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Topographic and atmospheric corrections of gravimetric geoid determination with special emphasis on the effects of harmonics of degrees zero and one

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Abstract. The topographic and atmospheric effects of gravimetric geoid determination by the modified Stokes formula, which combines terrestrial gravity and a global geopotential model, are presented. Special emphasis is given to the zero- and first-degree effects. The normal potential is defined in the traditional way, such that the disturbing potential in the exterior of the masses contains no zero- and first-degree harmonics. In contrast, it is shown that, as a result of the topographic masses, the gravimetric geoid includes such harmonics of the order of several centimetres. In addition, the atmosphere contributes with a zero-degree harmonic of magnitude within 1 cm.

Key words: Atmospheric Effects – Geoid – Topographic Effects

1 Introduction

Today, the estimation of the geoid from gravity typically relies on the well-known Stokes formula with surface gravity anomalies and a global gravity model (GGM) or a combination of the two data sets. Usually the terrestrial gravity data is corrected for the attraction of the 'forbidden' masses of the topography and the atmosphere (direct gravity effects) by the removal or reduction of these masses, and the surface anomaly is downward continued to sea level prior to the application of the Stokes integral. Finally, the quasi-geoid so determined is corrected to the geoid by the indirect effect, implying the restoration of the masses exterior to the geoid (i.e. topography and atmosphere). As it is also of great interest to compare and combine the gravimetric geoid model with a GPS-levelling-derived geoid model, information on its absolute position by its zero- and first-degree harmonics is gaining increased interest.

This article deals specifically with the zero- and first-degree effects on the geoid estimation. Discussion of the higher-degree effects on Stokes formula can, in the case of topography, be found in, for example, Sjöberg (1997, 2000), and, in the case of the atmosphere, be found in Sjöberg (1998, 1999a), but these effects are also included here for completeness. However, the paper does not treat the small corrections from the spherical to ellipsoidal shape of the reference surface; such effects can be found in, for example, Martinec (1998).

2 The modification of Stokes' formula

The combination of Stokes formula and a GGM can be represented by the modified Stokes formula originating with M. S. Molodensky (cf. Molodensky et al. 1962; Sjöberg 1991)

$$\tilde{N} = \frac{T_0^{\mathrm{H}}}{\gamma} + \frac{T_1^{\mathrm{H}}}{\gamma} + \frac{R}{4\pi\gamma} \iint\limits_{\sigma_0} S_L(\psi) \left(\Delta g^{\mathrm{H}}\right)^* \mathrm{d}\sigma + c \sum_{n=0}^{M} \left(s_n^* + Q_{Ln}\right) \Delta g_n^{\mathrm{H}} + \delta N_{\mathrm{I}} \tag{1}$$

where R is the mean surface radius of the Earth, γ is normal gravity at the reference ellipsoid, σ_0 is the spherical cap at the unit sphere of geocentric angle ψ_0, L is the degree of modification of Stokes' kernel, M is the upper limit of degree of the GGM, T_0^{H} and $T_1^{\mathrm{H}}$are the Helmert disturbing potential harmonics of degrees zero and one, and δN_{I} is the indirect effect (i.e. the effect on the geoidal height of restoring the topography). Furthermore, we use the following notation:

$$S_L(\psi) = S(\psi) - \sum_{n=0}^{L} \frac{2n+1}{2} s_n P_n(\cos\psi)$$

$S(\psi) =$ Stokes' kernel function

$\Delta g^{\mathrm{H}} = \Delta g + \delta\Delta g_{\mathrm{dir}}$

which means the Helmert anomaly defined by the surface gravity anomaly plus its direct effect. Furthermore, ( )* is the function in the bracket ( ) downward continued to sea level

$$\Delta g_n^{\mathrm{H}} = \Delta g_n^{\mathrm{S}} + (\delta \Delta g_{\mathrm{dir}}^{\mathrm{S}})_n$$

which is the geopotential-model-derived gravity anomaly harmonic plus its direct effect downward continued to sea level

$$c = R/(2\gamma)$$

$$s_n^* = \begin{cases} s_n & \text{if } 0 \le n \le L \\ 0 & \text{if } n > L \end{cases}$$

where s_n are selected parameters of the modified Stokes kernel and

$$Q_{Ln} = Q_n - \sum_{k=0}^{L} \frac{2k+1}{2} e_{nk} s_k$$

$$Q_n = \int_{\psi_0}^{\pi} S(\psi) \sin\psi \, \mathrm{d}\psi$$

$$e_{nk} = \int_{\psi_0}^{\pi} P_n(\cos\psi) P_k(\cos\psi) \sin\psi \, \mathrm{d}\psi$$

Both Δg^{H} and Δg_n^{H} are corrected for the reductions of the topography and the atmosphere. Most frequently the mass of the reference ellipsoid, related to the normal gravity field, is chosen to be equal to the mass of the Earth (including the atmosphere), and the origin of the reference ellipsoid is chosen to be at the gravity centre of the Earth. In this way both harmonics T_0 and T_1 vanish in the exterior space. However, this is not necessarily the case after the application of the direct topographic and atmospheric effects, i.e. for T_0^{H} and T_1^{H}, as will be considered below.

In the reduction of the exterior masses for T_0^{H}, T_1^{H}, Δg^{H} and Δg_n^{H} we will use Helmert's second method of condensation (Heiskanen and Moritz 1967, p. 145), and consequently the indirect effect δN_{I} also refers to Helmert's method.

