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Exploring gravity field determination from orbit perturbations of the European Gravity Mission GOCE

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Abstract. A comparison was made between two methods for gravity field recovery from orbit perturbations that can be derived from global positioning system satellite-to-satellite tracking observations of the future European gravity field mission GOCE (Gravity Field and Steady-State Ocean Circulation Explorer). The first method is based on the analytical linear orbit perturbation theory that leads under certain conditions to a block-diagonal normal matrix for the gravity unknowns, significantly reducing the required computation time. The second method makes use of numerical integration to derive the observation equations, leading to a full set of normal equations requiring powerful computer facilities. Simulations were carried out for gravity field recovery experiments up to spherical harmonic degree and order 80 from 10 days of observation. It was found that the first method leads to large approximation errors as soon as the maximum degree surpasses the first resonance orders and great care has to be taken with modeling resonance orbit perturbations, thereby loosing the block-diagonal structure. The second method proved to be successful, provided a proper division of the data period into orbital arcs that are not too long.

Key words: GOCE – GPS – Gravity field determination – Linear perturbation theory – Numerical integration – Orbit perturbations

1 Introduction

The Gravity Field and Steady-State Ocean Circulation Explorer (GOCE) will be the first satellite to fly in the framework of the European Space Agency (ESA) Earth Explorer Mission program (ESA 1999). The projected launch is in 2004 and the mission duration will be 20 months, consisting of a 2-month commissioning phase and two 6-month measurement collection periods with a 5-month hibernation period in between. GOCE will fly in either a dawn–dusk (launch in winter) or dusk–dawn (launch in summer) sun-synchronous orbit at a mean altitude of 250 km and an inclination of 96.5°. The orbit will be near-circular with the eccentricity always smaller than 0.0045. GOCE will be equipped with a drag-free control (DFC) system eliminating the larger part of non-conservative forces and allowing GOCE to fly a (near) repeat orbit. The repeat period will be 2 months or longer in order to guarantee a sufficiently dense pattern of ground tracks for high-resolution gravity field sampling. The objective of the mission is the determination of a model of the static gravity field with an accuracy of better than 1–2 mGal and 1-cm accuracy in terms of gravity anomalies and geoid heights, respectively, at a resolution of 100 km. GOCE will be equipped with a high-quality, dual-frequency global positioning system (GPS)/(GLONASS) Global Navigation Satellite System receiver and a gravity gradient gradiometer providing satellite-to- satellite tracking (SST) observations between GOCE and the GPS satellites and observations of the full gravity gradient tensor (SGG, satellite gravity gradient) at satellite altitude, respectively.

The GPS/GLONASS receiver plays a dual role in that it both enables a high-precision orbit determination and long-to-medium-wavelength gravity field recovery and supports the gradiometer to very accurately geolocate the SGG observations in an Earth-fixed reference frame. The capability of precise orbit determination (POD) has been assessed in detail by Visser and van den IJssel (2000). Gravity field recovery up to high degree and order is a demanding task from a computational point of view. With a total mission duration of 12 months and a rich observation environment, which is the case with tracking by GPS of GOCE, the total amount of observations is of the order of tens of millions (the foreseen GPS observation time interval is 10 s or less). The GOCE orbit is sensitive to gravity field terms of degree 80 and even higher (SID 2000; Visser and van

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den IJssel 2000), leading to a total number of gravity unknowns of the order of 6500 or more. For SGG observations the maximum resolvable degree lies around 300, leading to about 90,000 unknowns. An efficient algorithm, referred to as iterative block-diagonal gravity field estimation, was implemented to estimate this amount of unknowns from SGG observations based on the peculiar observation geometry that exists if GOCE flies a repeat orbit (Klees et al. 2000). This algorithm results in significant reductions in both the computer time and memory required. The subject of the present paper is assessment of a similar algorithm for gravity field recovery from orbit perturbations that can be observed by SST of GOCE based on the linear perturbation theory (LPT) (Kaula 1966; Rosborough 1987; Sect. 2.1). If such an algorithm proves to be feasible, this will allow a simple integration with the method outlined in Klees et al. (2000) to obtain combined SST/SGG gravity field solutions or GOCE.

