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An optimality property of the integer least-squares estimator

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Abstract. A probabilistic justification is given for using the integer least-squares (LS) estimator. The class of admissible integer estimators is introduced and classical adjustment theory is extended by proving that the integer LS estimator is best in the sense of maximizing the probability of correct integer estimation. For global positioning system ambiguity resolution, this implies that the success rate of any other integer estimator of the carrier phase ambiguities will be smaller than or at the most equal to the ambiguity success rate of the integer LS estimator. The success rates of any one of these estimators may therefore be used to provide lower bounds for the LS success rate. This is particularly useful in case of the bootstrapped estimator.

Key words. Integer LS · GPS ambiguity resolution · Ambiguity success rate

1 Introduction

Ambiguity resolution applies to a great variety of global positioning system (GPS) models currently in use. These range from single-baseline models used for kinematic positioning to multi-baseline models used as a tool for studying geodynamic phenomena. An overview of these and other GPS models, together with their application in surveying, navigation and geodesy, can be found in textbooks such as those of Leick (1995), Parkinson and Spilker (1996), Hofmann-Wellenhof et al. (1997), Strang and Borre (1997) and Teunissen and Kleusberg (1998). Despite the differences in application of the various GPS models, it is important to understand that their ambiguity resolution problems are intrinsically the same. That is, the GPS models on which ambiguity resolution is based can all be cast in the following conceptual frame of linear(ized) observation equations

$$y = Aa + Bb + e \tag{1}$$

where y is the given GPS data vector of order m, a and b are the unknown parameter vectors respectively of order n and o, and e is the noise vector. The matrices A and B are the corresponding design matrices. The data vector y will usually consist of the 'observed minus computed' single- or dual-frequency double-difference (DD) phase and/or pseudorange (code) observations accumulated over all observation epochs. The entries of vector a are then the DD carrier phase ambiguities, expressed in units of cycles rather than range. They are known to be *integers*, $a \in Z^n$. The entries of the vector b will consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere). They are known to be real-valued, $b \in R^o$.

The procedure which is usually followed for solving the GPS model of Eq. (1) can be divided into three steps (for more details we refer to e.g. Teunissen 1993 or de Jonge and Tiberius 1996). In the *first* step we simply disregard the integer constraints $a \in Z^n$ on the ambiguities and perform a standard adjustment. As a result we obtain the (real-valued) estimates of a and b, together with their variance–covariance matrix

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}, \quad \begin{bmatrix} Q_{\hat{a}} & Q_{\hat{a}\hat{b}} \\ Q_{\hat{b}\hat{a}} & Q_{\hat{b}} \end{bmatrix} \tag{2}$$

This solution is referred to as the 'float' solution. In the *second* step the 'float' ambiguity estimate $\hat{a}$ is used to compute the corresponding integer ambiguity estimate $\check{a}$. This implies that a mapping $F : R^n \mapsto Z^n$, from the n-dimensional space of real numbers to the n-dimensional space of integers, is introduced such that

$$\check{a} = F(\hat{a}) \tag{3}$$

Once the integer ambiguities are computed, they are used in the *third* step to finally correct the float estimate of b. As a result we obtain the 'fixed' solution

$$\check{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - \check{a}) \tag{4}$$

The ambiguity residual $(\hat{a} - \check{a})$ is thus used to adjust the float solution so as to obtain the fixed solution.

It is of course not enough to compute the fixed solution and be done with it. We can always compute such a solution, whether it is of good quality or not. We therefore still need to address the question whether we have enough confidence in the computed integer ambiguity solution. After all, unsuccessful ambiguity resolution, when passed unnoticed, will all too often lead to unacceptable errors in the positioning results. We therefore need to have a way of knowing how often we can expect the computed ambiguity solution to coincide with the correct, but unknown, solution. Is this 9 out of 10 times, 99 out of a 100, or a higher percentage? It will certainly never equal 100%. After all, the integer ambiguities are computed from the data: they are therefore subject to uncertainty just like the data are.

In order to obtain such a description, we require the probability distribution of the integer ambiguities (Teunissen 1997). This distribution will be a probability *mass* function, due to the integer nature of the ambiguities. Of this probability mass function, the probability of correct integer ambiguity estimation is of particular interest. This probability will be denoted as $P(\check{a} = a)$. It describes the frequency with which one can expect to have a successful ambiguity resolution. It equals the expected ambiguity success rate. This probability depends on three contributing factors: the functional model (the observation equations), the stochastic model (the distribution and precision of the observables) and the chosen method of integer ambiguity estimation. Changes in any one of these will affect the success rate.

