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World Geodetic Datum 2000

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Abstract. Based on the current best estimates of fundamental geodetic parameters $\{W_0, GM, J_2, \Omega\}$ the form parameters of a Somigliana-Pizzetti level ellipsoid, namely the semi-major axis a and semi-minor axis b (or equivalently the linear eccentricity $\varepsilon = \sqrt{a^2 - b^2}$) are computed and proposed as a new World Geodetic Datum 2000. There are six parameters namely the four fundamental geodetic parameters $\{W_0, GM, J_2, \Omega\}$ and the two form parameters $\{a, b\}$ or $\{a, \varepsilon\}$, which determine the ellipsoidal reference gravity field of Somigliana-Pizzetti type constraint to two nonlinear condition equations. Their iterative solution leads to best estimates $a = (6\ 378\ 136.572 \pm 0.053)$m, $b = (6\ 356\ 751.920 \pm 0.052)$m, $\varepsilon = (521\ 853.580 \pm 0.013)$m for the tide-free geoide of reference and $a = (6\ 378\ 136.602 \pm 0.053)$m, $b = (6\ 356\ 751.860 \pm 0.052)$m, $\varepsilon = (521\ 854.674 \pm 0.015)$m for the zero-frequency tide geoid of reference. The best estimates of the form parameters of a Somigliana-Pizzetti level ellipsoid, $\{a, b\}$, differ significantly by -0.39 m, -0.454 m, respectively, from the data of the Geodetic Reference System 1980.

Key words. Form parameters of Somigliana-Pizzetti level ellipsoid · Somigliana-Pizzetti gravity field · Spheroidal harmonics · Spheroidal coordinates

Introduction

The recent years have seen a tremendous progress in the high resolution of the terrestrial gravity field which demands an update of the form parameters of the Somigliana-Pizzetti level ellipsoid, in particular its semi-major axis a and its semi-minor axis b (or, equivalently the linear eccentricity $\varepsilon = \sqrt{a^2 - b^2}$) from current best estimates of fundamental geodetic parameters $\{W_0, GM, J_2, \Omega\}$, as a fundamental reference ellipsoid approximating the physical surface of the Earth, namely the Gauss-Listing geoid. Such a new parameter set will be proposed here as World Geodetic Datum 2000.

Here we intend to compute a new data set of the form parameters $\{a, b\}$ of the Somigliana-Pizzetti level ellipsoid based on current best estimates of the fundamental geodetic parameters $\{W_0, GM, J_2, \Omega\}$. Section 1 provides directly with such a data set of optimal form parameters $\{a, b\}$ by solving the two nonlinear condition equations (47′), (48′) by Newton iteration, e.g. following Saupe (1988). The best estimate off those form parameters $\{a, b\}$ is completed by nonlinear error propagation, namely by computing the variance-covariance matrix of optimal form parameters $\{a, b\}$ from the variance of the pseudo-observations of type $\{W_0, GM, J_2, \Omega\}$. In contrast, Sect. 2 reviews in all detail the Somigliana-Pizzetti gravity field of a level ellipsoid. As a transplant of functional analysis we emphasise the genesis of scalar-valued harmonic functions which are orthonormal on an ellipsoid-of-revolution. In particular we succeed to constrain the spheroidal harmonic coefficients of a harmonic gravitational potential, namely of degree/order zero/zero and two/zero, to produce a level ellipsoid-of-revolution. In contrast to previous representation of the Somigliana-Pizzetti gravity field we express the radial dependence by base functions of type Legendre polynomials of the first kind $P^*_{lm}(u/\varepsilon)$ as being postulated by functional analysis, namely the separation solution of the three dimensional Laplace equation in spheroidal coordinates $\{\lambda, \phi, u\}$ following Thong and Grafarend (1989). Section 3 on spherical coordinates and spherical gravity field is preparatory in order to succeed to transforming spheroidal harmonic coefficients into spherical harmonic coefficients (degree/order of type zero/zero and two/zero and vice versa within Sect. 4, e.g. Hotine (1969, page 194, (22.59)). Such a chapter has to be placed in a contribution of spheroidal gravity field of Somigliana-Pizzetti type since it is common practice to present the spheroidal gravity field

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not in scalar-valued spheroidal harmonics (the natural choice when dealing with spheroidal geometry), but instead in scalar-valued spherical harmonics. One argument for such a surprising representation may be given by the fact the physical geodesists as well as satellite geodesists have developed high resolution Standard Gravity Field Earth Models of coefficients of a spherical harmonic expansion of the gravitational potential, e.g. up to degree 1800 (Wenzel 1998). Here, the impact of the gauge $R = a$ (the identity of the average Earth radius R and of semi-major axis of the International Reference Ellipsoid) as well as $w_0 = W_0,\ \omega = \Omega$ within the above quoted transformation is finally highlighted. As historical references for spheroidal geometry and spheroidal gravity we finally quote Heiskanen (1951), Hirvonen (1960) and Lambert (1961) among others and our presentation in Sects. 2–4 should be compared with Vermeer and Poutanen (1997) already studied the influence of the permanent tide and the atmosphere on a geodetic reference system we are dealing with.

1 Best estimates of the form parameters $\{a, b\}$ of a Somigliana-Pizzetti related level ellipsoid

By means of (47), (48) we shall establish the conditional equations of spheroidal harmonic coefficients of degree/order (0,0) and (2,0) in case of a level ellipsoid of the Somigliana-Pizzetti gravity field. As soon as we take advantage of the transformation of spheroidal harmonic coefficients into spherical harmonic coefficients we arrive at (47′), (48′) as the conditional equations, now in terms of the spherical harmonic coefficients GM and J_2, namely for the conventional datum $R = a$ as well as $w_0 = W_0, \omega = \Omega$. These final conditional equations (47′), (48′) are nonlinearly relating those six parameters of the Somigliana-Pizzetti gravity field $a, b, W_0, GM, J_2, \Omega$. Given the four parameters $\{W_0, GM, J_2^*, \Omega\} = \{y_1, y_2, y_3, y_4\}$ called pseudo-observations the conditional equations (47′), (48′) are linearized with respect to the semi-major axis a and semi-minor axis b of the level ellipsoid, $\{a, b\} = \{x_1, x_2\}$, namely

$$f_1(x_1, x_2, y_1, y_2, y_3, y_4) = \frac{y_2}{\sqrt{x_1^2 - x_2^2}} \operatorname{arccot}\left(\frac{x_2}{\sqrt{x_1^2 - x_2^2}}\right) + \frac{1}{3} y_4^2 x_1^2 - y_1 = 0 \tag{1}$$

$$\begin{aligned} f_2(x_1, x_2, y_1, y_2, y_3, y_4) &= \frac{1}{4}\frac{y_2}{\sqrt{x_1^2 - x_2^2}}\left(15 y_3 \frac{x_1^2}{x_1^2 - x_2^2} + \sqrt{5}\right) \\ &\times \left[\left(3\frac{x_2^2}{x_1^2 - x_2^2} + 1\right)\operatorname{arccot}\left(\frac{x_2}{\sqrt{x_1^2 - x_2^2}}\right) - 3\frac{x_2}{\sqrt{x_1^2 - x_2^2}}\right] \\ &- \frac{1}{3\sqrt{5}} y_4^2 x_1^2 = 0\ . \end{aligned} \tag{2}$$

A Taylor expansion of $\mathbf{f}(\mathbf{x}, \mathbf{y})$ is

$$\begin{aligned} \mathbf{f}(\mathbf{x}, \mathbf{y}) &= \mathbf{f}(\mathbf{x}_0, \mathbf{y}) + \frac{1}{1!}\mathbf{f}'(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \\ &+ \frac{1}{2!}\mathbf{f}''(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \otimes (\mathbf{x} - \mathbf{x}_0) \\ &+ \mathcal{O}_3((\mathbf{x} - \mathbf{x}_0) \otimes (\mathbf{x} - \mathbf{x}_0) \otimes (\mathbf{x} - \mathbf{x}_0)) \\ &= \mathbf{f}_0 + \mathbf{J}_0(\mathbf{x} - \mathbf{x}_0) + \tfrac{1}{2}\mathbf{H}_0(\mathbf{x} - \mathbf{x}_0) \otimes (\mathbf{x} - \mathbf{x}_0) + \mathcal{O}_3 \end{aligned} \tag{3}$$

with respect to the Jacobi-Matrix $\mathbf{J}_0$ of the first order partial derivatives at the approximation point $\mathbf{x}_0$ as well as the Hesse matrix $\mathbf{H}_0 = [\mathbf{vec}\,\mathbf{H}_1; \mathbf{vec}\,\mathbf{H}_2]$ of second order partial derivatives at the approximation point $\mathbf{x}_0$. $\mathbf{H}_1$ denotes the Hesse matrix of $\mathbf{f}_1$ while $\mathbf{H}_2$ the Hesse matrix of $\mathbf{f}_2$. Newton iteration starts with the linearized Taylor expansion (for details we refer to Saupe (1988))

$$\Delta\mathbf{f} := \mathbf{f}(\mathbf{x}, \mathbf{y}) - \mathbf{f}(\mathbf{x}_0, \mathbf{y}) = \mathbf{J}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) = \mathbf{J}_0\Delta\mathbf{x} \tag{4}$$

subject to

$$\mathbf{J} := \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \tag{5}$$

where the partials of the Jacobi matrix are collected in Table 1.

$$\mathbf{x} - \mathbf{x_0} = \Delta\mathbf{x} = \mathbf{J}_0^{-1}(\mathbf{f} - \mathbf{f}_0) = (\mathbf{J}(\mathbf{x}_0))^{-1}(\mathbf{f} - \mathbf{f}_0) \tag{6}$$

holds. Newton iteration updates are generated by the n-sequence

$$\mathbf{x} - \mathbf{x}_0 = \mathbf{J}_0^{-1}(\mathbf{f} - \mathbf{f}_0)$$

$$\Rightarrow \quad \mathbf{x}_1 = \mathbf{x}_0 + \mathbf{J}_0^{-1}(\mathbf{f} - \mathbf{f}_0) \tag{7}$$

$$\Rightarrow \quad \mathbf{x}_2 = \mathbf{x}_1 + \mathbf{J}_1^{-1}(\mathbf{f} - \mathbf{f}_1) \tag{8}$$

$$\Rightarrow \quad \cdots \quad \Rightarrow \quad \mathbf{x}_n = \mathbf{x}_{n-1} \tag{9}$$

and stops at the reproducing point ("fix-point") $\mathbf{x}_n = \mathbf{x}_{n-1}$. Here we needed $n = 1$ Newton iteration step.