In addition, a method based on numerical integration was investigated (Sect. 2.2). The latter method is more demanding from a computational point of view. For example, 35 minutes of CPU time were required on a CRAY-J90 computer for one iteration using the block-diagonal method for a gravity field recovery complete to degree and order 80 for a 10-day data period, compared to 50 hours for the approach using numerical integration, i.e. a reduction by a factor of 85. For higher truncation degrees, this reduction will be larger. The memory requirement for the first approach was around 1 Mb compared to 175 Mb for the second approach, i.e. a reduction by a factor of 175. This reduction will also be larger for higher truncation degrees. In fact, it can be shown that this reduction is equal to six times the truncation degree.

Gravity field recovery experiments were conducted with both methods (Sect. 3) and evaluated (Sect. 4).

2 Methodology

The methods were based on gravity field parameter adjustment from orbit perturbations that can be observed using GPS. The strong geometry of the GPS system allows a high-precision orbit determination in which dynamic model errors, predominantly in the a priori gravity field model, can effectively be minimized to the few-centimetre level by, for example, a kinematic or reduced dynamic orbit determination technique (SID 2000; Visser and van den IJssel 2000). In other words, after a proper processing of the GPS observations, the GOCE orbit may be considered to be known at this accuracy level. In the SGG data analysis process, this accuracy is high enough to regard the orbit as a prior known quantity, allowing us not to include unknown position parameters in the estimation procedure (ESA 1999). The GOCE orbit solution can be used as pseudo observations, e.g. Cartesian x, y, and z coordinates in an Earth-centered pseudo-inertial reference frame, for a gravity field model adjustment.

In this adjustment a simulated state-of-the-art a priori gravity field model that differs from the simulated true gravity field model is used to compute a dynamic orbit that fits best with these observations. In this orbit computation no attempt is made to reduce the effect of gravity field model errors and the total effect of these errors will be included in the dynamic orbit. Thus, in this case, the differences between the pseudo observations and the dynamic orbit reflect the differences between the a priori and true gravity field models. The differences can then be used to adjust the coefficients of this a priori model.

Two methods were investigated for recovering the gravity field information from the orbit differences, an analytical method leading to a sparse, block-diagonal normal matrix for the gravity field unknowns (Sect. 2.1) and a method based on numerical integration leading to a full normal matrix (Sect. 2.2).

2.1 Analytical block-diagonal approach

It was investigated whether a block-diagonal matrix approach similar to a successful method implemented for gravity field recovery from SGG observations outlined in Klees et al. (2000) can be used for orbit differences. In this case, the orbit differences were differences between the Cartesian x, y, and z coordinates of the true GOCE orbit (simulated with the true gravity field model) and a dynamic orbit computed with an a priori gravity field model. An important aspect of this technique is the computation of the partial derivatives of the coordinate differences $\Delta x, \Delta y$, and Δz to the gravity field parameters. In a first step, these differences are transformed to perturbations in the radial, along-track and cross-track directions. The partial derivatives can then be computed using an analytical orbit perturbation theory, in the following referred to as LPT (Kaula 1966; Rosborough 1987; Schrama 1989; Visser 1992; Visser et al. 1994). With the partial derivatives, linear observation equations can be derived linking the coordinate differences to the gravity field unknowns (Kaula 1966; Schrama 1989; Visser 1992)

$$\Delta r = a \sum_{l=2}^{l_{\max}} \sum_{m=0}^{l} \sum_{p=0}^{l} \left(\frac{a_e}{a}\right)^l \times F_{lmp}\left[\frac{2(l-2p)}{f_{lmp}} + \frac{4p-3l-1}{2(f_{lmp}+1)} + \frac{4p-l+1}{2(f_{lmp}-1)}\right] S_{lmp} \tag{1}$$

