Determination of the boundary values for the Stokes–Helmert problem

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Abstract. The definition of the mean Helmert anomaly is reviewed and the theoretically correct procedure for computing this quantity on the Earth's surface and on the Helmert co-geoid is suggested. This includes a discussion of the role of the direct topographical and atmospherical effects, primary and secondary indirect topographical and atmospherical effects, ellipsoidal corrections to the gravity anomaly, its downward continuation and other effects. For the rigorous derivations it was found necessary to treat the gravity anomaly systematically as a point function, defined by means of the fundamental gravimetric equation. It is this treatment that allows one to formulate the corrections necessary for computing the 'one-centimetre geoid'. Compared to the standard treatment, it is shown that a 'correction for the quasigeoid-to-geoid separation', amounting to about 3 cm for our area of interest, has to be considered. It is also shown that the 'secondary indirect effect' has to be evaluated at the topography rather than at the geoid level. This results in another difference of the order of several centimetres in the area of interest. An approach is then proposed for determining the mean Helmert anomalies from gravity data observed on the Earth's surface. This approach is based on the widely-held belief that complete Bouguer anomalies are generally fairly smooth and thus particularly useful for interpolation, approximation and averaging. Numerical results from the Canadian Rocky Mountains for all the corrections as well as the downward continuation are shown.

Key words. Precise geoid determinations · Gravity anomaly · Geodetic boundary value problem · Downward continuation of gravity

1 Introduction

The investigation described in this contribution represents a continuation of the effort to come up with sufficiently accurate boundary values on Helmert's co-geoid for the solution of the Stokes–Helmert boundary value problem. This problem can be formulated as the determination of the geoid by means of a transformation of the Stokes boundary value problem into Helmert's space, where the Stokes problem represents the standard free boundary value problem of geodesy. The term 'sufficiently accurate' is to be interpreted to mean such boundary values that, given accurate and sufficiently dense observed values of gravity, heights and topographical density, would guarantee a solution, the geoid, accurate to one centimetre, notwithstanding the inaccurate knowledge of the Earth's mass and thus the zero-degree term. The working tool adopted in this contribution is the spherical Stokes's theory (Stokes 1849). The boundary values will thus be formulated for the spherical model. The basic theory of the Stokes–Helmert scheme was described by Vaníček and Martinec (1994) and used, in its embryonic form, in a geoid evaluation for western Canada (Vaníček et al. 1995).

This contribution will be working with quantities from the real and Helmert spaces; the latter will be denoted by a superscript 'h'. For the sake of brevity, wherever possible, this superscript will be omitted. Not all the derived formulae are original; there are other investigators who have derived either identical, or very similar expressions for some of the quantities discussed here. The fact that their contributions are not cited here should not be taken as a lack of respect: we refer to our own publications only to demonstrate that all the formulae used here were derived by us from first principles.

2 Gravity disturbance

Let us begin with the known definition of the disturbing (real, Helmert or other) potential

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$$T(r, \Omega) = W(r, \Omega) - U(r, \Omega) \tag{1}$$

where W is the gravity (real, Helmert or other) potential, U is the normal potential of Somigliana–Pizzetti type, r is the geocentric distance, and Ω is the geocentric angle denoting the pair (ϑ, λ), the spherical co-latitude and longitude. In the sequel, the arguments (r, Ω), that is the position in three dimensions, will sometimes be omitted also for the sake of brevity, if there is no danger of misinterpretation. Next, the radial derivative of the disturbing potential is constructed as

$$\frac{\partial T}{\partial r} = \frac{\partial W}{\partial r} - \frac{\partial U}{\partial r} \tag{2}$$

Realizing that for any scalar S it can be written as

$$\frac{\partial S}{\partial r} = |\mathbf{grad}\ S| \cos(\mathbf{grad}\ S, \mathbf{e_r}) \tag{3}$$

where $\mathbf{e_r}$ is the unit vector in the radial direction, the radial derivatives can be expressed as follows:

$$\frac{\partial W}{\partial r} = |\mathbf{grad}\ W| \cos(\mathbf{grad}\ W, \mathbf{e_r}) = g \cos(\mathbf{grad}\ W, \mathbf{e_r}) \tag{4}$$

$$\frac{\partial U}{\partial r} = |\mathbf{grad}\ U| \cos(\mathbf{grad}\ U, \mathbf{e_r}) = \gamma \cos(\mathbf{grad}\ U, \mathbf{e_r}) \tag{5}$$

In Eqs. (4) and (5) g stands for gravity (real, Helmert or other) and γ stands for normal gravity generated by the Somigliana–Pizzetti field. Let us now denote the angle between $-\mathbf{g}$ and $\mathbf{e_r}$ by β_g, and the angle between $-\gamma$ and $\mathbf{e_r}$ by β_γ. As these two angles will be sufficiently small, in both real and Helmert spaces, they can be written to a sufficient accuracy

$$\cos(-\mathbf{grad}\ W, \mathbf{e_r}) \doteq 1 - \frac{\beta_g^2}{2} \tag{6}$$

$$\cos(-\mathbf{grad}\ U, \mathbf{e_r}) \doteq 1 - \frac{\beta_\gamma^2}{2} \tag{7}$$

Substituting Eqs. (4) and (5) back into Eq. (2) yields

$$\frac{\partial T}{\partial r} \doteq -g + \gamma + \frac{g}{2}\beta_g^2 - \frac{\gamma}{2}\beta_\gamma^2 \tag{8}$$

Here, it is recognized that the first difference on the right-hand side of Eq. (8) is the negative gravity disturbance δg and the second difference is the *ellipsoidal correction to the gravity disturbance*, i.e.

$$\frac{\partial T}{\partial r} \doteq -\delta g + \epsilon_{\delta g} \tag{9}$$

This correction was deemed too small to be significant by Vaníček and Martinec (1994) in the original paper describing the Stokes–Helmert technique. However, in view of the adopted more stringent accuracy requirements it is now appropriate to revisit and re-evaluate

this ellipsoidal correction. Figure 1 shows the intersections of vectors $\mathbf{e_r}$, $-\mathbf{g}$, and $-\gamma$ at the point of interest, with a unit sphere drawn around the point of interest, where all vectors are projected onto a tangent plane to the sphere at point $\mathbf{e_r}$. These projected intersections are denoted by the same symbols as the vectors. As the angles of interest are very small, less than three minutes of arc, to a sufficient accuracy they equal to the distances between the projected intersections. Note that the angles ξ and η are nothing other than the meridian and prime vertical components of the deflection of the vertical (Vaníček and Krakiwsky 1986, Fig. 6.20). Referring to Eq. (8) and Fig. 1, the ellipsoidal correction to the gravity disturbance can be written as

$$\epsilon_{\delta g} = \frac{g}{2}\left[(\beta_\gamma + \xi)^2 + \eta^2\right] - \frac{\gamma}{2}\beta_\gamma^2 \tag{10}$$

which can also be written as

$$\epsilon_{\delta g} = \delta g \frac{\beta_\gamma^2}{2} + g\beta_\gamma\xi + \frac{g}{2}(\xi^2 + \eta^2) \tag{11}$$

The angle β_γ is simply the difference between geodetic latitude φ and geocentric latitude $(\pi/2 - \vartheta)$, and is given by Bomford [1971, Eq. (A.48)]

$$\beta_\gamma = f \sin 2\varphi \tag{12}$$

where f is the geometrical flattening of the reference ellipsoid, and can reach a value of 3.35×10^{-3}. Hence, the first term in Eq. (11) is less than $5.63 \times 10^{-6} \times \delta g$. Even for a very large value of δg, this will amount only to a few µGal, and the first term can thus be safely neglected.

