Journal of Geodesy (1997) 71: 149–159

Journal of Geodesy

Analytical solution of perturbed circular motion: application to satellite geodesy

P. Exertier, P. Bonnefond

Observatoire de la Côte d'Azur, Dept. C.e.r.g.a., Avenue Copernic, F-06130 Grasse, France

Received 22 January 1996; Accepted 19 September 1996

Abstract. Starting from the analytical theory of perturbed circular motions presented in Celestial Mechanics (Bois 1994) and from specific extended formulations of the perturbations in a uniformly rotating plane of constant inclination, this paper presents an extended formulation of the solution. The actual gain made through this extension is the establishment of a first-order predictive theory written in spherical coordinates and thus free of singularities, whose perturbations are directly expressed in the local orbital frame generally used in satellite geodesy. This new formulation improves the generality, the precision and the field of applications of the theory. It is particularly devoted to the analysis of satellite position perturbations for satellites in low eccentricity orbits usually used for many Earth observation applications. An application to the TOPEX/Poseidon (T/P) orbit is performed. In particular, contour maps are provided which show the geographical location of orbit differences coming from geopotential coefficient differences of two recent gravity field models. Comparison of predicted radial and along-track orbit differences with respect to numerical results provided by the French group (CNES, in Toulouse) in charge of the T/P orbit are convincing.

Key words. satellite theory – circular motion – spherical coordinates – geopotential

1. Introduction

The a priori analysis of satellite orbit perturbations induced by the Earth's gravity field (static part) is usually required for planning space missions. In this field, the representation of orbital perturbations has been given traditionally by different satellite theories in terms of geometrical elements (Brouwer 1959; Kaula 1966). However, in many applications, it is necessary that the perturbations be expressed in coordinate variables rather than in element variables. In most cases, the requirement is actually imposed by the nature of the satellite mission, as in the case of satellite altimetry. Recently, interest in this problem has been notably augmented by the high precision requirements for the TOPEX/Poseidon orbit (Rosborough 1986; Melvin 1987; Balmino 1992). A recall of previous analyses of satellite position perturbations due to the complete geopotential can be found in Rosborough and Tapley (1987). In this last case, as it is the case more recently in Casotto (1993), the authors have expressed the solutions thanks to a transformation of the orbital element perturbations to the coordinate perturbations. The results available for various orbits contain however singularities for zero eccentricity and inclination. Since satellites in low eccentricity orbits are usually used for many Earth observation applications, it may be judicious to expand predictive theories directly in coordinate representations free of singularities ($e = 0$ and/or $I = 0$) rather than in orbital elements. The interest in Space Geodesy is obvious. As an example, Hill equations have been used to derive the radial perturbations on a satellite orbit due to the geopotential (Schrama 1989). And, more recently, Balmino et al. (1996) showed the compatibility of the classical linear perturbation theory based on Lagrange's planetary equations with Hill approach. Following the linear orbit perturbation technique, as developed by Kaula (1966), we show that it is possible to adopt spherical coordinates instead of Kepler elements. The theory is thus based on second order differential equations with respect to time, as given by Brouwer and Clemence (1961), describing the motion in a global geocentric reference frame. These equations are more general than the linearized Hill equations, which describe motion in a local Cartesian reference frame, co-moving along a strictly circular orbit. In addition, the set of spherical

Correspondence to: P. Exertier

coordinates is more suited than local Cartesian coordinates to express most of the gravitational perturbations. The circular motion is a particular solution of the zero-order equations of motion written in spherical coordinates. It has been adopted here as mean motion allowing for the linearization to be carried out with respect to this particular reference orbit. But in order to describe weakly eccentric orbital motion, it is also possible to adopt an approximate elliptical Keplerian solution of undisturbed motion, as showed by Breiter and Bois (1994).

The construction and the expansion of a new predictive analytical theory have been motivated by these above considerations. The demonstration of the theory and the whole formulation of the method of resolution can be found in Bois (1994). The main advantage of the first-order literal solution, expanded in Fourier series and non-singular variables, is the presence of iterative formation laws for its coefficients. The theory is then particularly accurate and suitable whatever the conservative forces, see (Exertier and Bois 1995), these forces being also expanded in Fourier series. As a consequence, this predictive analytical theory whose solution is given in a simple and compact form is very efficient to describe perturbed circular and quasi-circular motions. Now, the aim of the present paper is to extend the formulation of this previous solution to more possibilities of applications, particularly in the field of satellite geodesy. Given the secular effects that affect notably the ascending node of an inclined satellite orbit, it may be judicious to expand and to solve the equations of motion written in spherical coordinates in the precessing mean orbital plane. The actual gain made through our extension is the introduction of a uniformly rotating plane of constant inclination (I) as a new reference plane of the theory. From a geometrical point of view, this extension allows to connect permanently the spherical coordinate system used in this theory with the orbital frame usually used in satellite geodesy, whose coordinate axes are along the radial (R), the transverse (T) and the normal (N) directions. The advantage is twofold. First, the first-order literal relations given in section 4 between the terms of the perturbation and the solution are directly expressed in the local orbital frame using geocentric coordinate variables; the compact form which has been adopted in addition facilitates implementation and interpretation. Second, the present new formulation improves generality, field of validity, and precision of the predictive solution notably valid for $e = 0$, or very faint values, and for $0 \leq I \leq \pi$.

In order to explicit the field of the present extension, we briefly recall in section 2 some characteristics about the initial theory (Bois 1994). Section 3 contains the extension of the first-order equations of motion permitting to adopt a precessing plane of constant inclination as reference plane. Section 4 shows an application of the method of resolution in case of an extended central potential: the static geopotential. It leads to the complete solution permitting to give different physical descriptions of its terms. In section 5, a practical and concrete application is performed in the case of the TOPEX/Poseidon (T/P) orbit in order to show the validity of the analytical solution and its interest in satellite geodesy. Comparisons of the analytical solution with the numerical integration are certainly efficient for qualifying the first-order literal relations (see, e.g., Fig. 1 in Bois (ibid)). However, in order to make our application different and original, we focus on the description of radial and along-track orbit differences expressed in geographical coordinates. Actually, the Centre National d'Etudes Spatiales (CNES) precise orbit determination system computed T/P 10 day orbits twice using JGM-2 and JGM-3 models (Nouel et al. 1994). As a consequence, this has formed a very interesting basis of orbit differences which are only caused by different gravitational models. And this has lead us to compare, in the spatial domain – along theT/P ground tracks –, the mean part of these orbit differences with the coordinate perturbations given by our analytical theory using difference coefficients between JGM-2 and JGM-3 (Nerem et al. 1994; Tapley et al. 1994). After eliminating the frequencies very close to once per revolution in the analytical solution, comparisons of predicted orbit differences with those resulting from the numerical integrations show the satisfactory level of consistency of the analytical solution, always at the first-order.

The characteristics of the radial orbit errors for near circular satellite trajectories caused by uncertainties of geopotential coefficients have been discussed by many authors (e.g., Rosborough 1986; Balmino 1992; Schrama 1992). For information only, these errors may be evaluated grossly by applying the analytical formulas for the radial orbit perturbations to difference coefficients between two different geopotential models. This is certainly a too crude approximation, the evaluation of the actual radial orbit errors from geopotential coeffi-