Next we implement the error propagation from the pseudo-observations $\{y_1, y_2, y_3, y_4\}$ to the derived parameters $\{x_1, x_2\}$ namely characterised by the first moments, the expectation $\mathbf{E}\{\mathbf{x}\} = \xi$ and $\mathbf{E}\{\mathbf{y}\} = \eta$, as well as by the second moments, the variance-covariance-matrices/dispersion matrices $\mathbf{D}\{\mathbf{x}\} = \Sigma_\mathbf{x}$ and $\mathbf{D}\{\mathbf{y}\} = \Sigma_\mathbf{y}$ following Grafarend and Schaffrin (1993 pages 470–471). Up to nonlinear terms we derive

$$\begin{aligned} \mathbf{D}\{\mathbf{x}\} = \Sigma_\mathbf{x} &= \mathbf{J}_{\xi_0}^{-1}\mathbf{J}_{\eta_0}\mathbf{D}\{\mathbf{y}\}\mathbf{J}'_{\eta_0}(\mathbf{J}_{\xi_0}^{-1})' \\ &= \mathbf{J}_\mathbf{x}^{-1}\mathbf{J}_\mathbf{y}\Sigma_\mathbf{y}\mathbf{J}'_\mathbf{y}(\mathbf{J}_\mathbf{x}^{-1})' \end{aligned} \tag{10}$$

where $\mathbf{J}_{\xi_0}$, $\mathbf{J}_{\eta_0}$ respectively, represent the Jacobi matrices of partial derivatives of the function $\mathbf{f}(\mathbf{x}, \mathbf{y})$ with respect to $\mathbf{x}$, $\mathbf{y}$, respectively, at the evaluation point (ξ_0, η_0). $\mathbf{J}_\mathbf{x} \sim \mathbf{J}_{\xi_0}$ is given by Table 1, $\mathbf{J}_\mathbf{y} \sim \mathbf{J}_{\eta_0}$ is given by (11) and Table 2.

Table 1. Jacobian matrix of two constraints of the Somigliana-Pizzetti gravity field, partial derivatives with respect to the unknowns (x_1, x_2)

$$\frac{\partial f_1}{\partial x_1} = \frac{x_2 y_2}{x_1^2 - x_1 x_2^2} + \frac{2}{3} x_1 y_4^2 - \frac{x_1 y_2 \operatorname{arccot}\left(\frac{x_2}{\sqrt{x_1^2 - x_2^2}}\right)}{(x_1^2 - x_2^2)^{3/2}}$$

$$\frac{\partial f_1}{\partial x_2} = \frac{y_2}{-x_1^2 + x_2^2} + \frac{x_2 y_2 \operatorname{arccot}\left(\frac{x_2}{\sqrt{x_1^2 - x_2^2}}\right)}{(x_1^2 - x_2^2)^{3/2}}$$

$$\frac{\partial f_2}{\partial x_1} = \frac{-2x_1 y_4^2}{3\sqrt{5}} + \frac{x_2 y_2 \left(\sqrt{5}x_1^2 - \sqrt{5}x_2^2 + 15x_1^2 y_3\right)}{2x_1(-x_1^2 + x_2^2)^5} \left[-2x_1^6 + 3x_1^4 x_2^2 - x_2^6 + 3x_1^2 x_2 (x_1^2 - x_2^2)^{\frac{3}{2}} \operatorname{arccot}\left(\frac{x_2}{\sqrt{x_1^2 - x_2^2}}\right)\right]$$
$$+ \frac{x_1 y_2}{4(x_1^2 - x_2^2)^{5/2}} (-\sqrt{5}x_1^2 + \sqrt{5}x_2^2 - 15x_1^2 y_3 - 30x_2^2 y_3) \left[\frac{-3x_2}{\sqrt{x_1^2 - x_2^2}} + \left(1 + \frac{3x_2^2}{x_1^2 - x_2^2}\right) \operatorname{arccot}\left(\frac{x_2}{\sqrt{x_1^2 - x_2^2}}\right)\right]$$

$$\frac{\partial f_2}{\partial x_2} = \frac{y_2 \left(\sqrt{5}(x_1^2 - x_2^2) + 15x_1^2 y_3\right)}{4(x_1^2 - x_2^2)^{3/2}} \left[-\frac{x_1^2}{(x_1^2 - x_2^2)^{3/2}} - \frac{5x_2^2}{(x_1^2 - x_2^2)^{3/2}} - \frac{3}{\sqrt{x_1^2 - x_2^2}} + \frac{6x_1^2 x_2 \operatorname{arccot}\left(\frac{x_2}{\sqrt{x_1^2 - x_2^2}}\right)}{(x_1^2 - x_2^2)^2} \right]$$
$$+ \frac{x_2 y_2 (\sqrt{5}(x_1^2 - x_2^2) + 45x_1^2 y_3)}{4(x_1^2 - x_2^2)^{5/2}} \left[\frac{-3x_2}{\sqrt{x_1^2 - x_2^2}} + \left(1 + \frac{3x_2^2}{x_1^2 - x_2^2}\right) \operatorname{arccot}\left(\frac{x_2}{\sqrt{x_1^2 - x_2^2}}\right)\right]$$

Table 2. Jacobi matrix of the two constraints of the Somigliana-Pizzetti gravity field, partial derivatives with respect to the pseudo-observations (y_1, y_2, y_3, y_4)

$$\frac{\partial f_1}{\partial y_1} = -1 \quad \frac{\partial f_1}{\partial y_2} = \frac{\operatorname{arccot}\left(\frac{x_2}{\sqrt{x_1^2 - x_2^2}}\right)}{\sqrt{x_1^2 - x_2^2}} \qquad \frac{\partial f_1}{\partial y_3} = 0 \qquad \frac{\partial f_1}{\partial y_4} = \frac{2}{3} y_4 x_1^2$$

$$\frac{\partial f_2}{\partial y_1} = 0 \quad \frac{\partial f_2}{y_2} = \frac{\sqrt{5}(x_1^2 - x_2^2) + 15x_1^2 y_3}{4(x_1^2 - x_2^2)^{3/2}} \times \left[\frac{-3x_2}{\sqrt{x_1^2 - x_2^2}} + \left(1 + \frac{3x_2^2}{x_1^2 - x_2^2}\right) \operatorname{arccot}\left(\frac{x_2}{\sqrt{x_1^2 - x_2^2}}\right)\right]$$

$$\frac{\partial f_2}{\partial y_3} = \frac{15x_1^2 y_2}{4(x_1^2 - x_2^2)^{\frac{3}{2}}} \times \left[\frac{-3x_2}{\sqrt{x_1^2 - x_2^2}} + \left(1 + \frac{3x_2^2}{x_1^2 - x_2^2}\right) \operatorname{arccot}\left(\frac{x_2}{\sqrt{x_1^2 - x_2^2}}\right)\right] \qquad \frac{\partial f_2}{\partial y_4} = \frac{-2x_1^2 y_4}{3\sqrt{5}}$$

$$\mathbf{J_y} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial y_3} & \frac{\partial f_1}{\partial y_4} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial y_3} & \frac{\partial f_2}{\partial y_4} \end{bmatrix} (\xi_0, \eta_0) \tag{11}$$

$$\mathbf{D}\{\mathbf{x}\} = \mathbf{\Sigma_x} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix},$$

$$\mathbf{D}\{\mathbf{y}\} = \mathbf{\Sigma_y} = \operatorname{Diag}(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2) \tag{12}$$

Indeed for the error propagation in linearized form, namely (10), we have assumed that the variance-covariance-matrix/dispersion matrix/$\mathbf{D}\{\mathbf{y}\}$ of the pseudo-observations contains only variances: The correlation of the pseudo-observations $\{y_1, y_2, y_3, y_4\} = \{W_0, GM, J_2^*, \Omega\}$ is unknown, building up the argument why they have been neglected.

In the following, based upon current best estimates of the four fundamental parameters $\{W_0, GM, J_2, \Omega\}$ we will obtain numerical values of the form parameters $\{a, b\}$ of $\mathbb{E}_{a,b}^2$ of the level ellipsoid of Somigliana-Pizzetti type by Newton iteration of the linearized two conditional equations (47′), (48′), in particular dependent of the way that the indirect permanent tide is being implemented. Table 3 refers to the data $\{W_0, GM, J_2, \Omega\}$ to be applied within our computation. For instance, the zero-frequency-tide reference system is contained in the spherical harmonics coefficients J_2, while in the tide-free reference system J_2 is reduced for the effect of the indirect tide. Note that in both cases J_2 is free of direct tidal effects.

Table 3. Best estimates of pseudo-observations $\{W_0, GM, J_2^*, \Omega\}$

Author	Parameter	Value	Sources
Burša et al., 1997a	W_0	$(62636855.72 \pm 0.5)\ m^2/s^2$	Satellite altimetric data, gauge stations values
Burša et al., 1997b	W_0	$(62636855.80 \pm 0.5)\ m^2/s^2$	Satellite altimetric data, gauge stations values
Grafarend and Ardalan, 1997	W_0	$(62636855.80 \pm 3.6)\ m^2/s^2$	Gauge station data, GPS, Baltic Sea Level Project
Ries et al., 1992	GM	$(398600.4418 \pm 0.0008)\ km^3/s^2$	Satellite Laser Ranging, satellite altimetric data; in SI units, including mass of earth's atmosphere
Ries et al., 1992	GM	$(398600.4415 \pm 0.0008)\ km^3/s^2$	Satellite Laser Ranging, satellite altimetric data; in TDT units, including mass of earth's atmosphere
Tapley et al., 1996	J_2^*	$-4.8416954845647 \times 10^{-4} \pm 4.66 \times 10^{-11}$	JGM3 geopotential model; in zero frequency tide system
Lemoine et al., 1996	J_2^*	$-4.84165371736 \times 10^{-4} \pm 3.56 \times 10^{-11}$	EGM96 geopotential model; in tide-free system
Rapp et al., 1991	J_2^*	$-4.84165532804 \times 10^{-4} \pm 4.47 \times 10^{-11}$	OSU91A geopotential model; in tide-free system
Groten, 1997	Ω	$(7.292115 \times 10^{-5} \pm 10^{-12})$ rad/s	

Our calculations of the two form parameters $\{a, b\}$ of $\mathbb{E}^2_{a,b}$ of the level ellipsoid of Somigliana-Pizzetti type in the zero-frequency-tide reference system are collected in Table 4 while Table 5 is devoted to the results $\{a, b\}$ in the tide-free reference system. In both the reference systems different values of W_0 from Burša et al. (1997a), Burša et al. (1997b) as well as Grafarend and Ardalan (1997) and of GM from Ries et al. (1992) in SI and TDT units are tested. The effect of various J_2 values from JGM3 (Tapley et al., 1996), EGM96 (Lemoine et al., 1996) and OSU91A (Rapp et al., 1991) on the form parameters $\{a, b\}$ of $\mathbb{E}^2_{a,b}$ of the level ellipsoid of Somigliana-Pizzetti type is evaluated. In addition based upon error propagation the variance-covariance matrix of $\{a, b\}$ with respect to the best estimates of $\{W_0, GM, J_2, \Omega\}$ is outlined in Table 6.