In this contribution the choice of integer ambiguity estimator will be considered. In general, we would like to have the highest success rate possible. We therefore would like to know which integer ambiguity estimator maximizes $P(\check{a} = a)$. For this purpose we first need to introduce a class of candidate integer estimators. Such a class of admissible integer estimators is introduced in Sect. 2. In constructing this class, we are led by practical considerations such as the following: the estimator should map any float solution to a unique integer solution and when the float solution is perturbed by an integer amount, the integer solution should be perturbed by the same integer amount. In Sect. 3 we give three examples of integer estimators which belong to this class of admissible estimators. They are the 'rounding' estimator, the 'bootstrapped' estimator and the integer least-squares (LS) estimator.

Section 4 contains the main result of this contribution. It is proven that the integer LS estimator has the largest ambiguity success rate of all admissible estimators. The integer LS estimator is therefore the best estimator in the sense of maximizing the probability of correct integer estimation. Any other integer ambiguity estimator, such as for instance the 'rounding' estimator or the 'bootstrapped' estimator, will have a smaller success rate. GPS ambiguity resolution will therefore be less successful when integer estimators other than the LS estimator are used.

2 A class of integer estimators

There are many ways of computing an integer ambiguity vector $\check{a}$ from its real-valued counterpart $\hat{a}$. To each such method belongs a mapping $F : R^n \mapsto Z^n$ from the n-dimensional space of real numbers to the n-dimensional space of integers. Once this map has been defined, the integer ambiguity vector follows from its real-valued counterpart as $\check{a} = F(\hat{a})$. Due to the discrete nature of Z^n, the map F will not be one-to-one, but instead a many-to-one map. This implies that different real-valued ambiguity vectors may be mapped to the same integer vector. We can therefore assign a subset $S_z \subset R^n$ to each integer vector $z \in Z^n$:

$$S_z = \{x \in R^n \mid z = F(x)\}, \quad z \in Z^n \tag{5}$$

The subset S_z contains all real-valued ambiguity vectors that will be mapped by F to the same integer vector $z \in Z^n$. This subset is referred to as the *pull-in region* of z (Jonkman 1998; Teunissen 1998a). It is the region in which all ambiguity float solutions are pulled to the same fixed ambiguity vector z.

Having defined the pull-in regions, we are now in a position to give an explicit expression for the corresponding integer ambiguity estimator. It reads

$$\check{a} = \sum_{z \in Z^n} z s_z(\hat{a}) \text{ with } s_z(\hat{a}) = \begin{cases} 1 & \text{if } \hat{a} \in S_z \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Since the pull-in regions define the integer estimator completely, we can define a class of integer estimators by listing properties of these pull-in regions. In this section we introduce three properties for which it seems reasonable that they are possessed by the pull-in regions.

It seems reasonable to ask of the pull-in regions that their union covers the n-dimensional space completely

$$\bigcup_{z \in Z^n} S_z = R^n \tag{7}$$

Otherwise we would have gaps, in which case not every $\hat{a} \in R^n$ could be assigned to a corresponding integer ambiguity vector.

Another property that we require of the pull-in-regions is that any two distinct regions should not have an overlap. Otherwise we could end up in a situation where a float solution $\hat{a} \in R^n$ cannot be assigned uniquely to a single integer vector. For the interior points of two distinct pull-in regions we therefore require

$$S_{z_1} \bigcap S_{z_2} = \emptyset, \quad \forall z_1, z_2 \in Z^n, \quad z_1 \neq z_2 \tag{8}$$

We allow the pull-in regions to have common boundaries, however. This is permitted if we assume zero probability that $\hat{a}$ lies on one of the boundaries. This will be the case when the probability density function (pdf) of $\hat{a}$ is continuous.

The third and last property asked is that the integer map F possesses the property that $F(x+z) = F(x) + z, \quad \forall x \in R^n, \quad z \in Z^n$. This property is also a reasonable one to request. It states that when the float solution is moved by an integer amount z, the corresponding integer solution is moved by the same integer amount. This property allows one to use the 'integer remove–restore' technique: $F(\hat{a}-z)+z=F(\hat{a})$. It therefore allows us to work with the fractional parts of the entries of $\hat{a}$, instead of with its complete entries, which may sometimes be large numbers.