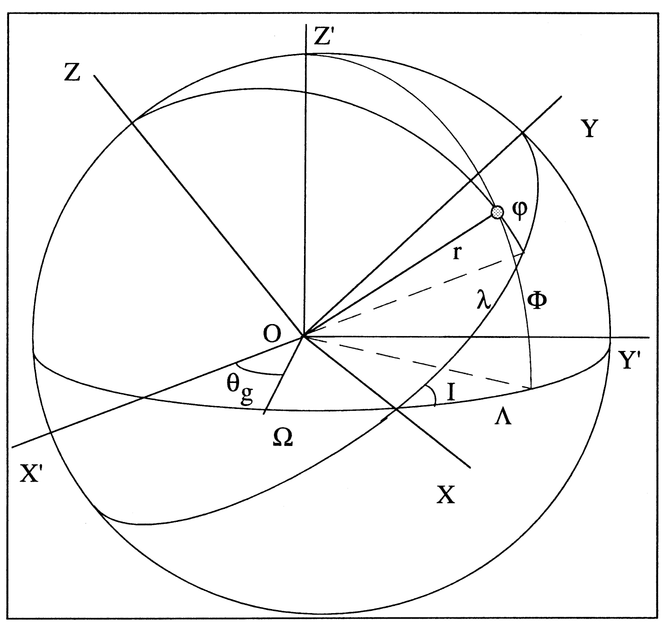

Fig. 1. Reference frames and associated spherical coordinate systems

cient covariances (if calibrated) being more realistic (Balmino 1992). Thus, prediction of orbit errors by means of difference coefficients in first-order analytical theories is certainly a good estimate, but it is not absolute. However, on a practical point of view it gives easily a general idea of the positive improvements seen in the Earth's gravity field models. As an example, the geopotential coefficient differences between JGM-2 and JGM-3 have been used again in order to predict the radial orbit differences to be expected along the ERS-1 ground tracks. We used intentionally the same conditions and gravity fields than in the case of T/P where the analytical solution proved to be an efficient tool for representing orbit errors; this has especially permitted to show the effects of the altitude difference between both satellites. Results show that the main part of the radial orbit differences are due to difference coefficients of degree and order higher than fifty at least for the altitude and inclination of ERS-1. On the other hand, gravity model differences that one gets between JGM-3 and GRIM4-C4 have been used also to derive geographically correlated radial orbit differences. In this last case, results have been compared to the ERS-1 radial orbit error propagated from the calibrated covariance matrix of GRIM4-C4 (Schwintzer et al. 1996).

2. Recall of the basic frame

Let us recall some characteristics about the initial theory developed by Bois (1994). Let $OXYZ$ be a fixed reference frame and let r, φ, λ be some spherical coordinates referred to it. Using this set of coordinates, and in the fixed frame, the Lagrangian of the motion of the center of mass of a body, submitted to any perturbation $W(r, \varphi, \lambda)$ is written as follows:

$$L = \frac{1}{2}\left[\dot{r}^2 + r^2\dot{\varphi}^2 + r^2\cos^2\varphi\dot{\lambda}^2\right] + W(r, \varphi, \lambda) \tag{1}$$

The classical general equations of the motion written in $OXYZ$ are then obtained by application of the algorithm of Lagrange (see, e.g., (Brouwer and Clemence 1961)). Starting from these equations, the components of the disturbing force and the solution are expanded, according to the Poincaré theorem, in powers of a small parameter ε reflecting the order of magnitude of the disturbing force. The principle of the method consists in exactly solving the successive differential sub-systems obtained at the zero order, first order, and so on. The circular motion, as a particular case of the classical Keplerian solution, is choosen as the zero-order solution. Then, the disturbing force and the solution are expanded in the form of Fourier series. The solution series $(r_1^*, \varphi_1^*, \lambda_1^*)$ representing the periodic perturbations are written as follows:

$$\left|\begin{aligned} r_1^* &= \sum_{i>0,i_n} \frac{as_{ii_n}\sin\Psi_{ii_n} + ac_{ii_n}\cos\Psi_{ii_n}}{\dot{\Psi}_{ii_n}} \\ \varphi_1^* &= \sum_{i>0,i_n} \frac{bs_{ii_n}\sin\Psi_{ii_n} + bc_{ii_n}\cos\Psi_{ii_n}}{\dot{\Psi}_{ii_n}} \\ \lambda_1^* &= \sum_{i>0,i_n} \frac{cs_{ii_n}\sin\Psi_{ii_n} + cc_{ii_n}\cos\Psi_{ii_n}}{\dot{\Psi}_{ii_n}} \end{aligned}\right. \tag{2}$$

with $\Psi_{ii_n} = i\lambda_0 + \sum_n i_n a_n$, where a_n are time dependent functions, and $\dot{\Psi}_{ii_n} = i\,\overline{n} + \sum_n i_n\dot{a}_n$ containing the fundamental frequencies involved in the problem. Each coefficient of the solution, as as_{ii_n} or ac_{ii_n}, is associated to a given frequency $\dot{\Psi}_{ii_n}$ for porperly separating all the periodic effects. The only requirement due to this global form is that $\dot{\Psi}_{ii_n}$ be not zero. In such a case, the case of the resonance (e.g., $\overline{n} = -i_1/i\,\dot{a}_1$), the above solution would require a specific formulation.

The nature of forces may be very different, the only assumption being that their components $(F, G, H$, see eq. 3) are periodical with respect to φ and λ. A similar treatment is performed on the perturbations as on the solution (the disturbing force and the solution are expanded in Fourier series) in function of the effective form of the zero-order solution (r_0 = constant, $\varphi_0 = 0$, λ_0 = linear function of time). Let us recall that at the zero order, the motion being in a plane, a simple choice of the reference plane OXY in the mean orbital plane permits to obtain $\varphi_0 = 0$ still remaining in the general case. After some trigonometric expansions, the components of the disturbing force take the following form:

$$\left|\begin{aligned} F_1^* &= K_F + \sum_{k>0,k_n}(p_{kk_n}\sin\Psi_{kk_n} + q_{kk_n}\cos\Psi_{kk_n}) \\ G_1^* &= K_G + \sum_{k>0,k_n}(r_{kk_n}\sin\Psi_{kk_n} + t_{kk_n}\cos\Psi_{kk_n}) \\ H_1^* &= \sum_{k>0,k_n}(u_{kk_n}\sin\Psi_{kk_n} + v_{kk_n}\cos\Psi_{kk_n}) \end{aligned}\right. \tag{3}$$

with $\Psi_{kk_n} = k\lambda_0 + \sum_n k_n a_n$. The resulting numerical coefficients $(p, q, r, t, u, v)_{kk_n}$ are associated to a given argument Ψ_{kk_n}, while (K_F, K_G) are two constant terms (independently of λ).

The first-order resolution leads to algorithmic solutions using few parameters and suitable for $e = 0$, or very faint values, and $I = 0$ (Bois 1994). Besides, an extended formulation of the initial first-order solution has been obtained by Exertier and Bois (1995), in order to generalize the capacity of the theory to take into account different periodic perturbations of various physical nature. Most gravitational perturbations are usually represented by spherical harmonic expansions depending on spherical coordinates. Such perturbations are directly expressed in the variables of the theory, thus avoiding the heavy developments in elliptic elements. This last idea obviously reinforces the interest in the use of spherical coordinates instead of Cartesian, e.g. in the so-called Hill equations. The coefficients $(as, ac, bs, bc, cs, cc)_{ii_n}$ of the solution (2) are given as literal functions of the disturbing parameters (3) by way of extended iterative formation laws (Exertier and Bois ibid):

$$S_1(x_{kk_n}, y_{kk_n}) = \frac{\dot{\Psi}_{kk_n} r_0\, x_{kk_n} + 2\overline{n} y_{kk_n}}{\left[\overline{n}^2 - \dot{\Psi}^2_{kk_n}\right] r_0}$$

$$S_2(x_{kk_n}) = \frac{\dot{\Psi}_{kk_n}\, x_{kk_n}}{\left[\overline{n}^2 - \dot{\Psi}^2_{kk_n}\right] r_0^2} \tag{4}$$

$$S_3(x_{kk_n}, y_{kk_n}) = \frac{2\overline{n}\, x_{kk_n} + [\dot{\Psi}_{kk_n} + 3\overline{n}^2/\dot{\Psi}_{kk_n}] y_{kk_n}/r_0}{\left[\overline{n}^2 - \dot{\Psi}^2_{kk_n}\right] r_0}$$

The indexes (k, k_n) depend on the boundaries of variation indexes of the disturbing force. Ψ_{kk_n} is the common trigonometric argument of all series. In the above equations, the case : $k = \pm 1, n = 1, k_1 = 0$, $\dot{\Psi}_{\pm 1,0} = \pm \overline{n}$ has not to be considered as what we call a resonance in celestial mechanics. It is a particular case to be discussed for each application of the first-order theory. In section 4, the application of the theory to the geopotential will permit to show these aspects in a concrete case.