The estimator of Eq. (1) can also be written

$$\tilde{N} = \frac{T_0}{\gamma} + \frac{T_1}{\gamma} + \frac{R}{4\pi\gamma} \iint_{\sigma_0} S_L(\psi) \Delta g \, \mathrm{d}\sigma + c \sum_{n=0}^{M} \left(s_n^* + Q_{Ln}\right) \Delta g_n^{\mathrm{S}} + \delta N_{\mathrm{tot}} \qquad (2)$$

where δN_{tot} is the total topographic and atmospheric correction consisting of the sum of the direct, indirect and downward-continuation effects

$$\delta N_{\mathrm{tot}} = \delta N_{\mathrm{dir}} + \delta N_{\mathrm{I}} + \delta N_{\mathrm{dwc}} \qquad (3)$$

This formula clearly shows that there are three steps in correcting Stokes' formula for the influences of the topography and the atmosphere. Comparing Eqs. (1) and (2), we obtain (cf. Sjöberg 1997)

$$\delta N_{\mathrm{dir}} = \frac{(T_0)_{\mathrm{dir}}}{\gamma} + \frac{(T_1)_{\mathrm{dir}}}{\gamma} + \frac{R}{4\pi\gamma} \iint_{\sigma_0} S_L(\psi) \delta \, \Delta g_{\mathrm{dir}} \, \mathrm{d}\sigma + c \sum_{n=0}^{M} \left(s_n^* + Q_{Ln}\right) \left(\delta \Delta g_{\mathrm{dir}}^{\mathrm{S}}\right)_n \qquad (4a)$$

and

$$\delta N_{\mathrm{dwc}} = \frac{R}{4\pi\gamma} \iint_{\sigma_0} S_L(\psi) \left(\left(\Delta g^{\mathrm{H}}\right)^* - \Delta g^{\mathrm{H}} \right) \mathrm{d}\sigma \qquad (4b)$$

where

$$(T_i)_{\mathrm{dir}} = \left(T_i^{\mathrm{H}}\right) - T_i; \quad i = 0, 1$$

Moreover,

$$\delta \Delta g_{\mathrm{dir}} = \Delta g^{\mathrm{H}} - \Delta g$$

is the direct effect on surface gravity anomaly, and $(\delta \Delta g_{\mathrm{dir}}^{\mathrm{S}})_n$ is the direct effect on the GGM derived anomaly (at sea level).

The indirect effect on the geoid (the restoration step) is nothing but the negative difference of the topographic potential and the reduction potential, both potentials computed at sea level and divided by normal gravity at the reference ellipsoid (see Eq. (18) below).

Alternatively, as

$$\Delta g^{\mathrm{H}} = \Delta g + \delta \Delta g_{\mathrm{dir}} \qquad (5)$$

we may also replace Eqs. (4a) and (4b) by the formulas

$$\delta N_{\mathrm{dir}} = \frac{(T_0)_{\mathrm{dir}}}{\gamma} + \frac{(T_1)_{\mathrm{dir}}}{\gamma} + \frac{R}{4\pi\gamma} \iint_{\sigma_0} S_L(\psi) \delta \, \Delta g_{\mathrm{dir}}^* \, \mathrm{d}\sigma + c \sum_{n=0}^{\infty} \left(s_n^* + Q_{Ln}\right) \left(\delta \Delta g_{\mathrm{dir}}^{\mathrm{S}}\right)_n \qquad (6a)$$

and

$$\delta N_{\mathrm{dwc}} = \frac{R}{4\pi\gamma} \iint_{\sigma_0} S_L(\psi) \left(\Delta g^* - \Delta g\right) \mathrm{d}\sigma \qquad (6b)$$

This latter approach will be used in the present study. The direct and indirect effects and their sum, the combined effect, will be treated separately for the topography and the atmosphere.

The present paper does not deal with the corrections from the spherical to the ellipsoidal shape of the reference surface, but such effects can be found in, for example, Martinec (1998).

3 The topographic effects

There are three types of topographic effects to be considered, namely the direct and indirect effects (or the effects of reduction and restoration of topography) and the effect of downward continuation of the surface gravity anomaly to sea level.

In what follows the topographic density is assumed to be constant, but all formulas could easily be extended to a laterally variable topographic density simply by putting the density under the surface integral.

3.1 The direct topographic effect (the topographic reduction step)

The direct topographic effect implies that the topography is substituted by a condensed layer at sea level of surface density ρH, where ρ is the topographic density and H is the elevation of topography. The direct effect on potential can be written (for a constant topographic density $\mu = G\rho$ and G = gravitational constant)

$$\delta V_{\rm dir}(P) = -\mu \iint\limits_{\sigma} \int\limits_{R}^{r_S} \frac{r^2\,{\rm d}r\,{\rm d}\sigma}{\ell_P} + \mu R^2 \iint\limits_{\sigma} \frac{H}{\ell_{Z0}}\,{\rm d}\sigma \qquad (7)$$

where

$$r_S = R + H$$

$$r_P = R + H_P$$

$$\ell_P = \left(r_P^2 + r^2 - 2r_P r\cos\psi\right)^{1/2}$$

$$\ell_{Z0} = \left(r_P^2 + R^2 - 2r_P R\cos\psi\right)^{1/2}$$

and σ is the unit sphere.