From the Tables 3 and 4 the following conclusions can be made: The change of the GM value from TDT to SI units produces a millimeter variation in the form parameters $\{a, b\}$ of $\mathbb{E}^2_{a,b}$ which is below the level of the calculated root-mean-square error values. J_2^* data from the geopotential models JGM3, EGM96 and OSU91A result in almost the same values of the form parameters $\{a, b\}$ of $\mathbb{E}^2_{a,b}$. The root-mean-square errors of the form parameters $\{a, b\}$ of $\mathbb{E}^2_{a,b}$ are directly proportional to the root-mean-square error of the gauge value of the potential W_0, the ruling parameter for the variance budget. Indeed, we have documented that the tidal potential which is constant in time in its indirect effect with a properly chosen secular Love number ("zero-frequency tidal reference system") flattens the reference ellipsoid as a level ellipsoid of Somigliana-Pizzetti type. The abstract contains the final data set of the form parameters $\{a, b\}$ of level ellipsoid-of-revolution of Somigliana-Pizzetti type (tide-free as well as zero-frequency tide reference).

2 Spheroidal coordinates, spheroidal gravity field

In order to derive the two conditions (47), (48) which characterise the Somigliana-Pizzetti gravity potential of a level Spheroid-of-revolution we focus here on spheroidal coordinates as well as spheroidal harmonics. Thong and Grafarend (1989) have given an extensive review of different types of spheroidal coordinates and the spheroidal eigenvalues/eigenfunctions which span the spheroidal solution space of the three-dimensional Laplace partial differential equation for the external gravity field of the Earth. Here we have chosen the variant $\{\lambda, \phi, u\}$ of spheroidal coordinates which are generated as elliptic coordinates by the intersection of a family of confocal oblate spheroids, a family of confocal half hyperboloids, and a family of half planes according to the following definition

Table 4. Best estimates of the form parameters $\{a, b\}$ of the Somigliana-Pizzetti level ellipsoid in zero frequency tide system, compared to H. Moritz (1992)

a (m)	b (m)	$a - a_{GRS80}$ (m)	$b - b_{GRS80}$ (m)	W_0 (m^2/s^2)	GM (km^3/s^2)	J_2^*
6378136.610 ± 0.053	6356751.868 ± 0.052	−0.390	−0.446	62636855.72[i] ± 0.5	398600.4418[iv] ± 0.0008	$-4.8416954845647 \times 10^{-4}$ [vi] $\pm 4.66 \times 10^{-11}$
6378136.605 ± 0.053	6356751.863 ± 0.052	−0.395	−0.451	62636855.72[i] ± 0.5	398600.4415[v] ± 0.0008	$-4.8416954845647 \times 10^{-4}$ [vi] $\pm 4.66 \times 10^{-11}$
6378136.602 ± 0.053	6356751.860 ± 0.052	−0.398	−0.454	62636855.80[ii] ± 0.5	398600.4418[iv] ± 0.0008	$-4.8416954845647 \times 10^{-4}$ [vi] $\pm 4.66 \times 10^{-11}$
6378136.602 ± 0.369	6356751.860 ± 0.366	−0.398	−0.454	62636855.80[iii] ± 3.6	398600.4418[iv] ± 0.0008	$-4.8416954845647 \times 10^{-4}$ [vi] $\pm 4.66 \times 10^{-11}$

[i] Burša et al., 1997a
[ii] Burša et al., 1997b
[iii] Grafarend and Ardalan, 1997
[iv] Ries et al., 1992 (in SI units)
[v] Ries et al., 1992 (in TDT units)
[vi] Tapley et al., 1996
$\Omega = 7.292115 \times 10^{-5} \pm 10^{-12}$ (rad/s) (Groten, 1997)

Table 5. Best estimates of the form parameters $\{a, b\}$ of the Somigliana-Pizzetti level ellipsoid in tide-free system, compared to H. Moritz (1992)

a (m)	b (m)	$a - a_{\mathrm{GRS80}}$ (m)	$b - b_{\mathrm{GRS80}}$ (m)	W_0 ($\mathrm{m^2/s^2}$)	GM ($\mathrm{km^3/s^2}$)	J_2^*
6378136.580 $\pm$ 0.053	6356751.928 $\pm$ 0.052	−0.420	−0.386	62636855.72[i] $\pm$ 0.5	398600.4418[iv] $\pm$ 0.0008	-8416537×10^{-4} [vi] $\pm$ 3.561×10^{-11}
6378136.575 $\pm$ 0.053	6356751.923 $\pm$ 0.052	−0.425	−0.391	62636855.72[i] $\pm$ 0.5	398600.4415[v] $\pm$ 0.0008	$-4.8416537 \times 10^{-4}$ [vi] $\pm$ 3.561×10^{-11}
6378136.572 $\pm$ 0.053	6356751.920 $\pm$ 0.052	−0.428	−0.394	62636855.80[ii] $\pm$ 0.5	398600.4418[iv] $\pm$ 0.0008	$-4.8416537 \times 10^{-4}$ [vi] $\pm$ 3.561×10^{-11}
6378136.572 $\pm$ 0.369	6356751.920 $\pm$ 0.366	−0.428	−0.394	62636855.80[iii] $\pm$ 3.6	398600.4418[iv] $\pm$ 0.0008	$-4.8416537 \times 10^{-4}$ [vi] $\pm$ 3.561×10^{-11}
6378136.581 $\pm$ 0.053	6356751.926 $\pm$ 0.052	−0.419	−0.388	62636855.72[i] $\pm$ 0.5	398600.4418[iv] $\pm$ 0.0008	$-4.8416553 \times 10^{-4}$ [vii] $\pm$ 4.472×10^{-11}
6378136.573 $\pm$ 0.369	6356751.917 $\pm$ 0.366	−0.427	−0.397	62636855.80[iii] $\pm$ 3.6	398600.4418[iv] $\pm$ 0.0008	$-4.8416553 \times 10^{-4}$ [vii] $\pm$ 4.472×10^{-11}

[i] Burša et al., 1997a
[ii] Burša et al., 1997b
[iii] Grafarend and Ardalan, 1997
[iv] Ries et al., 1992 (in SI units)
[v] Ries et al., 1992 (in TDT units)
[vi] Tapley et al., 1996
[vii] Rapp et al., 1991
$\Omega = 7.292115 \times 10^{-5} \pm 10^{-12}$ (rad/s) (Groten, 1997)

Table 6. Variance-covariance matrix of optimal $\{a, b\}$ of $\mathbb{E}^2_{a,b}$, of Somigliana-Pizzetti type, via error propagation from variances of the pseudo-observations $\{W_0, GM, J_2^*, \Omega\}$

σ_{W_0} ($\mathrm{m^2/s^2}$)	σ_{GM} ($\mathrm{km^3/s^2}$)	$\sigma_{J_2^*}$	σ_Ω (rad/s)	$\hat{\sigma}_a$ (m)	$\hat{\sigma}_b$ (m)	$\hat{\sigma}_{ab}$	$\rho = \frac{\hat{\sigma}_{ab}}{\hat{\sigma}_a \hat{\sigma}_b}$
0.5	0.0008	4.5×10^{-11}	10^{-12}	0.053	0.052	2.738×10^{-3}	0.99984 (%99.98)

Definition 1. (spheroidal coordinates $\{\lambda, \phi, u\}$)

The mixed elliptic-trigonometric elliptic coordinates generated by the intersection of

(i) the family of confocal, oblate spheroids

$$\mathbb{E}^2_{\sqrt{u^2+\varepsilon^2},u} := \left\{ \mathbf{x} \in \mathbb{R}^3 \middle| \frac{x^2+y^2}{u^2+\varepsilon^2} + \frac{z^2}{u^2} = 1, \right.$$
$$\left. u \in (0, +\infty), \varepsilon^2 := a^2 - b^2 \right\} \tag{13}$$

(ii) the family of confocal half hyperboloids

$$\mathbb{H}^2_{\varepsilon\cos\phi,\varepsilon\sin\phi} := \left\{ \mathbf{x} \in \mathbb{R}^3 \middle| \frac{x^2+y^2}{\varepsilon^2\cos^2\phi} - \frac{z^2}{\varepsilon^2\sin^2\phi} = 1, \right.$$
$$\left. \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \phi \neq 0 \right\} \tag{14}$$

(iii) the family of half planes

$$\mathbb{P}^2_{\cos\lambda,\sin\lambda} := \{\mathbf{x} \in \mathbb{R}^3 | y = x \tan\lambda,\ \lambda \in [0, 2\pi]\} \tag{15}$$

are called spheroidal. The longitude λ gives orientation to the half planes; the latitude ϕ relates to the inclination of the asymptotes of confocal half hyperboloids; the elliptic coordinate u coincides with the semi-minor axis of confocal oblate spheroids (confocal, oblate ellipsoids of revolution). □

In addition, let us assume that the pre-relativistic space near the Earth is three-dimensional Euclidean $\{\mathbb{R}^3, g_{k\ell}\}$ with the matrix of the metric $[g_{k\ell}]$. If $\mathbb{R}^3$ is covered by Cartesian coordinates, the three-dimensional Euclidean space $\{\mathbb{R}^3, \delta_{k\ell}\}$ is completely covered by one chart. In contrast, if $\mathbb{R}^3$ is covered by curvilinear coordinates of type spheroidal, a minimal atlas of the three-dimensional Euclidean space is established by three charts. Accordingly, the following corollary generates only one chart of $\{\mathbb{R}^3, g_{k\ell}\}$ due to the demanded domain $\lambda \in \{\lambda \in \mathbb{R} | 0 < \lambda < 2\pi\}$, $\phi \in \{\phi \in \mathbb{R} | -\pi/2 < \phi < +\pi/2\}$, $u \in \{u \in \mathbb{R} | u > 0\}$ by avoiding singularities at the North-pole as well as at the South-pole of the confocal oblate spheroids.