The integer remove–restore property implies that

$$\begin{aligned} S_{z_1+z_2} &= \{x \in R^n \mid z_1+z_2 = F(x)\} \\ &= \{x \in R^n \mid z_1 = F(x) - z_2 = F(x-z_2)\} \\ &= \{x \in R^n \mid z_1 = F(y), x = y+z_2\} \\ &= S_{z_1} + z_2, \quad \forall z_1, z_2 \in Z^n. \end{aligned}$$

Hence, it means that the pull-in regions are translated copies of one another. This third property may therefore also be stated as

$$S_z = z + S_0, \quad \forall z \in Z^n \tag{9}$$

with S_0 being the pull-in region of the origin of Z^n.

Integer ambiguity estimators that possess all three of the above stated properties form a class. This class will be referred to as the class of admissible integer ambiguity estimators. It is defined as follows.

Definition. The integer estimator $\check{a} = \sum_{z \in Z^n} z s_z(\hat{a})$ is said to be *admissible* if

1. $\bigcup_{z \in Z^n} S_z = R^n$
2. $S_{z_1} \bigcap S_{z_2} = \emptyset, \;\; \forall z_1, z_2 \in Z^n, z_1 \neq z_2$
3. $S_z = z + S_0, \;\; \forall z \in Z^n$

Various integer estimators exist that belong to this class. As the definition shows, one way of constructing admissible estimators is to choose a subset S_0 such that its translated copies cover R^n without gaps and overlaps. In two dimensions this can be achieved, for instance, by choosing S_0 as the unit square centred at the origin.

3 Examples of admissible estimators

In this section three different admissible integer estimators are considered. All three of them have been in use, in one way or another, for GPS ambiguity resolution. They are the 'rounding' estimator, the 'bootstrapped' estimator and the LS estimator.

3.1 Integer rounding

The simplest way to obtain an integer vector from the real-valued float solution is to round each of the entries of $\hat{a}$ to its nearest integer. The corresponding integer estimator reads therefore

$$\check{a}_R = ([\hat{a}_1], \ldots, [\hat{a}_n])^T \tag{10}$$

where '[·]' denotes rounding to the nearest integer. This estimator is clearly admissible. The first two conditions of the definition are satisfied, since – apart from ties in rounding – any float solution $\hat{a} \in R^n$ gets mapped to a unique integer vector. The third condition is also satisfied since rounding admits the integer remove–restore technique, that is, $[x-z]+z=[x], \forall x \in R, z \in Z$.

Since componentwise rounding implies that each real-valued ambiguity estimate $\hat{a}_i, i=1,\ldots, n$, is mapped to its nearest integer, the absolute value of the difference between the two is at most 1/2. The pull-in regions $S_{R,z}$ that belong to this integer estimator are therefore given as

$$S_{R,z} = \cap_{i=1}^{n} \{x \in R^n \mid \mid x_i - z_i \mid \leq \tfrac{1}{2}\}, \quad \forall z \in Z^n \tag{11}$$

They are n-dimensional cubes, centred at $z \in Z^n$, all having sides of length one.

3.2 Integer bootstrapping

Another relatively simple integer ambiguity estimator is the bootstrapped estimator (Blewitt 1989; Dong and Bock 1989). The bootstrapped estimator can be seen as a generalization of the previous estimator. It still makes use of integer rounding, but it also takes some of the correlation between the ambiguities into account. The bootstrapped estimator follows from a sequential conditional LS adjustment and it is computed as follows. If n ambiguities are available, we start with the first ambiguity $\hat{a}_1$, and round its value to the nearest integer. Having obtained the integer value of this first ambiguity, the real-valued estimates of all remaining ambiguities are then corrected by virtue of their correlation with the first ambiguity. Then the second, but now corrected, real-valued ambiguity estimate is rounded to its nearest integer. Having obtained the integer value of the second ambiguity, the real-valued estimates of all remaining $n-2$ ambiguities are then again corrected, but now by virtue of their correlation with the second ambiguity. This process is continued until all ambiguities are considered. The components of the bootstrapped estimator $\check{a}_B$ are given as