The first term of Eq. (7) is the effect of removing the topographic potential, and the second term is the Helmert layer potential. Formula (7) can be expanded into Legendre's polynomials $[P_n(t)]$ as follows:

$$\delta V_{\rm dir}(P) = -\frac{\mu}{r_P}\sum_{n=0}^{\infty} \iint\limits_{\sigma}\int\limits_{R}^{r_S}\left(\frac{r}{r_P}\right)^n r^2\,{\rm d}r\,P_n(t){\rm d}\sigma + \frac{\mu R^2}{r_P}\sum_{n=0}^{\infty}\iint\limits_{\sigma} H\left(\frac{R}{r_P}\right)^n P_n(t){\rm d}\sigma \qquad (8)$$

where

$$t = \cos\psi$$

From Eq. (8) the direct effect on the surface gravity anomaly is obtained by the *fundamental equation*

$$\delta\Delta g_{\rm dir}(P) = -\frac{\partial \delta V_{\rm dir}(P)}{\partial r_P} - \frac{2\delta V_{\rm dir}(P)}{r_P} \qquad (9)$$

The result is

$$\delta\Delta g_{\rm dir}(P) = -\mu\sum_{n=0}^{\infty}(n-1)\iint\limits_{\sigma}\int\limits_{R}^{r_S}\left(\frac{r}{r_P}\right)^{n+2}{\rm d}r\,P_n(t){\rm d}\sigma + \mu\sum_{n=0}^{\infty}(n-1)\iint\limits_{\sigma} H\left(\frac{R}{r_P}\right)^{n+2} P_n(t){\rm d}\sigma \qquad (10)$$

Equations (8) and (10) yield the zero- and first-degree direct effects on potential and gravity anomaly

$$(\delta V_{\rm dir}(P))_0 = -\mu\iint\limits_{\sigma}\left(\frac{r_S^3 - R^3}{3r_P} - H\frac{R^2}{r_P}\right){\rm d}\sigma \qquad (11a)$$

$$(\delta V_{\rm dir}(P))_1 = -\mu\iint\limits_{\sigma}\left(\frac{r_S^4 - R^4}{4r_P^2} - \frac{R^3 H}{r_P^2}\right)t\,{\rm d}\sigma \qquad (11b)$$

$$(\delta\Delta g_{\rm dir}(P))_0 = \mu\iint\limits_{\sigma}\left(\frac{r_S^3 - R^3}{3r_P^2} - \frac{HR^2}{r_P^2}\right){\rm d}\sigma \qquad (12a)$$

and

$$(\delta\Delta g_{\rm dir}(P))_1 = 0 \qquad (12b)$$

The corresponding effects on the geoid are obtained from Eqs. (11a) and (11b) by setting $r_P = R$ and using Bruns' well-known formula, i.e. dividing the disturbing potential by normal gravity (γ) at the reference ellipsoid. The result is

$$(\delta N_{\rm dir})_0 = -\frac{\mu}{\gamma}\iint\limits_{\sigma}\frac{r_S^3 - R^3 - 3R^2H}{3R}\,{\rm d}\sigma = -\frac{4\pi\mu}{\gamma}\left(H_0^2 + \frac{H_0^3}{3R}\right) \qquad (13a)$$

and

$$(\delta N_{\rm dir})_1 = -\frac{\mu}{\gamma}\iint\limits_{\sigma}\frac{r_S^4 - R^4 - 4R^3H}{4R^2}\,t\,{\rm d}\sigma = -\frac{4\pi\mu}{\gamma}\left(\frac{3}{2}H_1^2 + \frac{H_1^3}{R} + \frac{H_1^4}{4R^2}\right) \qquad (13b)$$

where

$$H_n^v = \frac{1}{4\pi}\iint\limits_{\sigma} H^v P_n(t){\rm d}\sigma; \quad v = 2, 3, 4, \; n = 0, 1$$

Hence, the zero- and first-degree effects on the geoid differ from zero.

We now consider the (total) direct effect on $\tilde{N}$ given by Eq. (6a)

$$\delta\tilde{N}_{\text{dir}} = \frac{(\delta T_{\text{dir}})_0}{\gamma} + \frac{(\delta T_{\text{dir}})_1}{\gamma} + \frac{R}{4\pi\gamma}\iint_{\sigma_0} S_L(\psi)\,\delta\Delta g^*_{\text{dir}}\,\mathrm{d}\sigma + c\sum_{n=0}^{M}\left(s_n^* + Q_{Ln}\right)\left(\delta\Delta g^{\mathrm{S}}_{\text{dir}}\right)_n \tag{14}$$

or, in spectral form

$$\delta\tilde{N}_{\text{dir}} = \frac{(\delta T_{\text{dir}})_0}{\gamma} + \frac{(\delta T_{\text{dir}})_1}{\gamma} + c\sum_{n=0}^{\infty}\left(k_n - s_n^* - Q_{Ln}\right)\left(\delta\Delta g^*_{\text{dir}}\right)_n + c\sum_{n=0}^{M}\left(s_n^* + Q_{Ln}\right)\left(\delta\Delta g^{\mathrm{S}}_{\text{dir}}\right)_n \tag{15}$$

where

$$k_n = \begin{cases} 0 & \text{if } n < 2 \\ 2/(n-1) & \text{if } n \geq 2 \end{cases}$$

As

$$\left(\delta\Delta g^*_{\text{dir}}\right)_n = \left(\delta\Delta g^{\mathrm{S}}_{\text{dir}}\right)_n$$

and

$$(\delta T_{\text{dir}})_{0,1} = (\delta V_{\text{dir}})_{0,1} \quad (\text{with } r_P = R)$$

we obtain

$$\delta\tilde{N}_{\text{dir}} = \frac{(\delta V_{\text{dir}})_0}{\gamma} + \frac{(\delta V_{\text{dir}})_1}{\gamma} + c\sum_{n=2}^{M}\frac{2}{n-1}\left(\delta\Delta g^*_{\text{dir}}\right)_n + c\sum_{n=M+1}^{\infty}\left(\frac{2}{n-1} - s_n^* - Q_{Ln}\right)\left(\delta\Delta g^*_{\text{dir}}\right)_n \tag{16}$$

