c_t.$$

Since $p_0 = 0$, at t' another most efficient firm comes to the industry. Similarly, it dies at some instant t'' and we obtain that $c_{t''} \leq \epsilon c_{t'}$ or $c_{t''} \leq \epsilon^2 c_t$.

If $\epsilon < 1$, we conclude that, since c_t is uniformly bounded from above (cf. Lemma 2), there is a sequence $t_k,\ k \geq 1$, of random time instants such that with certainty $c_{t_k} \to 0$ as $k \to \infty$. This contradicts (a2). Hence, it is not possible that there is a firm surviving infinitely long with positive probability.

If $\epsilon = 1$, we notice that, since θ is not deterministic, there is $\sigma > 0$ such that $P\{\theta \geq b + \sigma\} > 0$. By an argument similar to the one given in the proof of Lemma 1, we conclude that with probability one there is a sequence of random time instants $t'_k,\ k \geq 1$, such that at least one of the most efficient firms born at t'_k has initial capital exceeding $b + \sigma$. Then for $t_k \geq \tau$ we have that the capital of the infinitely long living firm at least does not grow (for the previous argument), but it shrinks at least $1 + \sigma/b$ times during the life time of every new most efficient entrant whose initial capital is equal or greater than $b + \sigma$. Since with probability one there are infinitely many of the latter firms, this again contradicts the assumption that such firm can live forever.

Now let us turn to the case when $p_0 > 0$. One accounts for the life cycle of a firm (living a finite time), entering the market, performing some number of production cycles, and finally exiting.

First let $\epsilon < 1$. Set M for the smallest natural number such that $Q_* A_k \epsilon^{M-1} < b$. When at least M subsequent life cycles of the most efficient firms take place, the physical capital any alive firm in the industry is pushed below the death threshold, ϵb. Consequently, if such a chain of life cycles occurs in a finite time with probability one, there may not be in the industry firms living infinitely long with positive probability. Let us implement this idea.

The physical capital of any alive firm is bounded from above by the total capital of the industry. By Lemma 2, the latter is smaller than $Q_* A_k$ with certainty. Shrinking at least ϵ^{-1} times during a single life cycle of a most efficient firm, after M such subsequent cycles, this individual physical capital will fall below $Q_* A_k \epsilon^M < \epsilon b$. Thus, no firm in the industry may survive M or more subsequent life cycles of most efficient firms.

Set S for the smallest natural number such that $S\epsilon b \geq Q_* A_k$. As the minimal capital size of an alive firm is ϵb, at any given time instant, the industry may not have S or more alive firms. Consequently, if during a time period, at every time instant at least one firm enters the industry, every S time instants at least one firm has to exit.

Let T be the event that during each of SM subsequent time instants at least one most efficient firm enters the industry. Then T implies at least M subsequent life cycles of most efficient firms. Indeed, during each of SM subsequent time instants at least one of them comes to the market and, thus, every S time instants at least one of them has to leave. Such life cycles are subsequent because at each time instant out of those SM at least one most efficient firm enters.

Since the entry decisions are independent across firms, the probability of T is $P_0 = (1 - p_0)^{SM} > 0$. Arguing like in the proof of Lemma 1, we conclude that T occurs in a finite time with with probability one. This accomplishes the argument for the case when $p_0 > 0$ and $\epsilon < 1$. Considering life cycles of the most efficient firms whose initial capitals do not fall below $b + \sigma$ for some $\sigma > 0$ such that $P\{\theta \geq b + \sigma\} > 0$, we can adjust the argument given above for the situation when $\epsilon = 1$.

The theorem is proved.

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