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Regulation of the evolution of the architecture of a network by tensors operating on coalitions of actors

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Abstract. Socio-economic networks, neural networks and genetic networks describe collective phenomena through constraints relating actions of several actors, coalitions of these actors and multilinear connectionist operators acting on the set of actions of each coalition. We provide a class of control systems governing the evolution of actions, coalitions and multilinear connectionist operators under which the architecture of the network remains viable. The controls are the "viability multipliers" of the "resource space" in which the constraints are defined. They are involved as "tensor products" of the actions of the coalitions and the viability multiplier, allowing us to encapsulate in this dynamical and multilinear framework the concept of Hebbian learning rules in neural networks in the form of "multi-Hebbian" dynamics in the evolution of connectionist operators. They are also involved in the evolution of coalitions through the "cost" of the constraints under the viability multiplier regarded as a price.

Key words: Dynamic network – Evolutionary economics – Viability multipliers – Fuzzy coalitions – Connectionist operators

JEL Classification: A1, C6, D5

1 Introduction

Collective phenomena deal with the coordination of actions by a finite number n of actors labelled $i = 1, \dots, n$ using the architecture of a network of actors, such as socio-economic networks (see, for instance, Ioannides [43], Aubin [6,8], Aubin and Foray [20], Bonneuil [28,27]), neural networks (see, for instance, Aubin [9,10, 7], Aubin and Burnod [17]) and genetic networks (see, for instance, Bonneuil [31, 30], Bonneuil and Saint-Pierre [32]). This coordinated activity requires a network of communications or connections of actions $x_i \in X_i$ ranging over n finite dimensional vector spaces X_i.

1.1 Definition of the architecture of a network

The simplest general form of a coordination is the requirement that a relation between actions of the form $g(A(x_1, \ldots, x_n)) \in M$ must be satisfied. Here

1. $A : \prod_{i=1}^n X_i \mapsto Y$ is a connectionist operator relating the individual actions in a collective way,
2. $M \subset Y$ is the subset of the resource space Y and g is a map, regarded as a propagation map.

We shall study this coordination problem in a dynamic environment, by allowing actions $x(t)$ and connectionist operators $A(t)$ to evolve[1] according to dynamical systems we shall construct later. In this case, the coordination problem takes the form

$$\forall\, t \geq 0, \;\; g(A(t)(x_1(t), \ldots, x_n(t))) \;\in\; M$$

However, in the fields of motivation under investigation, the number n of variables may be very large. Even though the connectionist operators $A(t)$ defining the "architecture" of the network are allowed to operate *a priori* on all variables $x_i(t)$, they actually operate at each instant t on a coalition $S(t) \subset N := \{1, \ldots, n\}$ of such variables, varying naturally with time according to the nature of the coordination problem (see Aubin [3], Petrosjan [53], Petrosjan and Zenkevitch [54] and Filar and Petrosjan [40]) for closely related issues in dynamic cooperative game theory). Therefore, our coordination problem in a dynamic environment is defined in the following way

Definition 1.1 *The* architecture of dynamical network *involves the evolution*

1. *of actions* $x(t) := (x_1(t), \ldots, x_n(t)) \in \prod_{i=1}^n X_i$,
2. *of connectionist operators* $A_{S(t)}(t) : \prod_{i=1}^n X_i \mapsto Y$,
3. *acting on* coalitions $S(t) \subset N := \{1, \ldots, n\}$ *of the* n *actors*

and requires that

$$\forall\, t \geq 0, \;\; g\left(\{A_S(t)(x(t))\}_{S \subset N}\right) \;\in\; M$$

where $g : \prod_{S \subset N} Y_S \mapsto Y$.

So, a network is not only any kind of a relationship between variables, but involves both connectionist operators operating on coalitions of actors.

[1] For simplicity, the set $M(t)$ is assumed to be constant. But they could also evolve through *mutational equations* and the following results can be adapted to this case. Curiously, the overall architecture is not changed when the set of available resources evolves under a mutational equation. See Aubin [4] for more details on mutational equations.

1.2 Constructing the dynamics

The question we raise is the following: Assume that we know the intrinsic laws of evolution of the variables x_i (independently of the constraints), of the connectionist operator $A_S(t)$ and of the coalitions $S(t)$. Is the above architecture viable under these dynamics, in the sense that the collective constraints defining the architecture of the dynamical network are satisfied at each instant.

There is no reason why let on his own, collective constraints defining the above architecture are viable under these dynamics. Then the question arises how to reestablish the viability of the system.

One may

1. either delineate those states (actions, connectionist operators, coalitions) from which begin viable evolutions
2. or correct the dynamics of the system in order that the architecture of the dynamical network is viable under the altered dynamical system

The first approach leads to take the viability kernel of the constrained subset of K of states (x_i, A_S, S) satisfying the constraints defining the architecture of the network. We refer to Aubin [6, 8] for this approach. We devote this paper to an exhibition of a class of methods for correcting the dynamics without touching on the architecture of the network.

One may indeed be able, with a lot of ingeniousness and intimate knowledge of a given problem, and for "simple constraints", to derive dynamics under which the constraints are viable.

However, we can investigate whether there is a kind of mathematical factory providing classes of dynamics "correcting" the initial (intrinsic) ones in such a way that the viability of the constraints is guaranteed. One way to achieve this aim is to use the concept of "viability multipliers" $q(t)$ ranging over the dual Y^* of the resource space Y that can be used as "controls" involved for modifying the initial dynamics. This allows us to provide an explanation of the formation and the evolution of the architecture of the network and of the active coalitions as well as the evolution of the actions themselves.

A few words about viability multipliers are in order here: If a constrained set K is of the form

$$K \ := \ \{x \in X \quad \text{such that} \ \ h(x) \in M\}$$

where $h : X \mapsto Z := \mathbf{R}^m$ is the constrained map form the state space X to the resource space Z and $M \subset Z$ is a subset of available resources, we regard elements $u \in Z^\star = Z$ in the dual of the resource space Z (identified with Z) as viability multipliers, since they play a role analogous to Lagrange multipliers in optimization under constraints.

Recall that the minimization of a function $x \mapsto J(x)$ over a constrained set K is equivalent to the minimization without constraints of the function

$$x \mapsto J(x) + \sum_{k=1}^{m} \frac{\partial h_k(x)}{\partial x_j} u_k$$

for an adequate Lagrange multiplier $u \in Z^\star = Z$ in the dual of the resource space Z (identified with Z). See, for instance, Aubin [5], Rockafellar and Wets [55] among many other references on this topic.

In an analogous way, but with unrelated methods, it has been proved that a closed convex subset K is viable under the control system

$$x'_j(t) = f_j(x(t)) + \sum_{k=1}^{m} \frac{\partial h_k(x(t))}{\partial x_j} u_k(t)$$

obtained by adding to the initial dynamics a term involving regulons that belong to the dual of the same resource space Z. See, for instance, Aubin and Cellina [18] and Aubin [12, 8] and Section 6 below for more details.

Therefore, these viability multipliers used as regulons benefit from the same economic interpretation of virtual prices as the ones provided for Lagrange multipliers in optimization theory.

1.3 Description of the typical results

The results presented here use this approach in the case of the above specific constraints. We show that by correcting dynamical systems with viability multipliers, the dynamics of the evolution of connectionist operators and coalitions present some interesting features.

We associate with any coalition $S \subset N$ the product $X^S := \prod_{i \in S} X_i$ and denote by $A_S \in \mathcal{L}_S(X^S, Y)$ the space of S-linear operators $A_S : X^S \mapsto Y$, i.e., operators that are linear with respect to each variable x_i, $(i \in S)$ when the other ones are fixed. Linear operators $A_i \in \mathcal{L}(X_i, Y)$ are obtained when the coalition $S := \{i\}$ is reduced to a singleton, and we identify $\mathcal{L}_\emptyset(X^\emptyset, Y) := Y$ with the vector space Y. See Section 2.1 for more details.

In order to tackle mathematically this problem, we shall

1. restrict the connectionist operators $A := \sum_{S \subset N} A_S$ to be multiaffine, i.e., the sum over all coalitions of S-linear operators[2] $A_S \in \mathcal{L}_S(X^S, Y)$,
2. allow coalitions S to become fuzzy coalitions so that they can evolve continuously.

Fuzzy coalitions $\chi = (\chi_1, \ldots, \chi_n)$ are defined by memberships $\chi_i \in [0, 1]$ between 0 and 1, instead of being equal to either 0 or 1 as in the case of usual coalitions. The membership $\gamma_S(\chi) := \prod_{i \in S} \chi_i$ is by definition the product of the memberships of the members $i \in S$ of the coalitions. Using fuzzy coalitions allows us to define their velocities and study their evolution.

The viability multipliers $q(t) \in Y^*$ can be regarded as regulons, i.e., regulation controls or parameters, or virtual prices in the language of economists. These are chosen at each instant in order that the viability constraints describing the network

[2] Also called (or regarded as) tensors.They are nothing other than matrices when the operators are linear instead of multilinear. Tensors are the matrices of multilinear operators, so to speak, and their "entries" depend upon several indexes instead of the two involved in matrices.

can be satisfied at each instant. The main theorem of this paper guarantees this possibility. Another theorem tells us how to choose at each instant such regulons (the regulation law).

Even though viability multipliers do not provide all the dynamics under which a constrained set is viable, they do provide important and noticeable classes of dynamics exhibiting interesting structures that deserve to be investigated and tested in concrete situations.

1.4 An economic interpretation

Although the theory applies to general networks, the problem we face has an economic interpretation that may help the reader in interpreting the main results that we summarize below.

Actors here are economic agents (producers) $i = 1, \ldots, n$ ranging over the set $N := \{1, \ldots, n\}$. Each coalition $S \subset N$ of economic agents is regarded as a production unit (a firm) using resources of their agents to produce (or not produce) commodities. Each agent $i \in N$ provides a resource vector (capital, competencies, etc.) $x_i \in X$ in a resource space $X_i := \mathbb{R}^{m_i}$ used in production processes involving coalitions $S \subset N$ of economic agents (regarded as firms employing economic agents)

We describe the $\mathrm{production\ process}$ of a firm $S \subset N$ by a S-linear operator $A_S : \prod_{i=1}^{n} X_i \mapsto Y$ associating with the resources $x := (x_1, \ldots, x_n)$ provided by the economic agents a commodity $A_S(x)$. The supply constraints are described by a subset $M \subset Y$ of the commodity space, representing the set of commodities that must be produced by the firms: Condition

$$\sum_{S \subset N} A_S(t)(x(t)) \ \in \ M$$

express that $\mathrm{at\ each\ instant}$, the total production must belong to M.

The connectionist operators among economic agents are the input-output production processes operating on the resources provided by the economic agents to the production units described by coalitions of economic agents. The architecture of the network is then described by the supply constraints requiring that $\mathrm{at\ each\ instant}$, agents supply adequate resources to the firms in order that the production objectives are fulfilled.

When fuzzy coalitions χ_i of economic agents[3] are involved, the supply constraints are described by

$$\sum_{S \subset N} \left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x(t)) \ \in \ M \qquad (1)$$

since the production operators are assumed to be multilinear.