Corollary 1. (conversion of Cartesian coordinates $\{x, y, z\}$ into spheroidal coordinates $\{\lambda, \phi, u\}$)

The forward transformation of spheroidal coordinates $\{\lambda, \phi, u\}$ into Cartesian coordinates $\{x, y, z\}$, namely

$$\begin{aligned} x &= \sqrt{u^2+\varepsilon^2}\cos\phi\cos\lambda \\ y &= \sqrt{u^2+\varepsilon^2}\cos\phi\sin\lambda \\ z &= u\sin\phi \end{aligned} \tag{16}$$

can be uniquely inverted into the backward transformation of Cartesian coordinates $\{x, y, z\}$ into spheroidal coordinates $\{\lambda, \phi, u\}$, namely

$$\lambda = \arctan\left(\frac{y}{x}\right) + \left(-\frac{1}{2}\operatorname{sgn} y - \frac{1}{2}\operatorname{sgn} y \operatorname{sgn} x + 1\right) \cdot \pi$$
$$\phi = (\operatorname{sgn} z)\arcsin\left\{\frac{1}{2\varepsilon^2}\left[\varepsilon^2 - (x^2+y^2+z^2) + \sqrt{(x^2+y^2+z^2-\varepsilon^2)^2 + 4\varepsilon^2 z^2}\right]\right\}^{1/2} \tag{17}$$

$$u = \left\{ \frac{1}{2} \left[x^2 + y^2 + z^2 - \varepsilon^2 + \sqrt{(x^2 + y^2 + z^2 - \varepsilon^2)^2 + 4\varepsilon^2 z^2} \right] \right\}^{1/2}$$

if

$$\begin{aligned} &\lambda \in \{\lambda \in \mathbb{R} | 0 < \lambda < 2\pi\} \\ &\phi \in \left\{ \phi \in \mathbb{R} | -\frac{\pi}{2} < \phi < +\frac{\phi}{2} \right\} \\ &u \in \{u \in \mathbb{R} | u > 0\} \end{aligned} \tag{18}$$

holds. □

Spheroidal coordinates enjoy the property to decomposing the three-dimensional Laplace partial differential equation into separable functions. We collect this basic result in

Lemma 1. (spheroidal eigenspace of the three-dimensional Laplace partial differential equation, external gravity field of the Earth)

For a static, uniformly rotating Earth the gravity potential field $W(\lambda, \phi, u)$ can be additively decomposed into the gravitational potential field $U(\lambda, \phi, u)$ and the centrifugal potential field $V(\lambda, \phi, u)$, namely

$$W(\lambda, \phi, u) = U(\lambda, \phi, u) + V(\lambda, \phi, u) \tag{19}$$

The multiplicative decomposition of the gravitational potential field into separable functions $U(\lambda, \phi, u) = \Lambda(\lambda)\Phi(\phi)U(u)$ generates the solution of the three dimensional Laplace partial differential equation

$$U(\lambda, \phi, u) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} u_{nm} \frac{Q^*_{n|m|}\left(\frac{u}{\varepsilon}\right)}{Q^*_{n|m|}\left(\frac{b}{\varepsilon}\right)} e_{nm}(\lambda, \phi) \tag{20}$$

in terms of surface spheroidal harmonics

$$e_{nm}(\lambda, \phi) = P^*_{n|m|}(\sin \phi) \left[\begin{array}{ll} \cos m\lambda & \forall\, m \geq 0 \\ \sin |m|\lambda & \forall\, m < 0 \end{array} \right. \tag{21}$$

in the space $\{\mathbb{R}^3 / \mathbb{E}^2_{a,b}\}$ which is external to the ellipsoid of reference $(x^2 + y^2)/(b^2 + \varepsilon^2) + z^2/b^2 = 1$. The eigenfunctions $e_{nm}(\lambda, \phi)$ are orthonormal on the ellipsoid of reference with a proper choice of a weighting function.

A representation of the centrifugal potential in (i) Cartesian coordinates, (ii) spheroidal coordinates and (iii) surface spheroidal harmonics is

$$\begin{aligned} V(\phi, u) &= \tfrac{1}{2}\, \omega^2 (x^2 + y^2) = \tfrac{1}{2}\, \omega^2 (u^2 + \varepsilon^2) \cos^2 \phi \\ &= \frac{1}{3} \omega^2 \left(P^*_{20}\left(\frac{u}{\varepsilon}\right) + \varepsilon^2 \right) \cos^2 \phi \end{aligned} \tag{22}$$

$$\begin{aligned} \cos^2 \phi &= \frac{2}{3} \left(P^*_{00}(\sin \phi) - \frac{1}{\sqrt{5}} P^*_{20}(\sin \phi) \right), \\ u^2 + \varepsilon^2 &= \frac{2}{3} \left(P^*_{20}\left(\frac{u}{\varepsilon}\right) + \varepsilon^2 \right) \end{aligned} \tag{23}$$

$$\begin{aligned} V(\lambda, \phi) &= \frac{2}{9} \omega^2 \left(P^*_{20}\left(\frac{u}{\varepsilon}\right) + \varepsilon^2 \right) P^*_{00}(\sin \phi) \\ &\quad - \frac{2}{9\sqrt{5}} \omega^2 \left(P^*_{20}\left(\frac{u}{\varepsilon}\right) + \varepsilon^2 \right) P^*_{20}(\sin \phi) \\ &= \frac{2}{9} \omega^2 \left(P^*_{20}\left(\frac{u}{\varepsilon}\right) + \varepsilon^2 \right) e_{00} \\ &\quad - \frac{2}{9\sqrt{5}} \omega^2 \left(P^*_{20}\left(\frac{u}{\varepsilon}\right) + \varepsilon^2 \right) e_{20} \end{aligned} \tag{24}$$

□

So far we have not defined the normalised associated Legendre functions of the first kind as well as of the second kind as they appear in (20)–(24).

Definition 2. (normalised associated Legendre functions of the first and second kind)

The fully normalised associated Legendre functions of the first kind are defined by means of recurrence relations of type

$$P^*_{nn}(\sin \phi) = \frac{\sqrt{2n+1}}{\sqrt{2n}} \cos \phi\, P^*_{n-1,n-1}(\sin \phi) \tag{25}$$

$$P^*_{n,n-1}(\sin \phi) = \frac{\sqrt{2n+1}}{\sqrt{2(n-1)}} \cos \phi\, P^*_{n-1,n-2}(\sin \phi) \tag{26}$$

$$\begin{aligned} P^*_{nm}(\sin \phi) &= \frac{\sqrt{4n^2 - 1}}{\sqrt{n^2 - m^2}} \sin \phi\, P^*_{n-1,m}(\sin \phi) \\ &\quad - \frac{\sqrt{(2n+1)(n+m-1)(n-m-1)}}{\sqrt{(n^2 - m^2)(2n-3)}} \\ &\quad \times P^*_{n-2,m}(\sin \phi) \end{aligned} \tag{27}$$

subject to

$$\forall\, n \in [3, \infty) \quad \text{and} \quad m \in [0, n-2]$$

with starting values

$$P^*_{00}(\sin \phi) = 1 \tag{28}$$

$$P^*_{10}(\sin \phi) = \sqrt{3} \sin \phi \tag{29}$$

$$P^*_{11}(\sin \phi) = \sqrt{3} \cos \phi \tag{30}$$

$$P^*_{20}(\sin \phi) = \frac{\sqrt{5}}{2} (3 \sin^2 \phi - 1) \tag{31}$$

$$P^*_{21}(\sin \phi) = \sqrt{15} \sin \phi \cos \phi \tag{32}$$

$$P^*_{22}(\sin \phi) = \frac{\sqrt{15}}{2} \cos^2 \phi \tag{33}$$

If i is the imaginary root of minus unity, the associated Legendre functions of the second kind are defined by an integral relation of type

$$Q^*_{nm}\left(\frac{u}{\varepsilon}\right) = i^{n+1} Q_{nm}\left(i \frac{u}{\varepsilon}\right) \tag{34}$$

$$Q_{nm}\left(i\frac{u}{\varepsilon}\right) = \frac{(-1)^m 2^n (n+m)! m!}{i^{n+1}(n-m)!(2m)!}\left(\frac{u^2}{\varepsilon^2}+1\right)^{m/2} \times \int_0^\infty \frac{\sinh^{2m}\tau \, \mathrm{d}\tau}{\left(\frac{u}{\varepsilon}+\sqrt{\frac{u^2}{\varepsilon^2}+1}\cosh\tau\right)^{n+m+1}} \quad (35)$$

with starting values for $n = 0, 1, 2$ and $m = 0$,

$$Q_0^*\left(\frac{u}{\varepsilon}\right) = \operatorname{arccot}\left(\frac{u}{\varepsilon}\right) \quad (36)$$

$$Q_1^*\left(\frac{u}{\varepsilon}\right) = 1 - \frac{u}{\varepsilon}\operatorname{arccot}\left(\frac{u}{\varepsilon}\right) \quad (37)$$

$$Q_2^*\left(\frac{u}{\varepsilon}\right) = \frac{1}{2}\left[\left(3\frac{u^2}{\varepsilon^2}+1\right)\operatorname{arccot}\left(\frac{u}{\varepsilon}\right) - 3\frac{u}{\varepsilon}\right] \quad (38)$$

The reader may wonder about the ratio $Q_{nm}^*(u/\varepsilon)/Q_{nm}^*(b/\varepsilon)$ of normalised associated Legendre functions of the second kind as they appear in the series expansion of the gravitational potential field $U(\lambda, \phi, u)$ of type (20) with respect to spheroidal coordinates. Indeed this ratio is motivated by the definition of "weighted orthonormality" of the base functions or eigenfunctions on the ellipsoid of revolution $u_0 = b$. Corollary 2 is a résumé of the global area element of the reference ellipsoid of revolution $\mathbb{E}^2_{a,b}$ which enables us in Corollary 3 to formulate "weighted orthonormality" as well as the reproducing property of the "weighted scalar product".