$$\begin{aligned} \check{a}_{B,1} &= [\hat{a}_1] \\ \check{a}_{B,2} &= [\hat{a}_{2|1}] = [\hat{a}_2 - \sigma_{\hat{a}_2\hat{a}_1}\sigma_{\hat{a}_1}^{-2}(\hat{a}_1 - \check{a}_{B,1})] \\ &\;\;\vdots \\ \check{a}_{B,n} &= [\hat{a}_{n|N}] = \left[\hat{a}_n - \sum_{i=1}^{n-1} \sigma_{\hat{a}_n\hat{a}_{i|I}}\sigma_{\hat{a}_{i|I}}^{-2}(\hat{a}_{i|I} - \check{a}_{B,i})\right] \end{aligned} \tag{12}$$

where the shorthand notation $\hat{a}_{i|I}$ stands for the ith LS ambiguity obtained through a conditioning on the previous $I = \{1, \ldots, (i-1)\}$ sequentially rounded ambiguities.

The bootstrapped estimator is admissible. The first two conditions of the definition are satisfied, since – apart from ties in rounding – any float solution gets mapped to a unique integer ambiguity vector. The integer remove–restore technique again applies. To see this, let $\check{a}'_B$ be the bootstrapped estimator which corresponds with $\hat{a}' = \hat{a} - z$. It then follows from Eq. (12) that $\check{a}_B = \check{a}'_B + z$.

The real-valued sequential conditional LS solution can be obtained by means of the triangular decomposition of the ambiguity variance–covariance matrix. Let the LDU decomposition of the variance–covariance matrix be given as $Q_{\hat{a}} = LDL^T$, with L a unit lower triangular matrix and D a diagonal matrix. Then $(\hat{a} - z) = L(\hat{a}^c - z)$, where $\hat{a}^c$ denotes the conditional LS solution obtained from a sequential conditioning on the entries of z. The variance–covariance matrix of $\hat{a}^c$ is given by the diagonal matrix D. This shows, when a componentwise rounding is applied to $\hat{a}^c$, that z is the integer solution of the bootstrapped method. Thus $\check{a}_B$ satisfies $[L^{-1}(\hat{a} - \check{a}_B)] = 0$. Hence, if c_i denotes the ith canonical unit vector having a 1 as its ith entry, the pull-in regions $S_{B,z}$ that belong to the bootstrapped estimator follow as

$$S_{B,z} = \cap_{i=1}^{n} \left\{ x \in R^n \mid \mid c_i^T L^{-1}(x - z) \mid \leq \tfrac{1}{2} \right\} , \quad \forall z \in Z^n \tag{13}$$

Note that these subsets reduce to the ones of Eq. (11) when L becomes diagonal. This is the case when the ambiguity variance–covariance matrix is diagonal. In that case the two integer estimators $\check{a}_R$ and $\check{a}_B$ are identical.

3.3 Integer LS

The integer LS estimator is defined as

$$\check{a}_{\mathrm{LS}} = \arg\min_{z \in Z^n} \parallel \hat{a} - z \parallel_{Q_{\hat{a}}}^2 \tag{14}$$

where $\parallel \cdot \parallel_{Q_{\hat{a}}}^2 = (\cdot)^T Q_{\hat{a}}^{-1}(\cdot)$. This ambiguity estimator was introduced for the first time in Teunissen (1993). This estimator is also admissible. Apart from boundary ties, it produces a unique integer vector for any float solution $\hat{a} \in R^n$. And since $\check{a}_{\mathrm{LS}} = \arg\min_{z \in Z^n} \parallel \hat{a} - u - z \parallel_{Q_{\hat{a}}}^2 + u$ holds true for any integer u, the integer remove–restore technique again applies.

It follows from Eq. (14) that the float solutions $\hat{a} \in R^n$ which are mapped to the same integer vector $\check{a}_{\mathrm{LS}}$ are those that lie closer to this integer vector than to any other integer vector $z \in Z^n$. This shows that the LS pull-in regions $S_{\mathrm{LS},z}$ consist of intersecting half-spaces, each one of which is bounded by the plane orthogonal to $(c - z)$, $c \in Z^n$ and passing through the mid-point $\frac{1}{2}(z + c)$. Here, orthogonality is taken with respect to the metric as defined by the ambiguity variance–covariance matrix. Since $\hat{a}$ lies in one of these half-spaces when the length of the orthogonal projection of $(\hat{a} - z)$ onto $(c - z)$ is less than or equal to half the distance between c and z, it follows that