[3] Whenever the resources involved in production processes are proportional to the intensity of labor, one could interpret in such specific economic models the rate of participation χ_i of economic agent i as (the rate of) labor he uses in the production activity.

Let us describe the main results in the framework of the economic example we have presented.

Assume that the dynamical behaviors of the economic agents are described by differential equations

$$\begin{cases} (i) & x_i'(t) \ = \ c_i(x_i(t)), \ i = 1, \ldots, n \\ & \text{describing the evolution of the resources put at the firm's disposal} \\ (ii) & \chi_i'(t) \ = \ \kappa_i(\chi(t)), \ i = 1, \ldots, n \\ & \text{describing the evolution of the rates of participation} \\ (iii) & A_S'(t) \ = \ \alpha_S(A(t)), \ S \subset N \\ & \text{describing the evolution of the production processes} \end{cases}$$

The viability multipliers $p(t)$ range over the dual $Y^\star$ of the commodity space Y, and can be interpreted economically as a space of shadow (or virtual) prices.

We denote by $A_S\,(x_{-i}) \in \mathcal{L}(X_i, Y)$ the linear operator defined by $u_i \mapsto A_S(x_{-i})u_i := A_S(x_{-i}, u_i)$ and by $A_S\,(x_{-i})^* \in \mathcal{L}(Y^*, X_i^*)$ its transpose defined by

$$\forall\, q \in Y^*, \ \forall\, u_i \in X_i, \ \ \big\langle A_S\,(x_{-i})^*\, q, u_i \big\rangle \ = \ \big\langle q, A_S\,(x_{-i})\, u_i \big\rangle$$

The operator $(\bigotimes_{i \in S} x_i) \otimes p \ \in \ \mathcal{L}_S(X^S, Y)$ associates with the resources $(x_i)_{i \in S}$ put at the disposal of the coalition S and the multiplier $p \in X$ the multilinear operator (called the tensor product of the x_i ($i \in S$) and p): it associates with $(q_i)_{i \in S} \in Y^S$ the element

$$\left(\bigotimes_{i \in S} x_i \otimes p\right)((q_i)_{i \in S}) := \left(\prod_{i \in S} \langle q_i, x_i\rangle\right) p$$

We shall prove that the constrained subset is viable under the above system of differential equations corrected by prices in the following way

$$\begin{cases} (i) & x_i'(t) \ = \ c_i(x_i(t)) + \sum_{S \ni i} \left(\prod_{j \in S} \chi_j(t)\right) A_S(t)(x_{-i}(t))^* p(t), \\ & i = 1, \ldots, n \quad \text{describing the evolution of the resources} \\ (ii) & \chi_i'(t) \ = \ \kappa_i(\chi(t)) + \sum_{S \ni i} \left(\prod_{j \in S \setminus i} \chi_j(t)\right) \langle p(t), A_S(t)\,(x(t))\rangle\,, \\ & i = 1, \ldots, n \quad \text{describing the evolution of the rates of participation} \\ (iii) & A_S'(t) \ = \ \alpha_S(A(t)) + \left(\prod_{j \in S} \chi_j(t)\right) \left(\bigotimes_{j \in S} x_j(t)\right) \otimes p(t), \ S \subset N \\ & \text{describing the evolution of the production processes} \end{cases}$$

This means that starting from an initial state $((x_i, \chi_i)_{i \in N}, (A_S)_{S \subset N})$ satisfying the constraints, we can find at each instant $t \geq 0$ prices $p(t) \in Y^*$ such that the evolutions

$$t \mapsto ((x_i(t), \chi_i(t))_{i \in N}, (A_S(t))_{S \subset N})$$

governed by the corrected system of differential equations satisfy the supply constraints

$$\sum_{S \subset N} \gamma_S(\chi(t)) A_S(t)(x(t)) \ \in \ M$$

In other words, for economic agents putting resources $x_i(t) \in X$ at production disposal with a participation rate $\chi_i(t) \in [0,1]$ and for economic processes modifying the production processes $A_S(t)$, they have to

1. increase their resources $x_i(t)$ by adding to their intrinsic dynamics $c_i(x_i(t))$ the "input prices" made of the sum over the firms S of the prices
$$\gamma_S(\chi(t))A_S(t)(x_{-i}(t))^* p(t)$$
weighted by the membership $\gamma_S(\chi(t))A_S(t)$ of the firm S in the fuzzy coalition $\chi(t)$,
2. increase their membership $\chi_i(t)$ by adding to their intrinsic dynamics $\kappa_i(\chi(t))$ the sum over the firms S of the costs
$$\gamma_{S\backslash i}(\chi(t))\, \langle p(t), A_S(t)\,(x(t))\rangle$$
of the production processes weighted by membership $\gamma_{S\backslash i}(\chi(t))$ in the fuzzy coalition $\chi(t)$ of the coalition $S\backslash i$ made of the agents of the coalition S other than i (mimetic behavior),
3. increase the multilinear operators $A_S(t)$ by adding to their intrinsic dynamics $\alpha_S(A(t))$ the tensor product $\gamma_S(\chi(t))\left(\bigotimes_{j\in S} x_j(t)\right) \otimes p(t)$ (multi-Hebbian learning rule).

The first rule is familiar in (mathematical) economics. The price is the message sent to the economic agents for allowing them to (algebraically) increase the resources provided to the firms, without knowing the behavior of the other economic agents. The second rule needs to be interpreted as an incentive for economic agents to increase or decrease his participation in the economy in terms of the cost of constraints and of the membership of other economic agents, encapsulating a mimetic – or "herd", panurgean [4] – behavior.

The third system seems new in the economic literature, but is the backbone of the investigations on Hebbian learning algorithms in the field of neural networks, where the connectionist matrices are synaptic matrices. The correction of the velocities of the connectionist tensors A_S is actually a weighted "multi-Hebbian" rule: for each component of the connectionist tensor, the correction term is the product of the membership $\gamma_S(\chi(t))$ of the coalition S and of the "tensor product" of the resources of the coalition Z and the price. In other words, the viability multipliers appear in the regulation of the multiaffine connectionist operators under the form of tensor products, implementing the Hebbian rule for affine constraints (see Aubin [9, 10, 7]), and "multi-Hebbian" rules for the multiaffine ones, as in Aubin and Burnod [17]. This can be regarded as a learning rule, actually an "adaptation rule", "reinforcing" the increase of the production process "proportionally" to the "product" of the resources put at the disposal of production by economic agents and the market price.

Let us emphasize that these dynamics are not proposed a priori, as Hebbian rules were in 1949, but derive from the theorem on viability multipliers

[4] From a famous story by François Rabelais (1483-1553), where Panurge sent overboard the head sheep, followed by the whole herd.

associated with the supply and demand constraints. Indeed, this rule emerged naturally from the mathematical properties of viability theorems, as well as the mimetic law. They were no part of the behavioral assumptions, described only by the dynamics c_i, κ_i and α_S subjected to the supply constraints.

Naturally, this paper being mainly devoted to a general model dealing with general networks, this is the description of a very simple model that is enriched and further commented upon in Section 5. The point made in this paper is to show how the mathematical methods presented in a general way can be useful in designing other models, as the Lagrange multiplier rule does in the static framework. By comparison, we see that if we minimize a collective utility function:

$$\sum_{i=1}^{n} \mathbf{u}_i(x_i) + \sum_{i=1}^{n} \mathbf{v}_i(\chi_i) + \sum_{S \subset N} \mathbf{w}_S(A_S)$$

under constraints (1), then first-order optimality conditions at a optimum $((x_i)_i, (\chi_i)_i, (A_S)_{S\subset N})$ imply the existence of Lagrange multipliers p such that:

$$\begin{cases} \nabla \mathbf{u}_i(x_i) & = \displaystyle\sum_{S \ni i} \left(\prod_{j \in S} \chi_j \right) A_S(x_{-i}(t))^* p, \ i = 1, \dots, n \\ \nabla \mathbf{v}_i(\chi_i) & = \displaystyle\sum_{S \ni i} \left(\prod_{j \in S \setminus i} \chi_j \right) \langle p, A_S(x) \rangle, \ i = 1, \dots, n \\ \nabla \mathbf{w}_S(A_S) & = \displaystyle\left(\prod_{j \in S} \chi_j \right) \left(\bigotimes_{j \in S} x_j \right) \otimes p, \ S \subset N \end{cases}$$

1.5 Outline

We shall present examples of network structures in order of increasing difficulty. We present in this introduction known results obtained for affine constraints (case of one actor), and expose them when there are only two actors and when bilinear constraints are involved.

In the first section, we next exhibit the results for n actors for multiaffine constraints without evolving coalitions, while in the second section, we introduce fuzzy coalitions and show how they may evolve to maintain the viability of the architecture of the network. We continue by allowing some nonlinearities in the constraints, and observing that the mains structures are not fundamentally modified. We thus present a more sophisticated economic model in which agents are both consumers and producers. A short review of viability multipliers and the proof of the main theorem conclude the paper.

1.6 Case of affine constraints

We summarize the case in which there is only one actor and the operator $A : X \mapsto Y$ is affine studied in Aubin [7,8,6]:

$$\forall\, x \in X, \;\; A(x) \; := \; Wx + y \;\; \text{where} \;\; W \in \mathcal{L}(X,Y) \;\&\; y \in Y$$

The coordination problem takes the form:

$$\forall\, t \geq 0, \;\; W(t)x(t) + y(t) \;\in\; M$$

where both the state x, the resource y and the connectionist operator W evolve. These constraints are not necessarily viable under an arbitrary dynamic system of the form

$$\begin{cases} (i) \;\;\; x'(t) \;=\; c(x(t)) \\ (ii) \;\; y'(t) \;=\; d(y(t)) \\ (iii) \; W'(t) \;=\; \alpha(W(t)) \end{cases} \tag{2}$$

We can reestablish viability by involving multipliers $q \in Y^*$ ranging over the dual $Y^* := Y$ of the resource space Y to correct the initial dynamics. We denote by $W^* \in \mathcal{L}(Y^*, X^*)$ the transpose of W:

$$\forall\, q \in Y^*, \; \forall\, x \in X, \;\; \langle W^*q, x\rangle \;:=\; \langle q, Wx\rangle$$

by $x \otimes q \in \mathcal{L}(X, Y^*)$ the tensor product defined by

$$x \otimes q \;:\; p \in X^* := X \mapsto (x \otimes q)(p) \;:=\; \langle p, x\rangle q$$

the matrix of which is made of entries $(x \otimes q)_i^j = x_i q^j$.

The contingent cone $T_M(x)$ to $M \subset Y$ at $y \in M$ is the set of directions $v \in Y$ such that there exist sequences $h_n > 0$ converging to 0, and v_n converging to v satisfying $y + h_n v_n \in M$ for every n. The (regular) normal cone to $M \subset Y$ at $y \in M$ is defined by

$$N_M(y) := \{q \in Y^* \mid \forall\, v \in T_M(y), \; \langle q, v\rangle \leq 0\}$$

(see Aubin and Frankowska [21] and Rockafellar and Wets [55] for more details on these topics).