3. Extension to a rotating system

The purpose of this section is to show how the motion equations have been modified when introducing the concept of a precessing reference plane of constant inclination in the theory. As a matter of fact, it permits to eliminate the secular out of plane perturbations (on φ) which arise from the even zonal harmonics of the central body.

3.1 Reference frames and systems of coordinates

Let $OX'Y'Z'$ and $OXYZ$ be a fixed and a relative reference frame respectively. $OX'Y'Z'$ represents the inertial reference system of the theory. It is connected to the central body such as $OX'Y'$ is put in its equator. Let us define some spherical coordinates r, ϕ, Λ referred to this fixed reference frame. On the other hand, $OXYZ$ rotates around OZ' with an uniform velocity $\dot{\Omega}$ the OXY plane having a constant inclination I relative to the planet's equator. The equatorial angle Ω between the OX' and OX directions (Figure 1) is assumed to be a linear function of time. As it is the case in the previous formulation (section 2), let r, φ, λ be some spherical coordinates referred to $OXYZ$; they are the variables used in the theory. Let us recall however that $OXYZ$ will be a rotating frame in the following.

Now, in the relative frame, the Lagrangian of the motion of the centre of mass of a body, submitted to any perturbation $W(r, \varphi, \lambda, I, \Omega)$ takes the following form:

$$\begin{aligned} L = \frac{1}{2}\Big[\dot{r}^2 &+ \left(r\dot{\varphi} - r\dot{\Omega}\sin I\cos\lambda\right)^2 \\ &+ \left(r\cos\varphi\ (\dot{\lambda} + \dot{\Omega}\cos I) - r\sin\varphi\dot{\Omega}\sin I\sin\lambda\right)^2\Big] \\ &+ W(r, \varphi, \lambda, I, \Omega) \end{aligned} \tag{5}$$

In this expression obviously, the components of the velocity expressed in $OXYZ$ relatively to $OX'Y'Z'$ contain some complicated terms due to the precession $\dot{\Omega}$ of OXY around OZ'.

3.2 Equations of motion

By application of the algorithm of Lagrange, we obtain the classical general equations of motion. Then, differential equations for the first order in ε can be established assuming the rate of rotation $\dot{\Omega}$ of the OXY plane is a first-order quantity: $\Omega = \Omega_0 + \varepsilon\dot{\Omega}_1(t - t_0)$, where Ω_0 is an initial constant phase angle. At this point however, no hypothesis is made on $\dot{\Omega}_1$. On the contrary, section 4 shows that this last term is completely determined in the resolution of the first-order equations of motion.

The zero-order solution $(r_0, \varphi_0, \lambda_0)$ due to the right hand side $(F_0 = -\mu/r_0^2, G_0 = 0, H_0 = 0)$ is a particular case of the general plane motion; indeed we adopt the circular motion permitting to write λ_0 as linear function of time. On the other hand, the adopted form for the first-order terms $(r_1, \varphi_1, \lambda_1)$ is as general as possible. These literal solutions are finally written as follows:

$$\begin{cases} r_0 &= \overline{r}_0 \\ \varphi_0 &= 0 \\ \lambda_0 &= \overline{n}(t - t_0) + \overline{\lambda}_0 \end{cases} \qquad \begin{cases} r_1 &= \varrho_1 + r_1^* \\ \varphi_1 &= \sigma_1 + \varphi_1^* \\ \lambda_1 &= \overline{n}_{\lambda_1}(t - t_0) + \lambda_1^* \end{cases} \tag{6}$$

where $\overline{n}$ is the mean motion of the body in the relative reference frame $OXYZ, \overline{r}_0$ a constant, and $\overline{\lambda}_0$ a constant phase angle. In addition, ϱ_1 and σ_1 are constant terms, $\overline{n}_{\lambda_1}$ represents a secular term, and $(r_1^*, \varphi_1^*, \lambda_1^*)$ represent the periodic perturbations (2). The first-order equations of motion have the following expression:

$$\begin{cases} \ddot{r}_1 - \overline{n}^2 r_1 - 2r_0\overline{n}\dot{\lambda}_1 - 2r_0\overline{n}\ \dot{\Omega}\cos I = \\ \qquad\qquad 2\dfrac{\mu}{r_0^3} r_1 + F_1^*(r_0, \varphi_0, \lambda_0, I, \Omega) \\ \ddot{\varphi}_1 + \overline{n}^2\varphi_1 + 2\overline{n}\ \dot{\Omega}_1 \sin I \sin\lambda = \\ \qquad\qquad \dfrac{1}{r_0^2} G_1^*(r_0, \varphi_0, \lambda_0, I, \Omega) \\ \ddot{\lambda}_1 + \dfrac{2}{r_0}\overline{n}\dot{r}_1 = \dfrac{1}{r_0^2} H_1^*(r_0, \varphi_0, \lambda_0, I, \Omega) \end{cases} \tag{7}$$

These equations depart from those given by Melvin (1987) mainly by the form of the zero- and first-order solutions and consequently by the notion of order adopted in our theory. The extended iterative formation laws given equations (4) are unchanged. However, when identifying the Fourier series terms of both perturbation and solution in these equations, we have to take into account for the new left hand side terms (in r and φ) due to $\dot{\Omega}_1$.

The linear orbit perturbation technique we applied to obtain and then to solve the first-order system (7) is very similar to the classical linear perturbation theory, as developed by Kaula (1966). The linearization is carried out with respect to a reference orbit; here, let us recall that it is a circular motion of constant inclination I

which precess with an uniform velocity $\dot{\Omega}$. As a consequence, the elements of the reference orbit are substituted on the right hand side of the equations of motion and result in the perturbations in the spherical coordinates. There is thus no interest to consider the homogeneous solution of the first-order equations (7). The general solution $(r(t) = r_0 + r_1, \varphi(t) = \varphi_0 + \varphi_1, \lambda(t) = \lambda_0 + \lambda_1)$ - see eq. (6) and (2) - contains already what we could call the force free and forced solutions of the equations of motion.

4. Application to the Geopotential

The formalism of the theory can be applied to various physical phenomena. Now, by taking into account the extended potential of the Earth as main perturbation of the problem, the aim is to obtain the expression of the disturbing function and then its partial derivatives with respect to r, φ, λ. These components of the disturbing force are expanded to the first-order in the small parameter ε by injecting the zero-order solution $(r_0 = \text{constant}, \varphi_0 = 0, \lambda_0)$ in the expressions of the partial derivatives. The constant and purely periodic terms are separated, these treatments being of different nature. In addition, terms in $\sin\lambda$ and $\cos\lambda$ in the components of the disturbing force are isolated (i.e., the particular case $\dot{\Psi}_{\pm1,0} = \pm\overline{n}$), these last terms being treated separately.

The gravitational field of a planet is expressed by the following standard expression:

$$U = \frac{\mu}{r} + U^*$$

where:

$$U^* = \frac{\mu}{r}\sum_{l=2}^{l_{Max}}\sum_{m=0}^{l}\left(\frac{a_e}{r}\right)^l P_{l,m}(\sin\phi) \left[C_{l,m}\cos m(\Lambda - \theta_g) + S_{l,m}\sin m(\Lambda - \theta_g)\right] \tag{8}$$

μ is the planet's gravity constant, a_e its equatorial radius, $P_{l,m}$ Legendre associated polynomials and $(C_{l,m}, S_{l,m})$ unnormalized coefficients depending on the physical properties of the planet's gravity field. These last coefficients are determined in the rotating reference frame with the central body, $\dot{\theta}_g$ being its uniform velocity around OZ'. Let us recall that r, ϕ, Λ are the spherical coordinates of a body in the planet's reference frame $(OX'Y'Z')$.