$$S_{\mathrm{LS},z} = \cap_{c \in Z^n} \left\{ x \in R^n \mid \mid w_c(x) \mid \leq \tfrac{1}{2} \parallel c \parallel_{Q_{\hat{a}}} \right\} , \quad \forall z \in Z^n \tag{15}$$

with

$$w_c(x) = \frac{c^T Q_{\hat{a}}^{-1}(x - z)}{\sqrt{c^T Q_{\hat{a}}^{-1} c}}$$

Note that $(c - z)$ has been replaced by c in Eq. (15). This is permitted since the intersection is taken with respect to all $c \in Z^n$. Also note that w_c is an example of the well-known w-test statistic for testing one-dimensional alternative hypotheses (Baarda 1968; Teunissen 1985). The absolute values of w_c are thus required to be no larger than the ‘critical values’ $\frac{1}{2} \parallel c \parallel_{Q_{\hat{a}}}$.

In our comparison of $\check{a}_R$ and $\check{a}_B$, we noted that the two estimators became identical in the case that the unit triangular matrix L reduced to the identity matrix. The same holds true in the case of $\check{a}_{\mathrm{LS}}$. Hence, all three estimators become identical in the case that the ambiguity variance–covariance matrix is diagonal. This condition can be relaxed, however, when comparing $\check{a}_B$ with $\check{a}_{\mathrm{LS}}$. These two estimators will already have become identical when all matrix entries of L are integer. This is the case when L is an admissible ambiguity transformation (Teunissen 1995). Thus, $\check{a}_{\mathrm{LS}} = \check{a}_B = L[L^{-1}\hat{a}]$.

4 Maximizing the ambiguity success rate

In this section the main result of this contribution is discussed. So far, we have introduced a class of admissible integer estimators and discussed some of its members. We thus have now a variety of reasonable integer estimators available. The question which arises next is which of these estimators to choose? Does an estimator exist which one can single out as being the ‘best’? And how do we want to define the qualification ‘best’? The approach that will be followed here is a probabilistic one. That is, we will use the probability distribution of the integer estimator in order to decide which estimator to choose. Since the integer estimator $\check{a}$ is by definition of the discrete type, its distribution will be a probability mass function (pmf). It will be denoted as $P(\check{a} = z)$, with $z \in Z^n$. In order to determine this distribution, we first need the pdf of $\hat{a}$. The pdf of $\hat{a}$ will be denoted as $p_a(x)$, with $x \in R^n$. The subscript is used to show that the pdf still depends on the unknown parameter vector $a \in Z^n$.

The pmf of $\check{a}$ can now be obtained as follows. Since the integer estimator is defined as

$$\check{a} = z \quad \Leftrightarrow \quad \hat{a} \in S_z \tag{16}$$

it follows that $P(\check{a} = z) = P(\hat{a} \in S_z)$. The pmf of $\check{a}$ therefore follows as

$$P(\check{a} = z) = \int_{S_z} p_a(x) \mathrm{d}x , \quad \forall z \in Z^n \tag{17}$$

The probability that $\check{a}$ coincides with z is therefore given by the integral of the pdf $p_a(x)$ over the pull-in region $S_z \subset R^n$.

The pmf of $\check{a}$ can now be used to study various properties of the integer estimator. In Teunissen (1998b), for instance, the first moments of the integer ambiguity estimators were studied. It was shown that $\check{a}$ is an *unbiased* estimator if $p_a(x)$ is symmetric about a and S_z reflection-symmetric about z. All three estimators $\check{a}_R$, $\check{a}_B$ and $\check{a}_{\mathrm{LS}}$ are therefore unbiased in the case that the pdf possesses the necessary property of symmetry. This is the case, for instance, when $p_a(x)$ is a member of the family of multivariate normal distributions. Note that the reflection-symmetric property of the pull-in region is not necessary for an integer estimator to be admissible. Hence, the unbiased estimators are admissible, but admissible estimators need not be unbiased.