Corollary 2. (local and global area element of the reference ellipsoid of revolution $\mathbb{E}^2_{a,b}$)

The local area element of the spheroid $\mathbb{E}^2_{a,b}$ is given by

$$\mathrm{d}S = \mathrm{d}\{\text{area } (\mathbb{E}^2_{a,b})\} = \sqrt{g_{\lambda\lambda}g_{\phi\phi}}\,\mathrm{d}\phi\,\mathrm{d}\lambda \quad (39)$$

$$\mathrm{d}S = \mathrm{d}\{\text{area } (\mathbb{E}^2_{a,b})\} = a \cdot \sqrt{b^2+\varepsilon^2\sin^2\phi}\,\cos\phi\,\mathrm{d}\lambda\,\mathrm{d}\phi \quad (40)$$

while the global area element of the spheroid of the spheroid $\mathbb{E}^2_{a,b}$ amounts to

$$S = \text{area}\left(\mathbb{E}^2_{a,b}\right) = 4\pi a \cdot \left\{\frac{1}{2}+\frac{1}{4}\frac{b^2}{a\varepsilon}\ln\frac{a+\varepsilon}{a-\varepsilon}\right\} \quad (41)$$

□

The proof of Corollary 2 follows from straight forward integration of dS.

Corollary 3. ("weighted orthonormality", "weighted scalar product" with respect to the reference ellipsoid of revolution $\mathbb{E}^2_{a,b}$)

The base functions or eigenfunctions $e_{nm}(\lambda, \phi)$ are orthonormal with respect to the weighted scalar product

$$\langle e_{pq}(\lambda,\phi)|e_{nm}(\lambda,\phi)\rangle_w := \frac{1}{S}\int_{\mathbb{E}^2_{\sqrt{u^2+\varepsilon^2},u}} \mathrm{d}S w(\phi) e_{pq}(\lambda,\phi)e_{mn}(\lambda,\phi) = \delta_{pn}\delta_{qm} \quad (42)$$

and the "quantum numbers"

$p, n = 0, 1, \ldots, \infty;$
$q = -p, -p+1, \ldots, -1, 0, +1, \ldots, p-1, p;$
$m = -n, -n+1, \ldots, -1, 0, +1, \ldots, n-1, n$.

The weighting function is defined by

$$w(\phi) := \frac{a}{\sqrt{b^2+\varepsilon^2\sin^2\phi}}\left(\frac{1}{2}+\frac{1}{4}\frac{b^2}{a\varepsilon}\cdot\ln\frac{a+\varepsilon}{a-\varepsilon}\right) \quad (43)$$

The weighted scalar product

$$\begin{aligned}
&\langle U(\lambda,\phi,u=b)|e_{nm}(\lambda,\phi)\rangle_w \\
&= \left[4\pi a\left(\frac{1}{2}+\frac{1}{4}\frac{b^2}{a\varepsilon}\cdot\ln\frac{a+\varepsilon}{a-\varepsilon}\right)\right]^{-1} \\
&\quad \times \int_0^{2\pi}\mathrm{d}\lambda\int_{-\pi/2}^{+\pi/2}\mathrm{d}\phi a\sqrt{b^2+\varepsilon^2\sin^2\phi}\cos\phi\, w(\phi) \\
&\quad \times U(\lambda,\phi,u=b)e_{nm}(\lambda,\phi) \\
&= \sum_{p=0}^{\infty}\sum_{q=-m}^{+m} u_{pq}\langle e_{pq}(\lambda,\phi)|e_{nm}(\lambda,\phi)\rangle_w \\
&= \sum_{p=0}^{\infty}\sum_{q=-m}^{+m} u_{pq}\delta_{pn}\delta_{qm} = u_{nm} \quad (44)
\end{aligned}$$

has the reproducing property. □

3 The Somigliana-Pizzetti gravity of a level ellipsoid, the two constraints

The idea of generating the gravity field of a rotating level ellipsoid according to Somigliana (1930) and Pizzetti (1894) is the following. Given the general representation of the gravity field (19), (20), (22), (24) in terms of surface spheroidal harmonics, namely normalised associated Legendre functions of the first and second kind. Find the gravity field in terms of spheroidal coordinates which is specified to a particular ellipsoid of revolution $\mathbb{E}^2_{a,b}$ of semi-major axis $a = \sqrt{b^2+\varepsilon^2}$, and semi-minor axis $b = u$ which is at the same time a level ellipsoid of revolution gauged to the gravity potential value $w_0 = W_0$ of the geoid. The solution of this problem in eigenspace of surface spheroidal harmonics is presented in

Lemma 2. (the gravity field of a level ellipsoid, gauge to the geoid)

If the spheroidal gravity potential field (19), (20), (22), (24) is specified to the level ellipsoid (i) $u = b$ and (ii) $w_0 = W_0$ constant, the eigenfunctions of the three dimensional Laplace partial differential equation are restricted according to

$$\begin{aligned}
W(\lambda,\phi,b) &= \sum_{n=0}^{\infty}\sum_{m=-n}^{+n} u_{nm}e_{nm}(\lambda,\phi) \\
&\quad + \frac{1}{3}\omega^2a^2e_{00} - \frac{1}{3\sqrt{5}}\omega^2a^2e_{20}(\lambda,\phi) \\
&= w_0 \quad (45)
\end{aligned}$$

$$W(\lambda,\phi,b) = \left(u_{00}+\frac{1}{3}\omega^2 a^2\right)e_{00}+\left(u_{20}-\frac{1}{3\sqrt{5}}\omega^2 a^2\right)e_{20}(\lambda,\phi) + \sum_{n=1}^{\infty}\sum_{\substack{m=-n\\(n,m)\neq(2,0)}}^{+n} u_{nm}e_{nm}(\lambda,\phi) = w_0 = \text{const.} \tag{46}$$

$$u_{00} + \tfrac{1}{3}\omega^2 a^2 = w_0 \tag{47}$$

$$u_{20} - \frac{1}{3\sqrt{5}}\omega^2 a^2 = 0 \tag{48}$$

$$u_{nm} = 0 \quad \forall\, n \geq 1,\ \ (n,m) \neq (2,0) \tag{49}$$

Since only the first term on the left-hand side is a constant, the constant w_0 of the level ellipsoid is balanced by (47). The terms of degree/order (2,0), as well as $(n,m) \neq (2,0), n \geq 1$ are not constant. Accordingly by means of (48), (49) they have to vanish. □

The proof of Lemma 2 is straight-forward.

In order to identify the terms of degree/order (0,0), (2,0) u_{00} and u_{20} respectively, we introduce the Newton gravitational potential field in terms of spheroidal coordinates – namely its spheroidal harmonic expansion – generated by the mass density field $\rho(\lambda',\phi',u')$.

Corollary 4. (spheroidal harmonic expansion of the inverse distance function in the domain $u' < u$)

If the inverse distance function, represented in Cartesian coordinates as well as in spheroidal coordinates,

$$\frac{1}{\|\mathbf{x}-\mathbf{x}'\|} = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} = \left\{\left[\sqrt{u^2+\varepsilon^2}\cos\phi\cos\lambda - \sqrt{u'^2+\varepsilon^2}\cos\phi'\cos\lambda'\right]^2 + \left[\sqrt{u^2+\varepsilon^2}\cos\phi\sin\lambda - \sqrt{u'^2+\varepsilon^2}\cos\phi'\sin\lambda'\right]^2 + [u\sin\phi - u'\sin\phi']^2\right\}^{-1/2} \tag{50}$$

is expanded in the domain $u' > u$ into surface spheroidal harmonics, we receive

$$\frac{1}{\|\mathbf{x}-\mathbf{x}'\|} = \frac{i}{\varepsilon}\sum_{n=0}^{\infty}(2n+1)\Bigg[P_n(\sin\phi)P_n(\sin\phi')Q_n\left(i\frac{u}{\varepsilon}\right)P_n\left(i\frac{u'}{\varepsilon}\right) + 2\sum_{m=1}^{n}(-1)^m\left(\frac{(n-m)!}{(n+m)!}\right)P_{nm}(\sin\phi) \times P_{nm}(\sin\phi')Q_{nm}\left(i\frac{u}{\varepsilon}\right)P_{nm}\left(i\frac{u'}{\epsilon}\right)\cos m(\lambda-\lambda')\Bigg] \tag{51}$$

subject to

$$e_{nm}(\lambda,\phi) = P^*_{n|m|}(\sin\phi)\begin{bmatrix}\cos m\lambda & \forall\, m \geq 0\\ \sin|m|\lambda & \forall\, m < 0\end{bmatrix} \tag{52}$$

$$P^*_n(\sin\phi) = \sqrt{2n+1}P_n(\sin\phi) \tag{53}$$

$$P^*_{nm}(\sin\phi) = \sqrt{2(2n+1)\frac{(n-|m|)!}{(n+|m|)!}}P_{nm}(\sin\phi) \tag{54}$$

$$P^*_{nm}\left(\frac{u}{\varepsilon}\right) = i^{-n}P_{nm}\left(i\frac{u}{\varepsilon}\right) \tag{55}$$

$$Q^*_{nm}\left(\frac{u}{\varepsilon}\right) = i^{n+1}Q_{nm}\left(i\frac{u}{\varepsilon}\right) \tag{56}$$

such that

$$\frac{1}{\|\mathbf{x}-\mathbf{x}'\|} = \frac{1}{\varepsilon}\sum_{n=0}^{\infty}\sum_{m=-n}^{+n}(-1)^m\frac{(n-|m|)!}{(n+|m|)!} \times P^*_{n|m|}\left(\frac{u'}{\varepsilon}\right)Q^*_{n|m|}\left(\frac{u}{\varepsilon}\right)e_{mn}(\lambda',\phi')e_{nm}(\lambda,\phi) \tag{57}$$

holds. □

For the proof of Corollary 4 we refer to Neumann (1848) or Hobson (1965, pp. 424–430).