It can be seen that the expression in the brackets in the third term of Eq. (11) is the square of the deflection of the vertical θ (real, Helmert or other), which, in turn, is unlikely to be larger than $0.5'$ (it is assumed that Helmert's deflections of the vertical are probably smoother than the real deflections of the vertical – therefore, $0.5'$ may be too large an estimate of their maximum value). Hence, the third term in Eq. (11) may reach at most a magnitude of $10\ \mu$Gal, and can be also
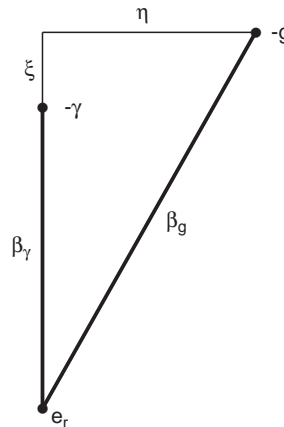

Fig. 1. Angles involved in derivation of the ellipsoidal correction to Helmert gravity disturbance

neglected. Thus, to an accuracy of $10\,\mu$Gal, the ellipsoidal correction to the gravity disturbance may be written as

$$\epsilon_{\delta g}(r,\Omega) \doteq g(r,\Omega)\beta_\gamma(\Omega)\xi(r,\Omega) \qquad (13)$$

and this quantity may reach a magnitude of up to $500\,\mu$Gal.

For the test area in the Canadian Rocky Mountains ($43° \leq \varphi \leq 60°$N, $224° \leq \lambda \leq 258°$E), the ellipsoidal correction to the gravity disturbance, estimated at the geoid from the global geopotential model GFZ93a (Gruber and Anzenhofer 1993), is between -118 and $+157\,\mu$Gal. This quantity has been computed from Eq. (13) for the test area and is shown in Fig. 2. This translates approximately into an effect on the geoid of between -0.7 and $+1.4$ cm as determined from Stokes' integration of the ellipsoidal correction to the gravity disturbance. A use of a different geopotential model may change these values slightly.

3 Gravity anomaly

The gravity disturbance δg is usually not considered a 'measurable' quantity on the surface of the Earth: normal gravity $\gamma(r_t,\Omega)$ cannot routinely be evaluated on the surface of the Earth (r_t,Ω) as this requires the knowledge of the geodetic height h of the point (r_t,Ω) above the reference ellipsoid. This height is, of course, available from space-based techniques, but still only very few gravity observations have a geodetic height associated with them. Therefore, δg is transformed to a more commonly available quantity, the gravity anomaly Δg (real, Helmert or other).

The transformation of δg to Δg is usually achieved by adding a term $\Gamma(r,\Omega)$ to the gravity disturbance. This term accounts for the change in normal gravity due to the difference between the geodetic height h and the commonly available orthometric height H^o. More correctly, the term $\Gamma(r,\Omega)$ should account for the change in normal gravity due to a (vertical) displacement – let us call this displacement $Z(r,\Omega)$ – of the actual equipotential surface (real, Helmert or other) with respect to the equivalent normal equipotential surface at point (r,Ω). Here, it is assumed once again that the normal gravity is generated by a field the potential of which on the reference ellipsoid is equal to the value of the actual potential on the geoid.

In the previous paragraph, the term 'more correctly' is used to mean: to satisfy the so-called fundamental gravimetric equation, Eq. (18). It is argued that the fundamental gravimetric equation for the disturbing potential (real, Helmert or other) should really be regarded as the defining equation for the corresponding (real, Helmert or other) gravity anomaly. This is the way in which this equation will be used here, and this is the main motivation for the approach to the derivation of the boundary values chosen here. Clearly, it does not quite conform to the definition commonly used in the geodetic literature.

It should be pointed out that the displacement $Z(r,\Omega)$ has been employed in geodesy in different contexts. Thus, for instance, considered at points (r_g,Ω) on the Helmert co-geoid, $Z^h(r_g,\Omega)$ is the height $N^h(\Omega)$ of Helmert's co-geoid above the reference ellipsoid. Considered at points (r_t,Ω) on the Earth's surface in the real space, $Z(r_t,\Omega)$ is the height anomaly $\zeta(\Omega)$ used in Molodenskij's approach (Molodenskij et al. 1960). As this
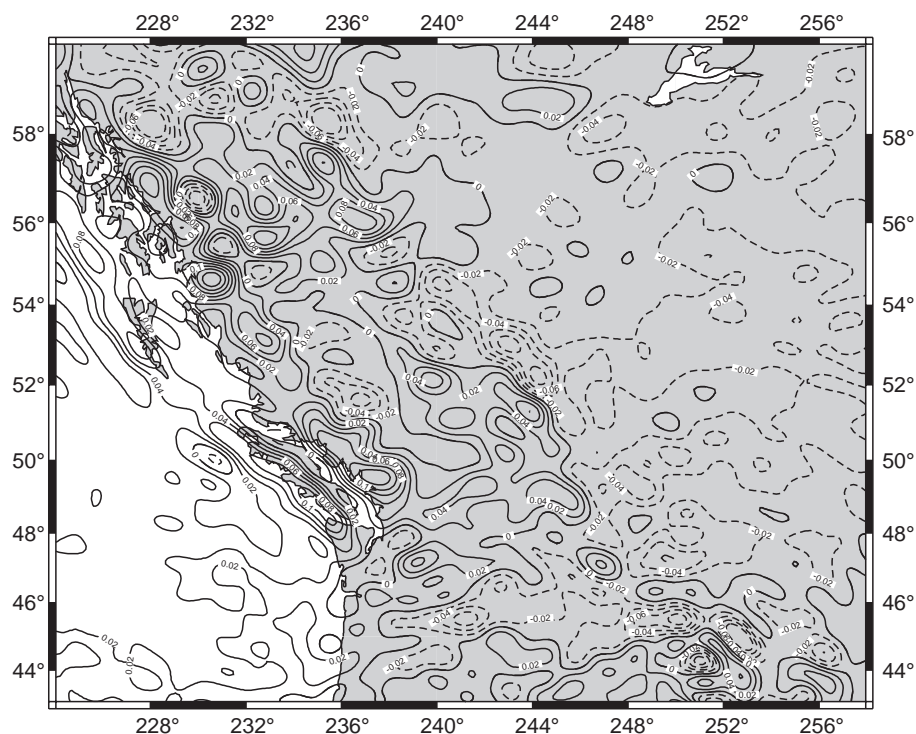

Fig. 2. Magnitude of the ellipsoidal correction to Helmert gravity disturbance (in mGal)

displacement would be in absolute value at most 100 m, it can be written as

$$\Gamma(r,\Omega) = |\mathbf{grad}\ \gamma|\, Z(r,\Omega) \doteq \frac{\partial\gamma(r,\Omega)}{\partial n} Z(r,\Omega) \tag{14}$$

where, with a sufficient accuracy, the derivative of normal gravity is taken with respect to the normal **n** to the reference ellipsoid.

Using Bruns's formula [Vaníček and Krakiwsky 1986, Eq. (21.4)], Eq. (14) can be rewritten to a sufficient accuracy (Vaníček and Martinec 1994) as

$$\Gamma(r,\Omega) = \frac{T(r,\Omega)}{\gamma(r,\Omega)} \frac{\partial\gamma(r,\Omega)}{\partial n} \tag{15}$$

In solving the boundary value problem of geodesy, it is convenient to introduce into Eq. (15) the following approximation:

$$\frac{1}{\gamma(r,\Omega)} \frac{\partial\gamma(r,\Omega)}{\partial n} \doteq -\frac{2}{R} \tag{16}$$

This is called the 'spherical approximation' and the error committed by this replacement requires another correction, called in the geodetic literature the *ellipsoidal correction for the spherical approximation*; let us denote this correction by ϵ_n. Vaníček and Martinec [1994, Eq. (29)] and others have shown that this correction has the following form:

$$\epsilon_n(r,\Omega) \doteq 2\left[m + f\left(\cos 2\varphi - \frac{1}{3}\right)\right] \frac{T(r,\Omega)}{R} \tag{17}$$

where m is the 'geodetic parameter', also called the 'Clairaut constant'. This correction may reach up to a few hundreds of μGal, and must therefore be considered in precise geoid determination. For the test area in the Canadian Rocky Mountains, the correction, as estimated at the geoid from the global geopotential model GFZ93a, is somewhat smaller, between −10 and +18 μGal (see Fig. 3). This translates approximately into an effect of up to 0.2 cm on the geoid as computed from Stokes's integral. This correction is very small in this area because the term in the square brackets in Eq. (17) happens to tend to zero for the geodetic latitude of about 55°.