Inserting Eq. (10) with $r_P = R$, we finally arrive at

$$\delta\tilde{N}_{\text{dir}} = \frac{(\delta V_{\text{dir}})_0}{\gamma} + \frac{(\delta V_{\text{dir}})_1}{\gamma} - \frac{\mu}{R\gamma}\sum_{n=2}^{\infty}\iint_{\sigma}\int_{R}^{r_S}\left(\frac{r}{R}\right)^n r^2 P_n(t)\,\mathrm{d}\sigma + \frac{\mu R}{\gamma}\sum_{n=2}^{\infty}\iint_{\sigma} H P_n(t)\mathrm{d}\sigma - c\sum_{n=M+1}^{\infty}\left(s_n^* + Q_{Ln}\right)\left(\delta\Delta g^*_{\text{dir}}\right)_n \tag{17}$$

Again, we notice that the zero- and first-degree direct effects on the geoid are given by Eqs. (13a) and (13b), implying small effects of a few centimetres as the result of the reduction of the topography to a Helmert layer of condensation at sea level. Martinec (1998, Chap. 2) selected a mass-conserving coating for the Helmert potential, which permits the elimination of the zero-degree direct effect. Alternatively, we might consider the modification to eliminate the first-degree effect. However, the direct effects alone are not very interesting to consider; it is the total effect, i.e. the sum of the direct, indirect and downward continuation effects, that is our concern. We will therefore return to this question at the end of Sect. 3.3.

3.2 The indirect effect (the restoration step)

The indirect effect is the result of restoring the topographic masses. For the potential it is given by the minus of the direct effect [Eq. (7)]. At sea level (and below) it can be expanded in an internal-type harmonic series [Eq. (25)]. In particular, for $r_P = R$ the expansion for the geoid becomes

$$\delta N_{\mathrm{I}} = -\frac{(\delta V_{\text{dir}})_{r_P=R}}{\gamma} = \frac{\mu}{R\gamma}\sum_{n=0}^{\infty}\iint_{\sigma}\left\{\int_{R}^{r_S}\left(\frac{R}{r}\right)^{n+1} r^2\mathrm{d}r - R^2 H\right\}P_n(t)\mathrm{d}\sigma \tag{18}$$

with the zero- and first-degree effects

$$(\delta N_{\mathrm{I}})_0 = \frac{\mu}{R\gamma}\iint_{\sigma}\left\{\frac{R\left(r_S^2 - R^2\right)}{2} - R^2 H\right\}\mathrm{d}\sigma = \frac{2\pi\mu}{\gamma}H_0^2 \tag{19a}$$

and

$$(\delta N_{\mathrm{I}})_1 = 0 \tag{19b}$$

3.3 The combined and total effects on the geoid

Adding the direct and indirect effects on the geoid we obtain the combined effect. If the modified Stokes formula of Eq. (1) is used for the geoid estimation, the direct effect is given by Eq. (17) and the indirect effect is given by Eq. (18). Thus we obtain the combined effect

$$\delta\tilde{N}_{\text{comb}} = \delta\tilde{N}_{\text{dir}} + \delta N_{\mathrm{I}} = (\delta N_{\text{comb}})_0 + (\delta N_{\text{comb}})_1 + \frac{\mu}{R\gamma}\sum_{n=2}^{\infty}\iint_{\sigma}\int_{R}^{r_S}\left\{\left(\frac{R}{r}\right)^{n+1} - \left(\frac{r}{R}\right)^n\right\}r^2\,\mathrm{d}r\,P_n(t)\mathrm{d}\sigma - c\sum_{n=M+1}^{\infty}\left(s_n^* + Q_{Ln}\right)\left(\delta\Delta g^*_{\text{dir}}\right)_n \tag{20a}$$

where

$$(\delta N_{\text{comb}})_0 = -\frac{2\pi\mu}{\gamma}\left(H_0^2 + 2\frac{H_0^3}{3R}\right) \tag{20b}$$

and

$$(\delta N_{\text{comb}})_1 = -\frac{4\pi\mu}{\gamma}\left(\frac{3}{2}H_1^2 + \frac{H_1^3}{R} + \frac{H_1^4}{4R^2}\right) \tag{20c}$$

As numerically

$$H_0^2 = 0.445\,\mathrm{km}^2$$

and

$$H_1^2 = -0.024\,Y_{10} + 0.085\,Y_{11} + 0.191\,Y_{1-1}\,\left[\mathrm{km}^2\right]$$

where Y_{nm} are fully normalized spherical harmonics, we arrive at

$\left(\delta\tilde{N}_{\text{comb}}\right)_0 \approx -5.1$ cm

and

$\left|\left(\delta\tilde{N}_{\text{comb}}\right)_1\right| \leq 10.3$ cm

i.e. the zero- and first-degree terms of $\delta\tilde{N}_{\text{comb}}$ are each within ± 11 cm.

The last sum of Eq. (20a)

$$-c\sum_{n=M+1}^{\infty}\left(s_n^* + Q_{Ln}\right)(\delta\Delta g_{\text{dir}})_n$$

is the direct effect of the bias of the modified Stokes formula. This bias is caused by the finite M and/or the limited cap size under Stokes' integral. Consequently, it vanishes in the original Stokes formula. As suggested by Sjöberg (1996), this term is eliminated by applying the direct effect for the whole anomaly spectrum ($M \to \infty$).