We proved that the viability of the constraints can be reestablished when the initial system (2) is replaced by the control system

$$\begin{cases} (i) \;\;\; x'(t) \;=\; c(x(t)) - W^*(t)q(t) \\ (ii) \;\; y'(t) \;=\; d(y(t)) - q(t) \\ (iii) \; W'(t) \;=\; \alpha(W(t)) - x(t) \otimes q(t) \\ \qquad\quad \text{where } q(t) \;\in\; N_M(W(t)x(t) + y(t)) \end{cases}$$

where $N_M(y) \subset Y^*$ denotes the normal cone to M at $y \in M \subset Y$ and $x \otimes q \in \mathcal{L}(X, Y^*)$ denotes the tensor product defined by

$$x \otimes q \;:\; p \in X^* := X \mapsto (x \otimes q)(x) \;:=\; \langle p, x\rangle q$$

the matrix of which is made of entries $(x \otimes q)_i^j = x_i q^j$. In other words, the correction of a dynamical system for reestablishing the viability of constraints of the form $W(t)x(t) + y(t) \in M$ involves the rule proposed by Hebb in his classic book *The organization of behavior* in 1949 as the basic learning process of synaptic weight and called the Hebbian rule: Taking $\alpha(W) = 0$, the evolution of the synaptic matrix $W := (w_i^j)$ obeys the differential equation

$$\frac{d}{dt} w_i^j(t) \;=\; -x_i(t) q^j(t)$$

The Hebbian rule states that the velocity of the synaptic weight is the product of pre-synaptic activity and post-synaptic activity. Such a learning rule "pops up" (or, more pedantically, emerges) whenever the synaptic matrices are involved in regulating the system in order to maintain the "homeostatic" constraint $W(t)x(t)+y(t) \in M$. (See Aubin [9] for more details on the relations between Hebbian rules and tensor products in the framework of neural networks).

We introduce a coefficient $\chi(t) \in [0,1]$ aimed at "tuning" the action $x(t)$, regarded as a potential action that is not wholly implemented. In this framework, the constraint becomes

$$\forall\, t \geq 0, \;\; W(t)\chi(t)x(t) + y(t) \;\in\; M$$

Again, one can correct a differential system of the form

$$\begin{cases} (i) & x'(t) \;=\; c(x(t)) \\ (ii) & y'(t) \;=\; d(y(t)) \\ (iii) & \chi'(t) \;=\; \kappa(\chi(t)) \\ (iv) & W'(t) \;=\; \alpha(W(t)) \end{cases}$$

by introducing $\text{viability multipliers}$ as controls in a system of the form

$$\begin{cases} (i) & x'(t) \;=\; c(x(t)) - W^*(t)q(t) \\ (ii) & y'(t) \;=\; d(y(t)) - q(t) \\ (iii) & \chi'(t) \;=\; \kappa(\chi(t)) - \langle q(t), W(t)x(t)\rangle \\ (iv) & W'(t) \;=\; \alpha(W(t)) - x(t) \otimes q(t) \\ & \text{where } q(t) \;\in\; N_M(W(t)\chi(t)x(t) + y(t)) \end{cases}$$

The correction term is the "cost of the linear constraint" $\langle q(t), W(t)x(t)\rangle$ in the law of evolution of $\chi(t)$.

1.7 Case of bi-affine constraints

Before investigating the general case and confronting notational difficulties, let us explain how we go from the affine case to the bi-affine case.

Here, we assume that $X := X_1 \times X_2$ is the product of two vector spaces. Affine constraints take the form

$$\forall\, t \geq 0, \;\; A_1(t)x_1(t) + A_2(t)x_2(t) + A_\emptyset(t) \;\in\; M$$

where $A_i \in \mathcal{L}(X_i, Y)$ $(i = 1, 2)$ and $A_\emptyset \in Y$. But we can also involve a bilinear operator $A_{\{1,2\}} \in \mathcal{L}_2(X_1 \times X_2, Y)$ and consider bi-affine constraints of the form:

$$\forall\, t \geq 0, \;\; A_{\{1,2\}}(t)(x_1(t), x_2(t)) + A_1(t)x_1(t) + A_2(t)x_2(t) + A_\emptyset(t) \;\in\; M$$

We introduce the linear operators $A_{\{1,2\}}(x_i) \in \mathcal{L}(X_{-i}, Y)$ defined by

$$A_{\{1,2\}}(x_1) : x_2 \mapsto A_{\{1,2\}}(x_1)x_2 := A_{\{1,2\}}(x_1, x_2)$$

and

$$A_{\{1,2\}}(x_2) : x_1 \mapsto A_{\{1,2\}}(x_2)x_1 := A_{\{1,2\}}(x_1, x_2)$$

We shall prove that when these constraints are not viable under an arbitrary dynamic system of the form

$$\begin{cases} (i) & x_i'(t) \;=\; c_i(x(t)),\; i = 1, 2 \\ (ii) & A_\emptyset'(t) \;=\; \alpha_\emptyset(A_\emptyset(t)) \\ (iii) & A_1'(t) \;=\; \alpha_1(A_1(t)) \\ (iv) & A_2'(t) \;=\; \alpha_2(A_2(t)) \\ (v) & A_{\{1,2\}}'(t) \;=\; \alpha_{\{1,2\}}(A_{\{1,2\}}(t)) \end{cases}$$

we can still reestablish viability by involving multipliers $q \in Y^*$ and correct the above system by the control system

$$\begin{cases} (i) & x_1'(t) = c_1(x(t)) - A_1(t)^*q(t) - A_{\{1,2\}}(t)(x_2(t))^*q(t) \\ (ii) & x_2'(t) = c_2(x(t)) - A_2(t)^*q(t) - A_{\{1,2\}}(t)(x_1(t))^*q(t) \\ (iii) & A_\emptyset'(t) = \alpha_\emptyset(A_\emptyset(t)) - q(t) \\ (iv) & A_1'(t) = \alpha_1(A_1(t)) - x_1(t) \otimes q(t) \\ (v) & A_2'(t) = \alpha_2(A_2(t)) - x_2(t) \otimes q(t) \\ (vi) & A_{\{1,2\}}'(t) = \alpha_{\{1,2\}}(A_{\{1,2\}}(t)) - x_1(t) \otimes x_2(t) \otimes q(t) \text{ where} \\ & q(t) \in N_M(A_{\{1,2\}}(t)(x_1(t), x_2(t)) + A_1(t)x_1(t) + A_2(t)x_2(t) + A_\emptyset(t)) \end{cases}$$

Hence, the structure of this control system involves the transposes $A_i^*(t)q(t)$ and $A_{\{1,2\}}(t)(x_j(t))^*(t)q(t)$ $(i = 1, 2)$ in the evolution of the variables $x_i(t)$, and the tensor products $x_i(t) \otimes q(t)$ (Hebbian rules) in the evolution of the linear operators $A_i(t)$, and the tensor product $x_1(t) \otimes x_2(t) \otimes q(t)$ in the evolution of the bilinear form $A_{\{1,2\}}$.

The tensor product $x_1 \otimes x_2 \otimes q$ is a bilinear operator from $X_1^* \times X_2^*$ to Y^* associating with any pair $(p_1, p_2) \in X_1^* \times X_2^*$ the element

$$(x_1 \otimes x_2 \otimes q)(p_1, p_2) \;:=\; \langle p_1, x_1 \rangle \langle p_2, x_2 \rangle q$$

If the vector spaces are supplied with bases, the components of this bilinear form – the "tensors" – can be written

$$(x_1 \otimes x_2 \otimes q)^j_{i_1, i_2} \;=\; x_{1_{i_1}} x_{2_{i_2}} q^j$$

as the products of the components of the three factors of this tensor product. Taking $\alpha_{1,2}(A) = 0$, the evolution of the bi-synaptic tensor $A_{\{1,2\}} := (a^j_{i_1, i_2})$ obeys the differential equation

$$\frac{d}{dt} a^j_{i_1, i_2}(t) \;=\; -\, x_{1_{i_1}}(t) x_{2_{i_2}}(t) q^j(t)$$

This states that the velocity of the synaptic tensor is the product of the presynaptic activities of the neurons arriving at the synapse (i_1, i_2, j) and the postsynaptic activity (see Aubin & Burnod [17]).

We may enrich this problem by introducing coefficients $\chi_i(t) \in [0, 1]$ aimed at tuning the action $x_i(t)$ $(i = 1, 2)$ that we shall later regard as the components of a fuzzy coalition. In this framework, the constraint becomes: $\forall\ t \geq 0$,

$$\begin{array}{l}\chi_1(t)\chi_2(t)A_{\{1,2\}}(t)(x_1(t), x_2(t))+\chi_1(t)A_1(t)x_1(t)\\ +\chi_2(t)A_2(t)x_2(t)+A_\emptyset(t) \in M\end{array}$$

If we assume that the evolutions of these $\chi_i(t)$ are governed by differential equations

$$\chi_i'(t) \ = \ \kappa_i(\chi_i(t)), \ \ i = 1, 2$$

we shall prove that the above constraints are viable under the control system

$$\begin{cases}(i) & x_1'(t) = c_1(x(t))-\chi_1(t)A_1(t)^\star q(t)-\chi_1(t)\chi_2(t)A_{\{1,2\}}(t)(x_2(t))^\star q(t)\\ (ii) & x_2'(t) = c_2(x(t))-\chi_2(t)A_2(t)^\star q(t)-\chi_1(t)\chi_2(t)A_{\{1,2\}}(t)(x_1(t))^\star q(t)\\ (iii) & \chi_1'(t) = \kappa_1(\chi_1(t))-\langle q(t), A_1(t)x_1(t)+\chi_2(t)A_{\{1,2\}}(t)(x_1(t), x_2(t))\rangle\\ (iv) & \chi_2'(t) = \kappa_2(\chi_2(t))-\langle q(t), A_2(t)x_2(t)+\chi_1(t)A_{\{1,2\}}(t)(x_1(t), x_2(t))\rangle\\ (v) & A_\emptyset'(t) = \alpha_\emptyset(A_\emptyset(t))-q(t)\\ (vi) & A_1'(t) = \alpha_1(A_1(t))-\chi_1(t)x_1(t) \otimes q(t)\\ (vii) & A_2'(t) = \alpha_2(A_2(t))-\chi_2(t)x_2(t)\otimes q(t)\\ (viii) & A_{\{1,2\}}'(t) = \alpha_{\{1,2\}}(A_{\{1,2\}}(t))-\chi_1(t)\chi_2(t)x_1(t)\otimes x_2(t)\otimes q(t) \text{ where}\\ & q(t) \in N_M(\chi_1(t)\chi_2(t)A_{\{1,2\}}(t)(x_1(t), x_2(t))+\chi_1(t)A_1(t)x_1(t)\\ & +\chi_2(t)A_2(t)x_2(t)+A_\emptyset(t))\end{cases}$$

2 Regulation by connectionist tensors

2.1 Connectionist tensors

In order to handle more explicit and tractable formulas and results, we shall assume that the connectionist operator $A : X := \prod_{i=1}^n X_i \rightsquigarrow Y$ is multiaffine.