In order to express the perturbation U^* in the variables r, φ, λ of the theory, it is necessary to introduce the transformation of spherical functions under rotation for using these coordinates instead of r, ϕ, Λ coordinates. The general expression of the transformation which depends on a sequence of three constant Euler angles can be found in Borderies (1978) or more recently in Sneeuw (1992). For the particular sequence $(\Omega - \theta_g, I, 0)$ which is not exactly an Eulerian sequence according to the definition given by Sneeuw (1992), let us recall the form of the disturbing function (e.g., in (Kaula 1966)):

$$U^* = \frac{\mu}{r}\sum_{l\geq 2}\left(\frac{a_e}{r}\right)^l \sum_{m=0}^{l}(C_{l,m} - jS_{l,m})\frac{(-1)^{l-m}}{(l-m)!} \sum_{m'=-l}^{+l}(l-m')!C_{l,m}^{m'}(I)P_{l,m'}(\sin\varphi) \exp j[m'\lambda + (m'-m)\pi/2 + m(\Omega - \theta_g)] \tag{9}$$

where $j^2 = -1$ and $C_{lm}^{m'}(I)$ are trigonometric polynomials in $(I/2)$ (Borderies 1978). The three components of the disturbing force (F_1^*, G_1^*, H_1^*) are obtained by differentiation of U^* in r, φ, λ, and then by making $r = r_0 = \text{constant}$, $\varphi = 0$ and $\lambda = \lambda_0$. The argument of these series is: $\Psi_{m'm} = m'\lambda_0 + m(\Omega - \theta_g) + (m' - m)\pi/2$ with: $\dot{\Psi}_{m'm} = m'\overline{n} - m\dot{\theta}_g$.

In particular, the constant terms in $\partial U^*/\partial r$ and $\partial U^*/\partial\varphi$ are obtained by making $m' = m = 0$. They give the following expressions of ϱ_1, σ_1, and $\overline{n}_{\lambda_1}$:

$$\left|\begin{aligned} \varrho_1 &= 0 \text{ or any corrective value} \\ \sigma_1 &= \sum_{l=3}^{l_{Max}}\left(\frac{a_e}{r_0}\right)^l(-1)^l C_{l,0}^0(I)P_{l,0}'(0)C_{l,0} \\ \overline{n}_{\lambda_1} &= -\dot{\Omega}_1\cos I \\ &\quad + \frac{\overline{n}}{2}\sum_{l=2}^{l_{Max}}(l+1)\left(\frac{a_e}{r_0}\right)^l C_{l,0}^0(I)P_{l,0}(0)C_{l,0} \end{aligned}\right. \tag{10}$$

where $P_{l,m}'(0)$ is the derivative of $P_{l,m}(0)$ for $\sin\varphi = 0$. Their values are zero respectively for $(l-m)$ even and $(l-m)$ odd. Now, $\dot{\Omega}_1$ has to be defined in order to complete the expression of $\overline{n}_{\lambda_1}$. To do that, let us consider the second differential equation of system (7). For the particular case $m' = \pm1, m = 0$ ($\dot{\Psi}_{\pm1,0} = \pm\overline{n}$) in this equation, and after equating terms in $\sin\lambda$ it is found that:

$$\dot{\Omega}_1 = -\frac{\overline{n}}{2\sin I}\sum_{l=2}^{l_{Max}}\left(\frac{a_e}{r_0}\right)^l \sum_{\pm}\frac{(l\mp1)!}{l!}C_{l,0}^{\pm1}(I)P_{l,\pm1}'(0)C_{l,0} \tag{11}$$

The advantage is two-fold. First, this permits to completely determine the precession rate $\dot{\Omega}_1$ without making an a priori hypothesis on the nature of this secular term. As an example, the rotating rate of the reference plane OXY of the theory has the following expression for $l = 2: \dot{\Omega}_1 = 3/2\overline{n}a_e^2/r_0^2\cos I C_{2,0}$. It is in agreement with other classical results of satellite theory (e.g., Kovalevsky 1963), considering here the motion is circular. Second, it avoids to take into account the case $m' = \pm1, m = 0$ in using the expression of S_2 (4), since this case is already treated here. On the other hand, it is clear that no first-order term in the form of $\sin\lambda$ and $\cos\lambda$ is determined for the periodic solution in the latitude component.

Considering the value of l must be even in order to make $(l \pm 1)$ odd and then $P_{l,\pm1}'(0) \neq 0$ in (11), the secular terms $\overline{n}_{\lambda_1}$ and $\dot{\Omega}_1$ are connected to even zonal harmonic coefficients, properly. It represents the secular effects arising in the classical argument of latitude and ascending node of a satellite orbit. On the other hand, the constant term σ_1 depending on odd zonal coefficients leads to motions that are slightly displaced out of the reference plane OXY. Besides, the division by sin I in (11) is not singular as the trigonometric polynomials $C^{\pm 1}_{l,0}(I)$ are proportional to sin I. In fact, these expressions can be computed whatever the inclination value $(0 \leq I \leq \pi)$.

The coefficients $(r_1^*, \varphi_1^*, \lambda_1^*)$ of periodic solutions series are obtained from the disturbing series (F_1^*, G_1^*, H_1^*) with both m' and $m \neq 0$, and using the iterative formation laws given in (4). It gives the following first-order periodic terms:

$$\begin{aligned}
(r_1^*)_{l,m,m'} &= \overline{n}^2 r_0 \left(\frac{a_e}{r_0}\right)^l (-1)^{l-m} \frac{(l-m')!}{(l-m)!} C^{m'}_{l,m}(I) \\
&\quad P_{l,m'}(0) \frac{-(l+1) + 2m'\overline{n}/\dot{\Psi}_{mm'}}{\overline{n}^2 - \dot{\Psi}^2_{mm'}} \mathscr{A}_{l,m} \\
(\varphi_1^*)_{l,m,m'} &= \overline{n}^2 \left(\frac{a_e}{r_0}\right)^l (-1)^{l-m} \frac{(l-m')!}{(l-m)!} C^{m'}_{l,m}(I) \\
&\quad P'_{l,m'}(0) \frac{1}{\overline{n}^2 - \dot{\Psi}^2_{mm'}} \mathscr{A}_{l,m} \\
(\lambda_1^*)_{l,m,m'} &= \overline{n}^2 \left(\frac{a_e}{r_0}\right)^l (-1)^{l-m} \frac{(l-m')!}{(l-m)!} C^{m'}_{l,m}(I) \\
&\quad \left[2(l+1)\frac{\overline{n}}{\dot{\Psi}_{mm'}} - m' - 3m'\frac{\overline{n}^2}{\dot{\Psi}^2_{mm'}}\right] \\
&\quad P_{l,m'}(0) \frac{1}{\overline{n}^2 - \dot{\Psi}^2_{mm'}} \mathscr{B}_{l,m}
\end{aligned} \tag{12}$$

with:

$$\mathscr{A}_{l,m} = [C_{l,m} \cos \Psi_{mm'} + S_{l,m} \sin \Psi_{mm'}] \tag{13}$$

$$\mathscr{B}_{l,m} = [C_{l,m} \sin \Psi_{mm'} - S_{l,m} \cos \Psi_{mm'}] \tag{14}$$

where $l = 2, 3, \ldots, m = [0, l]$, and $m' = [-l, +l]$. In addition, $(l - m')$ is even in the expressions of r_1^* and λ_1^* but is odd in φ_1^*. As already shown in section 2, the particular case $m' = \pm 1, m = 0$ has to be discussed. Concerning φ_1^* these last terms are now excluded; this has been expected from the beginning with the introduction of the nodal rate $\dot{\Omega}_1$ in the equations. Concerning r_1^* and λ_1^*, the singularity can be avoided here by taking into account the first-order secular terms $\overline{n}_{\lambda_1}$ in the expression of $\dot{\Psi}_{m'm}$. This technique has been introduced, e.g. by Kaula (1966), in its first-order solution of Lagrange equations. Thus, the common denominator $(\overline{n}^2 - \dot{\Psi}^2_{\pm 1,0})$ of equations (12) takes the following form, for $l = 2$ as an example:

$$\begin{aligned}
\overline{n}^2 - \dot{\Psi}^2_{\pm 1,0} &= 2\overline{n}\dot{\Omega}_1 \cos I \\
&\quad + \frac{3}{2}\overline{n}^2 \left(\frac{a_e}{r_0}\right)^2 C_{2,0}\left(1 - \frac{3}{2}\sin^2 I\right)
\end{aligned} \tag{15}$$

These terms produce short period perturbations of large amplitude notably on r; they are induced primarily by the odd degree zonals. An explanation of this result can be found e.g. in Balmino (1992).