Now, Eqs. (9) and (15) are combined to yield

$$\frac{\partial T(r,\Omega)}{\partial r} + \frac{2}{R} T(r,\Omega) \doteq -\delta g(r,\Omega) - \Gamma(r,\Omega) - \epsilon_n(r,\Omega)$$

$$+ \epsilon_{\delta g}(r,\Omega) \tag{18}$$

which is valid for the real, Helmert or any other meaningful gravity field. In Eq. (18), $\Gamma(r,\Omega)$ represents the change in normal gravity when moving from point (r,Ω) to the 'corresponding' point $(r-Z,\Omega)$ at which U has the same value as W has at the original point (r,Ω), i.e.

$$U[r - Z(r,\Omega),\Omega] = W(r,\Omega) \tag{19}$$

Thus, the first two terms on the right-hand side of Eq. (18) can be written as

$$-\delta g(r,\Omega) - \Gamma(r,\Omega) = -g(r,\Omega) + \gamma[r - Z(r,\Omega),\Omega] \tag{20}$$

which is, by common understanding, nothing other than the negative gravity anomaly Δg (real, Helmert or other) at the point (r,Ω). It has to be emphasized that the argument developed above is valid for any point (r,Ω).

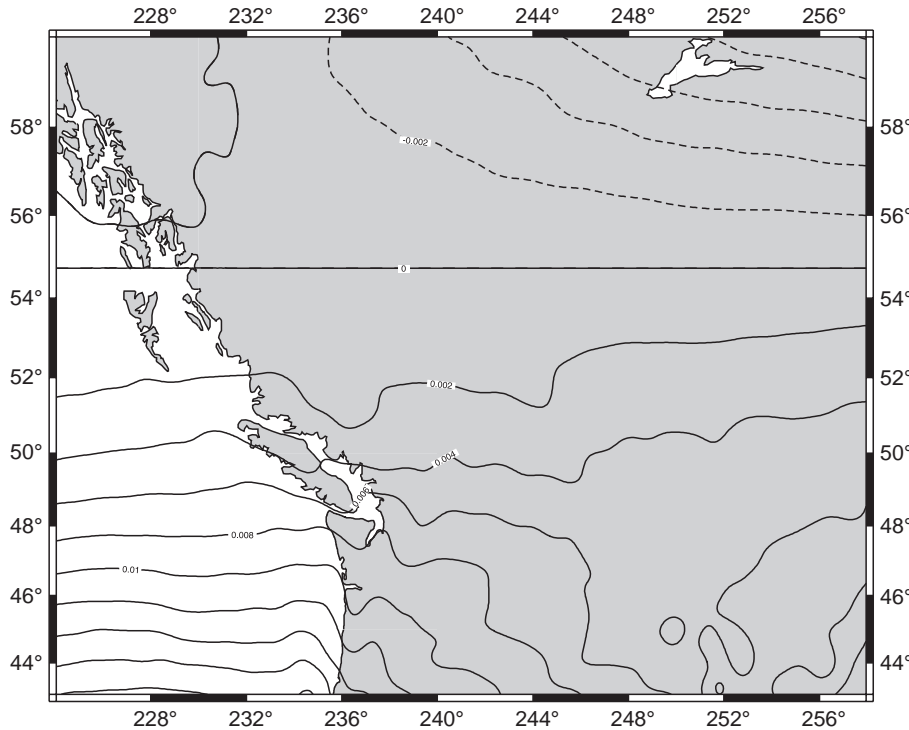Equation (18) can be rewritten in the form in which it is usually presented, i.e.

Fig. 3. Magnitude of the ellipsoidal correction to spherical approximation (in mGal)

$$-\frac{\partial T(r,\Omega)}{\partial r}-\frac{2}{R}T(r,\Omega) \doteq \Delta g(r,\Omega)+\epsilon_n(r,\Omega)-\epsilon_{\delta g}(r,\Omega) \tag{21}$$

where $\Delta g(r,\Omega)$ is a quantity that can be and has been measured on the Earth's surface, and the two 'ellipsoidal correction terms' are given by Eqs. (13) and (17). It is the quantity described by Eq. (21) on the Helmert co-geoid that is used as the *boundary value* in the (*spherical*) *Stokes–Helmert formulation*.

Note that $\epsilon_n(r,\Omega)$ in Eq. (21) is a linear function of the disturbing potential $T(r,\Omega)$, the unknown quantity in the boundary value problem of geodesy. Thus, this ellipsoidal correction cannot be evaluated ahead of the solution of the boundary value problem. However, it can be evaluated iteratively, if such iterations converge to the correct solution. It is normally assumed that since these corrections are fairly small, one can use the existing knowledge of the Earth's gravity field (in terms of a global geopotential model) to obtain sufficiently accurate estimates of their real values. This approach allows them to be treated as a priori quantities so they can remain on the right-hand side of Eq. (21). However, the validity of this assumption has not yet been proved. Alternatively, the ellipsoidal corrections may be added to the left-hand side of Eq. (21) so they become part of the boundary operator (i.e. one which transforms T into the boundary values).

The second ellipsoidal correction, $\epsilon_{\delta g}(r,\Omega)$, is a linear function of the meridian component $\xi(r,\Omega)$ of the deflection of the vertical and of gravity g. As $\xi(r,\Omega)$ is not generally known, it can be evaluated from the disturbing potential $T(r,\Omega)$. It can be written, to a sufficient accuracy [Vaníček and Krakiwsky 1986, Eq. (21.18)], as

$$\xi(r,\Omega) \doteq -\frac{1}{R}\frac{\partial Z(r,\Omega)}{\partial \varphi} \tag{22}$$

and applying Bruns's formula gives

$$\xi(r,\Omega) \doteq -\frac{1}{Rg(r,\Omega)}\frac{\partial T(r,\Omega)}{\partial \varphi} \tag{23}$$

Substitution of Eq. (23) in Eq. (13) then yields

$$\epsilon_{\delta g}(r,\Omega) \doteq -\frac{\beta_\gamma(\Omega)}{R}\frac{\partial T(r,\Omega)}{\partial \varphi} \tag{24}$$

giving $\epsilon_{\delta g}(r,\Omega)$ also as a function of T. In this case, $\epsilon_{\delta g}(r,\Omega)$ can once again be evaluated from a global geopotential model, or has to be added to the left-hand side of Eq. (21) and treated in the same way as the first ellipsoidal correction (i.e. to be made part of the boundary operator).

4 Evaluation of the Helmert gravity anomaly on the surface of the Earth

Let us now concentrate on the evaluation of Helmert's gravity anomaly on the Earth's surface from Eq. (20).

After inserting a superscript h to denote the quantities in Helmert's space, the Helmert gravity anomaly is defined as

$$\Delta g^h(r_t,\Omega) = g^h(r_t,\Omega) - \gamma[r_t - Z^h(r_t,\Omega),\Omega] \tag{25}$$

where (r_t,Ω) denotes a point on the Earth's surface, and $[r_t - Z^h(r_t,\Omega),\Omega]$ denotes the corresponding point (point in the same direction Ω) on the telluroid (Molodenskij et al. 1960) in Helmert space. Helmert's gravity $g^h(r_t,\Omega)$ on the Earth's surface is obtained from the observed gravity $g(r_t,\Omega)$ on the Earth's surface, by adding to it the *direct topographical effect* $\delta A^t(r_t,\Omega)$ and the *direct atmospherical effect* $\delta A^a(r_t,\Omega)$

$$g^h(r_t,\Omega) = g(r_t,\Omega) + \delta A^t(r_t,\Omega) + \delta A^a(r_t,\Omega)$$

$$= g(r_t,\Omega) + \frac{\partial \delta V^t(r,\Omega)}{\partial r}\bigg|_{r_t} + \frac{\partial \delta V^a(r,\Omega)}{\partial r}\bigg|_{r_t} \tag{26}$$

where the *residual topographical potential* δV^t being defined as a difference between the Newton potential generated by the topographical masses and the potential of the condensed layer [Martinec and Vaníček 1994a, Eq. (2)]

$$\delta V^t(r,\Omega) = V^t(r,\Omega) - V^{ct}(r,\Omega) \tag{27}$$

and similarly defined *residual atmospherical potential* δV^a

$$\delta V^a(r,\Omega) = V^a(r,\Omega) - V^{ca}(r,\Omega) \tag{28}$$

can easily be recognized. For details on the direct topographical effect (DTE), see Martinec and Vaníček (1994b). The direct atmospherical effect will not be discussed here either. Suffice it to state that it can be approximated by the atmospherical gravity correction recommended by I.A.G. (1971).