The total topographic effect on the geoid estimator [Eq. (1)] is given by the sum of the combined effect and the effect of downward continuation of the Helmert gravity anomaly to sea level under Stokes' integral

$$\delta\tilde{N}_{\text{tot}} = \delta\tilde{N}_{\text{comb}} + \delta\tilde{N}_{\text{dwc}} \tag{21}$$

where

$$\delta\tilde{N}_{\text{dwc}} = \frac{R}{4\pi\gamma}\iint_{\sigma_0} S_L(\psi)(\Delta g^* - \Delta g)\mathrm{d}\sigma \tag{22}$$

If we assume that the normal gravity field is defined in such a way that Δg contains no zero- and first-degree harmonics, this will be the case also for Δg^* (the harmonic reduction of Δg to sea level) and, from Eq. (22), for $\delta\tilde{N}_{\text{dwc}}$. Consequently, the total topographic effects of degrees zero and one are given by Eqs. (20b) and (20c).

4 The atmospheric effects

The reduction of the atmospheric masses is most frequently applied according to the approach of the International Association of Geodesy (IAG) described in Moritz (1980, p. 422). An alternative approach was presented by Sjöberg (1998, 1999a). Following the latter approach, the potential of the atmosphere at an arbitrary point P can be written

$$V^{\text{a}}(P) = \iint_{\sigma}\int_{r_S}^{\infty}\frac{\rho_{\text{a}}^* r^2\mathrm{d}r}{\ell_P}\mathrm{d}\sigma \tag{23}$$

where

$\rho_{\text{a}}^* = G\rho_{\text{a}};\ \rho_{\text{a}} =$ atmospheric density

$\ell_P^2 = r_P^2 + r^2 - 2r_P r\cos\psi$

Let us assume that the atmospheric density is radial-symmetrically layered, i.e.

$$\rho_{\text{a}}^* = \rho_0^*(R/r)^{\nu} \tag{24}$$

where ρ_0^* is the density at sea level and $\nu \gg 2$.

Expanding ℓ_P^{-1} as an internal-type Legendre series

$$\ell_P^{-1} = \frac{1}{r_P}\sum_{n=0}^{\infty}\left(\frac{r_P}{r}\right)^{n+1}P_n(t);\quad r_P \leq r \tag{25}$$

and inserting Eqs. (24) and (25) into Eq. (23), we arrive at the following series expansion after integration with respect to r:

$$V^{\text{a}}(P) = \rho_0^* R^2\sum_{n=0}^{\infty}\frac{1}{n+\nu-2}\left(\frac{r_P}{R}\right)^n \times\iint_{\sigma}\left(\frac{R}{r_S}\right)^{n+\nu-2}P_n(\psi)\mathrm{d}\sigma \tag{26}$$

or, after using the first-order expansion

$$\left(\frac{R}{r_s}\right)^{n+\nu-2} = \left(1+\frac{H}{R}\right)^{-(n+\nu-2)} = 1-(n+\nu-2)\frac{H}{R} \tag{27}$$

and

$$P_n(t) = \frac{1}{2n+1}\sum_{m=-n}^{\infty}Y_{nm}(P)Y_{nm}(Q)$$

$$V^{\text{a}}(P) = 4\pi\rho_0^* R^2\left\{\frac{1}{\nu-2} - \sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{r_P}{R}\right)^n\frac{H_n(P)}{R}\right\} \tag{28a}$$

where

$$H_n(P) = \sum_{m=-n}^{n}H_{nm}Y_{nm}(P) \tag{28b}$$

$$H_{nm} = \frac{1}{4\pi}\iint_{\sigma}H\ Y_{nm}\,\mathrm{d}\sigma \tag{28c}$$

It should be emphasised that the right-hand side of Eq. (27) is limited to a first-order approximation of the topographic impact on the atmospheric effect. Sjöberg (1999a) showed that the second-order term of the expansion of Eq. (27) contributes to the atmospheric effect within 1 mm. Knickmeyer (1984) discussed the convergence of the expansion of Eq. (27) when inserted into an integral formula such as Eq. (26), and Sun and Sjöberg (2001) considered the optimum degree of truncation of the series. However, as the atmospheric effect of the higher-degree terms is certainly small, we will not further dwell upon this question here.

4.1 The direct effects (the reductions)

The direct atmospheric effect on the gravity anomaly at a point P is given by

$$\Delta g_{\text{dir}}^{\text{a}}(P) = \frac{\partial}{\partial r_P}V^{\text{a}}(P) + \frac{2V^{\text{a}}(P)}{r_P} \tag{29}$$

Inserting Eq. (28a) we arrive at

$$\Delta g^{\mathrm{a}}_{\mathrm{dir}}(P) = 4\pi\rho_0^*\left\{\frac{2R^2}{r_P(\nu-2)} - \sum_{n=0}^{\infty}\frac{n+2}{2n+1}\left(\frac{r_P}{R}\right)^{n-1}H_n(P)\right\} \tag{30}$$

This effect includes both zero- and first-degree harmonics.

The direct effect on the geoid estimator of Eq. (1) becomes

$$\delta\tilde{N}^{\mathrm{a}}_{\mathrm{dir}} = -\frac{V_0^{\mathrm{a}}}{\gamma} - \frac{V_1^{\mathrm{a}}}{\gamma} + \frac{R}{4\pi\gamma}\iint_{\sigma_0} S_L(\psi)\Delta g^{\mathrm{a}}_{\mathrm{dir}}\,\mathrm{d}\sigma + c\sum_{n=0}^{M}\left(s_n^* + Q_{Ln}\right)\left(\Delta g_n^{\mathrm{S}}\right)^{\mathrm{a}} \tag{31}$$

where $\left(\Delta g_n^{\mathrm{S}}\right)^{\mathrm{a}}$ = direct effect of the atmosphere on the anomaly derived by the GGM.