The above solution (12) based on series expansions proves to be a very compact expression of the complete problem. As an example, the ''M-dailies'' are simply obtained from these expressions for the subscripts $m \neq 0$, $m' = 0$. Besides, the solution is presented in a form analogous to Kaula's solution form, which facilitates the interpretation. Finally, the position perturbations $(r_1^*, \varphi_1^*, \lambda_1^*)$ given by the solution are directly expressed in the local orbital frame used in satellite geodesy, the reference plane OXY containing the satellite mean orbital plane. Moreover, literal expressions of the velocities $(\dot{r}_1^*, \dot{\varphi}_1^*, \dot{\lambda}_1^*)$ are easily obtained by taking derivatives of the coefficients $\mathscr{A}$ and $\mathscr{B}$ with respect to time.

5. Validation from TOPEX/Poseidon orbit

The aim of this section is to show the validity and the interest in space geodesy of the analytical solution of perturbed circular motion presented above. To this end, this application to the T/P orbit consists in comparing the analytical solution with numerical results obtained in the framework of the altimetric mission. However, we have chosen to map the mean orbit differences coming respectively from the analytical and numerical solutions.

Concerning the numerical solutions, orbit differences have been performed using CNES precise orbit ephemeris (POE). Indeed, the precise orbit determination service at CNES in Toulouse is in charge of the T/P precise orbit computation from different sources of tracking observations using ZOOM software. It provided, notably in 1995, two kinds of orbit: the JGM-2 and JGM-3 orbits (Nouel et al. 1994). This double set of orbits has constituted the background of our validation. From these purely numerical results, the aim is to properly separate orbit differences induced by the potential coefficient differences and other sources of orbit differences of variable nature in time and space. To this end, the following procedure is applied within a period of time of 100 days, corresponding to the 10-day cycles 96 to 105. For each cycle, orbit differences computed every minute are projected onto the T/P ground tracks (ascending and descending) resulting in 5×5 deg grid after a 2D interpolation. Finally, by averaging of the 10 cycles used, results in the radial, along-track (Figures 2 and 3, respectively), and across-track components are obtained on the basis of a 5×5 deg mean grid in order to highlight the large-scale features. We have to underline that this period of time has been chosen because no more significant information appears on the mean grid for a longer period.

On the other hand, analytical orbit differences have been determined using potential coefficient differences between the JGM-2 and JGM-3 models given to the degree and order 70 (Nerem et al. 1994; Tapley et al. 1994). These differences have been introduced in (12) in order to determine the satellite coordinate differences

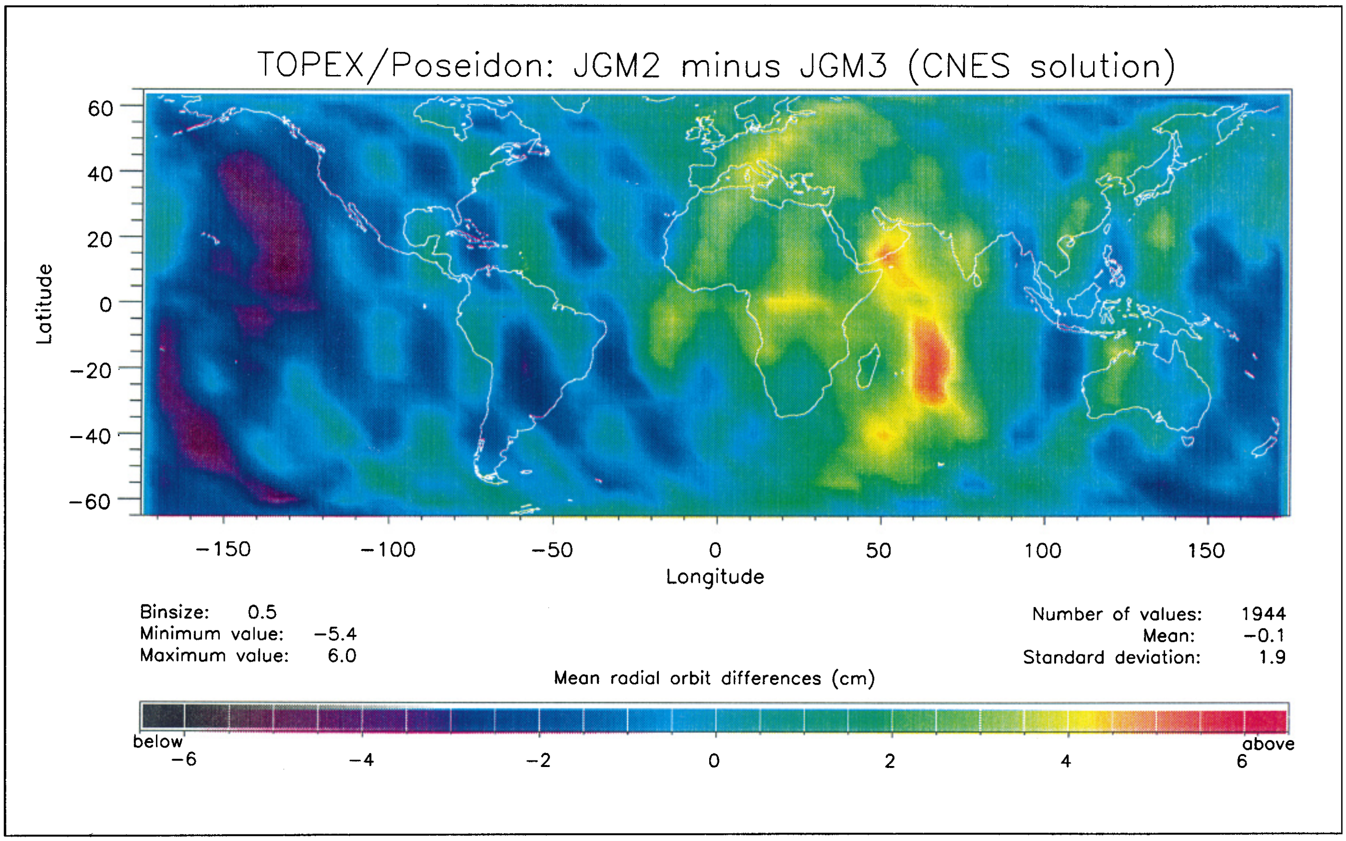

Fig. 2. Mean values of orbit differences (radial component) in geographical coordinates computed from two numerical POE solutions of TOPEX/Poseidon orbit over a period of 10 cycles: JGM-2 orbit minus JGM-3 orbit