It must be noted at this point that the DTE term is also a function of topographical mass density. In the first approximation, it is sufficient to replace the density by its approximate mean value $\varrho_0 = 2670$ kg m^{-3}. For an accurate result, however, the actual topographical mass density must be used (Martinec 1993), but this problem is considered outside our interests in this paper.

Normal gravity on the telluroid in the Helmert space is obtained by developing the Somigliana–Pizzetti formula for normal gravity into a Taylor series. This is carried out at the corresponding point on the reference ellipsoid (Vaníček and Martinec 1994) for the height of the Helmert telluroid above the reference ellipsoid, i.e. for the normal height $(H^N)^h$. This is shown in Fig. 4. In practice, however, heights of gravity observations taken on the Earth's surface are given in the orthometric height system, i.e. as H^o, and the upward continuation of normal gravity from the reference ellipsoid is computed for H^o, instead of $(H^N)^h$. This is indeed the case in Canada. The difference between the correct value of normal gravity reckoned on the Helmert telluroid

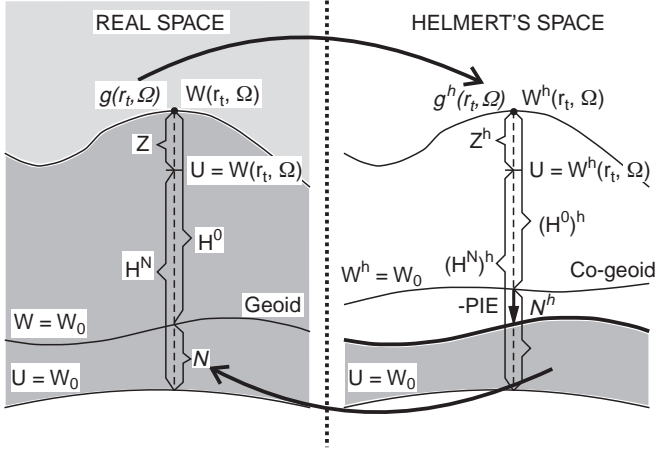$$\gamma[r_t - Z^h(r_t,\Omega),\Omega] = \gamma[r_0 + (H^N)^h(\Omega),\Omega] \tag{29}$$

Fig. 4. Real and Helmert spaces involved in the formulation

and the usually computed value $\gamma[r_0 + H^o(\Omega), \Omega]$ at the height H^o above the reference ellipsoid can be expressed as yet another correction. This will be called the *correction for orthometric height*, which is to be added to the standard computed value $\gamma[r_0 + H^o(\Omega), \Omega]$. This correction now has to be evaluated.

Recalling the definition of the normal height (Molodenskij et al. 1960)

$$H^N(\Omega) = \frac{W_o - W(r_t, \Omega)}{\overline{\gamma}(\Omega)} \tag{30}$$

where W_o is the potential on the geoid and $\overline{\gamma}(\Omega)$ is the mean value, in the integral sense, of normal gravity between the reference ellipsoid and the telluroid. To ensure that the Helmert normal height behaves in the Helmert space in the same way as the normal height behaves in the real space, the former is defined the same way as the latter, i.e. as

$$(H^N)^h(\Omega) = \frac{W_o - W^h(r_t, \Omega)}{\overline{\gamma}^h(\Omega)} \tag{31}$$

where $\overline{\gamma}^h(\Omega)$ is the mean value of normal gravity between the reference ellipsoid and the Helmert telluroid. Note that the co-geoid in the Helmert space has the same gravitational potential as the geoid in the real space, thus the term W_o is the same in both Eqs. (30) and (31).

In Eq. (31), $\overline{\gamma}^h(\Omega)$ can be approximated by $\overline{\gamma}(\Omega)$, which results in an error in $(H^N)^h(\Omega)$ of much less than 1 mm. Thus

$$(H^N)^h(\Omega) - H^N(\Omega) \doteq \frac{W(r_t, \Omega) - W^h(r_t, \Omega)}{\overline{\gamma}(\Omega)} \tag{32}$$

where the difference $W(r_t, \Omega) - W^h(r_t, \Omega)$ is easily recognized to be the sum of the residual topographical potential [see Eq. (27)] and the residual atmospherical potential [see Eq. (28)]. Therefore

$$(H^N)^h(\Omega) \doteq H^N(\Omega) + \frac{\delta V^t(r_t, \Omega)}{\gamma(r_t, \Omega)} + \frac{\delta V^a(r_t, \Omega)}{\gamma(r_t, \Omega)} \tag{33}$$

where $\overline{\gamma}(\Omega)$ has been replaced by $\gamma(r_t, \Omega)$ with an associated error of less than 1 mm.

As it is intended to evaluate the correction to $\gamma[r_o + H^o(\Omega), \Omega]$, it is still necessary to relate the normal height $H^N(\Omega)$ in Eq. (33) to the orthometric height $H^o(\Omega)$. According to Heiskanen and Moritz [1967, Eq. (8-103)], the difference in these two heights is, to a sufficient accuracy, equal to

$$H^N(\Omega) - H^o(\Omega) \doteq H^o(\Omega) \frac{\Delta g^B(\Omega)}{\gamma_0(\Omega)} \tag{34}$$

where $\Delta g^B(\Omega)$ is the simple Bouguer anomaly [which will be defined in Eq. (38)] and $\gamma_0(\Omega)$ is the value of normal gravity on the reference ellipsoid. Note that Eq. (34) also provides an estimate of the separation between the geoid and quasigeoid (Molodenskij et al. 1960).

The correction for the orthometric height can now be evaluated from Eqs. (33) and (34)

$$\left. \frac{\partial \gamma(\Omega)}{\partial n} \right|_0 [(H^N)^h(\Omega) - H^o(\Omega)] \doteq \left. \frac{\partial \gamma(\Omega)}{\partial n} \right|_0$$
$$\times \left[H^o(\Omega) \frac{\Delta g^B(\Omega)}{\gamma_0(\Omega)} + \frac{\delta V^t(\Omega)}{\gamma(r_t, \Omega)} + \frac{\delta V^a(\Omega)}{\gamma(r_t, \Omega)} \right] \tag{35}$$

where the subscript 0 denotes that the quantity is evaluated on the reference ellipsoid. This expression can then be simplified, with a sufficient accuracy, to

$$\left. \frac{\partial \gamma(\Omega)}{\partial n} \right|_0 [(H^N)^h(\Omega) - H^o(\Omega)] \doteq -\frac{2}{R} H^o(\Omega) \Delta g^B(\Omega)$$
$$- \frac{2}{R} \delta V^t(r_t, \Omega) - \frac{2}{R} \delta V^a(r_t, \Omega) \tag{36}$$

Closer inspection of Eq. (36) reveals that the correction for the orthometric height is nothing other than a sum of a correction for the *difference between the quasigeoid and the geoid* plus the correction for the *secondary indirect topographical (and atmospherical) effects* on gravity at the Earth's surface, SITE(r_t, Ω) and SIAE(r_t, Ω), denoted by $\delta \gamma^t(r_t, \Omega)$ and $\delta \gamma^a(r_t, \Omega)$. The mathematical expression for the first of these two effects on the geoid under a spherical approximation was derived by Vaníček and Martinec [1994, Eq. (40)]. (It is acknowledged that the first term in Eq. (36) was neglected in the cited paper.) The expression for the $\delta \gamma^t(r_t, \Omega)$ term on the Earth's surface under a planar approximation is derived in Appendix 1, Eq. (A9).

The values of the quasigeoid to geoid separation correction for the test areas are plotted in Fig. 5, using simple Bouguer anomalies and orthometric heights available from the Geodetic Survey Division of Canada. These values are between -189 and $+23 \mu$Gal, which translates approximately to a geoid correction between -2.9 and -0.8, as obtained again from the Stokes integral. It is of interest to compare Fig. 5 with the topographical map of the area of interest, shown in Fig. 6.