$$\Delta \tau = a \sum_{l=2}^{l_{\max}} \sum_{m=0}^{l} \sum_{p=0}^{l} \left(\frac{a_e}{a}\right)^l F_{lmp}\left[\frac{2(l+1) - 3(l-2p)\frac{1}{f_{lmp}}}{f_{lmp}} + \frac{4p-3l-1}{f_{lmp}+1} + \frac{l-4p-1}{f_{lmp}-1}\right] S^*_{lmp} \tag{2}$$

$$\Delta c = \frac{1}{2} a \sum_{l=2}^{l_{\max}} \sum_{m=0}^{l} \sum_{p=0}^{l} \left(\frac{a_e}{a}\right)^l \frac{1}{f_{lmp}} \times \left[\left(\frac{F_{lmp}}{\sin i}((l-2p)\cos i - m) - F'_{lmp} \right) S^*_{(l+1)mp} - \left(\frac{F_{lmp}}{\sin i}((l-2p)\cos i - m) + F'_{lmp} \right) S^*_{(l-1)mp} \right] \quad (3)$$

$$f_{lmp} = l - 2p - m \frac{n_{\text{day}}}{n_{\text{rev}}} \quad (4)$$

$$S_{lmp}(\omega + M, \Omega - \theta) = \begin{bmatrix} \Delta\bar{C}_{lm} \\ -\Delta\bar{S}_{lm} \end{bmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} \cos((l-2p)(\omega + M) + m(\Omega - \theta)) + \begin{bmatrix} \Delta\bar{S}_{lm} \\ \Delta\bar{C}_{lm} \end{bmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} \sin((l-2p)(\omega + M) + m(\Omega - \theta)) \quad (5)$$

$$S^*_{lmp}(\omega + M, \Omega - \theta) = \begin{bmatrix} \Delta\bar{C}_{lm} \\ -\Delta\bar{S}_{lm} \end{bmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} \sin((l-2p)(\omega + M) + m(\Omega - \theta)) - \begin{bmatrix} \Delta\bar{S}_{lm} \\ \Delta\bar{C}_{lm} \end{bmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} \cos((l-2p)(\omega + M) + m(\Omega - \theta)) \quad (6)$$

where r, τ, and c denote the orbit perturbations in the radial, along-track, and cross-track directions, respectively. Use has been made of the Kepler elements: the orbital semi-major axis a, eccentricity e (assumed zero for GOCE orbit), argument of perigee ω, inclination i right ascension of ascending node Ω, and the mean anomaly M. The Greenwich hour angle is denoted by θ, while F_{lmp} is a function depending on the orbital inclination i only. Furthermore, F'_{lmp} is the derivative of F_{lmp} with respect to the inclination i. It is assumed that the satellite flies in a repeat orbit with a duration of n_{day} nodal days in which n_{rev} orbital revolutions are completed (n_{day} and n_{rev} are relative primes). The gravity field unknowns are represented by $\Delta\bar{S}_{lm}$ and $\Delta\bar{C}_{lm}$, where l and m denote the degree and order. The gravity field is represented by a spherical harmonic model complete to degree and order $l_{\max}$.

For an exact repeat orbit, it can be shown that these equations become Fourier series with the basic frequency equal to $1/n_{\text{rev}}$ cycles per orbital revolution (CPR) (Colombo 1984). The method of least squares (LS) is used to solve these equations. With the above equations, it can be shown that the normal equations become block-diagonal when organized per order m, provided there are a continuous data stream of orbit differences and a data period equal to, or an integer multiple of, the repeat period of the GOCE orbit. The normal matrix elements are equal to zero outside these blocks on the diagonal.