For defining such a multiaffine operator, we associate with any coalition $S \subset N$ its characteristic function $\chi_S : N \mapsto \mathbf{R}$ associating with any $i \in N$

$$\chi_S(i) \ := \begin{cases} 1 \text{ if } i \in S\\ 0 \text{ if } i \notin S\end{cases}$$

It defines a linear operator $\chi_S \circ \in \mathcal{L}\left(\prod_{i=1}^n X_i, \prod_{i=1}^n X_i\right)$ that associates with any $x = (x_1, \ldots, x_n) \in \prod_{i=1}^n X_i$ the sequence $\chi_S \circ x \in \mathbf{R}^n$ defined by

$$\forall\ i = 1, \ldots, n, \ \ (\chi_S \circ x)_i \ := \begin{cases} x_i \text{ if } i \in S\\ 0 \ \ \text{if } i \notin S\end{cases}$$

We associate with any coalition $S \subset N$ the subspace

$$X^S \ := \ x_S \circ \prod_{i=1}^n X_i \ = \ \left\{x \in \prod_{i=1}^n X_i \ \text{ such that } \ \forall\ i \notin S, \ \ x_i = 0\right\}$$

since $x_S\circ$ is nothing other that the canonical projector from $\prod_{i=1}^n X_i$ onto X^S. In particular, $X^N := \prod_{i=1}^n X_i$ and $X^\emptyset := \{0\}$.

Let Y be another finite dimensional vector space. We associate with any coalition $S \subset N$ the space $\mathcal{L}_S(X^S, Y)$ of S-linear operators A_S. We extend such a S-linear operator A_S to a n-linear operator (again denoted by) $A_S \in \mathcal{L}_n\left(\prod_{i=1}^n X_i, Y\right)$ defined by:

$$\forall\, x \in \prod_{i=1}^n X_i,\;\; A_S(x) \;=\; A_S(x_1, \dots, x_n) \;:=\; A_S(\chi_S \circ x)$$

A multiaffine operator $A \in \mathcal{A}_n\left(\prod_{i=1}^n X_i, Y\right)$ is a sum of S-linear operators $A_S \in \mathcal{L}_S(X^S, Y)$ when S ranges over the family of coalitions:

$$A(x_1, \dots, x_n) \;:=\; \sum_{S\subset N} A_S(\chi_S \circ x) \;=\; \sum_{S\subset N} A_S(x)$$

We identify $A_\emptyset$ with a constant $A_\emptyset \in Y$.

Hence the collective constraint linking multiaffine operators and actions can be written in the form

$$\forall\, t \geq 0,\;\; \sum_{S\subset N} A_S(t)(x(t)) \;\in\; M$$

For any $i \in S$, we shall denote by $(x_{-i}, u_i) \in X^N$ the sequence $y \in X^N$ where $y_j := x_j$ when $j \neq i$ and $y_i = u_i$ when $j = i$.

We shall denote by $A_S\,(x_{-i}) \in \mathcal{L}(X_i, Y)$ the linear operator defined by $u_i \mapsto A_S(x_{-i})u_i := A_S(x_{-i}, u_i)$. We shall use its transpose $A_S\,(x_{-i})^* \in \mathcal{L}(Y^*, X_i^*)$ defined by

$$\forall\, q \in Y^*,\; \forall\, u_i \in X_i,\;\; \left\langle A_S\,(x_{-i})^*\, q, u_i\right\rangle \;=\; \left\langle q, A_S\,(x_{-i})\, u_i\right\rangle$$

We associate with $q \in Y^*$ and elements $x_i \in X_i$ the multilinear operator[5]

$$x_1 \otimes \cdots \otimes x_n \otimes q \;\in\; \mathcal{L}_n\left(\prod_{i=1}^n X_i^*, Y^*\right)$$

associating with any $p := (p_1, \dots, p_n) \in \prod_{i=1}^n X_i^*$ the element $\left(\prod_{i=1}^n \langle p_i, x_i\rangle\right)\; q$:

$$x_1 \otimes \cdots \otimes x_n \otimes q : p := (p_1, \dots, p_n) \in \prod_{i=1}^n X_i^* \mapsto (x_1 \otimes \cdots \otimes x_n \otimes q)(p)$$
$$:= \left(\prod_{i=1}^n \langle p_i, x_i\rangle\right) q \in Y^*$$

[5] We recall that the space $\mathcal{L}_n\left(\prod_{i=1}^n X_i, Y\right)$ of n-linear operators from $\prod_{i=1}^n X_i$ to Y is isometric to the tensor product $\bigotimes_{i=1}^n X_i^* \otimes Y$, the dual of which is $\bigotimes_{i=1}^n X_i \otimes Y^*$, that is isometric with $\mathcal{L}_n\left(\prod_{i=1}^n X_i^*, Y^*\right)$.

This multilinear operator $x_1 \otimes \cdots \otimes x_n \otimes q$ is called the tensor product of the x_i's and q.

We recall that the duality product on $\mathcal{L}_n\left(\prod_{i=1}^n X_i^\star, Y^\star\right) \times \mathcal{L}_n\left(\prod_{i=1}^n X_i, Y\right)$ for pairs $(x_1 \otimes \cdots \otimes x_n \otimes q, A)$ can be written in the form:

$$\langle x_1 \otimes \cdots \otimes x_n \otimes q, A\rangle \ := \ \langle q, A(x_1, \ldots, x_n)\rangle$$

2.2 Multi-Hebbian learning process

Assume that we start with intrinsic dynamics of the actions x_i, the resources y, the connectionist matrices W and the fuzzy coalitions χ:

$$\begin{cases} (i) \ \ x_i'(t) \ = \ c_i(x(t)), \ i = 1, \ldots, n \\ (ii) \ A_S'(t) \ = \ \alpha_S(A(t)), \ S \subset N \end{cases}$$

Using viability multipliers, we can modify the above dynamics by introducing regulons that are elements $q \in Y^\star$ of the dual $Y^\star$ of the space Y:

Theorem 2.1 *Assume that the functions c_i, κ_i and α_S are continuous and that $M \subset Y$ are closed. Then the constraints*

$$\forall\, t \geq 0, \ \sum_{S \subset N} A_S(t)(x(t)) \ \in \ M$$

are viable under the control system

$$\begin{cases} (i) \ \ x_i'(t) \ = \ c_i(x_i(t)) - \displaystyle\sum_{S \ni i} A_S(t)(x_{-i}(t))^\star q(t), \ i = 1, \ldots, n \\ \\ (ii) \ A_S'(t) \ = \ \alpha_S(A(t)) - \left(\displaystyle\bigotimes_{j \in S} x_j(t)\right) \otimes q(t), \ S \subset N \\ \qquad \text{where } q(t) \in N_M\left(\sum_{S \subset N} A_S(t)(x(t))\right) \end{cases}$$

Remark: Multi-Hebbian Rule – When we regard the multilinear operator A_S as a tensor product of components $A^j_{S_{\Pi_{i \in S} i_k}}$, $j = 1, \ldots, p, \ i_k = 1, \ldots, n_i, \ i \in S$, differential equation (ii) can be written in the form: $\forall\, i \in S, \ j = 1, \ldots, p, \ k = 1, \ldots, n_i$,

$$\frac{d}{dt} A^j_{S_{\Pi_{i \in S} i_k}} \ = \ \alpha_{S_{\Pi_{i \in S} i_k}}(A(t)) - \left(\prod_{i \in S} x_{i_k}(t)\right) q^j(t)$$

The correction term of the component $A^j_{S_{\Pi_{i \in S} i_k}}$ of the S-linear operator is the product of the components $x_{i_k}(t)$ actions x_i in the coalition S and of the component q^j of the viability multiplier. *This can be regarded as a multi-Hebbian rule in neural network learning algorithms*, since for linear operators, we find the product of the component x_k of the pre-synaptic action and the component q^j of the post-synaptic action. □

Indeed, when the vector spaces $X_i := \mathbf{R}^{n_i}$ are supplied with basis e^{i_k}, $k = 1, \ldots, n_i$, when we denote by $e^*_{i_k}$ their dual basis, and when $Y := \mathbf{R}^p$ is supplied with a basis f^j, and its dual supplied with the dual basis f^*_j, then the tensor products $\left(\bigotimes_{i \in S} e^{i_k}\right) \otimes f^*_j$ $(j = 1, \ldots, p, k = 1, \ldots, n_i)$ form a basis of $\mathcal{L}_S\left(X^{S^*}, Y^*\right)$.

Hence the components of the tensor product $\left(\bigotimes_{i \in S} x_i\right) \otimes q$ in this basis are the products $\left(\prod_{i \in S} x_{i_k}\right) q^j$ of the components q^j of q and x_{i_k} of the x_i's, where $q^j := \langle q, f^j \rangle$ and $x_{i_k} := \langle e^*_{i_k}, x_i \rangle$. Indeed, we can write

$$\left(\bigotimes_{i \in S} x_i\right) \otimes q \; = \; \sum_{j=1}^{p} \sum_{i \in S} \sum_{k=1}^{n_i} \left(\langle q, f^j \rangle \prod_{i \in S} \langle e^*_{i_k}, x_i \rangle \right) \left(\bigotimes_{i=1}^{n} e^{i_k} \right) \otimes f^*_j$$

3 Regulation involving fuzzy coalitions

3.1 Fuzzy coalitions

The first definition of a coalition which comes to mind, being that of a subset of players $S \subset N$, is not adequate for tackling dynamical models of evolution of coalitions since the 2^n coalitions range over a finite set, preventing us from using analytical techniques.

One way to overcome this difficulty is to embed the family of subsets of a (discrete) set N of n players to the space $\mathbf{R}^n$ through the map χ associating with any coalition $S \in \mathcal{P}(N)$ its characteristic function[6] $\chi_S \in \{0,1\}^n \subset \mathbf{R}^n$, since $\mathbf{R}^n$ can be regarded as the set of functions from N to $\mathbf{R}$.

By definition, the family of fuzzy sets[7] is the convex hull $[0,1]^n$ of the power set $\{0,1\}^n$ in $\mathbf{R}^n$. Therefore, we can write any fuzzy set in the form

$$\chi \; = \; \sum_{S \in \mathcal{P}(N)} m_S \chi_S \;\text{ where }\; m_S \geq 0 \;\; \& \;\; \sum_{S \in \mathcal{P}(N)} m_S = 1$$

The memberships are then equal to

$$\forall\, i \in N, \;\; \chi_i \; = \; \sum_{S \ni i} m_S$$

[6] This canonical embedding is more adapted to the nature of the power set $\mathcal{P}(N)$ than to the universal embedding of a discrete set M of m elements to $\mathbf{R}^m$ by the Dirac measure associating with any $j \in M$ the jth element of the canonical basis of $\mathbf{R}^m$. The convex hull of the image of M by this embedding is the probability simplex of $\mathbf{R}^m$. Hence fuzzy sets offer a "dedicated convexification" procedure of the discrete power set $M := \mathcal{P}(N)$ instead of the universal convexification procedure of frequencies, probabilities, mixed strategies derived from its embedding in $\mathbf{R}^m = \mathbf{R}^{2^n}$.

[7] This concept of fuzzy set was introduced in 1965 by L. A. Zadeh. Since then, it has been wildly successful, even in many areas outside mathematics!. We found in *"La lutte finale"*, Michel Lafon (1994), p.69 by A. Bercoff the following quotation of the late François Mitterand, president of the French Republic (1981–1995): *"Aujourd'hui, nous nageons dans la poésie pure des sous ensembles flous"* ... (Today, we swim in the pure poetry of fuzzy subsets)!