The values of the secondary indirect topographical effect (SITE) on Helmert gravity on the Earth's surface for our area of interest range between -20 and $+17 \mu$Gal, and its effect on the geoid is in absolute

value smaller than 0.1 cm. Clearly, only the first correction is significant at the accuracy level we are interested in, and has to be considered in any computation of the geoid at the one-centimetre level of accuracy. As the maximum value of $\delta\gamma^a(r_t,\Omega)$ in our area of interest is in absolute value smaller than $2\,\mu$Gal, it does not have to be considered and is not going to be shown here.

Finally, Eq. (25) can be rewritten as follows:

$$\begin{aligned}
\Delta g^h(r_t,\Omega) &= g^h(r_t,\Omega) - \gamma[r_t - Z^h(r_t,\Omega),\Omega] \\
&= g(r_t,\Omega) - \gamma[r_o + H^o(\Omega),\Omega] \\
&\quad + \frac{2}{R}H^o(\Omega)\Delta g^B(\Omega) + \delta\gamma^t(r_t,\Omega) + \delta\gamma^a(r_t,\Omega) \\
&\quad + \delta A^t(r_t,\Omega) + \delta A^a(r_t,\Omega)
\end{aligned} \qquad (37)$$

where the difference in the first two terms on the right-hand side is easily recognized as the standard *free-air gravity* anomaly. Note that we neither need, nor want to specify just which surface the free-air anomaly is 'referred to'. Based on the understanding of the fundamental gravimetric equation employed here, this is considered to be a contentious point and is not addressed here. The direct effects originate in the transformation of the observed gravity from the real space to the Helmert space, while the quasigeoid-to-geoid correction and the secondary indirect effects result from the evaluation of normal gravity on the Helmert telluroid.

The upward continuation of normal gravity from the reference ellipsoid to $\gamma[r_o + H^o(\Omega),\Omega]$, which is needed in the evaluation of the free-air gravity anomaly, is sometimes carried out by means of the simple 'free-air gradient', which is really just the first term of the Taylor expansion of normal gravity. This simplification does not give accurate enough results for this application, and a higher-order approximation is required. As shown by Vaníček and Martinec (1994), at least the latitude and the altitude effects have to be considered. We shall not discuss this topic here; for a detailed discussion and numerical results, see Vaníček et al. (1995).

5 Making use of the complete Bouguer gravity anomaly

It is widely believed that the complete Bouguer anomaly is the smoothest of all gravity anomalies and, as such, the best suited for interpolation or averaging. This is why it is quite popular in practice to use it for gravity interpolation, prediction and averaging. The complete Bouguer anomaly is usually defined (as it is indeed the case in Canada) by the following equation:

$$\begin{aligned}
\Delta g^{CB}(\Omega) &= \Delta g^B(\Omega) + \delta g^{tc}(r_t,\Omega) \\
&= g(r_t,\Omega) - \gamma[r_o + H^o(\Omega),\Omega] \\
&\quad - 2\pi\varrho_0 GH^o(\Omega) + \delta g^{tc}(r_t,\Omega)
\end{aligned} \qquad (38)$$

where δg^{tc} is the gravimetric *terrain correction*, i.e. a correction for the attraction of the Earth's topography taken relative to the height of the evaluation point (r_t,Ω). The small curvature effect, $8\pi\varrho_0\,GH^o(\Omega)^2/R$ [Vaníček and Krakiwsky 1986, Eq. (21.43)], is usually not considered. As for the free-air gravity anomaly, the potentially contentious issue of which surface the simple Bouguer or complete Bouguer anomaly refers to is not considered. This is the reason why only the direction Ω is used as an independent argument in the expressions.
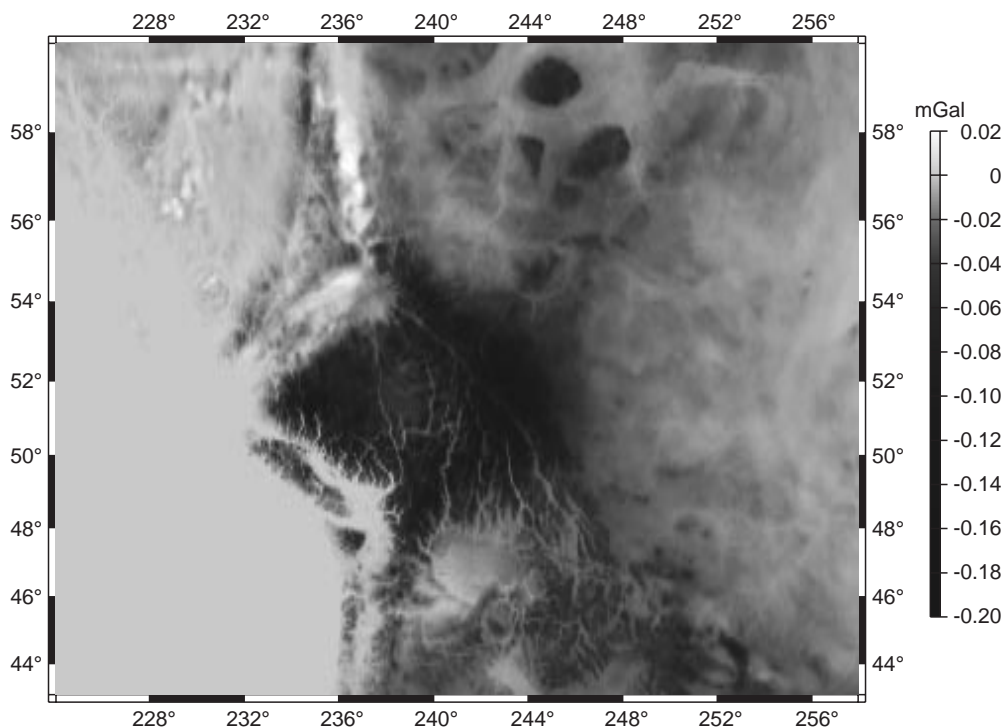

Fig. 5. Values of the quasi-geoid-to-geoid separation correction (in mGal)
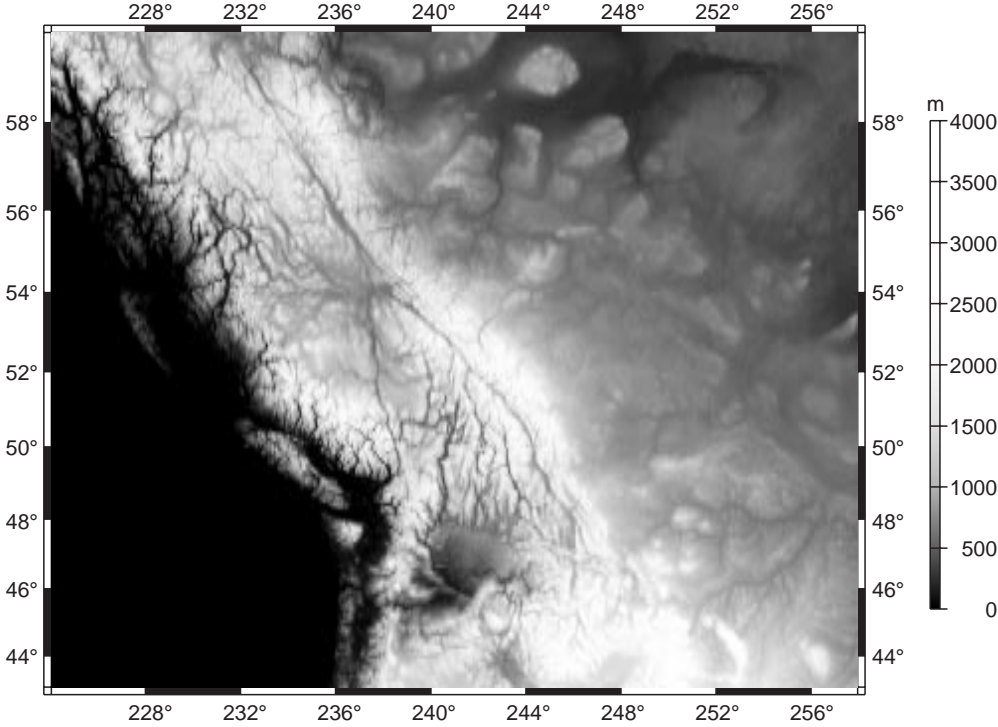
Fig. 6. Topography of the Canadian Rocky Mountains (in m) plotted from the $5 \times 5'$ elevation averages