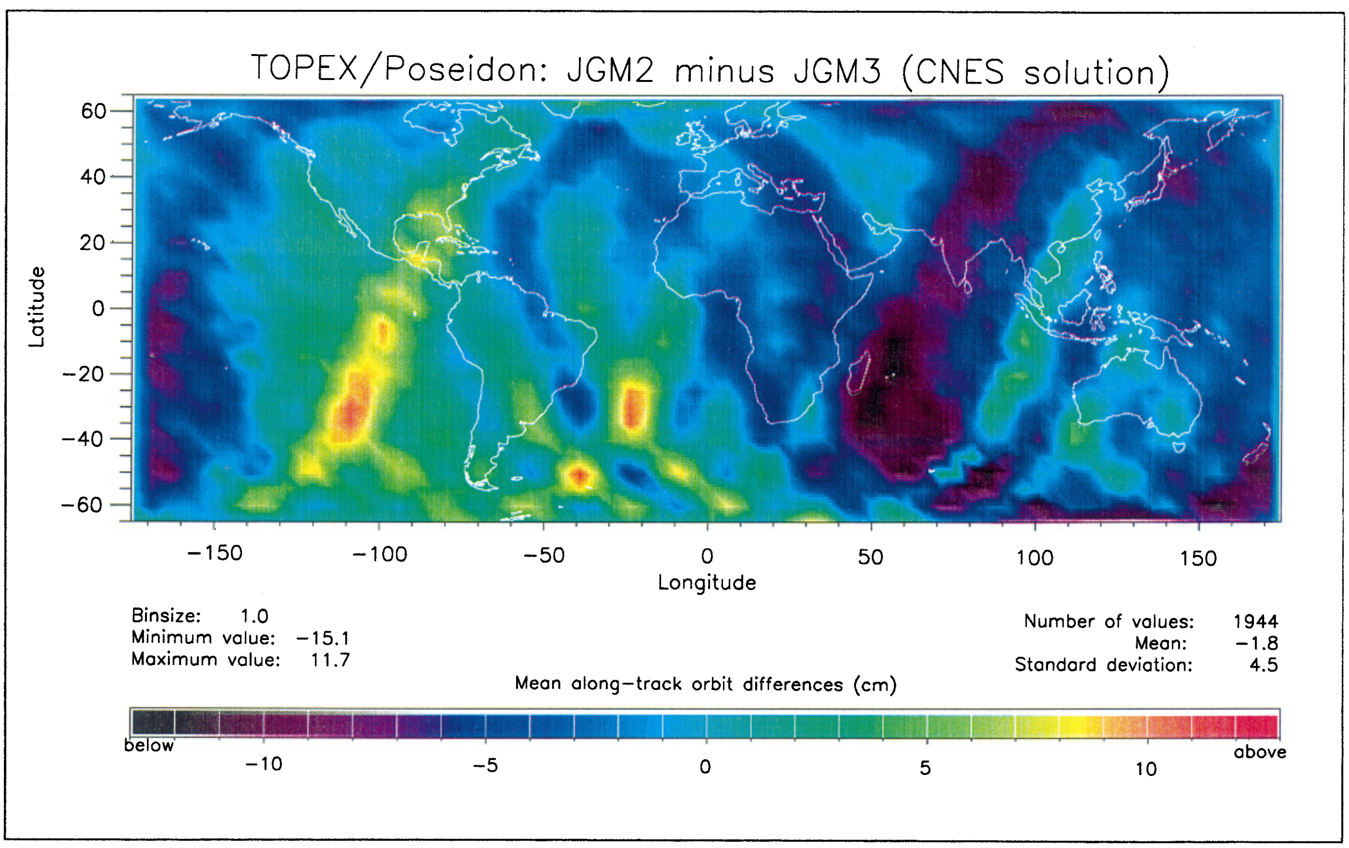

Fig. 3. Mean values of orbit differences (along-track component) in geographical coordinates computed from two numerical POE solutions of TOPEX/Poseidon orbit over a period of 10 cycles: JGM-2 orbit minus JGM-3 orbit

δr_1^*, $\delta\varphi_1^*$, and $\delta\lambda_1^*$. Then, by considering the sub-set of mean orbital elements of the T/P orbit ($r_0 = 7716682.0$ m, $I = 66.040^o$, $\Omega_0 = 7.404^o$, MJD $= 49830$), the analytical solution has been evaluated along a 10-day orbital arc and values of δr_1^*, $\delta\varphi_1^*$, and $\delta\lambda_1^*$ have been projected onto the map of the Earth every minute of time. Figures 4 and 5 show maps of the predicted radial (δr_1^*) and along-track ($\delta\lambda_1^*$) orbit differences (for both ascending and descending tracks) obtained by this method.

Now, these results have to be compared to orbit differences based on the numerical integration in order to validate expressions (12) of the analytical solution. The maps (Figures 2, 4 on the one hand, and 3, 5 on the other hand) are in good agreement permitting to validate the analytical solution, particularly the expressions of periodic series. The comparison concerning the across-track component ($\delta\varphi_1^*$) gives also a very good agreement, at the same level. However, the lack of information at the 1/rev (cycle per revolution) and 1/rev $\pm$ 1/day frequencies in the numerical solutions is obvious, due to the fit of empirical coefficients in the dynamics (see e.g. (Marshall et al. 1995)). As a consequence, the terms associated to these frequencies have been removed in the analytical solution, and thus have not been checked. Moreover, the radial orbit differences given by these two methods, numerical and analytical, have been compared to a third source, as an ultimate control: the T/P radial orbit differences derived by Haines et al. (1995) from two other dynamical solutions. Results are still in good agreement. For example, a comparison between the analytical and numerical grids in the case of the radial component gives differences with a mean value of 0.1 cm and a standard deviation of 1.0 cm, in a global sense. It is quite acceptable in the context of this application.

In order to give another example of the analytical computation of radial orbit differences the ERS-1 satellite orbit has been considered also. First, we started from gravity model differences between JGM-2 and JGM-3 as in the case of T/P (see above). Using the same conditions for both orbits should permit simply the effects of the altitude difference to be emphasized. However, the rms of the radial orbit differences we have obtained (31.5 cm) seems to be rather high and thus non realistic compared to the radial accuracy recently achieved on ERS-1 (about one decimeter). In fact, the quasi-systematic employment of empirical and/or stochastic accelerations adjustments in current precise orbit determination methods can absorb signals arising from geophysical processes. This idea is surely reinforced in the case of ERS-1. Accordingly, terms close to the once-per-orbital-revolution frequency have to be removed from analytical schemes as well as terms of periods in excess of few days as such long-periodic errors are absorbed to a large extent within numerical