The spectral form of Eq. (31) becomes

$$\delta\tilde{N}^{\mathrm{a}}_{\mathrm{dir}} = -\frac{V_0^{\mathrm{a}}}{\gamma} - \frac{V_1^{\mathrm{a}}}{\gamma} + c\sum_{n=0}^{\infty}\left(k_n - s_n^* - Q_{Ln}\right)\Delta g_n^{\mathrm{a}} + c\sum_{n=0}^{M}\left(s_n^* + Q_{Ln}\right)\left(\Delta g_n^{\mathrm{S}}\right)^{\mathrm{a}} \tag{32}$$

If the GGM is derived from terrestrial gravity, the harmonics Δg_n^{a} and $\left(\Delta g_n^{\mathrm{S}}\right)^{\mathrm{a}}$ are equal and Eq. (32) can be simplified to

$$\delta\tilde{N}^{\mathrm{a}}_{\mathrm{dir}} = -\frac{V_0^{\mathrm{a}}}{\gamma} - \frac{V_1^{\mathrm{a}}}{\gamma} + c\sum_{n=2}^{M}\frac{2}{n-1}\Delta g_n^{\mathrm{a}} + c\sum_{n=M+1}^{\infty}\left(\frac{2}{n-1} - s_n^* - Q_{Ln}\right)\Delta g_n^{\mathrm{a}} \tag{33}$$

where

$$\Delta g_n^{\mathrm{a}} = -4\pi\rho_0^*\frac{n+2}{2n+1}H_n(P); \quad n > 0 \tag{34a}$$

Also

$$\Delta g_0^{\mathrm{a}} = 8\pi\rho_0^*\left(\frac{R}{\nu-2} - H_0\right) \tag{34b}$$

If the GGM is derived from satellite data (in practice above the atmosphere) the atmospheric potential of Eq. (23) at a point P can be written as the external-type series

$$V^{\mathrm{a}}(P) = \frac{4\pi\rho_0^* R^3}{r_P}\left(\frac{1}{\nu-3} - \frac{H_0}{R}\right) - 4\pi\rho_0^*\sum_{n=1}^{\infty}\frac{R^{n+2}}{r_P^{n+1}}\frac{H_n(P)}{2n+1} \tag{35}$$

which yields the direct anomaly effect according to Eq. (29)

$$\Delta g^{\mathrm{a}}_{\mathrm{dir}}(P) = \frac{4\pi\rho_0^* R^3}{r_P^2}\left(\frac{1}{\nu-3} - \frac{H_0}{R}\right) + 4\pi\rho_0^*\sum_{n=2}^{\infty}\frac{R^{n+2}}{r_P^{n+2}}\frac{(n-1)}{2n+1}H_n(P) \tag{36}$$

The direct effect on the geoid is given by Eq. (31) with V_0^{a} and V_1^{a} given by the first harmonics of Eq. (35) and Δg_n^{a} given by Eq. (36).

4.2 The indirect (the restoration) and total effects

The indirect effect on the geoid is given by Eq. (28a) divided by γ

$$\delta N_I^{\mathrm{a}} = \frac{4\pi\rho_0^* R^2}{\gamma}\left\{\frac{1}{\nu-2} - \sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{H_n(P)}{R}\right\} \tag{37}$$

By adding the direct and indirect effects, we obtain the combined (or total) effect on the geoid.

If we assume that the GGM is determined from terrestrial data, we obtain

$$\delta N^{\mathrm{a}}_{\mathrm{total}} = \frac{-4\pi\rho_0^* R}{\gamma}\sum_{n=2}^{\infty}\frac{1}{n-1}H_n - c\sum_{n=M+1}^{\infty}\left(s_n^* + Q_{Ln}\right)\Delta g_n^{\mathrm{a}} \tag{38}$$

or

$$\delta N^{\mathrm{a}}_{\mathrm{total}} = \frac{-\rho_0^* R}{\gamma}\iint_{\sigma} S(\psi)H\,\mathrm{d}\sigma - c\sum_{n=M+1}^{\infty}\left(s_n^* + Q_{Ln}\right)\Delta g_n^{\mathrm{a}} \tag{39}$$

If we assume that V_0^{a}, V_1^{a} and $(\Delta g_n^{\mathrm{S}})^{\mathrm{a}}$ are all derived from satellite data, we arrive at the total atmospheric effect

$$\delta N^{\mathrm{a}}_{\mathrm{total}} = \frac{\delta V_0^{\mathrm{a}}}{\gamma} + \frac{\delta V_1^{\mathrm{a}}}{\gamma} - \frac{\rho_0^* R}{\gamma}\iint_{\sigma} S(\psi)H\,\mathrm{d}\sigma + c\sum_{n=0}^{M}\left(s_n^* + Q_{Ln}\right)\left((\Delta g_n^{\mathrm{S}})^{\mathrm{a}} - \Delta g_n^{\mathrm{a}}\right) - c\sum_{M+1}^{\infty}\left(s_n^* + Q_{Ln}\right)\Delta g_n^{\mathrm{a}} \tag{40}$$

where

$$\frac{\delta V_0^{\mathrm{a}}}{\gamma} = \frac{4\pi\rho_0^* R^2}{\gamma}\left(\frac{1}{\nu-2} - \frac{1}{\nu-3}\right) = -\frac{4\pi\rho_0^* R^2}{\gamma(\nu-2)(\nu-3)} \tag{41a}$$

and

$$\frac{\delta V_1^{\mathrm{a}}}{\gamma} = 0 \tag{41b}$$

From Eqs. (34b) and (36) we obtain

$$(\Delta g_0^{\mathrm{S}})^{\mathrm{a}} - \Delta g_0^{\mathrm{a}} = 4\pi\rho_0^* R\left(\frac{1}{v-3} - \frac{2}{v-2}\right) = -\frac{4\pi\rho_0^* R(v-4)}{(v-3)(v-2)} \tag{41c}$$

and from Eqs. (34a) and (36)

$$(\Delta g_n^{\mathrm{S}})^{\mathrm{a}} - \Delta g_n^{\mathrm{a}} = 4\pi\rho_0^* H_n; \quad n > 0 \tag{41d}$$