It should be noted that Eqs. (1)–(6) represent the non-resonant particular solution of the linearized equations of motion of a satellite with respect to a reference circular orbit. Not included are the homogeneous solution and resonance effects caused by errors in the initial condition, i.e. the initial position and velocity of the satellite, and those caused by the zonal coefficients of the gravity field. The homogeneous solution and resonance effects can be represented by (Schrama 1991)

$$\Delta r = a_{0,r} + a_{1,r}\omega_0 t + a_{2,r}\cos\omega_0 t + b_{2,r}\sin\omega_0 t + a_{3,r}\omega_0 t\cos\omega_0 t + b_{3,r}\omega_0 t\sin\omega_0 t \quad (7)$$

$$\Delta\tau = a_{0,\tau} + a_{1,\tau}\omega_0 t + a_{2,\tau}\cos\omega_0 t + b_{2,\tau}\sin\omega_0 t + a_{3,\tau}\omega_0 t\cos\omega_0 t + b_{3,\tau}\omega_0 t\sin\omega_0 t + a_{4,\tau}(\omega_0 t)^2 \quad (8)$$

$$\Delta c = a_{0,c} + a_{2,c}\cos\omega_0 t + b_{2,c}\sin\omega_0 t + a_{3,c}\omega_0 t\cos\omega_0 t + b_{3,c}\omega_0 t\sin\omega_0 t \quad (9)$$

where ω_0 represents the orbital angular velocity of the satellite, t represents time and $a_{j,k}, b_{j,k} (j = 0 \ldots 4, k = r, \tau, c)$ are constant amplitudes. These effects have to be eliminated or accounted for when using the orbit differences in the gravity field estimation. In the block-diagonal approach, this is done by separately estimating the amplitudes and subtracting the solution from the orbit differences (Sect. 3). If this cannot be done separately, the normal matrix looses its block-diagonal structure. The analytical block-diagonal approach is thus based on the assumption that the correlation between the amplitudes of Eqs. (7)–(9) and the gravity field unknowns is at such a low level that separate estimation is allowed, otherwise the normal matrix will also be filled outside the blocks on the diagonal. More attention will be paid to this aspect in Sect. 3.

The coordinate differences are assumed to be produced at a constant time interval, making it possible to perform a discrete fast Fourier transform (FFT) on the data. As mentioned previously, the orbit perturbations, correlated for the homogeneous solution and resonance effects, can be represented by a Fourier series with basis frequency $1/n_{\text{rev}}$ (CPR) if the satellite flies in a repeat orbit. In that case, the orbit perturbations can be written as

$$\Delta p = \frac{a_o}{2} + \sum_{j=1}^{n_{fr}} \left(a_{k,j}\cos\left(\frac{j\omega_0 t}{n_{\text{rev}}}\right) + b_{k,j}\sin\left(\frac{j\omega_0 t}{n_{\text{rev}}}\right) \right) \quad (10)$$

where $\{p \in \{r, \tau, c\} \mid k = 1, 2, 3\}, a_{k,j}$ and $b_{k,j}$ are the amplitudes of the perturbations with frequency j/n_{rev} (CPR), and n_{fr} indicates the maximum number of frequencies, which is limited by the Nyquist frequency imposed by the data time interval Δt.

If a complete repeat period of n_{rep} orbit perturbations at time t_1 with constant time interval is available, the amplitudes $a_{k,j}$ and $b_{k,j}$ can be obtained by

$$a_{k,j} = \frac{2}{n_{\text{rep}}} \sum_{l=1}^{n_{\text{rep}}} \Delta p(t_l) \cos\left(\frac{j\omega_0(l-1)\Delta t}{n_{\text{rev}}}\right) \tag{11}$$

$$b_{k,j} = \frac{2}{n_{\text{rep}}} \sum_{l=1}^{n_{\text{rep}}} \Delta p(t_l) \sin\left(\frac{j\omega_0(l-1)\Delta t}{n_{\text{rev}}}\right) \tag{12}$$

The amplitudes $a_{k,j}$ and $b_{k,j}$ can be connected to the gravity field unknowns $\Delta\bar{C}_{lm}$ and $\Delta\bar{S}_{l,m}$ by means of the LPT [Eqs. (1)–(6)].