Having the problem of GPS ambiguity resolution in mind, we focus our attention in this contribution to the chance of successful ambiguity resolution. That is, we consider the probability of correct integer estimation. This is given as

$$P(\check{a} = a) = \text{probability of correct integer estimation} \quad (18)$$

and it describes the reliability of ambiguity resolution in terms of its expected success rate. Since unsuccessful ambiguity resolution, when passed unnoticed, will all too often lead to unacceptable errors in the positioning results, we desire high success rates and therefore a large value for $P(\check{a} = a)$. It is therefore not only of theoretical interest, but also of practical interest, to know which integer estimator maximizes the ambiguity success rate. It will be proven, for a general family of pdfs, that of all admissible estimators it is the *integer LS* estimator which maximizes Eq. (18). The pdfs that we will consider all belong to the family of elliptically contoured distributions. They are defined as follows (Chmielewsky 1981).

Definition. The random vector $\hat{a} \in R^n$ is said to have an *elliptically contoured distribution* if its pdf is of the form

$$p_a(x) = \sqrt{\det(Q_{\hat{a}}^{-1})}\, G(\| x - a \|^2_{Q_{\hat{a}}}) \quad (19)$$

where $G : R \mapsto [0, \infty)$ is decreasing and $Q_{\hat{a}}$ is positive-definite.

Several important distributions belong to this family. The multivariate normal distribution can be shown to be a member of this family by choosing

$$G(x) = (2\pi)^{-\frac{n}{2}} \exp\left(-\tfrac{1}{2}x\right) , \quad x \in R$$

Another member is the multivariate t-distribution.

Since we can formulate the LS pull-in regions for all members of the family of elliptically contoured distributions as $S_{\mathrm{LS},z} = \{x \in R^n \mid p_z(x) \geq p_u(x) \ , \ \forall u \in Z^n\}$, it follows that

$$p_a(x) \geq \sum_{z \in Z^n} s_z(x) p_z(x) \ , \ \forall x \in S_{\mathrm{LS},a} \quad (20)$$

with the indicator function

$$s_z(x) = \begin{cases} 1 & x \in S_z \\ 0 & \text{otherwise} \end{cases}$$

where S_z are the pull-in regions of an arbitrary admissible integer estimator. When taking the integral of Eq. (20) over $S_{LS,a}$, we obtain

$$\int_{S_{\mathrm{LS},a}} p_a(x)\mathrm{d}x \geq \sum_{z \in Z^n} \int_{S_{\mathrm{LS},a} \cap S_z} p_z(x)\mathrm{d}x \quad (21)$$

We now apply the change of variable $y = x + a - z$ and obtain the following replacements: $p_z(x) \to p_z(y - a + z) = p_a(y)$, $S_{\mathrm{LS},a} \to S_{\mathrm{LS},2a-z}$ and $S_z \to S_a$. Hence

$$\int_{S_{\mathrm{LS},a}} p_a(x)\mathrm{d}x \geq \sum_{z \in Z^n} \int_{S_{\mathrm{LS},2a-z} \cap S_a} p_a(y)\mathrm{d}y = \int_{S_a} p_a(y)\mathrm{d}y \quad (22)$$

where the last equality is a consequence of $\cup_{z \in Z^n} S_{\mathrm{LS},2a-z} = R^n$. On the left side of Eq. (22) we recognize the probability of correct integer estimation of the LS estimator and on the right side the probability of correct integer estimation of any arbitrary admissible integer estimator. This concludes the proof that the integer LS estimator indeed maximizes the ambiguity success rate. This result is summarized in the following theorem.

Theorem. Let the integer LS estimator be given as

$$\check{a}_{\mathrm{LS}} = \arg\min_{z \in Z^n} \| \hat{a} - z \|^2_{Q_{\hat{a}}}$$

and the pdf of $\hat{a}$ as

$$p_a(x) = \sqrt{\det(Q_{\hat{a}}^{-1})}\, G(\| x - a \|^2_{Q_{\hat{a}}})$$

where $G : R \mapsto [0, \infty)$ is decreasing and $Q_{\hat{a}}$ is positive-definite. Then

$$P(\check{a}_{\mathrm{LS}} = a) \geq P(\check{a} = a) \quad (23)$$

for any admissible estimator $\check{a}$.