Consequently, if m_S is regarded as the probability for the set S to be formed, the membership of player i to the fuzzy set[8] χ is the sum of the probabilities of the coalitions to which player i belongs. Player i participates fully in χ if $\chi_i = 1$, does not participate at all if $\chi_i = 0$ and participates in a fuzzy way if $\chi_i \in]0,1[$. We associate with a fuzzy coalition χ the set $P(\chi) := \{i \in N \mid \chi_i \neq 0\} \subset N$ of actors i participating in the fuzzy coalition χ.

We also introduce the membership

$$\gamma_S(\chi) \ := \ \prod_{j \in S} \chi_j$$

of a coalition S in the fuzzy coalition χ as the product of the memberships of actors i in the coalition S. It vanishes whenever the membership of one actor does and reduces to individual memberships for one actor coalitions. When two coalitions are disjoint ($S \cap T = \emptyset$), then $\gamma_{S \cup T}(\chi) = \gamma_S(\chi)\gamma_T(\chi)$. In particular, for any actor $i \in S$, $\gamma_S(\chi) = \chi_i \gamma_{S \setminus i}(\chi)$

Let $A \in \mathcal{A}_n\left(\prod_{i=1}^n X_i, Y\right)$, a sum of S-linear operators $A_S \in \mathcal{L}_S(X^S, Y)$ when S ranges over the family of coalitions, be a multiaffine operator.

When χ is a fuzzy coalition, we observe that

$$A(\chi \circ x) \ = \ \sum_{S \subset P(\chi)} \gamma_S(\chi) A_S(x) \ = \ \sum_{S \subset P(\chi)} \left(\prod_{j \in S} \chi_j \right) A_S(x)$$

We wish to encapsulate the idea that at each instant, only a number of fuzzy coalitions χ are active. Hence the collective constraint linking multiaffine operators, fuzzy coalitions and actions can be written in the form

$$\begin{aligned} \forall\, t \geq 0, \quad & \sum_{S \subset P(\chi(t))} \gamma_S(\chi(t)) A_S(t)(x(t)) \\ = & \sum_{S \subset P(\chi(t))} \left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x(t)) \ \in \ M \end{aligned}$$

3.2 Constructing viable dynamics

Assume that we start with intrinsic dynamics of the actions x_i, the resources y, the connectionist matrices W and the fuzzy coalitions χ:

$$\begin{cases} (i) \quad x_i'(t) \ = \ c_i(x(t)), \ i = 1, \ldots, n \\ (ii) \ \ \chi_i'(t) \ = \ \kappa_i(\chi(t)), \ i = 1, \ldots, n \\ (iii) \ A_S'(t) \ = \ \alpha_S(A(t)), \ S \subset N \end{cases}$$

[8] Actually, this idea of using fuzzy coalitions has already been used in the framework of cooperative games with and without side-payments (see Aubin [14,15], Aubin, Chap. 12 [16] and Aubin, Chap. 13 [5], the books Mares [51] and Mishizaki and Sokawa [52], Basile [24–26], Basile, De Simone and Graziano [23], Florenzano [41]). Fuzzy coalitions have also been used in dynamical models of cooperative games in Aubin and Cellina, Chap. 4 [18] and of economic theory in Aubin, Chap. 5 [8].

Using viability multipliers, we can modify the above dynamics by introducing regulons that are elements $q \in Y^*$ of the dual Y^* of the space Y:

Theorem 3.1 *Assume that the functions c_i, κ_i and α_S are continuous and that $M \subset Y$ are closed. Then the constraints*

$$\forall\, t \geq 0, \quad \sum_{S \subset P(\chi(t))} A_S(t)(\chi(t) \circ x(t))$$

$$= \sum_{S \subset P(\chi(t))} \left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x(t)) \ \in \ M$$

are viable under the control system

$$\begin{cases} (i) \quad x_i'(t) = c_i(x_i(t)) - \sum_{S \ni i} \left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x_{-i}(t))^* q(t), \ i{=}1,\ldots,n \\ \\ (ii) \ \ \chi_i'(t) = \kappa_i(\chi(t)) - \sum_{S \ni i} \left(\prod_{j \in S \setminus i} \chi_j(t) \right) \langle q(t), A_S(t)\,(x(t)) \rangle\,, \ i{=}1,\ldots,n \\ \\ (iii) \ A_S'(t) = \alpha_S(A(t)) - \left(\prod_{j \in S} \chi_j(t) \right) \left(\bigotimes_{j \in S} x_j(t) \right) \otimes q(t), \ S \subset N \\ \qquad \text{where } q(t) \in N_M(\sum_{S \subset P(\chi(t))} \left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x(t)))) \end{cases}$$

Let us comment on these formulas. First, the viability multipliers $q(t) \in Y^*$ can be regarded as regulons, i.e., regulation controls or parameters, or virtual prices in the language of economists. They are chosen adequately at each instant in order that the viability constraints describing the network can be satisfied at each instant, and the above theorem guarantees this possibility. The next section tells us how to choose at each instant such regulons (the regulation law).

For each actor i, the velocities $x_i'(t)$ of the state and the velocities $\chi_i'(t)$ of its membership in the fuzzy coalition $\chi(t)$ are corrected by subtracting

1. the sum over all coalitions S to which he belongs of the $A_S(t)(x_{-i}(t))^* q(t)$ weighted by the membership $\gamma_S(\chi(t))$:

$$x_i'(t) \ = \ c_i(x_i(t)) - \sum_{S \ni i} \gamma_S(\chi(t)) A_S(t)(x_{-i}(t))^* q(t)$$

2. the sum over all coalitions S to which he belongs of the costs $\langle q(t), A_S(t)\,(x(t)) \rangle$ of the constraints associated with connectionist tensor A_S of the coalition S weighted by the membership $\gamma_{S \setminus i}(\chi(t))$:

$$\chi_i'(t) \ = \ \kappa_i(\chi(t)) - \sum_{S \ni i} \gamma_{S \setminus i}(\chi(t)) \langle q(t), A_S(t)\,(x(t)) \rangle$$

This type of dynamics describes a *mimetic or herd (or panurgean) effect*. The (algebraic) increase of actor i's membership in the fuzzy coalition aggregates over all coalitions to which he belongs the cost of their constraints weighted by the products of memberships of the other actors in the coalition.

As for the correction of the velocities of the connectionist tensors A_S, their correction is a weighted "multi-Hebbian" rule: for each component $A^j_{S_{\Pi_{i\in S} i_k}}$ of A_S, the correction term is the product of the membership $\gamma S(\chi(t))$ of the coalition S, of the components $x_{i_k}(t)$ and of the component $q^j(t)$ of the regulon:

$$\frac{d}{dt}A^j_{S_{\Pi_{i\in S} i_k}} = \alpha_{S_{\Pi_{i\in S} i_k}}(A(t)) - \gamma_S(\chi(t))\left(\prod_{i\in S} x_{i_k}(t)\right) q^j(t)$$

3.3 The regulation map

Actually, the viability multipliers $q(t)$ regulating viable evolutions of the actions $x_i(t)$, the fuzzy coalitions $\chi(t)$ and the multiaffine operators $A(t)$ obey the regulation law (an "adjustment law", in the vocabulary of economists) of the form

$$\forall\, t \geq 0,\ \ q(t) \in R_M(x(t), \chi(t), A(t))$$

where $R_M : X^N \times \mathbf{R}^n \times \mathcal{A}_n(X^N, Y) \rightsquigarrow Y^*$ is the $\mathrm{regulation\ map}$ R_M that we shall compute.

For this purpose, we introduce the operator $h : X^N \times \mathbf{R}^n \times \mathcal{A}_n(X^N, Y)$ defined by

$$h(x, \chi, A) := \sum_{S\subset N} A_S(\chi \circ x)$$

and the linear operator $H(x, \chi, A) : Y^* := Y \mapsto Y$ defined by:

$$\begin{cases} H(x, \chi, A) := \displaystyle\sum_{S\subset N}\left(\prod_{j\in S}\chi_j^2\|x_j\|^2\right)\mathbf{I} \\ + \displaystyle\sum_{R,S\subset N}\sum_{i\in R\cap S}(\gamma_R(\chi)\gamma_S(\chi)A_R(x_{-i})A_S(x_{-i})^* \\ +\gamma_{R\backslash i}(\chi)\gamma_{S\backslash i}(\chi)A_R(x)\otimes A_S(x)) \end{cases}$$

Then the regulation map is defined by

$$\begin{cases} R_M(x, \chi, A) := H(x, \chi, A)^{-1} \\ \left(\displaystyle\sum_{S\subset N}\left(\alpha_S(A)(x) + \sum_{i\in S}\left(\gamma_S(\chi)A_S(x_{-i}, c_i(x)) + \gamma_{S\backslash i}(\chi)\kappa_i(\chi)A_S(x)\right)\right) \\ -T_M(h(x, \chi, A)) \qquad \right) \end{cases}$$

Indeed, the regulation map R_M associates with any (x,χ,A) the subset $R_M(x,\chi,A)$ of $q \in Y^*$ such that

$$h'(x,\chi,A)((c(x),\kappa(\chi),\alpha(A)) - h'(x,\chi,A)^* q) \ \in \ \overline{\text{co}}\,(T_M(h(x)))$$

We next observe that

$$h'(x,\chi,A)h'(x,\chi,A)^* \ = \ H(x,\chi,A)$$

and that

$$\begin{cases} h'(x,\chi,A)(c(x),\kappa(\chi),\alpha(A)) \\ = \displaystyle\sum_{S\subset N} \left(\alpha_S(A)(x) + \sum_{i\in S} \left(\gamma_S(\chi) A_S(x_{-i}, c_i(x)) + \gamma_{S\setminus i}(\chi)\kappa_i(\chi) A_S(x) \right) \right) \end{cases}$$

4 Case of nonlinear constraints

We may complicate somewhat the structure of the constraints by introducing finite dimensional vector spaces Y_S indexed[9] by $S \subset N$, by requiring that the S-linear operators $A_S \in \mathcal{L}_S(X^S, Y_S)$ map X^S to the vector space Y_S and by involving nonlinearities defined by a map

$$g : \prod_{S\subset N} Y_S \mapsto Y$$

from the product of the vector spaces Y_S to the resource space Y.