From Ecker and Mittermayer (1969) we estimate the power v to be about 850. Also, using $\rho_0 = 1.23 \cdot 10^{-3}$ g/cm^3 from Ecker and Mittermayer (1969), we obtain the constant

$$\frac{4\pi\rho_0^* R}{\gamma} = 0.670 \text{ m/km}$$

This yields, with $R = 6371$ km and $\gamma = 981$ Gal

$$\left(\delta N_I^{\mathrm{a}}\right)_0 \approx \frac{4\pi\rho_0^* R^2}{\gamma(v-2)} = 5.03 \text{ m}$$

and

$$\frac{\delta V_0^{\mathrm{a}}}{\gamma} \approx -\frac{1}{v-3}\left(\delta N_I^{\mathrm{a}}\right)_0 = -6 \text{ mm}$$

The last term agrees well with Moritz (1980, p. 425). Furthermore

$$(\Delta g_0^{\mathrm{S}})^{\mathrm{a}} - \Delta g_0^{\mathrm{a}} = (v-4)\frac{\delta V_0^{\mathrm{a}}}{R} = 0.77 \text{ mGal}$$

Hence the zero-degree effect on the geoid $\left(\delta V_0^{\mathrm{a}}/\gamma\right)$ is within 1 cm and the first-degree effect $\left(\delta V_1^{\mathrm{a}}/\gamma\right)$ vanishes. In addition, the possible truncation errors of degrees zero and one of Eq. (40) are usually small. If all data are determined from terrestrial gravity, Eq. (39) shows that both the zero- and first-degree atmospheric effects vanish.

5 Corrections to a harmonic representation of the geoid

This section includes an alternative derivation of the combined effect on the geoid in the case of an external-type harmonic representation of the disturbing potential of the Earth. Considering Eqs. (7) and (23) we obtain the *topographic–atmospheric* potential by the formula

$$V^{\mathrm{ta}}(P) = \mu \iint_\sigma \int_R^{r_S} \frac{r^2\,\mathrm{d}r\,\mathrm{d}\sigma}{\ell_P} + \iint_\sigma \int_{r_S}^{\infty} \frac{\rho_{\mathrm{a}}^* r^2\,\mathrm{d}r\,\mathrm{d}\sigma}{\ell_P} \tag{42}$$

At satellite level most of the atmosphere is below that level. Hence the potential at such a level can in practice be expanded in an exterior-type harmonic series

$$V_{\mathrm{e}}^{\mathrm{ta}} = \frac{\mu}{r_P}\sum_{n=0}^{\infty} \iint_\sigma \int_R^{r_S} \left(\frac{r}{r_P}\right)^n r^2\,\mathrm{d}r\,P_n(t)\mathrm{d}\sigma + \sum_{n=0}^{\infty} \iint_\sigma \int_{r_S}^{\infty} \frac{\rho_{\mathrm{a}}^* r^{n+2}}{r_P^{n+1}}\,\mathrm{d}r\,P_n(t)\mathrm{d}\sigma \tag{43}$$

including the zero- and first-degree harmonics

$$\left(V_{\mathrm{e}}^{\mathrm{ta}}\right)_0 = \frac{\mu}{3r_P}\iint_\sigma \left(r_S^2 + Rr_S + R^2\right)H\,\mathrm{d}\sigma + \iint_\sigma \int_{r_S}^{\infty} \frac{\rho_{\mathrm{a}}^* r^2\,\mathrm{d}r}{r_P}\mathrm{d}\sigma \tag{44}$$

and

$$\left(V^{\mathrm{ta}}\right)_1 = \frac{\mu}{4r_P^2}\iint_\sigma \left(r_S^4 - R^4\right)t\,\mathrm{d}\sigma + \iint_\sigma \int_{r_S}^{\infty} \frac{\rho_{\mathrm{a}}^* r^3}{r_P^2}\mathrm{d}r\,t\,\mathrm{d}\sigma \tag{45}$$

Usually geodetic reference systems, such as GRS80, are defined in such a way that the normal potential includes the zero- and first-degree harmonics of Eqs. (44) and (45), implying that the disturbing potential in the exterior space lacks these harmonics. However, this is not necessarily the case at the continental geoid inside the topographic masses. A correct representation of the topographic potential between sea level and the bounding sphere including all topographic masses was given by, for example, Sjöberg (1999b, p. 218). In particular, at the geoid (with $r_P = R$) the internal-type series representation becomes

$$V_{\mathrm{i}}^{\mathrm{ta}} = \sum_{n=0}^{\infty} \iint_\sigma \left\{\frac{\mu}{R}\int_R^{r_S}\left(\frac{R}{r}\right)^{n+1} r^2\,\mathrm{d}r + \int_{r_S}^{\infty} \frac{\rho_{\mathrm{a}}^* R^n}{r^{n-1}}\mathrm{d}r\right\}P_n(t)\mathrm{d}\sigma \tag{46}$$