With this theorem and a result of Teunissen (1998c) we are now also in a position to order the three admissible estimators $\check{a}_R$, $\check{a}_B$ and $\check{a}_{\mathrm{LS}}$ in terms of their success rates. From the theorem it follows that $\check{a}_{\mathrm{LS}}$ is better than both $\check{a}_R$ and $\check{a}_B$, and in Teunissen (1998c) it was shown that $\check{a}_B$ is again better than $\check{a}_R$. We thus have the ordering

$$P(\check{a}_R = a) \leq P(\check{a}_B = a) \leq P(\check{a}_{\mathrm{LS}} = a) \quad (24)$$

A very useful application of this result is that it shows how one can *lower-bound* the probability of correct integer LS estimation. This is particularly useful when $P(\check{a}_B = a)$ is used as lower bound. This probability can be computed exactly and rather easily in case that the pdf $p_a(x)$ is normal. As was shown in Teunissen (1997), it can be computed as

$$P(\check{a}_B = a) = \prod_{i=1}^{n} \left(2\Phi\left(\frac{1}{2\sigma_{\hat{a}_{i|I}}} \right) - 1 \right) \quad (25)$$

with

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) \mathrm{d}y$$

Such an easy way of evaluating the success rate is usually not possible in case of the LS estimator.

When using Eq. (25) as the lower bound for $P(\check{a}_{\mathrm{LS}} = a)$, there is one important point to be recognized. Since Eq. (25) depends only on the sequential conditional standard deviations of $\hat{a}$, it is generally not invariant for the class of admissible ambiguity transformations (Teunissen 1995). On the other hand, the probability of correct integer LS estimation is invariant for the class of admissible ambiguity transformations. The lack of invariance in $P(\check{a}_B = a)$ implies that we still have some degrees of freedom for improving this probability so as to make it a sharper lower bound of $P(\check{a}_{\mathrm{LS}} = a)$.

The lower bound is usually particularly poor when applied to the DD ambiguities. The lower bound becomes much sharper, however, when it is applied to ambiguities which are almost decorrelated. Such ambiguities can be obtained by means of the decorrelating ambiguity transformation of the LAMBDA method (see e.g. Teunissen 1993; de Jonge and Tiberius 1996). Since the transformed ambiguities obtained by this method are far more precise than the original DD ambiguities, the lower bound becomes sharper due to its increase in value. Although other type of lower bounds for $P(\check{a}_{\mathrm{LS}} = a)$ can be given, it is our experience that $P(\check{a}_B = a)$, when applied to the decorrelated ambiguities, is usually the best lower bound we obtain and very sharp indeed. Finally, note that the additional computations required for evaluating the lower bound are minimal. Since the LAMBDA method is already used for efficiently solving the integer LS problem, the sequential conditional standard deviations of the transformed ambiguities are available at no extra cost.

5 Summary

In this contribution a probabilistic justification for using the integer LS estimator has been given. It was shown that when the pdf of the ambiguity float solution is a member of the elliptically contoured distributions, the integer LS estimator will have the largest success rate of all admissible ambiguity estimators. The class of admissible estimators was defined by means of the following three properties of their pull-in regions:

1. $\bigcup_{z \in Z^n} S_z = R^n$
2. $S_{z_1} \bigcap S_{z_2} = \emptyset, \ \ \forall z_1, z_2 \in Z^n, z_1 \neq z_2$
3. $S_z = z + S_0, \ \ \forall z \in Z^n$

The first condition states that the pull-in regions should not leave any gaps, the second that they should not overlap and the third that the integer estimators should admit the 'integer remove–restore' technique. Various admissible estimators exist, three of which are as follows:

1. $\check{a}_R = ([\hat{a}_1], \ldots, [\hat{a}_n])^T$
2. $\check{a}_B = ([\hat{a}_1], \ldots, [\hat{a}_{n|N}])^T$
3. $\check{a}_{LS} = \arg\min_{z \in Z^n} \| \hat{a} - z \|^2_{Q_{\hat{a}}}$

Their pull-in regions are shaped as the n-dimensional versions of a square, a parallelogram and a convex polygon respectively. The theorem given in this contribution shows that

$$P(\check{a}_{\mathrm{LS}} = a) \geq P(\check{a} = a) \ \text{ for all admissible } \check{a}$$

In maximizing the success of GPS ambiguity resolution we are thus better off when using the integer LS estimator than any other admissible ambiguity estimator. As a direct consequence of the theorem we have

$$P(\check{a}_{\mathrm{LS}} = a) \geq P(\check{a}_B = a) = \prod_{i=1}^{n} \left(2\Phi\left(\frac{1}{2\sigma_{\hat{a}_{i|I}}}\right) - 1\right)$$

where the last equality follows from Teunissen (1997). This shows how the easily computed success rate of the bootstrapped estimator can be used as the lower bound.

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