By taking $Y_S := Y$ for all $S \subset N$ and $g(\{y_S\}_{S\subset N}) := \displaystyle\sum_{S\subset N} y_S$, Theorem 3.1 is then a particular case of

Theorem 4.1 *Assume that the functions c_i, κ_i and α_S are continuous, that $M \subset Y$ are sleek and closed and that the map $g : \prod_{S\subset N} Y_S \mapsto Y$ is continuously differentiable. Then the constraints*

$$\forall\, t \geq 0, \ \ g\left(\{A_S(t)(\chi(t) \circ x(t))\}_{S\subset N}\right) \ \in \ M$$

[9] The space Y_S is not necessarily the product $Y^S := \prod_{i\in S} Y_i$. They can all be equal to a same resource space Y for instance.

are viable under the control system

$$\begin{cases} (i) \quad x_i'(t) = c_i(x_i(t)) - \sum_{S \ni i} \left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x_{-i}(t))^* q^S(t), \; i=1,\ldots,n \\ (ii) \; \chi_i'(t) = \kappa_i(\chi(t)) - \sum_{S \ni i} \left(\prod_{j \in S \setminus i} \chi_j(t) \right) \left\langle q^S(t), A_S(t)\left(x(t)\right) \right\rangle, \; i=1,\ldots,n \\ (iii) \; A_S'(t) = \alpha_S(A(t)) - \left(\prod_{j \in S} \chi_j(t) \right) \left(\bigotimes_{j \in S} x_j(t) \right) \otimes q^S(t), \; S \subset N \\ \quad \text{where } q(t) \in N_M \left(g\left(\{A_S(t)(\chi(t) \circ x(t))\}_{S \subset N} \right) \right) \\ \quad \text{and where} \forall S \subset N, \; q^S(t) := \left(\frac{\partial g}{\partial y_S} \left(\{A_S(t)(\chi(t) \circ x(t))\}_{S \subset N} \right) \right)^* q(t) \end{cases}$$

We can multiply the examples. For instance, by involving nonlinearities defined by maps

$$\forall \; S \subset N, \;\; g_S : Y_S \rightsquigarrow Y$$

from finite dimensional vector spaces Y_S indexed by S to the resource space Y and taking $Y : \prod_{S \subset N} Y_S$ and $g\left(\{y_S\}_{S \subset N}\right) := \{g_S(y_S)\}$, we obtain the following

Corollary 4.2 *Assume that the functions c_i, κ_i and α_S are continuous, that $M \subset Y$ are sleek and closed and that the maps $g_S : Y_S \mapsto Y$ are continuously differentiable. Then the constraints*

$$\forall \; t \geq 0, \quad \sum_{S \subset P(\chi(t))} g_S \left(\left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x(t)) \right) \; \in \; M$$

are viable under the control system

$$\begin{cases} (i) \quad x_i'(t) = c_i(x_i(t)) - \sum_{S \ni i} \left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x_{-i}(t))^* q^S(t), \; i=1,\ldots,n \\ (ii) \; \chi_i'(t) = \kappa_i(\chi(t)) - \sum_{S \ni i} \left(\prod_{j \in S \setminus i} \chi_j(t) \right) \left\langle q^S(t), A_S(t)\left(x(t)\right) \right\rangle, \; i=1,\ldots,n \\ (iii) \; A_S'(t) = \alpha_S(A(t)) - \left(\prod_{j \in S} \chi_j(t) \right) \left(\bigotimes_{j \in S} x_j(t) \right) \otimes q^S(t), \; S \subset N \\ \quad \text{where } q(t) \in N_M (\sum_{S \subset P(\chi(t))} g_S \left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x(t)))) \\ \quad \text{and where } \forall S \subset N, \; q^S(t) := \left(g_S' \left(\left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x(t)) \right) \right)^* q(t) \end{cases}$$

5 Example: regulation of the production processes of a dynamical economy

We introduce

1. $\mathbb{I}$ economic agents $i = 1, \ldots, \mathbb{I}$,
2. $\mathbb{K}$ firms $k = 1, \ldots, \mathbb{K}$ described by a subset $S_k \subset \{1, \ldots, \mathbb{I}\}$ of agents employed by firm k.

Agent i

1. consumes commodities $x_i^k \in X$ produced by firm k in the commodity space $X := \mathbb{R}^l$
2. whenever he is employed by firm k, $i \in S_k$ provides to firm k resources $y_i^k \in Y_i$ in a resource space $Y_i := \mathbb{R}^{m_i}$.

   We set
$$y^k := \{y_i^k\}_{i \in S_k} \in Y^k := Y^{S_k} := \prod_{i \in S_k} Y_i$$

The state $(x_i^k, y_i^k)_{k=1,\ldots,\mathbb{K}}$ of agent i is made up of his consumptions $(x_i^k)_{k=1,\ldots,\mathbb{K}}$ that he receives and the resources $(y_i^k)_{k=1,\ldots,\mathbb{K}}$ that he provides to the $\mathbb{K}$ firms.

Production processes of firms $k \in \mathbb{K}$ are described

1. by multi-affine input-output maps $A^k := \sum_{S \subset S_k} A_S^{S_k} : Y^{S_k} \mapsto X$ associating a commodity $x \in X$ with any resource $y^k \in Y^{S_k}$ provided to firm k by each agent. The production process $A^k := \sum_{S \subset S_k} A_S^{S_k} : Y^{S_k} \mapsto X$ aggregates the commodities produced by all the coalitions $S \subset S_k$ of employees of firm (naturally, we assume that $A_S^{S_k} = 0$ if coalition S does not use the resources of its members). The case when $A_\emptyset^k \in X$ represents available commodities not produced by firm k, the case when all the maps $A_S^k = 0$ for all coalitions having more than one agent represents affine production maps (linear plus a constant). We set $Y^S := \prod_{i \in S} Y$ and denote by $\mathcal{L}_S(Y^S, X)$ the space of S-linear $A^S : Y^S \mapsto X$ maps from the space Y^S of resources available to coalition S in the commodity space. We identify $\mathcal{L}_\emptyset(Y^S, X) =: X$ with the commodity space X.
2. for each agent $i = 1, \ldots, \mathbb{I}$, by a set-valued map $L_i : X \rightsquigarrow Y_i$ associating with its total consumption $\sum_{k=1}^{\mathbb{K}} x_i^k \in X$ the set of resources which he provides to the firms in which he participates.
3. fuzzy coalitions $\chi \in [0,1]^n$ that are involved in the production process only.

We describe “competition” among firms k by introducing $\mathbb{L}$ pairwise distinct coalitions $T_l \subset \{1, \ldots, \mathbb{K}\}$ of firms competing in a same “market” l, $(l = 1, \ldots, \mathbb{L})$.

The supply and demand constraints described by

$$\begin{cases} (i) \;\; \forall l = 1, \ldots, \mathbb{L}, \; \sum_{i=1}^{\mathbb{I}} \sum_{k \in T_l} x_i^k(t) \le \sum_{k \in T_l} \sum_{S \subset S_k} \gamma_S(\chi(t)) A_S^k(y^k(t)) \\ \\ (ii) \; \forall i = 1, \ldots, \mathbb{I}, \; \sum_{\{k | S_k \ni i\}} y_i^k(t) \in L_i \left(\sum_{k=1}^{\mathbb{K}} x_i^k(t) \right) \end{cases} \quad (3)$$

cannot be violated.

For instance, we can take

1. **One market**: We take $\mathbb{L} := 1$ and $T_1 := \{1, \dots, \mathbb{K}\}$. Therefore the supply constraint (3) (i) can be written

$$\sum_{i=1}^{\mathbb{I}} \sum_{k=1}^{\mathbb{K}} x_i^k(t) \leq \sum_{k=1}^{\mathbb{K}} \sum_{S \subset S_k} \gamma_S(\chi(t)) A_S^k(y^k(t))$$

2. **One market per firm**: We take $\mathbb{L}:=\mathbb{K}$ and $T_l:=\{l\}, l=1, \dots, \mathbb{K}$. Therefore the supply constraint (3) (i) can be written

$$\forall k = 1, \dots, \mathbb{K}, \ \sum_{i=1}^{\mathbb{I}} x_i^k(t) \leq \sum_{S \subset S_k} \gamma_S(\chi(t)) A_S^k(y^k(t))$$

Individual constraints (3) (ii) involve implicitly feasibility constraints on the total consumption ($\sum_{k=1}^{\mathbb{K}} x_i^k \in \mathrm{Dom}(L_i)$) and provide consumption dependent constraints on the total resources $\sum_{k=1}^{\mathbb{K}} x_i^k$ provided by the agent. Preferences preordering on the consumptions can also be taken into account as in Aubin [8].

Naturally, one can introduce other constraints that we shall not take into account here for the sake of simplicity. For instance, we can require that

$$\forall \ i = 1, \dots, \mathbb{I}, \ k = 1, \dots, \mathbb{K}, \ \ x_i^k \in A_i^k \subset X \ \& \ y_i^k \in B_i^k \subset Y_i$$

It is impossible to design, even qualitatively, the structure of such dynamical systems under which this set K of allocations is viable.

We thus start with *initial dynamics* describing the dynamical behavior of agents *in the absence of scarcity constraints*. This assumption is the dynamic analogue of the classical assumption describing the static behavior of an economic agent by its utility function. For instance, in the simplest case, the evolutionary behavior of agents is described by

1. $\mathbb{I} \cdot \mathbb{K}$ continuous maps $c_i^k : X \mapsto X$ governing the evolution of consumptions $x_i^{\cdot}$ bought by the ith agent to firm k,
2. $\mathbb{I} \cdot \mathbb{K}$ continuous maps $d_i^k : Y_i \mapsto Y_i$ governing the evolution of resources y_i^k brought to firm k by agent $i \in S_k$,
3. continuous maps $e_S^k : \mathcal{L}_S(Y^S, X) \mapsto \mathcal{L}_S(Y^S, X), \ k = 1, \dots, \mathbb{K}, \ S \subset S_k$, governing the evolution of the S-linear input-output maps,
4. $\mathbb{I}$ continuous maps $\kappa_i : [0, 1] \mapsto \mathbb{R}$ governing the evolution of the fuzzy coalition

by the system of differential equations

$$\begin{cases} (i) & \dfrac{d}{dt} x_i^k(t) \ = \ c_i^k(x_i^k(t)), \ (i = 1, \dots, \mathbb{I}) \\ (ii) & \dfrac{d}{dt} y_i^k(t) \ = \ d_i^k(y_i^k(t)), \ (i = 1, \dots, \mathbb{I}, \ k = 1, \dots, \mathbb{K}) \\ (iii) & \dfrac{d}{dt} \chi_i(t) \ = \ \kappa_i(\chi_i(t)), \ (i = 1, \dots, \mathbb{I}) \\ (iv) & \dfrac{d}{dt} A_S^k(t) \ = \ e_S^k(A_S^k(t)), \ (k = 1, \dots, \mathbb{K}, \ S \subset S_k) \end{cases}$$

Examples of such dynamics are provided by gradients $c_i^k(x_i^k) := \nabla \mathbf{u}_i^k(x_i^k)$ of utility functions $\mathbf{u}_i^k$ or gradients $d_i^k(y_i^k) := -\nabla \mathbf{v}_i^k(y_i^k)$ of disutility functions $\mathbf{v}_i^k$, or gradients $\kappa_i(\chi_i) := \nabla \mathbf{s}_i(\chi_i)$ of utility functions $\mathbf{s}_i$ or gradient $e_S^k(A_S^k) := \nabla \mathbf{w}_S^k(A_S^k)$ of functions $\mathbf{w}_S^k$ that must increase the consumption or decrease the resource, the memberships of fuzzy coalitions, the S-linear operators along their evolutions.