Lemma 3. (spheroidal harmonic expansion of the Newton gravitational potential)

With respect to the spheroidal harmonic expansion of the inverse distance function in the domain $u' < u$ the Newton gravitational potential field can be represented by

$$U(\lambda,\phi,u) = G\int_C^{2\pi}\mathrm{d}\lambda'\int_{-\pi/2}^{\pi/2}\mathrm{d}\phi'\cos\phi' \times \int_0^{u'(\lambda',\phi)}\mathrm{d}u'(u'^2+\varepsilon^2\sin\phi') \times \rho(\lambda',\phi',u')\frac{1}{\|\mathbf{x}(\lambda,\phi,u)-\mathbf{x}(\lambda',\phi',u')\|} \tag{58}$$

$$U(\lambda,\phi,u) = \frac{G}{\varepsilon}\int_0^{2\pi}\mathrm{d}\lambda'\int_{-\pi/2}^{\pi/2}\mathrm{d}\phi'\cos\phi' \times \int_0^{u'(\lambda',\phi)}\mathrm{d}u'(u'^2+\varepsilon^2\sin^2\ \phi')\rho(\lambda',\phi',u') \times \sum_{n=0}^{\infty}\sum_{m=-n}^{+n}(-1)^m\frac{(n-|m|)!}{(n+|m|)!}P^*_{n|m|}\left(\frac{u'}{\varepsilon}\right) \times Q^*_{n|m|}\left(\frac{u}{\varepsilon}\right)e_{nm}(\lambda',\phi')e_{nm}(\lambda,\phi) \tag{59}$$

namely with uniform convergence of the spheriodal harmonic expansion of the inverse distance function such that summation and integration can be interchanged.

$$U(\lambda,\phi,u) = \frac{G}{\varepsilon}\sum_{n=0}^{\infty}\sum_{m=-n}^{+n}\Bigg[(-1)^m\frac{(n-|m|)!}{(n+|m|)!}\Bigg\{\int_0^{2\pi}\mathrm{d}\lambda' \times \int_{-\pi/2}^{\pi/2}\mathrm{d}\phi'\cos\phi'\int_0^{u'(\lambda',\phi)}\mathrm{d}u'(u'^2+\varepsilon^2\sin^2\ \phi') \times \rho(\lambda',\phi',u')P^*_{n|m|}\left(\frac{u'}{\varepsilon}\right)e_{nm}(\lambda',\phi')\Bigg\} \times Q^*_{n|m|}\left(\frac{u}{\varepsilon}\right)e_{nm}(\lambda,\phi)\Bigg] \tag{60}$$

The spheriodal harmonic coefficients of (20) amount to

$$u_{nm} := \frac{G}{\varepsilon}(-1)^m \frac{(n-|m|)!}{(n+|m|)!} Q^*_{n|m|}\left(\frac{b}{\varepsilon}\right) \int_0^{2\pi} d\lambda' \int_{-\pi/2}^{+\pi/2} d\phi' \cos\phi' \times \int_0^{u'(\lambda',\phi')} du' (u'^2 + \varepsilon^2 \sin^2\phi') \times \rho(\lambda',\phi',u') P^*_{n|m|}\left(\frac{u'}{\varepsilon}\right) e_{nm}(\lambda',\phi') \quad \forall \begin{bmatrix} n = 0,1,\ldots,\infty \\ m = -n,-n+1,\ldots,n-1,n \end{bmatrix} \tag{61}$$

□

Here is a sketch of the proof. The Newton gravitational potential (58) contains four factorial elements. (i) G denotes the Newton gravitational constant, also called coupling constant between the inertial force and the gravitational force. (ii) The volume in spheroidal coordinates is generated by $\sqrt{\det G}$, $G \in \mathbb{R}^{3*3}$, the square-rooted determinant of the matrix G of the metric. In particular, $g_{11} = g_{\lambda\lambda} = (u^2+\varepsilon^2)\cos^2\varphi$, $g_{22} = g_{\phi\phi} = u^2 + \varepsilon^2\sin^2\varphi$, $g_{33} = g_{uu} = (u^2+\varepsilon^2\sin^2\varphi)/(u^2+\varepsilon^2)$, $g_{k\ell} = 0 \ \forall k \neq \ell$, $k,\ell \in \{1,2,3\}$, $\sqrt{\det G} = \cos\phi(u^2 + \varepsilon^2\sin^2\phi)$ holds. The volume of the terrestrial body is bounded by its surface, here represented by the function $u(\lambda,\phi)$. Accordingly, the integration over the third spheroidal coordinate u extends from zero to $u(\lambda,\phi)$. (iii) $\rho(\lambda,\phi,u)$ denotes mass density field, expressed in terms of the spheroidal coordinates $\{\lambda,\phi,u\}$. (iv) $1/\|\mathbf{x}(\lambda,\phi,u) - \mathbf{x}(\lambda',\phi',u')\|$ denotes the inverse of the Euclidean distance between the points $\mathbf{x} = \mathbf{x}(\lambda,\phi,u)$ and $\mathbf{x}' = \mathbf{x}(\lambda',\phi',u')$. Next, (59) is generated by implementing the inverse distance function as the spheroidal harmonic expansion with respect to the domain $u' < u$ from (57) into the kernel of the Newton gravitational potential integral. In the external domain $u' < u$ the series expansion of the Newton kernel is uniformly convergent, a prerequisite in order to interchange integration and summation. Accordingly (60) is generated by this operation. Finally, if we compare the external solution of the three-dimensional Laplace partial differential equation in terms of spheroidal coordinates, (20), and the spheroidal harmonic expansion of the Newton integral we are led to the spheroidal harmonic coefficients (61).

These spheroidal harmonic coefficients, also called spheroidal multipoles, are now specified for (degree/order) (0,0) and (2,0), respectively, since they appear in two constraints (47), (48) of the Somigliana-Pizzetti gravity field.

Corollary 5. (spheroidal multipoles of degree/order $(0,0)$ and (2,0), respectively)

By means of (28)–(33), (34)–(38) the spheroidal harmonic coefficients/spheroidal multiploes u_{00} and u_{20} are represented by

$$u_{00} = \frac{G}{\varepsilon} Q^*_{00}\left(\frac{b}{\varepsilon}\right) \int_0^{2\pi} d\lambda' \int_{-\pi/2}^{+\pi/2} d\phi' \cos\phi' \times \int_0^{u'(\lambda',\phi')} du'(u'^2 + \varepsilon^2\sin^2\phi') \times \rho(\lambda',\phi',u') P^*_{00}\left(\frac{u'}{\varepsilon}\right) e_{00}(\lambda',\phi') \tag{62}$$

$$u_{20} = \frac{G}{\varepsilon} Q^*_{20}\left(\frac{b}{\varepsilon}\right) \int_0^{2\pi} d\lambda' \int_{-\pi/2}^{+\pi/2} d\phi' \cos\phi' \times \int_0^{u'(\lambda',\phi')} du'(u'^2 + \varepsilon^2\sin^2\phi') \times \rho(\lambda',\phi',u') P^*_{00}\left(\frac{u'}{\varepsilon}\right) e_{20}(\lambda',\phi') \tag{63}$$

or

$$u_{00} = \frac{G}{\varepsilon} \operatorname{arc\,cot}\left(\frac{b}{\varepsilon}\right) \int_0^{2\pi} d\lambda' \int_{-\pi/2}^{+\pi/2} d\phi' \cos\phi' \times \int_0^{u'(\lambda',\phi')} du'(u'^2 + \varepsilon^2\sin^2\phi') \times \rho(\lambda',\phi',u') P^*_{00}\left(\frac{u'}{\varepsilon}\right) e_{00}(\lambda',\phi') \tag{64}$$

$$u_{00} = \frac{GM}{\varepsilon} \operatorname{arccot}\left(\frac{b}{\varepsilon}\right) \tag{65}$$

$$u_{20} = \frac{G}{\varepsilon}\frac{1}{2}\left[\left(3\frac{b^2}{\varepsilon^2}+1\right)\operatorname{arccot}\left(\frac{b}{\varepsilon}\right) - 3\frac{b}{\varepsilon}\right] \int_0^{2\pi} d\lambda' \times \int_{-\pi/2}^{+\pi/2} d\phi' \cos\phi' \int_0^{u'(\lambda',\phi')} du'(u'^2 + \varepsilon^2\sin^2\phi') \times \rho(\lambda',\phi',u') \frac{1}{2}\left(3\frac{u'^2}{\varepsilon^2}+1\right)\frac{\sqrt{5}}{2}(3\sin^2\phi' - 1) \tag{66}$$

$$u_{20} = \frac{\sqrt{5}}{8}\frac{G}{\varepsilon^3}\left[\left(3\frac{b^2}{\varepsilon^2}+1\right)\operatorname{arccot}\left(\frac{b}{\varepsilon}\right) - 3\frac{b}{\varepsilon}\right] \times \int_0^{2\pi} d\lambda' \int_{-\pi/2}^{+\pi/2} d\phi' \cos\phi' \times \int_0^{u'(\lambda',\phi')} du'(u'^2 + \varepsilon^2\sin^2\phi')\rho(\lambda',\phi',u') \times (9u'^2\sin^2\phi' + 3\varepsilon^2\sin^2\phi' - 3u' - \varepsilon^2) \tag{67}$$

$$u_{20} = \frac{\sqrt{5}}{8}\frac{G}{\varepsilon^3}\left[\left(3\frac{b^2}{\varepsilon^2}+1\right)\operatorname{arccot}\left(\frac{b}{\varepsilon}\right) - 3\frac{b}{\varepsilon}\right] \int_{x_1}^{x_2} dx' \times \int_{y_1}^{y_2} dy' \int_{z'_1(x',y')}^{z'_2(x',y')} dz' \rho(\lambda',\phi',u') \times [-3(x'^2+y'^2) + 6z'^2 + 2\varepsilon^2] \tag{68}$$

$$u_{20} = \frac{\sqrt{5}}{8}\frac{G}{\varepsilon^3}\left[\left(3\frac{b^2}{\varepsilon^2}+1\right)\operatorname{arccot}\left(\frac{b}{\varepsilon}\right) - 3\frac{b}{\varepsilon}\right] \times \left[6\left\{\tfrac{1}{2}(I^{11}+I^{22}) - I^{33}\right\} + 2M\varepsilon^2\right] \tag{69}$$

subject to

$$M := \int_0^{2\pi} d\lambda' \int_{-\pi/2}^{+\pi/2} d\phi' \cos\phi' \times \int_0^{u'(\lambda',\phi')} du'(u'^2 + \varepsilon^2 \sin^2\phi')\rho(\lambda',\phi',u') = \int_{x_1}^{x_2} dx' \int_{y_1}^{y_2} dy' \int_{z'_1(x',y')}^{z'_2(x',y')} dz' \rho(\lambda',\phi',u') \quad (70)$$

$$I^{11} := \int_{x_1}^{x_2} dx' \int_{y_1}^{y_2} dy' \int_{z'_1(x',y')}^{z'_2(x',y')} dz' \rho(\lambda',\phi',u') \cdot (y'^2 + z'^2) \quad (71)$$

$$I^{22} := \int_{x_1}^{x_2} dx' \int_{y_1}^{y_2} dy' \int_{z'_1(x',y')}^{z'_2(x',y')} dz' \rho(\lambda',\phi',u') \cdot (z'^2 + x'^2) \quad (72)$$

$$I^{33} := \int_{x_1}^{x_2} dx' \int_{y_1}^{y_2} dy' \int_{z'_1(x',y')}^{z'_2(x',y')} dz' \rho(\lambda',\phi',u') \cdot (x'^2 + y'^2) \quad (73)$$

$$9u'^2 \sin^2\phi' + 3\varepsilon^2 \sin^2\phi' - 3u' - \varepsilon^2 = -3(x'^2 + y'^2) + 6z'^2 + 2\varepsilon^2 \quad (74)$$

as the moment of zero order (70), called the mass of the body, and moments of second order (71)–(73) with respect to Cartesian coordinates. (74) generates the transformation of the spheroidal kernel of u_{20} into its Cartesian kernel. □

The proof of Corollary 5 is straight-forward. If we substitute (16) into right-hand side of (67), namely,

$$u^2 \sin^2\phi = z^2,$$
$$x^2 + y^2 = (u^2 + \varepsilon^2)\cos^2\phi = u^2 + \varepsilon^2 - (u^2 + \varepsilon^2)\sin^2\phi,$$
$$\varepsilon^2 \sin^2\phi = u^2 + \varepsilon^2 - u^2 \sin^2\phi - (u^2 + \varepsilon^2)\cos^2\phi \ ,$$

we arrive at the right-hand side of (68). In summarising, we have succeeded to represent the spheroidal harmonic coefficients (u_{00}, u_{20}) as they appear in the Somigliana-Pizzetti gravity potential field (47), (48) in terms of the mass of the terrestrial body and the spheroidal/Cartesian mass moments of second order. Why did we bother you with the Cartesian moment representation of the spheroidal harmonic coefficient of second order u_{20}? The reason is to finally bridge the gap to the conventional spherical harmonic expansion of the terrestrial gravitational potential.