In practice, it is easy to obtain the complete Bouguer anomalies, either in a point form, or as mean values for the cells used in the numerical integration. The latter is what we wish to use in numerical geoid computations. The question then arises: how can these complete Bouguer anomalies be best transformed to Helmert anomalies on the surface of the Earth? Equations (37) and (38) provide the answer, which is

$$\Delta g^{h}(r_{t}, \Omega) = \Delta g^{CB}(\Omega) + 2\pi\varrho_{0}GH^{o}(\Omega) - \delta g^{tc}(r_{t}, \Omega)$$
$$+ \frac{2}{R}H^{o}(\Omega)\Delta g^{B}(\Omega) + \delta\gamma^{t}(r_{t}, \Omega) + \delta\gamma^{a}(r_{t}, \Omega)$$
$$+ \delta A^{t}(r_{t}, \Omega) + \delta A^{a}(r_{t}, \Omega) \tag{39}$$

In Eq. (39), the dominant term to be added to the complete Bouguer anomaly is the Bouguer plate effect, $2\pi\varrho_{0}GH^{o}(\Omega)$. The direct topographical and atmospherical effects, and the terrain correction are much smaller (typically by one order of magnitude), and the other terms are smaller still.

It is interesting to realize that the terrain correction is in fact embedded in the DTE. The DTE can be thought of as the negative difference between the attraction of the whole topography (which is a very large effect) and the attraction of topography when condensed onto the geoid (which is also a very large effect) according to some condensation scheme (Martinec 1993).

It has been shown by Martinec [1993, Eq. (4.21)] that when using a condensation scheme that preserves the mass of the Earth in the Helmert space, the difference is exactly equal to the difference between the attraction of the terrain (topography referred to the height of the point of evaluation, or equivalently, attraction of the whole topography minus the attraction of the Bouguer shell) and the attraction of the terrain condensed on the geoid. The first term is nothing other than the negative

terrain correction, $-\delta g^{tc}(r_{t}, \Omega)$; the second term is the *condensed terrain effect* (CTE) $\delta A^{cr}(r_{t}, \Omega)$, called by Martinec and Vaníček (1994b) the 'topographical roughness' term. Thus, we obtain

$$\delta A^{t}(r_{t}, \Omega) - \delta g^{tc}(r_{t}, \Omega) = \delta A^{cr}(r_{t}, \Omega) \tag{40}$$

The mathematical expression for $\delta A^{cr}(r_{t}, \Omega)$ is derived in Appendix 2, Eq. (A12), and numerical results for the test area are shown in Fig. 8. These results were compiled from a combination of Canadian and US elevation data. The effect, computed from Eq. (A12), ranges between -25.136 and $+92.776$ mGal. Its influence on the geoid, computed from the Stokes integral, is between $+0.127$ and $+1.787$ m.

Insertion of Eq. (40) into Eq. (39) gives the final expression for the *Helmert gravity anomaly on the Earth's surface*, namely

$$\Delta g^{h}(r_{t}, \Omega) = \Delta g^{CB}(\Omega) + 2\pi\varrho_{0}GH^{o}(\Omega) + \frac{2}{R}H^{o}(\Omega)\Delta g^{B}(\Omega)$$
$$+ \delta\gamma^{t}(r_{t}, \Omega) + \delta\gamma^{a}(r_{t}, \Omega) + \delta A^{a}(r_{t}, \Omega)$$
$$+ \delta A^{cr}(r_{t}, \Omega) \tag{41}$$

It is Eq. (41) from which the mean Helmert gravity anomalies on the topography can be evaluated in practice, when mean values of the individual terms are considered. These mean integral values are approximated by arithmetic means of as many values in each compartment as are available.

6 Downward continuation of mean Helmert gravity anomalies onto the Helmert co-geoid

Once the mean Helmert gravity anomaly is assembled on the surface of the Earth – Eq. (41) – it has to be
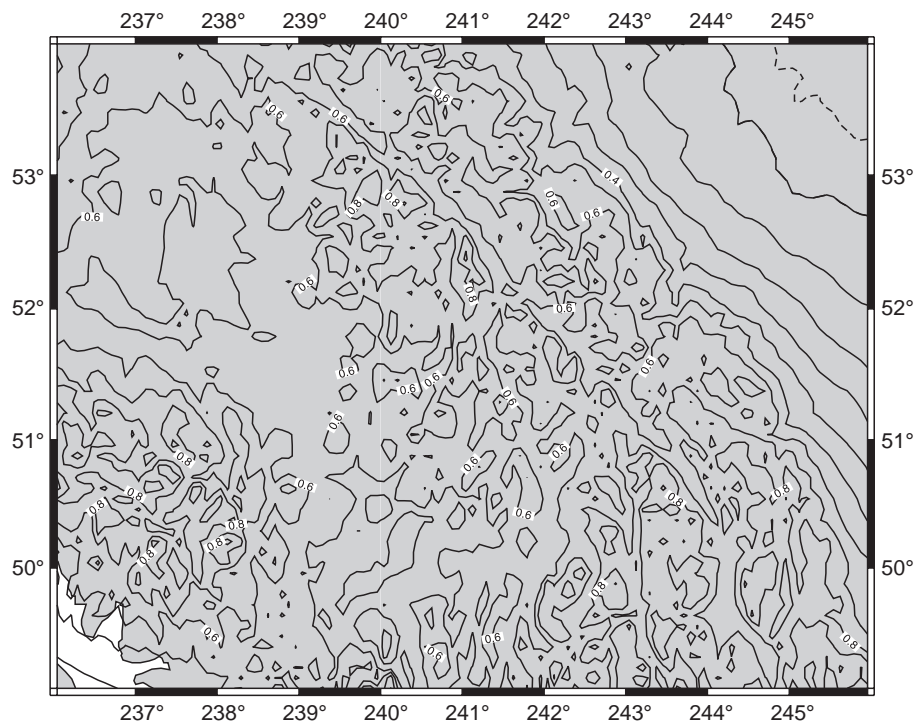
Fig. 7. Effect of the Poisson downward continuation of mean Helmert anomalies on the geoid (in m)

transferred to the real boundary, the Helmert co-geoid, where it is needed for the solution, $T^h(r,\Omega)$, of the Stokes–Helmert boundary value problem. This process is known as the downward continuation of the Helmert gravity anomaly. The Helmert disturbing potential $T^h(r,\Omega)$ is a harmonic function above the geoid and hence also above the Helmert co-geoid in the Helmert space, since the Helmert co-geoid is always above the

geoid (Vaníček et al. 1995). Thus, Poisson's solution to the Dirichlet problem of upward continuation of a harmonic function (Kellogg 1929) can be employed in seeking the downward continuation to the geoid in the first instance. The application of this process was described in detail by Vaníček et al. (1996), for mean Helmert gravity anomalies, and the description will not be repeated here. A few remarks, however, are in order.
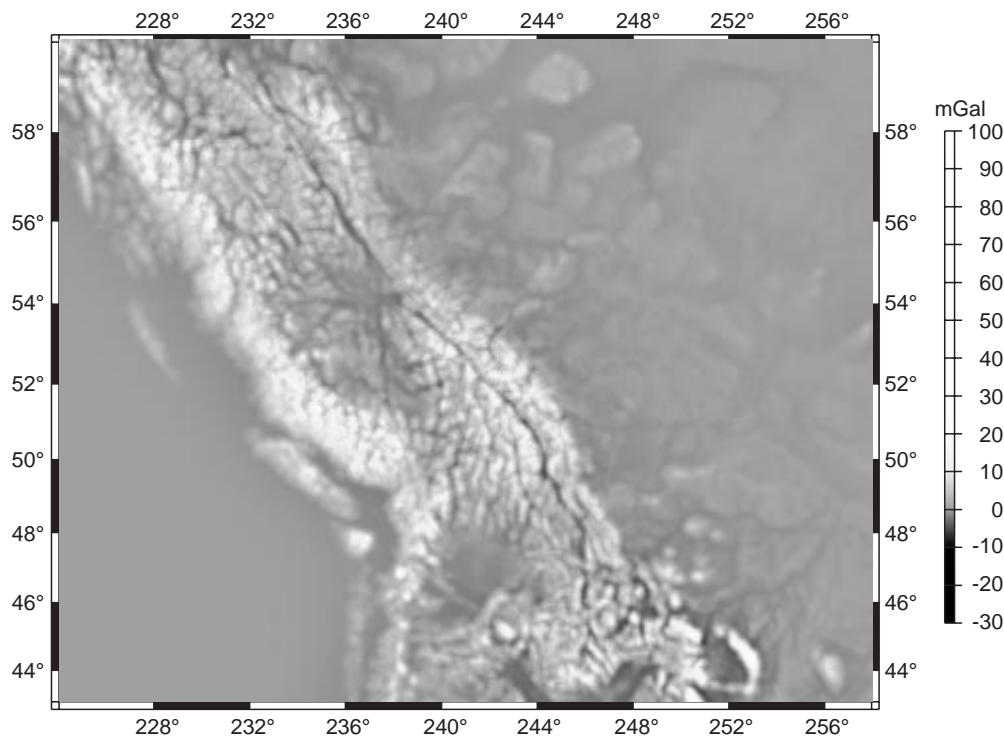