As stated before, the observation equations are solved by the LS method, leading to a block-diagonal matrix. This is the case when the matrix is organized per order m (Colombo 1984). It can also be shown that even and odd parities of $l-m$ become uncorrelated. The largest dimension of the blocks is thus equal to half the maximum degree of the gravity field harmonic expansion that is to be estimated. The solution of the normal equations transforms to solving block matrices with dimensions much smaller than the dimension of the full matrix. For example, for a gravity field recovery up to degree and order 80 the largest submatrix that has to be inverted has a dimension of 40, compares to about 6550 for the entire matrix. This prevents the necessity of long and costly computer runs and facilities. An additional advantage is that only a small part of the total normal matrix (only the blocks on the diagonal) has to be stored.

In summary, the analytical technique can be written in the following steps.

(1) High-precision (kinematic or reduced-dynamic) orbit determination from GPS SST measurements for a repeat period resulting in x, y, z coordinates that match the true orbit as closely as possible.
(2) Dynamic orbit determination from these coordinates and computation of residuals with an a priori gravity field model.
(3) Transformation of coordinate differences between the orbits of steps 1 and 2 into residual radial, along-track, and cross-track orbit perturbations.
(4) Estimation of homogeneous solution and resonance effects and elimination from the orbit perturbations.
(5) Discrete FFT of the remaining perturbations.
(6) Application of LPT to compute the normal equations, taking into account the block-diagonal structure of the normal matrix.

It is obvious that several approximations are introduced by using this method. First, the observation equations are obtained by a linearization of the equations of motion along a circular orbit. Such a linearization leads in general to a less accurate representation of (near-) resonance orbit perturbations (Visser 1992). In addition, the orbit will not be exactly circular and will have a small eccentricity e leading to higher-order errors $O(e)$. The deviation from circularity may be several or even tens of kilometers (SID 2000). Second, it has been assumed that the homogeneous solution and resonant orbit perturbations can be treated separately from the particular solution. Due to the linearization, an iterative procedure has to be adopted and steps 1–6 should be repeated until convergence.

Suppose that the exact observation and, in conjunction, normal equations can be derived and are represented by

$$A_t\bar{x} = \bar{y}, \quad N_t\bar{x} = A_t^T A_t\bar{x} = A_t^T\bar{y} \tag{13}$$

and that the approximated equations using the LPT can be written as

$$A_a\bar{x} = \bar{y}, \quad N_a\bar{x} = A_a^T A_a\bar{x} = A_a^T\bar{y} \tag{14}$$

where A_t and A_a represent the matrices with exact and approximated partial derivatives, respectively, $\bar{x}$ is the vector of unknowns $\Delta\bar{C}_{lm}, \Delta\bar{S}_{lm}$, and $\bar{y}$ the vector of orbit differences. Then the model or approximation error $\Delta\bar{x}$ becomes

$$\Delta\bar{x} = (N_t^{-1}A_t - N_a^{-1}A_a)\bar{y} \tag{15}$$

As will be shown in Sect. 3, the matrix N_t will not be purely block-diagonal. In Klees et al. (2000) an iterative scheme was adopted to eliminate the approximation errors. Applying the same scheme results in a re-computation after each iteration of the right-hand side of the normal equations using the same A_a matrix, but with updated orbit differences. The left-hand side remains constant, i.e. the block-diagonal matrix N_a. After some manipulations (Klees et al. 2000), it can be shown that the iterative process only converges when the following criterion is met:

$$\rho(I - N_a^{-1}A_a^T A_t) < 1 \tag{16}$$

where ρ is the spectral radius and I the identity matrix. If A_a is a close approximation of A_t, the following formula will give a good indication for convergence:

$$\rho(I - [N_t]_{\text{bd}}^{-1}N_t) < 1 \tag{17}$$

where $[N_t]_{\text{bd}}$ is the block-diagonal part of the true normal matrix. Actual computations of the spectral radius are discussed in Sect. 3.2.