Naturally, there is no reason why the *global* scarcity constraints are viable under such a *local* dynamical system, because the scarcity constraints are *collective* and the dynamics of the agents are *individual*. To assume that these maps c_i^k and d_i^k describing the dynamical behavior of the agents depend upon the consumptions or resources of the other agents would not complicate the mathematics, but to the contrary, would be totally unrealistic: *This is the decentralizing character of prices which is the main message of general equilibrium models that we choose to keep in the dynamic and connectionist framework we have chosen.*

Since these global constraints are not viable under individual dynamic behaviors, we use viability multipliers

$$\begin{cases} (i) & p^l(t) \in X^\star := X \ (l = 1, \ldots, \mathbb{L}) \\ (ii) & u_i(t) \in {}^\star := X \ (i = 1, \ldots, \mathbb{I}) \\ (iii) & v_i(t) \in Y_i^\star := Y_i \ (i = 1, \ldots, \mathbb{I}) \end{cases}$$

to correct the dynamics:

$$\begin{cases} (i) & \frac{d}{dt} x_i^k(t) = c_i^k(x_i^k(t)) - u_i(t) - \sum_{l | T_l \ni k} p^l(t), \ (i{=}1, \ldots, \mathbb{I}) \\ (ii) & \frac{d}{dt} y_i^k(t) = d_i^k(y_i^k(t)) + v_i(t) \\ & \quad + \sum_{\{S | i \in S \subset S_k\}} \gamma_S(\chi(t)) A_S^k \left(y_{-i}^k(t)\right)^\star \left(\sum_{l | T_l \ni k} p^l(t) \right), \\ & (i = 1, \ldots, \mathbb{I}, \ k = 1, \ldots, \mathbb{K}) \\ (iii) & \frac{d}{dt} \chi_i(t) = \kappa_i(\chi_i(t)) + \sum_{i \in S \subset S_k} \gamma_{S \setminus \{i\}}(\chi_i(t)) \left\langle \left(\sum_{l | T_l \ni k} p^l(t) \right), A_S^k(y^k(t)) \right\rangle, \\ & (i{=}1, \ldots, \mathbb{I}) \\ (iv) & \frac{d}{dt} A_S^k(t) = e_S^k(A_S^k(t)) + \gamma_S(\chi(t)) \left(\bigotimes_{i \in S} y_i^k(t) \right) \otimes \left(\sum_{l | T_l \ni k} p^l(t) \right), \\ & (k{=}1, \ldots, \mathbb{K}, S \subset S_k) \end{cases}$$

By comparison, we see that if we minimize a collective utility function:

$$\sum_{i=1}^{\mathbb{I}} \mathbf{u}_i(x_i) + \sum_{k=1}^{\mathbb{K}} \mathbf{v}_i^k(y_i^k) + \sum_{i=1}^{\mathbb{I}} \mathbf{s}_i(\chi_i) + \sum_{k=1}^{\mathbb{K}} \sum_{S \subset S_k} \mathbf{w}_S^k(A_S^k)$$

under constraints (3)(i) and (ii), then first-order optimality conditions at a optimum $((x_i)_i, (y_i^k)_{i,\,k}, (\chi_i)_i, (A_S^k)_{k,\,S \subset S_k})$ imply the existence of Lagrange multipliers

p^l, u_i, v_i such that:

$$\begin{cases} \nabla \mathbf{u}_i(x_i) \;=\; u_i + \sum_{l|T_l \ni k} p^l \\ \nabla \mathbf{v}_i^k(y^k) \;=\; -v_i - \sum_{i \in S \subset S_k} \gamma_S(\chi) A_S^k \left(y_{-i}^k\right)^\star \left(\sum_{l|T_l \ni k} p^l \right) \\ \nabla \mathbf{s}_i(x\chi_i) \;=\; u_i + \sum_{i \in S \subset S_k} \gamma_{S\backslash\{i\}}(\chi) \left\langle \left(\sum_{l|T_l \ni k} p^l \right), A_S^k(y^k) \right\rangle \\ \nabla \mathbf{w}_S^k(A_S^k) \;=\; -\gamma_S(\chi) \left(\bigotimes_{i \in S} y_i^k \right) \otimes \left(\sum_{l|T_l \ni k} p^l \right) \end{cases}$$

6 Viability multipliers and proof of the main theorem

The proof is an application of the basic theorem on viability multipliers. We summarize here the basic facts that can be found in Aubin [12, 8].

6.1 Differentiating constraints

Consider the initial – disconnected – dynamical system

$$x'(t) \;=\; f(x(t)) \tag{4}$$

subject to collective viability constraints of the form

$$\forall\, t \geq 0, \quad h(x(t)) \;\in\; M \tag{5}$$

Nagumo's Invariance Theorem provides a necessary and sufficient condition for the subset

$$K \;:=\; h^{-1}(M) \;=\; \{x \in X \quad \text{such that} \;\; h(x) \in M\}$$

to be viable in the sense that from any initial state $x_0 \in K$ starts a solution $x(\cdot)$ to the differential equation $x' = f(x)$ viable in K in the sense that

$$\forall\, t \geq 0, \quad h(x(t)) \;\in\; M$$

For that purpose, we shall need to "differentiate" these viability constraints by implementing the concept of tangency to any subset. The adequate choice to obtain viability theorems is the concept of *contingent cone*[10] introduced for the first time by Georges Bouligand in the thirties (and which happens to be the cornerstone to set-valued analysis).

[10] For a presentation of the ménagerie of tangent cones, we refer to chapter 4 of Aubin and Frankowska [21] and Rockafellar and Wets [55].

Definition 6.1 *When K is a subset of X and x belongs to K, the contingent cone $T_K(x)$ to K at x is the closed cone of elements v satisfying*

$$\liminf_{h \to 0+} \frac{d(x+hv, K)}{h} \;=\; 0$$

We observe that

$$\text{if} \quad x \in \operatorname{Int}(K), \quad \text{then} \;\; T_K(x) \;=\; X$$

and that if $K := \{\bar{x}\}$ is a singleton, then $T_{\{\bar{x}\}}(\bar{x}) = \{0\}$.

We recall that a differentiable function viable in K satisfies

$$\forall\, t \geq 0, \;\; x'(t) \;\in\; T_K(x(t))$$

Let us mention that the contingent cone coincides with the tangent space $T_K(x)$ of differential geometry when K is a "smooth manifold". Also, when K is convex, one can prove that the contingent cone coincides with the *tangent cone* $T_K(x)$ to K at $x \in K$ of convex analysis, which is the closed cone spanned by $K - x$:

$$T_K(x) \;=\; \overline{\bigcup_{h>0} \frac{K-x}{h}}$$

In this case, the tangent cone is convex.

One can prove that the contingent cone $T_K(x)$ is convex whenever K "is *sleek* at x", which means that the set-valued map $T_K(\cdot)$ is lower semicontinuous at x. Convex subsets are sleek at every elements.

6.2 Viability multipliers

For differential equations $x' = f(x)$, the Viability Theorem was proved in 1942 by Nagumo[11], stating that K is viable under $x' = f(x)$ if and only if for any $x \in K$, the dynamics and the constraints are linked by the following relation:

$$f(x) \;\in\; \overline{\text{co}}(T_K(x))$$

When K is not viable under g, the simple idea is to project $f(x)$ onto the $\overline{\text{co}}(T_K(x))$ and to replace the initial dynamic g by its projection $\Pi_{\overline{\text{co}}(T_K(x))} f(x)$. Hence, whenever the differential equation

$$x'(t) \;=\; \Pi_{\overline{\text{co}}(T_K(x(t)))} f(x(t))$$

[11] That stated only that the necessary and sufficient condition for viability is

$$f(x) \;\in\; T_K(x)$$

Recalling that the polar cone $P^- \subset X^*$ of a subset $P \subset X$ is defined by $P^- := \{p \in X^* \mid \forall\, x \in P,\ \langle p, x\rangle \le 0\}$, we introduce the $\mathrm{normal\ cone}$ $N_K(x)$ to K at $x \in K$ defined by

$$N_M(x) \;:=\; T_K(x)^- \;=\; (\overline{\mathrm{co}}(T_K(x)))^-$$

Moreau's Theorem states that

$$\Pi_{\overline{\mathrm{co}}(T_K(x))} f(x) \;=\; f(x) - \Pi_{N_M(f(x))}$$

Therefore, every solution to the differential equation

$$x'(t) \;=\; \Pi_{\overline{\mathrm{co}}(T_K(x(t)))} f(x(t))$$

(introduced in Henry [42], studied in Cornet [36,37] and also presented in Aubin and Cellina [18]) is a solution to the control system

$$\begin{cases} (i) \;\; x'(t) \;=\; f(x(t)) - p(t) \\ (ii) \; p(t) \;\in\; N_K(x(t)) \end{cases}$$

regulated by the controls $p(t) \in N_K(x(t))$. This is the reason we call elements $p \in N_K(x)$ $\mathrm{viability\ multipliers}$.

There is a calculus of contingent cones which allows us to "compute them", and thus, to "apply" the viability theorems[12].

We recall in particular that if $K := h^{-1}(M)$ where $h : X \mapsto Y$ is a continuously differentiable map such that $h'(x)$ is surjective[13] and M is closed and convex (or, more generally, sleek), then

$$T_K(x) \;=\; h'(x)^{-1} T_M(h(x))$$

and

$$N_K(x) \;=\; h'(x)^* N_M(h(x))$$

Hence the control system can be written in the form

$$\begin{cases} (i) \;\; x'(t) \;=\; f(x(t)) - h'(x)^* q(t) \\ (ii) \; q(t) \;\in\; N_M(h(x(t))) \end{cases}$$

regulated by the controls $q(t) \in N_M(h(x(t)))$. In this explicit case, we reserve the word $\mathrm{viability\ multipliers}$ for the elements $q \in N_M(h(x)) \subset Y^*$.

Since $\Pi_{\overline{\mathrm{co}}(T_K(x))} f(x) \;=\; f(x) - \Pi_{N_K(x)} f(x)$, we know that the projection

$$\Pi_{N_K(x)} f(x) \;:=\; h'(x)^* \varpi_M(x)$$

[12] As well as the "equilibrium theorems under constraints" and "optimization theorems under constraints".