Fig. 8. The condensed terrain effect (in mGal)

First, Eq. (21) is rewritten in a clearer form as follows:

$$-\frac{\partial T^{\mathrm{h}}(r,\Omega)}{\partial r}-\frac{2}{R}T^{\mathrm{h}}(r,\Omega)\doteq$$

$$\Delta g^{\mathrm{h}}(r,\Omega)+\frac{2}{R}\left\{\left[m+f\left(\cos 2\varphi-\frac{1}{3}\right)\right]T^{\mathrm{h}}(r,\Omega)\right.$$

$$\left.+\frac{f}{2}\sin 2\varphi\,\frac{\partial T^{\mathrm{h}}(r,\Omega)}{\partial\varphi}\right\} \quad (42)$$

where the first term on the right-hand side is given by Eq. (41). It is the left-hand side of Eq. (42) that can be continued downward. The downward continuation is a non-linear problem: different spatial wavelengths propagate differently. Therefore, the estimates given earlier of the influence of the individual effects evaluated on the Earth's surface, presented as the influence on the geoid, may be in error and should be interpreted only as first-order estimates and nothing else.

The Poisson downward continuation is also known to be an unstable problem. The stability of the formulation for the mean Helmert gravity anomalies has been investigated by Martinec (1996). It was proved that even in fairly high mountains (mean $H^{\mathrm{o}}=4$ km), the stability is good for mean anomalies representing geographical cells larger than $2\times2'$ (approximately 3.2×3.2 km).

Thus, we encounter no stability problem when dealing with the mean Helmert gravity anomalies of $5\times5'$ (approximately 9×9 km), which is what has been done in the computations described herein. In the unlikely case that the mean Helmert gravity anomalies are known on a significantly denser grid in the high mountains, the downward-continued anomalies in this area will be burdened with larger errors than their counterparts in lower-lying areas of the Earth's surface.

The change of the $5\times5'$ mean Helmert anomalies (corrected for the two estimated ellipsoidal corrections) in the test area due to the Poisson downward continuation to the geoid has been computed. The effect of this change on the geoid was obtained from the Stokes integration; it is shown in Fig. 7. The effect ranges between -0.163 and 1.037 m. It should be noted that the values within a few-degree strip along the eastern border of the area are affected by the edge effect on the downward continuation (Sun and Vaníček 1996) and should not be fully trusted.

As the heights used in the Poisson kernel are the orthometric heights (above the geoid), the results of the downward continuation are the Helmert gravity anomalies on the geoid, as already discussed above. However, the boundary values are required on the Helmert co-geoid. The further reduction from the geoid onto the Helmert co-geoid is achieved by adding to the Helmert gravity anomalies on the geoid the upward continuation (recalling that the co-geoid is everywhere above the geoid) of the Helmert gravity anomalies from the geoid to the co-geoid. Let the upward continuation of the Helmert gravity anomaly from the geoid to the topography be denoted by $D\Delta g^{\mathrm{h}}$

$$D\Delta g^{\mathrm{h}}(\Omega)=\Delta g^{\mathrm{h}}(r_g,\Omega)-\Delta g^{\mathrm{h}}(r_{\mathrm{t}},\Omega) \quad (43)$$

Then the average vertical gradient of the Helmert gravity anomaly between the geoid and the surface of the Earth is equal to

$$\forall H^{\mathrm{o}}(\Omega)\neq0:\overline{\mathrm{grad}}[\Delta g^{\mathrm{h}}(\Omega)]=\frac{D\Delta g^{\mathrm{h}}(\Omega)}{H^{\mathrm{o}}(\Omega)} \quad (44)$$

and the upward continuation $\delta D\Delta g^{\mathrm{h}}$, from the geoid to the Helmert co-geoid, can be estimated from the following expression:

$$\forall H^{\mathrm{o}}(\Omega)\neq0:\delta D\Delta g^{\mathrm{h}}(\Omega)=-\frac{D\Delta g^{\mathrm{h}}(\Omega)\delta N_{\mathrm{p}}(\Omega)}{H^{\mathrm{o}}(\Omega)} \quad (45)$$

where δN_{p} stands for the sum of the primary indirect topographical effect (PITE) and the primary indirect atmospherical effect (PIAE). The latter effect for our area of interest is in absolute value equal at most to 0.6 cm. The former (PITE) was discussed by Martinec and Vaníček [1994a, Eq. (50)], and the discussion will not be repeated here. (As the PIE terms represent the transformation of the co-geoid from the Helmert space to the geoid in the real space, it is irrelevant to the derivation of boundary values discussed in this paper and will not be addressed any further here.) In the test area, the term $\delta D\Delta g^{\mathrm{h}}(\Omega)$ was found to be between -11 and $+31$ μGal and its effect on the geoid is smaller than 0.1 cm. It can thus be safely neglected even for mountains as high as the Canadian Rockies.

7 Conclusions

A process has been formulated which yields the required boundary values on the Helmert co-geoid to an accuracy that would allow the determination of the geoid accurate to one centimetre. The actual accuracy will, of course, depend on the available data, their accuracy and their spatial distribution. At present, the density and the quality of the necessary data in Canada appear good enough for at best a one-decimetre solution. This situation is somewhat similar in other parts of the world. This fact, however, should not preclude the theory of calculations to a one-order-of-magnitude better accuracy.

The starting point of our formulation has been the 'fundamental gravimetric equation', which we have taken as a defining equation for a gravity anomaly as a function of disturbing potential. In contrast with standard practice, this definition leads to an interpretation of gravity anomaly as a function of the position (r,Ω) of the gravity value $g(r,\Omega)$. Normal gravity, which is to be subtracted from $g(r,\Omega)$, is then evaluated at a point $(r-Z,\Omega)$ at which the value of normal potential U is the same as the value of actual potential W at (r,Ω). Rigorous application of this definition in the Helmert space then leads to an expression for the Helmert gravity anomaly on the Earth's surface that can be evaluated to a desired accuracy. This expression includes two small

corrections to normal gravity for quasigeoid–geoid separation and for the secondary indirect effect evaluated at the surface of the earth. These two corrections are not considered in standard practice.

In the formulation, use is made of complete Bouguer gravity anomalies, which are easy to compute, interpolate and predict. This approach gives rise to a correction for the CTE that replaces the DTE. This CTE correction has the same order of magnitude as the DTE, and as such is about two orders of magnitude larger than the two corrections mentioned above. Only approximate values of this correction are presented here as the effect of actual topographical density and the effect of limited integration area have not yet been considered.

The effect of actual topographical density reaches a few decimetres in the Canadian Rockies (Martinec 1993), and the effect of limited integration is now being investigating by the authors. It is pointed out that the CTE is akin to the routinely-used downward continuation approximation, the g_1 term. Finally, the case is made for using the rigorous Poisson theory for the downward continuation to the geoid in the Helmert space. Numerical results for the test area in the Canadian Rocky Mountains are also presented. It is shown that, for the selected test area, the upward continuation from the geoid to the Helmert co-geoid, where the boundary values are needed, yields negligible values and can be safely neglected.