Hence, in view of the series of Eq. (46) the external-type representation $V_{\mathrm{e}}^{\mathrm{ta}}$ needs a correction

$$\delta V_{\mathrm{corr}}^{\mathrm{ta}} = V_{\mathrm{i}}^{\mathrm{ta}} - V_{\mathrm{e}}^{\mathrm{ta}} \tag{47}$$

when applied at the geoid. This correction becomes explicitly

$$\delta V_{\mathrm{corr}}^{\mathrm{ta}} = \frac{\mu}{R}\sum_{n=0}^{\infty} \iint_\sigma \int_R^{r_s} \left\{\left(\frac{R}{r}\right)^{n+1} - \left(\frac{r}{R}\right)^n\right\} r^2\,\mathrm{d}r\,P_n(t)\mathrm{d}\sigma + \sum_{n=0}^{\infty} \iint_\sigma \int_{r_S}^{\infty} \rho_{\mathrm{a}}^* \left\{\frac{R^n}{r^{n+1}} - \frac{r^n}{R^{n+1}}\right\} r^2\,\mathrm{d}r\,P_n(t)\mathrm{d}\sigma \tag{48}$$

including the zero- and first-degree terms

$$\left(\delta V_{\mathrm{corr}}^{\mathrm{ta}}\right)_0 = -2\pi\mu\left(H_0^2 + \frac{2H_0^3}{3R}\right) + \iint_\sigma \int_{R_S}^{\infty} \rho_{\mathrm{a}}^*\left(r - \frac{r^2}{R}\right)\mathrm{d}r\,\mathrm{d}\sigma \tag{49a}$$

and

$$\left(\delta V_{\mathrm{corr}}^{\mathrm{ta}}\right)_1 = -4\pi\mu\left(\frac{3}{2}H_1^2 + \frac{H_1^3}{R} + \frac{H_1^4}{4R^2}\right) + \iint_\sigma \int_{R_S}^{\infty} \rho_{\mathrm{a}}^* \left(R - \frac{r^3}{R^2}\right) \mathrm{d}r\, t\, \mathrm{d}\sigma \tag{49b}$$

Again, assuming that the atmospheric density distribution is given by Eq. (24) and expanding the atmospheric effect to first order of elevation, we arrive at the geoid corrections

$$\delta N_{\mathrm{corr}}^{\mathrm{ta}} = \frac{\mu}{R\gamma}\sum_{n=0}^{\infty} \iint_\sigma \int_R^{r_S} \left\{\left(\frac{R}{r}\right)^{n+1} - \left(\frac{r}{R}\right)^n\right\} r^2\, \mathrm{d}r P_n(t) \mathrm{d}\sigma - \frac{4\pi\rho_0^* R^2}{\gamma(v-2)(v-3)} - \frac{\rho_0^* R}{\gamma}\iint_\sigma S(\psi) H\, \mathrm{d}\sigma \tag{50a}$$

with the zero- and first-degree terms

$$\left(\delta N_{\mathrm{corr}}^{\mathrm{ta}}\right)_0 = \frac{-2\pi\mu}{\gamma}\left(H_0^2 + \frac{2H_0^3}{3R}\right) - \frac{4\pi\rho_0^* R^2}{\gamma(v-2)(v-3)} \tag{50b}$$

and

$$\left(\delta N_{\mathrm{corr}}^{\mathrm{ta}}\right)_1 = -\frac{4\pi\mu}{\gamma}\left(\frac{3}{2}H_1^2 + \frac{H_1^3}{R} + \frac{H_1^4}{4R^2}\right) \tag{50c}$$

where we have divided the potential correction by γ (Bruns' formula) to obtain the geoid correction. These formulas are the same as the combined topographic effect (Sect. 3) and the atmospheric effect (Sect. 4) in the case of no truncation of Stokes' integral; see Eqs. (20a), (40), (41a) and (41b).

6 Conclusions

Primarily, the topographic effects on the gravimetric geoid are of order two of topographic height, while the atmospheric effect is of order one. Considering the combined effect on the geoid (i.e. the sum of the direct and indirect effects), the topographic effect is of the order

$$\delta N_{\mathrm{comb}} \approx -\frac{2\pi\mu}{\gamma}H^2$$

reaching several metres and including the zero- and first-degree effects

$$(\delta N_{\mathrm{comb}})_0 = -5.1\,\mathrm{cm}$$

and

$$\left|(\delta N_{\mathrm{comb}})_1\right| \le 10.3\,\mathrm{cm}$$

The combined atmospheric effect on the geoid is given by Eq. (39) (implying no effects of degrees zero and one) in the case of pure terrestrial data, and Eq. (40) in the case of a satellite-derived GGM, including a small zero-degree effect of the order of −6 mm. The resulting zero- and first-degree effects on the geoid cannot be eliminated by changing the Helmert-layer coating, as this harmonic potential implies merely an intermediate step that is added in the direct effect and subtracted in the indirect effect. In conclusion, in defining the normal potential such that the disturbing potential outside the topography and the atmosphere has vanishing harmonics of degrees zero and one, the geoidal undulation will still have such harmonics of the order of several centimetres.

Recently Nahavandchi and Sjöberg (2001) computed a mean difference of 10.1 cm between the gravimetric geoid model KTH98 for Sweden and the GPS-derived geoidal heights at 23 permanent SWEPOS GPS sites spread over Sweden. As the model KTH98 does not include the above zero- and first-degree effects, the zero-degree effect alone of −5.1 cm could explain half of the discrepancy.

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