[13] A weaker requirement is the "transversality assumption"

$$\mathrm{Im}(h'(x)) + T_M(h(x)) \;=\; Y$$

and that, when $h'(x)$ is surjective,

$$\varpi_M(x) \; = \; \Pi^{h'(x)^*}_{N_M(h(x))}(h'(x)h'(x)^*)^{-1}h'(x)f(x)$$

where $\Pi^{h'(x)^*}_{N_M(h(x))}$ denotes the projection onto the normal cone $N_M(h(x))$ when Y^* is supplied with the scalar product

$$\langle q_1, q_2\rangle_{h'(x)^*} \; := \; \langle h'(x)^*q_1, h'(x)^*q_2\rangle$$

However the open loop control $q(t) := \varpi_M(x(t))$ obtained through this specific feedback is not the only control regulating viable evolutions of the control system

$$\begin{cases} (i) \;\; x'(t) \; = \; f(x(t)) - h'(x)^*q(t) \\ (ii) \; q(t) \; \in \; \Pi_M(x(t)) \end{cases}$$

where the $\mathrm{regulation\ map}$ Π_M is defined by

$$\Pi_M(x) \; := \; \{q \in Y^* \mid h'(x)h'(x)^*q \in h'(x)f(x) - \overline{\mathrm{co}}(T_M(h(x)))\}$$

6.3 Proof of the main theorem

We now derive the proof of Theorem 4.1. The constrained set $K \subset X^N \times [0,1]^n \times \mathcal{A}_n(X^N, Y)$ is of the form $h^{-1}(M)$ where $M \subset Y$ and where h is the map defined by

$$h(x, \chi, A) \; := \; g\left(\{A_S(\chi \circ x)\}_{S \subset N}\right)$$

This is a differentiable map the derivative $h'(x, \chi, A)$ of which is defined by

$$\begin{cases} h'(x, \chi, A)(dx, d\chi, dA) \\ \displaystyle\sum_{S \subset N} \frac{\partial g}{\partial y_S}\left(\{A_S(\chi \circ x)\}_{S \subset N}\right) \\ \displaystyle\left(dA_S(\chi \circ x) + \sum_{i \in S}\left(\prod_{j \in S \backslash i} \chi_j\right) A_S\left(x_{-i}\right)\left(\chi_i dx_i + d\chi_i x_i\right)\right) \end{cases}$$

Since $h'(x, \chi, A)$ is surjective, we know that the normal cone $N_K(x, \chi, A)$ is equal to

$$N_K(x, \chi, A) \; = \; h'(x, \chi, A)^* N_M(h(x, \chi, A))$$

Setting $q^S \; := \; \left(\frac{\partial g}{\partial y_S}\left(\{A_S(\chi \circ x)\}_{S \subset N}\right)\right)^* q$, simple algebraic manipulations show that

$$\begin{cases} \langle h'(x, \chi, A)^*q, (dx, d\chi, dA)\rangle \; = \; \langle q, h'(x, \chi, A)(dx, d\chi, dA)\rangle \\ \displaystyle\sum_{S \subset N}\left\langle q^S, dA_S(\chi \circ x)\right\rangle + \sum_{i=1}^{n}\sum_{S \ni i}\left(\prod_{j \in S}\chi_j\right)\left\langle A_S\left(x_{-i}\right)^* q^S, dx_i\right\rangle \\ \displaystyle + \sum_{i=1}^{n} d\chi_i \sum_{S \ni i}\left(\prod_{j \in S \backslash i}\chi_j\right)\left\langle q^S, A_S(\chi_S \circ x)\right\rangle \end{cases}$$

Therefore, we conclude that

$$\begin{cases} h'(x,\chi,A)^* q \;= \\ \left\{ \left(\sum_{S\ni i} \left(\prod_{j\in S} \chi_j \right) A_S\left(x_{-i}\right)^* q^S \right)_{i=1,\dots n}, \right. \\ \left(\sum_{S\ni i} \left(\prod_{j\in S\setminus i} \chi_j \right) \langle q^S, A_S(\chi_S \circ x)\rangle \right)_{i=1,\dots n}, \\ \left. \left(\left(\prod_{j\in S} \chi_j \right) \left(\bigotimes_{j\in S} x_j \right) \otimes q^S \right)_{S\subset N} \right\} \\ \in \prod_{i=1}^{n} X_i^* \times \mathbf{R}^n \times \prod_{S\subset N} \mathcal{L}_S(X^{S^*}, Y^*) \end{cases}$$

References

1. Allen B (2000) The future of microeconomic theory. Journal of Economic Perspectives 14: 143–150
2. Arthur WN, Durlauf SN, Lane D (eds) (1997) The economy as an evolving complex system, vol II. SFI Studies in the Sciences of Complexity, Addison- Wesley, New York
3. Aubin J-P (2002) Dynamic core of fuzzy dynamical cooperative games. Annals of Dynamic Games, Ninth International Symposium on Dynamical Games and Applications, Adelaide 2000
4. Aubin J-P (1999) Mutational and morphological analysis: tools for shape regulation and morphogenesis. Birkhäuser, Boston Basel Berlin
5. Aubin J-P (1998, 1993) Optima and equilibria (2nd ed). Springer, Berlin Heidelberg New York
6. Aubin J-P (1998) Connectionist complexity and its evolution. In: Equations aux dérivées partielles. Articles dédiés à J-L Lions, pp 50–79. Elsevier, Amsterdam
7. Aubin J-P (1998) *Minimal complexity and maximal decentralization*. In: Beckmann HJ, Johansson B, Snickars F, Thord D (eds) Knowledge and information in a dynamic economy, pp 83–104. Springer, Berlin Heidelberg New York
8. Aubin J-P (1997) Dynamic economic theory: a viability approach. Springer, Berlin Heidelberg New York
9. Aubin J-P (1996) Neural networks and qualitative physics: a viability approach. Cambridge University Press, Cambridge
10. Aubin J-P (1995) Learning as adaptive control of synaptic matrices. In: Arbib M (ed) The handbook of brain theory and neural networks, pp 527–530. Bradford Books and MIT Press
11. Aubin J-P (1993) Beyond neural networks: cognitive systems. In: Demongeot J, Capasso (eds) Mathematics Appliedto Biology and Medicine. Wuers, Winnipeg
12. Aubin J-P (1991) Viability theory. Birkhäuser, Boston Basel Berlin
13. Aubin J-P (1982) An alternative mathematical description of a player in game theory. IIASA WP-82-122
14. Aubin J-P (1981) Cooperative fuzzy games. Mathematical Operation Research 6: 1–13
15. Aubin J-P (1981) Locally lipchitz cooperative games. Journal of Mathematical Economics 8: 241–262
16. Aubin J-P (1979) Mathematical methods of game and economic theory. (Studies in Mathematics and its applications, Vol 7, pp 1–619. North Holland, Amsterdam
17. Aubin J-P, Burnod Y (1998) Hebbian learning in neural networks with gates. Cahiers du Centre de Recherche Viabilité, Jeux, Contrôle # 981
18. Aubin J-P, Cellina A (1984) Differential inclusions. Springer, Berlin Heidelberg New York

19. Aubin J-P, Dordan O (1996) Fuzzy systems, viability theory and toll sets. In: Nguyen H (ed) Handbook of fuzzy systems, modeling and control, pp 461–488. Kluwer, Amsterdam
20. Aubin J-P, Foray D (1998) The emergence of network organizations in processes of technological choice: a viability approach. In: Cohendet P, Llerena P, Stahn H, Umbhauer G (eds) The economics of networks, pp 283–290. Springer, Berlin Heidelberg New York
21. Aubin J-P, Frankowska H (1990) Set-valued analysis. Birkhäuser, Boston Basel Berlin
22. Aubin J-P, Louis-Guerin C, Zavalloni M (1979) Comptabilité entre conduites sociales réelles dans les groupes et les représentations symboliques de ces groupes : un essai de formalisation mathématique. Mathematical Sciences Hum 68: 27–61
23. Basile A, De Simone A, Graziano MG (1996) On the Aubin-like characterization of competitive equilibria in infinite-dimensional economies. Rivista di Matematica per le Scienze Economiche e Sociali 19: 187–213
24. Basile A (1993) Finitely additive nonatomic coalition production economies: Core-Walras equivalence. International Economic Review 34: 993–995
25. Basile A (1994) Finitely additive correpondences. Proceedings AMS 121: 883–891
26. Basile A (to appear) On the range of certain additive correspondences. Universita di Napoli, Napoli
27. Bonneuil N (2001) History, differential inclusions and narrative. History and Theory, Theme issue 40 "Agency after postmodernism", Wesleyan University, pp 101–115
28. Bonneuil N (2000) Viability in dynamic social networks. Journal of Mathematical Sociology 24: 175–182
29. Bonneuil N (1998) Games, equilibria, and population regulation under viability constraints: an interpretation of the work of the anthropologist Fredrik Barth. Population: An English selection. Special issue of: New methods in demography, pp 151–179
30. Bonneuil N (1998) Population paths implied by the mean number of pairwise nucleotide differences among mitochondrial sequences. Annals of Human Genetics 62: 61–73
31. Bonneuil N (in preparation) Possible coalescent populations implied by pairwise comparisons of mitochondrial DNA sequences. Theoretical Population Biology (forthcoming)
32. Bonneuil N, Saint-Pierre P (2000) Protected polymorphism in the theo-locus haploid model with unpredictable firnesses. Journal of Mathematical Biology 40: 251–377
33. Bonneuil N, Saint-Pierre P (1998) Domaine de victoire et stratégies viable dans le cas d'une correspondance non convexe : application à l'anthropologie des pêcheurs selon Fredrik Barth. Mathématiques et Sciences Humaines 132: 43–66
34. Bala V, Goyal S (2000) A noncooperative model of network formation. Econometrica 68(5): 1181–1229
35. Brock WA, Durlauf SN (2001) Discrete choice with social interactions. Review of Economic Studies 68(2) 235–260
36. Cornet B (1976) On planning procedures defined by multivalued differential equations. Système dynamiques et modèles économiques, C.N.R.S
37. Cornet B (1983) An existence theorem of slow solutions for a class of differential inclusions. Journal of Mathematical Analysis and Applications 96: 130–147
38. Deghdak M, Florenzano M (1999) Decentralizing Edgeworth equilibria in economies with many commodities. Economic Theory 14: 287–310
39. Durlauf SN (1997) Statistical mechanics approaches to socioeconomic behavior. In: Arthur WN, Durlauf SN, Lane D (eds) The economy as an evolving complex system, vol II, pp 81–104. SFI Studies in the Sciences of Complexity. Addison-Wesley, New York
40. Filar JA, Petrosjan LA (2000) Dynamic cooperative games. International Game Theory Review 2: 47–65
41. Florenzano M (1990) Edgeworth equilibria, fuzzy core and equilibria of a production economy without ordered preferences. Journal of Mathematical Analysis and Applications 153: 18–36
42. Henry C (1972) Differential equations with discontinuous right hand side. Journal of Economic Theory 4: 545–551
43. Ioannides YM (1997) Evolution of trading structures. In: Arthur WN, Durlauf SN, Lane D (eds) The economy as an evolving complex system, vol II, pp 129–167. SFI Studies in the sciences of complexity. Addison- Wesley, New York

44. Ioannides YM (1997) The evolution of trading structures. In: Arthur WN, Durlauf SN, Lane D (eds) The economy as an evolving complex system, vol II, pp 129–167. SFI Studies in the Sciences of Complexity. Addison-Wesley, New York
45. Ioannides YM (1999) Location decisions, social interactions in cities, and lifetime income inequality. Preprint
46. Ioannides YM (2000) Modelling complex production settings. Preprint
47. Ioannides YM (2000) The architecture of complex production organizations. Preprint
48. Ioannides YM (2001) Topologies of social interactions. Preprint
49. Ioannides YM, Loury LD (1999) Job information effects, and inequality networks, neighborhood effects and inequality. AEA Meetings
50. Kremer M, Maskin E (1996) Segregation by skill. NBER working paper No 5718
51. Mares M (2001) Fuzzy cooperative games. Cooperation with vague expectations. Physica, Heidelberg
52. Mishizaki I, Sokawa M (2001) Fuzzy and multiobjective games for conflict resolution. Physica, Heidelberg
53. Petrosjan LA (1996) The time consistency (dynamic stability) in differential games with a discount factor. Game theory and application, 4753. Nova Science, Commack, NY
54. Petrosjan LA, Zenkevitch NA (1996) Game theory. World Scientific, Singapore
55. Rockafellar RT, Wets R (1997) Variational analysis. Springer, Berlin Heidelberg New York