4 Spherical coordinates, spherical gravity field

Spherical coordinates as well as spherical harmonics are modern standard, in particular to represent the general reference gravity field. "Standard Earth Models" in terms of spherical harmonics are available to a high degree/order. In Lemma 4 we accordingly summarise the spherical eigenspace of the three-dimensional Laplace partial differential equation, namely the external gravity field of the Earth. Corollary 6 introduces the spherical harmonic expansion of the inverse distance function which is used in Lemma 5 for the external spherical harmonic expansion of the Newton gravitational potential. Of special importance is again the definition of orthonormality of spherical eigenfunctions with respect to the reference sphere $\mathbb{S}^2_R$ of radius R given in Corollary 9 as well as the representation of spherical multipoles of degree/order (0,0) and (2,0), respectively, of Corollary 7.

Let us note that spherical coordinates $\{\lambda, \phi_s, r\}$ can be generated by the intersection of the family of spheres $\mathbb{S}^2_r$, the family of circular cones $\mathbb{C}^2_{\cos\phi_s,\sin\phi_s}$, and the family of half planes $\mathbb{P}^2_{\cos\lambda,\sin\lambda}$.

Lemma 4. (spherical eigenspace of the three-dimensional Laplace partial differential equation, external gravity field of the Earth)

For a static, uniformly rotating Earth the gravity potential field $W(\lambda, \phi_s, r)$ with respect to spherical coordinates $\{\lambda, \phi_s, r\}$ can be additively decomposed into the gravitational potential field $U(\lambda, \phi_s, r)$ and the centrifugal potential field $V(\lambda, \phi_s, r)$, namely

$$W(\lambda, \phi_s, r) = U(\lambda, \phi_s, r) + V(\lambda, \phi_s, r) \quad (75)$$

The multiplicative decomposition of the gravitational potential field into separable functions $U(\lambda, \phi_s, r) = \Lambda(\lambda)\Phi(\phi)R(r)$ generates the solution of the three-dimensional Laplace partial differential equation

$$U(\lambda, \phi_s, r) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} u^s_{nm} \frac{R^{n+1}}{r^{n+1}} e_{nm}(\lambda, \phi_s) \quad (76)$$

in terms of surface spherical harmonics in the space $\{\mathbb{R}^3/\mathbb{S}^2_R\}$ which is external to the Brillouin sphere $\mathbb{S}^2_R$. A representation of the centrifugal potential in (i) Cartesian coordinates, (ii) spherical coordinates and (iii) surface spherical harmonics is

$$V(\phi_s, r) = \tfrac{1}{2}\omega^2(x^2 + y^2) = \tfrac{1}{2}\omega^2 r^2 \cos^2\phi_s = \frac{1}{3}\omega^2 r^2 e_{00} - \frac{1}{3\sqrt{5}}\omega^2 r^2 e_{20} \quad (77)$$

□

Corollary 6. (spherical harmonic expansion of the inverse distance function in the domain $r' < r$)

If the inverse distance function represented in Cartesian coordinates as well as in spherical coordinates,

$$\frac{1}{||\mathbf{x} - \mathbf{x}'||} = \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} = \{[r\cos\phi_s \cos\lambda - r'\cos\phi'_s \cos\lambda']^2 + [r\cos\phi_s, \sin\lambda - r'\cos\phi'_s \sin\lambda']^2 + [r\sin\phi_s - r'\sin\phi'_s]^2\}^{-1/2} \quad (78)$$

is expanded in the domain $r' < r$ into surface spherical harmonics, we receive

$$\frac{1}{||\mathbf{x}-\mathbf{x}'||} = \frac{1}{r}\sum_{n=0}^{\infty}\sum_{m=-n}^{+n}\left(\frac{r'}{r}\right)^{n}\frac{1}{2n+1}e_{nm}(\lambda,\phi_s)e_{nm}(\lambda',\phi_s') \tag{79}$$

□

Lemma 5. (spherical harmonic expansion of the Newton gravitational potential)

With respect to the spherical harmonic expansion of the inverse distance function in the domain $r' < r$ the Newton gravitational potential field can be represented by

$$\begin{aligned} U(\lambda,\phi_s,r) &= G\int_0^{2\pi}\mathrm{d}\lambda'\int_{-\pi/2}^{+\pi/2}\mathrm{d}\phi_s'\cos\phi_s' \\ &\quad\times\int_0^{r'(\lambda',\phi_s')}\mathrm{d}r' r'^2\frac{1}{||\mathbf{x}(\lambda,\phi_s,r)-\mathbf{x}'(\lambda',\phi_s',r')||} \\ &\quad\times\rho(\lambda',\phi_s',r') \end{aligned} \tag{80}$$

$$\begin{aligned} U(\lambda,\phi_s,r) &= G\int_0^{2\pi}\mathrm{d}\lambda'\int_{-\pi/2}^{+\pi/2}\mathrm{d}\phi_s'\cos\phi_s'\int_0^{r'(\lambda',\phi_s')}\mathrm{d}r' r'^2 \\ &\quad\times\frac{1}{r}\sum_{n=0}^{\infty}\sum_{m=-n}^{+n}\left(\frac{r'}{r}\right)^{n}\frac{1}{2n+1} \\ &\quad\times e_{nm}(\lambda,\phi_s)e_{nm}(\lambda',\phi_s')\rho(\lambda',\phi_s',r') \end{aligned} \tag{81}$$

namely with uniform convergence of the spherical harmonic expansion of the inverse distance function such that summation and integration can be interchanged. The spherical harmonic coefficients of (76) amount to

$$\begin{aligned} u_{nm}^s &= G\frac{1}{2n+1}\frac{1}{R^{n+1}}\int_0^{2\pi}\mathrm{d}\lambda'\int_{-\pi/2}^{+\pi/2}\mathrm{d}\phi_s'\cos\phi_s' \\ &\quad\times\int_0^{r'(\lambda',\phi_s')}\mathrm{d}r' r'^{(n+2)}\rho(\lambda',\phi_s',r')e_{nm}(\lambda',\phi_s') \end{aligned}$$

$$\left[\begin{array}{l} n = 0,1,\ldots,\infty \\ m = -n,-n+1,\ldots,n-1,n \end{array}\right. \tag{82}$$

□

Corollary 7. (spherical multipoles of degree/order (0,0) and (2,0), respectively)

By means of (28)–(33) the spherical harmonic coefficients/spherical multipoles u_{00}^s and u_{20}^s are presented by

$$\begin{aligned} u_{00}^s &= \frac{G}{R}\int_0^{2\pi}\mathrm{d}\lambda'\int_{-\pi/2}^{+\pi/2}\mathrm{d}\phi_s'\cos\phi_s' \\ &\quad\times\int_0^{r'(\lambda',\phi_s')}\mathrm{d}r' r'^2\rho(\lambda',\phi_s',r')e_{00}(\lambda',\phi_s') \end{aligned} \tag{83}$$

$$\begin{aligned} u_{20}^s &= G\frac{1}{5}\frac{1}{R^{31}}\int_0^{2\pi}\mathrm{d}\lambda'\int_{-\pi/2}^{+\pi/2}\mathrm{d}\phi_s'\cos\phi_s' \\ &\quad\times\int_0^{r'(\lambda',\phi_s')}\mathrm{d}r' r'^4\rho(\lambda',\phi_s',r')e_{20}(\lambda',\phi_s') \end{aligned} \tag{84}$$

or

$$u_{00}^s = \frac{GM}{R} \tag{85}$$

$$\begin{aligned} u_{20}^s &= \frac{G}{5R^3}\int_0^{2\pi}\mathrm{d}\lambda'\int_{-\pi/2}^{+\pi/2}\mathrm{d}\phi_s'\cos\phi_s' \\ &\quad\times\int_0^{r'(\lambda',\phi_s')}\mathrm{d}r' r'^4\rho(\lambda',\phi_s',r')\frac{\sqrt{5}}{2}(3\sin^2\phi_s - 1) \end{aligned} \tag{86}$$

$$\begin{aligned} u_{20}^s &= \frac{\sqrt{5}}{10}\frac{G}{R^3}\int_{x_1}^{x_2}\mathrm{d}x'\int_{y_1}^{y_2}\mathrm{d}y' \\ &\quad\times\int_{z_1}^{z_2}\mathrm{d}z'\rho(x',y',z')[-(x'^2+y'^2)+2z'^2] \end{aligned} \tag{87}$$

$$u_{20}^s = \frac{1}{\sqrt{5}}\frac{G}{R^3}\left[\frac{1}{2}(I^{11}+I^{22}) - I^{33}\right] \tag{88}$$

□

The reader may wonder about the ratio R^{n+1}/r^{n+1} of radii as they appear in the series expansion of the gravitational potential field $U(\lambda,\phi_s,r)$ of type (76) with respect to spherical coordinates. Indeed, this ratio is motivated by the definition of "orthonormality" of the base functions or eigenfunctions on the sphere $r_0 = R$. Corollary 8 is a résumé of the global area element of the reference sphere $\mathbb{S}_R^2$ which enables us in Corollary 9 to formulate "orthonormality" as well as the reproducing property of the "scalar product".