Appendix 1

Derivation of the SITE on the Earth's surface

The SITE which is needed for the proper upward continuation of the normal gravity from the ellipsoid to the telluroid in the Helmert space can be written simply from Eqs. (27) and (36)

$$\delta\gamma^{t}(r_t, \Omega) = \frac{2}{R}\left[V^{t}(r_t, \Omega) - V^{ct}(r_t, \Omega)\right] \qquad (A1)$$

As the Bouguer shell contributions to both these potentials are the same, they cancel in the difference (Martinec 1993) and we can rewrite Eq. (A1) as

$$\delta\gamma^{t}(r_t, \Omega) = \frac{2}{R}\left[V^{r}(r_t, \Omega) - V^{cr}(r_t, \Omega)\right] \qquad (A2)$$

where V^{r} stands for the terrain (topography referred to the height of the point of interest) potential, and V^{cr} stands for the potential of condensed terrain.

In planar approximation, the terrain potential can be written as a Newton integral

$$V^{r}(r_t, \Omega) = GR^2\varrho_0 \iint_{\Omega'} \int_0^{H^{o}(\Omega')-H^{o}(\Omega)} \frac{1}{L(r_t, \psi, r')}\, dz' d\Omega' \qquad (A3)$$

where we have taken the topographical density to be constant and equal to ϱ_0. In Eq. (A3), L stands for the straight distance between the point of interest (r_r, Ω) and the integration point $(r_t + z, \Omega')$, and the outside integration is carried over the whole Earth in a spherical approximation.

To evaluate the potential V^{cr}, we first have to select the way we want the topography condensed onto the geoid. Assuming the following condensation scheme:

$$\sigma(\Omega') = \frac{\varrho_0}{R^2} \int_R^{r_t(\Omega')} r'^2\, dr'$$
$$= \varrho_0 H^{o}(\Omega')\left[1 + \frac{H^{o}(\Omega')}{R} + \frac{H^{o2}(\Omega')}{3R^2}\right] \qquad (A4)$$

which preserves the mass of the Earth (Wichiencharoen 1982), the potential of the condensed terrain layer V^{cr} experienced at the surface of the Earth (r_t, Ω) is given by the following Newton integral:

$$V^{cr}(r_t, \Omega) = GR^2 \iint_{\Omega'} \frac{\sigma(\Omega') - \sigma(\Omega)}{L(r_t, \psi, R)}\, d\Omega' \qquad (A5)$$

where $L(r_t, \psi, R)$ is the straight distance between the point of interest (r_t, Ω) and the integration point (R, Ω'), and the integration is carried over the whole Earth in a spherical approximation. Substituting Eq. (A4) into Eq. (A5), taking the topographical density $\varrho(\Omega)$ to equal to ϱ_0, and neglecting terms of the order H^{o}/R and higher (compared to 1), we obtain

$$V^{cr}(r_t, \Omega) = GR^2\varrho_0 \iint_{\Omega'} \frac{H^{o}(\Omega') - H^{o}(\Omega)}{L(r_t, \psi, R)}\, d\Omega' \qquad (A6)$$

Now, Eqs. (A3) and (A6) can be substituted into Eq. (A1) to give us the final result. However, before we do this, we can evaluate the inside integral in Eq. (A3) to simplify the final expression. We obtain

$$\int_0^{H^{o}(\Omega')-H^{o}(\Omega)} \frac{1}{L(r_t, \psi, r')}\, dz'$$
$$= \ln \frac{H^{o}(\Omega') - H^{o}(\Omega) + \sqrt{R^2\psi^2 + [H^{o}(\Omega') - H^{o}(\Omega)]^2}}{R\psi} \qquad (A7)$$

where $\psi = \psi(\Omega, \Omega')$ is the spherical distance between the point of interest (r_t, Ω) and the integration point (R, Ω') in the outside integration. This solution is only a planar approximation since we have approximated the distance L by

$$L(r_t, \Omega; r_t + z, \Omega') \doteq$$
$$\sqrt{R^2\psi^2 + [H^{o}(\Omega') - H^{o}(\Omega)]^2} \qquad (A8)$$

where we have taken $R\psi$ as a planar approximation of the horizontal distance between the point of interest and the integration point. The final result can now be written in the following form:

$$\delta\gamma^{t}(r_{t},\Omega) \doteq 2GR\varrho_0$$

$$\iint_{\Omega'} \ln\frac{H^{o}(\Omega')-H^{o}(\Omega)+\sqrt{R^2\psi^2+[H^{o}(\Omega')-H^{o}(\Omega)]^2}}{R\psi}d\Omega'$$

$$-2GR\varrho_0 \iint_{\Omega'} \frac{H^{o}(\Omega')-H^{o}(\Omega)}{\sqrt{R^2\psi^2+[H^{o}(\Omega)]^2}}d\Omega' \qquad (A9)$$

where we use the same order of approximation for ℓ as we have used for the distance L in Eq. (A8), namely

$$\ell(r_{t},\Omega;R,\Omega') \doteq \sqrt{R^2\psi^2+[H^{o}(\Omega)]^2} \qquad (A10)$$

To conclude, we note that the effect derived here is very small indeed, so that an accuracy of 1% is all we need. Thus the approximations used in the derivations above are justified. For the same reason, the integration in Eq. (A9) does not have to be extended over the whole globe. Accurate enough results are obtained when the integration is confined to a cap of a radius of 3°.

Appendix 2

Derivation of the CTE

The CTE is the attraction of the terrain condensed on the geoid experienced on the Earth's surface, i.e. it is the negative radial derivative of the potential of the condensed topographical roughness which is defined in Eq. (A6) at $r=r_{t}$. Assuming again the constant density ϱ_0, and neglecting terms of the first and higher orders of H^{o}/R in Eq. (A4)

$$\sigma^{t}(\Omega') \doteq \varrho_o^{t}\Big[H^{o}(\Omega')-H^{o}(\Omega)\Big] \qquad (A11)$$

Having derived the radial derivative of Eq. (A6), we obtain

$$A^{cr}(r_{t},\Omega) =$$

$$-GR^2\varrho_0 \iint_{\Omega'} \Big[H^{o}(\Omega')-H^{o}(\Omega)\Big]K(r_{t},\psi,R)d\Omega' \qquad (A12)$$

The integration kernel, K, a function of the orthometric height of the point of interest and a spherical distance ψ between points Ω and Ω', is defined as the following function [Martinec 1993, Eq. (2.21)]:

$$K(r_{t},\psi,R) = \frac{\partial L^{-1}(r_{t},\psi,R)}{\partial r}\Big|_{r_{t}} = \frac{R\cos\psi-r_{t}(\Omega)}{L^3(r_{t},\psi,R)} \qquad (A13)$$

The integral (A12) defining the CTE is regular for $H^{o}(\Omega)\neq 0$, and the kernel given by Eq. (A13) tends to zero very rapidly with growing distance ψ. Therefore, the integration does not have to be taken over the entire

Earth even for large orthometric heights. Following Martinec's (1993) recommendation, it has been decided to integrate over a cap of a radius of 3°, corresponding to a linear distance of about 330 km. For the numerical integration, this cap was divided into two zones: the inner zone of a rectangle of $1\times1°$ and the rest of the cap. Within the inner zone, the densest available elevation data ($30\times60''$) in Canada were used. In the rest of the cap, the $5\times5'$ mean heights proved to give sufficiently accurate results. A word of caution is in order here: numerical tests have shown that even for a 3° cap, the contribution from the rest of the world can be significant. It may thus be necessary to evaluate the 'truncation effect' correction to the limited integration. This is currently under investigation.

The numerical results for the testing area are shown in Fig. A1. These results were compiled from a combination of Canadian and US elevation data. The effect ranges between -25.136 and $+92.776$ mGal. Its influence on the geoid is between $+0.127$ and $+1.787$ m. Interestingly, the numerical results for the CTE are somewhat similar to those for the so called 'downward continuation correction' g_1

$$g_1(\Omega) = -GR^2H^{o}(\Omega)\varrho_0 \iint_{\Omega'} \frac{H^{o}(\Omega')-H^{o}(\Omega)}{L^3(r_{t},\psi,R)}d\Omega' \qquad (A14)$$

as given by Moritz [1980, Eq. (48-14)].

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