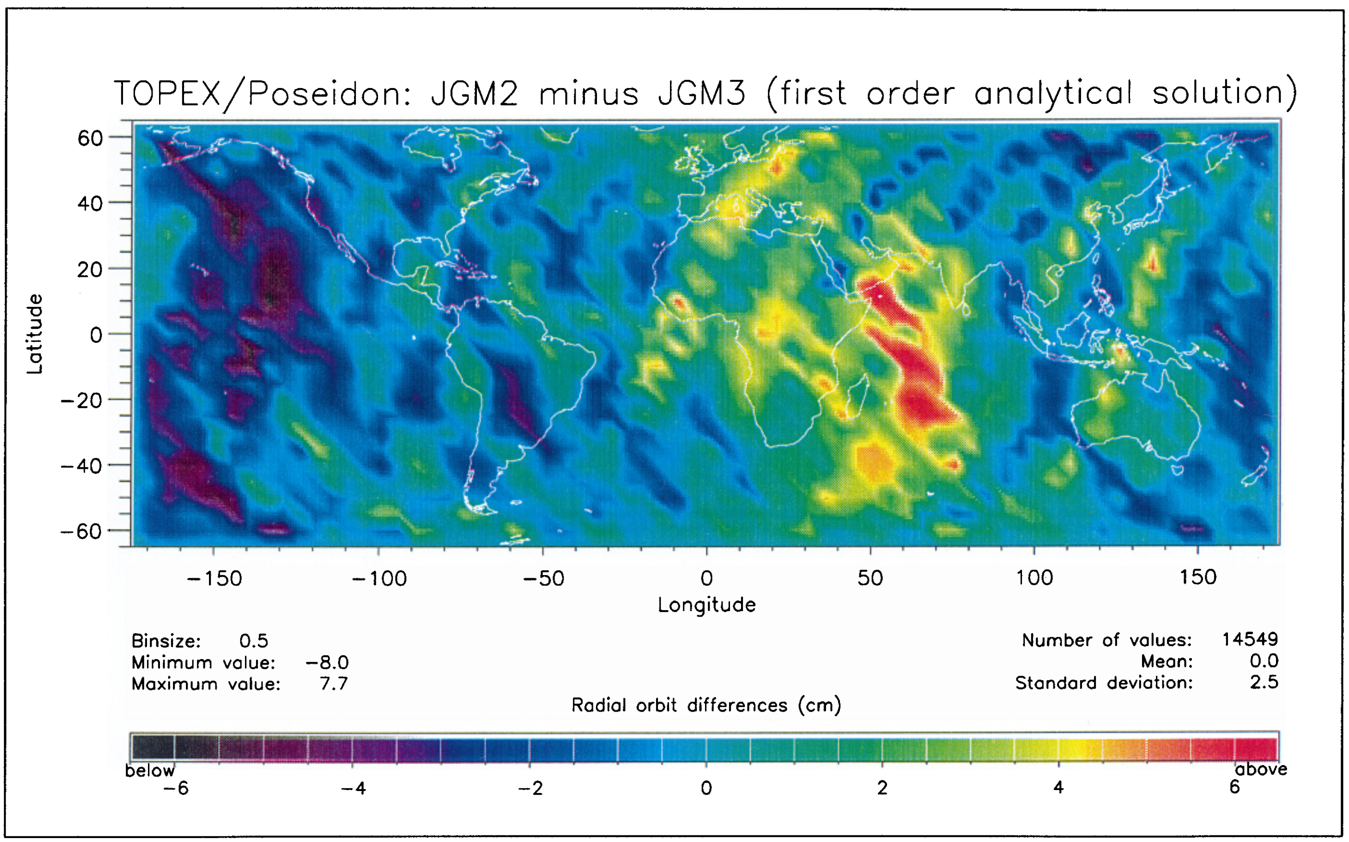

Fig. 4. Application of the first-order analytical solution of perturbed circular motion in spherical coordinates to the TOPEX/Poseidon orbit. Predicted radial orbit differences computed from the potential differences of JGM-2 minus JGM-3

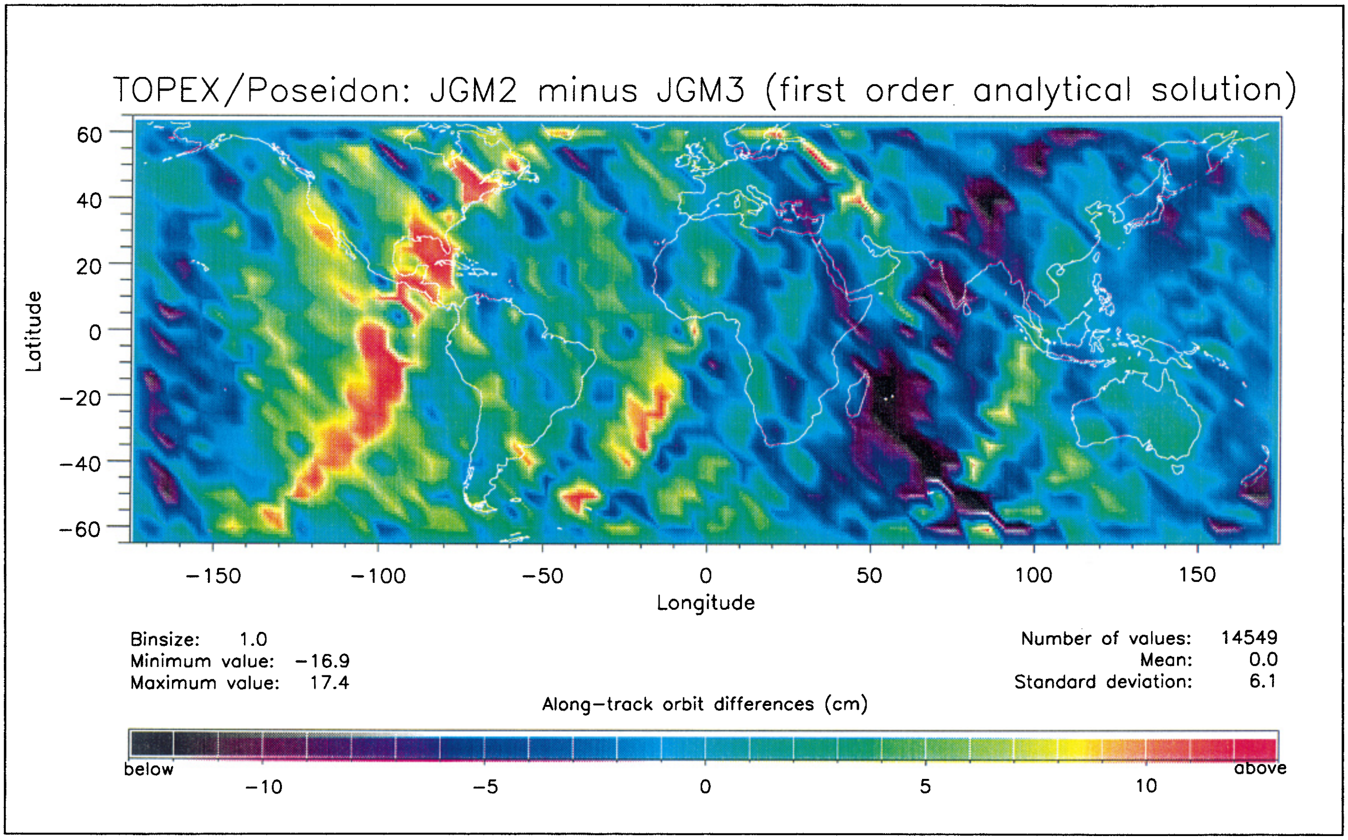

Fig. 5. Application of the first-order analytical solution of perturbed circular motion in spherical coordinates to the TOPEX/Poseidon orbit. Predicted along-track orbit differences computed from the potential differences of JGM-2 minus JGM-3