Corollary 8. (local and global area element of the reference sphere $\mathbb{S}_R^2$)

The local area element of sphere $\mathbb{S}_R^2$ is given by

$$\mathrm{d}S = \mathrm{d}\{\mathrm{area}(\mathbb{S}_R^2)\} = \sqrt{g_{\lambda\lambda}g_{\phi_s\phi_s}}\mathrm{d}\lambda\,\mathrm{d}\phi_s \tag{89}$$

$$\mathrm{d}S = \mathrm{d}\{\mathrm{area}(\mathbb{S}_R^2)\} = R^2\cos\phi_s\,\mathrm{d}\lambda\,\mathrm{d}\phi_s \tag{90}$$

while the global area element of $\mathbb{S}_R^2$ amounts to

$$S = \mathrm{area}(\mathbb{S}_R^2) = 4\pi R^2 \tag{91}$$

□

Corollary 9. ("orthonormality", "scalar product" with respect to the reference sphere $\mathbb{S}_R^2$)

The base functions or eigenfunctions $e_{nm}(\lambda,\phi_s)$ are orthonormal with respect to the scalar product

$$\begin{aligned} \langle e_{pq}(\lambda,\phi_s)|e_{nm}(\lambda,\phi_s)\rangle &:= \frac{1}{S}\int_{\mathbb{S}_R^2}\mathrm{d}S e_{pq}(\lambda,\phi_s)e_{mn}(\lambda,\phi_s) \\ &= \delta_{pn}\delta_{qm} \end{aligned} \tag{92}$$

and the "quantum numbers"

$p, n = 0, 1, \ldots, \infty;$

$q = -p, -p+1, \ldots, -1, 0, +1, \ldots, p-1, p;$

$m = -n, -n+1, \ldots, -1, 0, +1, \ldots, n-1, n \ .$

The scalar product

$$\begin{aligned}
&\langle U(\lambda, \phi_s, r = R) | e_{nm}(\lambda, \phi_s) \rangle \\
&= \frac{1}{4\pi R^2} \int_0^{2\pi} \mathrm{d}\lambda \\
&\quad \times \int_{-\pi/2}^{+\pi/2} \mathrm{d}\phi_s R^2 \cos\phi_s U(\lambda, \phi_s, r = R) e_m(\lambda, \phi_s) \\
&= \sum_{p=0}^{\infty} \sum_{q=-m}^{+m} u_{pq} \langle e_{pq}(\lambda, \phi_s) | e_{nm}(\lambda, \phi_s) \rangle \\
&= \sum_{p=0}^{\infty} \sum_{q=-m}^{+m} u_{pq} \delta_{pn} \delta_{qm} = u_{nm}
\end{aligned} \tag{93}$$

has the reproducing property. □

5 Transformation between spheroidal and spherical multipoles of degree/order (0,0) and (2,0), respectively

The two constraints of the Somigliana-Pizzetti gravity field of type (47), (48) contain the spheroidal multipoles of degree/order (0,0) and (2,0), respectively. $\{u_{00}, u_{20}\}$ has been expressed by means of (65), (66) in terms of a spherodial series expansion of the Newton gravitational field. Finally, we succeeded to represent u_{20} in terms of Cartesian mass multipoles I^{11}, I^{22} and I^{33} as well as M by means of (69). In contrast, the spherical multipoles of degree/order (0,0) and (2,0), namely $\{u^s_{00}, u^s_{20}\}$ within (83), (84) have been expressed in terms of a spherical series expansion of the Newton gravitational field. At the end we could represent u^s_{20} in terms of Cartesian mass multipoles I^{11}, I^{22}, and I^{33} by means of (88). These results enable us now to represent spheroidal and spherical multipoles of degree/order (0,0) and (2,0), respectively, with respect to each other. Corollary 10 is a collection of the results.

Corollary 10. (transformation between spheroidal and spherical multipoles of degree/order (0,0) and (2,0), respectively)

Spheroidal and spherical multipoles of degree/order (0,0) and (2,0), respectively, are related by

$$u_{00} = \frac{\operatorname{arccot}\left(\frac{b}{\varepsilon}\right)}{\varepsilon} R u^s_{00} \tag{94}$$

$\Longleftrightarrow$

$$u^s_{00} = \frac{\varepsilon}{R \operatorname{arccot}\left(\frac{b}{\varepsilon}\right)} u_{00} \tag{95}$$

$$\begin{aligned}
u_{20} &= \frac{\sqrt{5}}{8\varepsilon} \left[\left(3\frac{b^2}{\varepsilon^2} + 1 \right) \operatorname{arccot}\left(\frac{b}{\varepsilon}\right) - 3\frac{b}{\varepsilon} \right] \\
&\quad \times \left[6\sqrt{5} \frac{R^3}{\varepsilon^2} u^s_{20} + 2R u^s_{00} \right]
\end{aligned} \tag{96}$$

$\Longleftrightarrow$

$$\begin{aligned}
u^s_{20} &= \frac{4}{15} \left[\left(3\frac{b^2}{\varepsilon^2} + 1 \right) \operatorname{arccot}\left(\frac{b}{\varepsilon}\right) - 3\frac{b}{\varepsilon} \right]^{-1} \frac{\varepsilon^3}{R^3} u_{20} \\
&\quad - \frac{1}{3\sqrt{5}} \frac{\varepsilon^3}{R^3 \operatorname{arccot}\left(\frac{b}{\varepsilon}\right)} u_{00}
\end{aligned} \tag{97}$$

□

For the proof, we depart from (85) u^s_{00} and (65) u_{00} which directly leads to (94). Similarly $\frac{1}{2}(I^{11} + I^{22}) - I^{33}$ from (88) u^s_{20} is used in (69) u_{20} which together with (85) leads to (96). The inverse relations (95), (97) follow from direct inversion and substitution of u^s_{00} by means of u_{00} of type (95).

Obviously the transformation between spheroidal and spherical multipoles, namely, of degree/order (0,0) and (2,0), respectively, depend on the linear eccentricity $\varepsilon = \sqrt{a^2 - b^2}$ the semi-minor axis b of the reference ellipsoid of revolution $\mathbb{E}^2_{a,b}$ as well as on the radius R of the reference sphere $\mathbb{S}^2_R$. The spheroidal zonal coefficient of order two depends on the spherical zonal coefficients of order two and zero, as well-known result (Jekeli 1981, 1988). Similarly, spherical zonal coefficient of order two depends on the spheroidal zonal coefficient of order two and zero.

From "Standard Earth Models" the spherical zonal harmonic coefficients

$$GM = R u^s_{00} \tag{98}$$

$$\begin{aligned}
J_2 &= \sqrt{5} J^*_2 := \frac{1}{MR^2} \left(\frac{1}{2}(I^{11} + I^{22}) - I^{33} \right) \\
&= \sqrt{5} \frac{R}{GM} u^s_{20}
\end{aligned} \tag{99}$$

are given.

In terms of $\{W_0, GM, J_2, \Omega\}$ the two constraints (47), (48) of the Somigliana-Pizzetti gravity field can accordingly be represented by

Lemma 6. (the gravity field of a level ellipsoid, gauge to the geiod)

The gravity field of a level ellipsoid of type Somigliana-Pizzetti subject to the two constraints

$$\frac{GM}{\varepsilon} \operatorname{arccot}\left(\frac{b}{\varepsilon}\right) - \frac{1}{3}\Omega^2 a^2 = W_0 \tag{47'}$$

$$\begin{aligned}
&\frac{\sqrt{5}}{4} \frac{GM}{\varepsilon} \left[\left(3\frac{b^2}{\varepsilon^2} + 1 \right) \operatorname{arccot}\left(\frac{b}{\varepsilon}\right) - 3\frac{b}{\varepsilon} \right] \\
&\quad \times \left[3\frac{a^2}{\varepsilon^2} J_2 + 1 \right] - \frac{1}{3\sqrt{5}} \Omega^2 a^2 = 0
\end{aligned} \tag{48'}$$

gauged to $w_0 = W_0, \omega = \Omega, GM, J_2$ and $R = a$ is represented by

$$U(\lambda, \phi, u) = u_{00}\frac{Q^*_{00}\left(\frac{u}{\varepsilon}\right)}{Q^*_{00}\left(\frac{b}{\varepsilon}\right)}e_{00} + u_{20}\frac{Q^*_{20}\left(\frac{u}{\varepsilon}\right)}{Q^*_{20}\left(\frac{b}{\varepsilon}\right)}e_{20}(\phi) \tag{100}$$

subject to

$$u_{00} = \frac{GM}{\varepsilon}\operatorname{arccot}\left(\frac{b}{\varepsilon}\right) = W_0 - \frac{1}{3}\Omega^2 a^2 \tag{101}$$

$$u_{20} = \frac{\sqrt{5}}{4}\frac{GM}{\varepsilon}\left[\left(3\frac{b^2}{\varepsilon^2}+1\right)\operatorname{arccot}\left(\frac{b}{\varepsilon}\right) - 3\frac{b}{\varepsilon}\right]\cdot\left[3\frac{a^2}{\varepsilon^2}J_2+1\right] = \frac{1}{3\sqrt{5}}\Omega^2 a^2 \tag{102}$$

or

$$U(\lambda, \phi, u) = \frac{GM}{\varepsilon}\operatorname{arccot}\left(\frac{u}{\varepsilon}\right) + \frac{5}{8}\frac{GM}{\varepsilon}\left(3\frac{a^2}{\varepsilon^2}J_2+1\right) \times \left[\left(3\frac{u^2}{\varepsilon^2}+1\right)\operatorname{arccot}\left(\frac{u}{\varepsilon}\right) - 3\frac{u}{\varepsilon}\right](3\sin^2\phi - 1) \tag{103}$$

□

The two constraints (47′), (48′) are generated by means of (47), (48) as soon as we represent the spheroidal zonal coefficients of order two and zero by the spherical zonal coefficients (98), (99), namely J_0 and J_2, respectively, within (94), (96). In addition, we have fixed the fundamental parameters (i) of the level ellipsoid potential value w_0 to the Gauss-Listing geoid potential value W_0, (ii) of the rotational velocity ω of the level ellipsoid to the rotational velocity Ω of the Earth at some reference epoch, (iii) of the ‘‘gravitational mass’’ gm of the level ellipsoid to the ‘‘gravitational mass’’ GM of the Earth, (iv) of the spherical zonal coefficient J_2 and (v) of the semi-major axis a to the raius R of the Brillouin sphere.

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