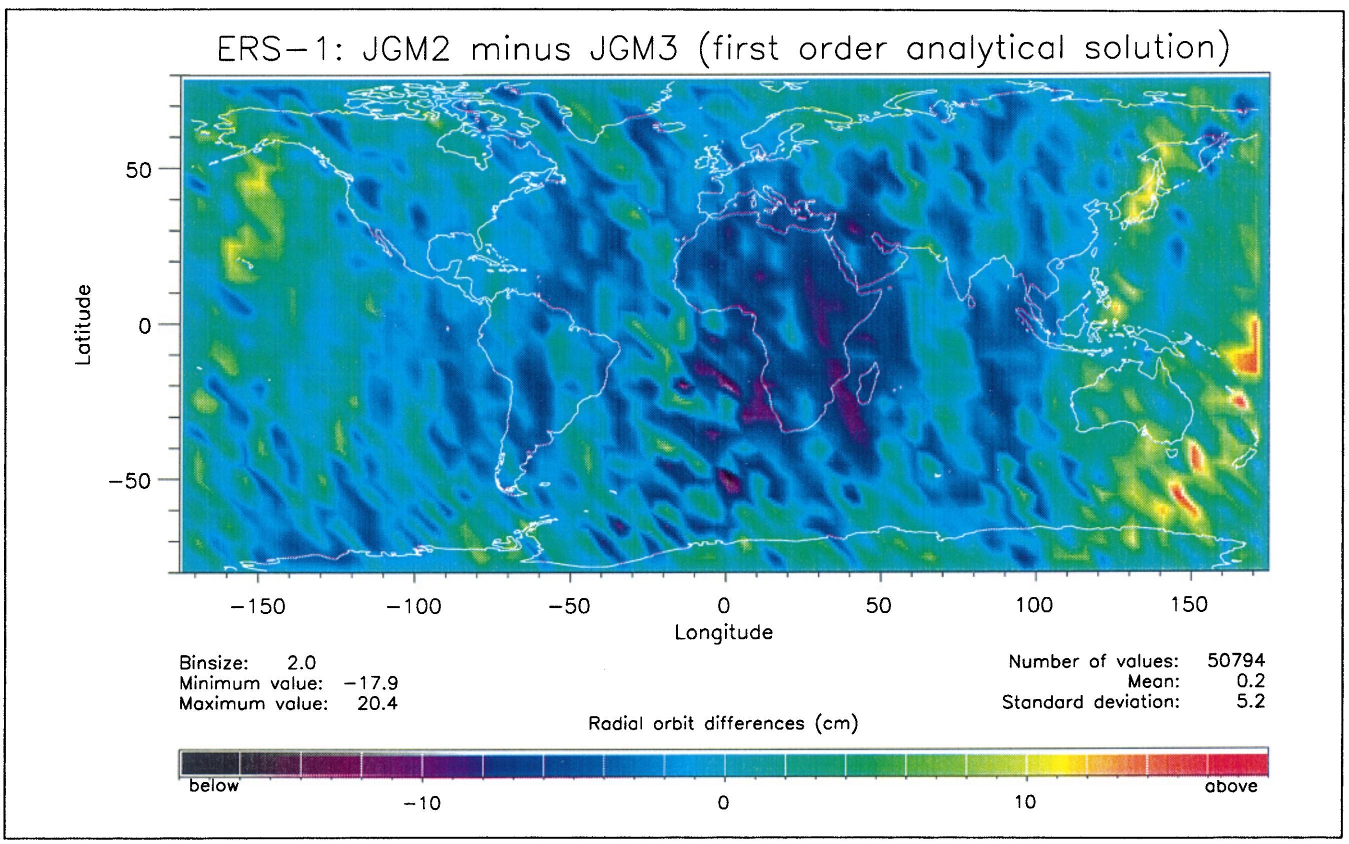

Fig. 6. Application of the first-order analytical solution of perturbed circular motion in spherical coordinates to the ERS-1 orbit. Predicted radial orbit differences computed from the potential differences of JGM-2 minus JGM-3 restricted to degree and order fifty

procedures. Thus, the geographically correlated radial orbit differences showed in Figure 6 have been derived with the JGM-2 and JGM-3 gravity fields restricted to degree and order fifty. Their rms is now at the level of 5.2 cm allowing to note the relatively large uncertainties in high degree and order geopotential coefficients at least at the altitude and inclination of ERS-1.

Secondly, gravity model differences that one gets between JGM-3 and GRIM4-C4 have been used also to derive geographically correlated radial orbit differences on ERS-1. An rms of 8.4 cm is obtained with both gravity fields restricted to degree and order fifty. Now, this last value can be compared to the ERS-1 radial orbit error – about 4 cm – propagated from the covariance matrix of the new GRIM4-C4 model (Schwintzer et al. 1996). As expected, geopotential coefficient differences form certainly an estimate of errors in the knowledge of the Earth's gravity field, but it is not absolute. Thus, satellite coordinate differences in geographical coordinates based on such coefficient differences allow to evaluate the order of magnitude of the so-called geographically correlated orbit errors. Figures 4 and 6 constitute such indicators. A more detailed discussion on realizations of orbit errors based upon gravity model differences can be found in (Schrama 1992).

6. Conclusion

The general goal and innovation of this work is the establishment of a purely analytical theory of orbital motions notably valid for $e = 0$ and $I = 0$ or π allowing easily very quick computation. It is based on a direct integration of the differential equations of motion written in spherical coordinates. Besides, by introduction of an uniformly rotating plane of constant inclination as a new reference plane of the theory, we produce new analytical relations representing the position perturbations due to the complete geopotential on a satellite orbit. In addition, expressions can be used to derive directly the coordinate perturbations in the local orbital frame usually used in satellite geodesy. The present extension improves thus the generality, the field of validity, the precision, and as a consequence the possibilities of applications of the solution.

The validation of the first-order solution is performed in the case of the TOPEX/Poseidon orbit. It shows its capability to predict orbital differences computed from potential coefficient differences of two recent Earth gravity field models: JGM-2 and JGM-3. Maps of the radial and along-track orbit differences in geographical coordinates obtained by the theory have been compared with corresponding numerical results (precise CNES orbit differences). These show a very good agreement, at the level of 1.0 cm. That confirms the interest of such a satellite theory whose equations have been treated in geocentric coordinate variables permitting to describe circular motions at any orbital inclination.

In addition, ERS-1 radial orbit differences due to geopotential coefficient differences (JGM-2 minus JGM-3, and JGM-3 minus GRIM4-C4) have been mapped in geographical coordinates. The first-order analytical solution applied to the geopotential coefficient differences between JGM-2 and JGM-3 provide an estimate of radial errors at the 5 cm level with both gravity fields restricted to degree and order fifty. On the other hand, the same estimate of the ERS-1 radial orbit errors from differences between JGM-3 and GRIM4-C4 is at the 8 cm level. These similar results prove the analytical solution to be an efficient tool for representing an estimate of orbit errors.

Acknowledgement. We want to thank S. Houry, from the Service d'Orbitographie DORIS (CNES) directed by F. Nouel, for the numerical computations of T/P orbits. Unknown referees have to be thanked for numerous suggestions concerning the paper, and R. Feldman is acknowledged for improving the English readability of the final document.

References

Balmino, G. (1992) Orbit Choice and the Theory of Radial Orbit Error for Altimetry, International Summer School of Theoretical Geodesy, "Satellite Altimetry in Geodesy and Oceanography", Lecture Notes, Trieste, Italy, May 25-June 6

Balmino, G., E. Schrama and N. Sneeuw (1996) Compatibility of first-order circular orbit perturbations theories; consequences for cross-track inclination functions, Journal of Geodesy, 70: 554–561

Bois, E. (1994) First-order Accurate Theory of Perturbed Circular Motion, Celestial Mechanics, 58: 125–138

Borderies, N. (1978) Mutual Gravitational Potential of N-Bodies, Celestial Mechanics 18: 295–307

Breiter, S. and E. Bois (1994) First-Order Theory of Weakly Eccentric Orbital Motion, Celestial Mechanics, 60: 191–206

Brouwer, D. (1959) Solution of the Problem of Artificial Satellite Theory without Drag, Astron. J., 64: 378–397

Brouwer, D. and G.M. Clemence (1961) Methods of Celestial Mechanics, Academic Press: New-York and London

Casotto, S. (1993) Position and Velocity Perturbations in the Orbital Frame in Terms of Classical Elements Perturbations, Celestial Mechanics, 55: 209–221

Exertier, P. and E. Bois (1995) Analytical Solution of the Perturbed Circular Motion: an Extended Formulation for Various Perturbations, Planet. Space Sci., 43: 863–874

Haines, B.J., E.J. Christensen, J.R. Guinn, R.A. Norman and A. Marshall (1995) Observations of TOPEX/Poseidon Orbit Errors Due to Gravitationnal and Tidal Modeling Errors Using the Global Positioning System, IUGG General Assembly, Jul 3–15, Boulder, CO

Kaula, W. (1966) Theory of Satellite Geodesy, Blaisdell, Waltham, Mass

Kovalevsky, J. (1963) Introduction à la Mécanique Céleste, Armand Colin

Marshall, J.A., S.M. Klosko, J.C. Ries (1995) Dynamics of SLR tracked satellites, Reviews of Geophysics, Supplement: 353–360

Melvin, P.J. (1987) Satellite Geodesy from Orlov's Plane, Advances in the Astronautical Sciences, 65: 1449–1472

Nerem, R.S., et al. (1994) Gravity model development for TOPEX/ Poseidon: Joint Gravity Models 1 and 2, J. Geophys. Res., 99(C12): 24421–24447

Nouel, F., et al. (1994) Precise Centre National dEtudes Spatiales orbits for TOPEX/ Poseidon: Is reaching 2cm still a challenge?, J. Geophys. Res., 99(C12): 24405–24419

Rosborough, G.W. (1986) Satellite Orbit Perturbations due to the Geopotential, Report No. CSR-86-1, Center for Space Research, The University of Texas at Austin, USA

Rosborough, G.W. and B.D. Tapley (1987) Radial, Transverse and Normal Satellite Position Perturbations due to the Geopotential, Celestial Mechanics, 40: 409–421

Sneeuw, N.J. (1992) Representation Coefficients and their Use in Satellite Geodesy, Manuscripta Geodaetica, 17: 117–123

Schrama, E.J.O. (1989) The Role of Orbit Errors in Processing of Satellite Altimeter Data, Netherlands Geodetic Commission, report 33

Schrama, E.J.O. (1992) Some remarks on several definitions of geographically correlated orbit errors: consequences for satellite altimetry, Manuscripta Geodaetica, 17: 282–294

Tapley, B.D., et al. (1994) The JGM-3 gravity model, Annales Geophysicae, 12, Suppl. 1, C192