2.2 Full matrix approach with numerical integration

In a second method, the partial derivatives are obtained by a numerical integration of the variational equations that link the orbit differences to the gravity field unknowns (SID 2000). The resulting observation equations are again linearized equations and are solved by the method of LS. However, a fundamental difference with the analytical approach is that the equations of motion are not approximated. In addition, the observation equations are set up along the orbit computed with the a priori gravity field model instead of a circular reference orbit. A full set of normal equations is built up and a completely filled normal matrix is obtained. It must be noted that in this approach the data period can be split into orbital arcs and for each arc an initial position and velocity is estimated in addition to the gravity field unknowns. The normal equations for the different arcs are accumulated by the method of partitioning, properly taking into account the effect of the estimation of the initial conditions

$$N\bar{x} = \begin{bmatrix} N_{aa} & N_{ag} \\ N_{ga} & N_{gg} \end{bmatrix} \begin{pmatrix} \bar{x}_a \\ \bar{x}_g \end{pmatrix} = \bar{r} = \begin{pmatrix} \bar{r}_a \\ \bar{r}_g \end{pmatrix} \tag{18}$$

$$[N_{gg} - N_{ga}N_{aa}^{-1}N_{ag}]\bar{x}_g = \bar{r}_g - N_{ga}N_{aa}^{-1}\bar{r}_a \tag{19}$$

where N is the normal matrix, $\bar{x}$ the vector of unknowns and $\bar{r}$ the right-hand side. The subscript a denotes the initial conditions and g the gravity field unknowns. The final normal equations are obtained by summing Eq. (19) for all arcs. Because of the linearization of the observation equations, this method has to be iterated until convergence (as is also the case for the analytical method), despite the effectively exact computation of the partial derivatives by numerical integration. However, due to the large computation time required, the simulations were limited to one iteration.

3 Results

Simulations were carried out in which the true orbit of GOCE was computed with the OSU91A model (Rapp et al. 1991) truncated at several degrees. Thus different cases were investigated in which the true gravity field was simulated by the OSU91A model truncated at different degrees in order to assess estimability of the gravity field as a function of the truncation degree with the two approaches outlined above. It was attempted to compute an orbit that matches a repeat orbit as closely as possible. In all cases, the difference between the Earth-fixed positions at the start and end of the repeat period was equal to 3 km or less, which seems to be a realistic value for maintaining such an orbit. From the true orbits the pseudo observations were derived, which were then used to compute a dynamic reference orbit with an a priori model for which JGM-3 (Tapley et al. 1996) was used truncated at the same degree as OSU91A, resulting in coordinate differences $\Delta x, \Delta y$, and Δz caused by the gravity field model coefficient differences. It has to be noted that no errors were added to the coordinate differences, thus these differences are caused purely by gravity field coefficient differences. The a priori models were truncated at the same degree to prevent aliasing of higher-degree terms in the gravity recovery. Both the true and dynamic reference orbits were computed with the GEODYN software kindly provided by the NASA Goddard Space Flight Center (Rowlands et al. 1995). The equations of motion and variational equations are solved by an 11th-order numerical integration procedure. The orbit was output in the form of Earth-centered pseudo- inertial Cartesian x, y, and z coordinates with a time interval of 10 s. A 10-day repeat period was selected in September 2003, close to one of the preliminary projected operation periods of GOCE. In these 10 days GOCE will complete 161 orbital revolutions. Although it was stated above that the repeat period for GOCE will not be shorter than 2 months, a 10-day period was selected for testing and analyzing the two gravity field recovery approaches and for keeping the necessary computer time within limits.

3.1 Analytical block-diagonal approach

The analytical model was verified by comparing radial, along-track, and cross-track orbit differences obtained with numerical integration with those predicted by Eqs. (1)–(9) and the gravity field coefficient differences (Table 1). The approximation errors of the analytical model are almost negligible for a truncation at degree 10: a few centimeters at most, or 0.4–1.7% of the total signal. For a truncation at degree 50, the errors are at the 8–13% level of the total signal. In terms of energy, this is still less than $(13/100)^2 \times 100 = 1.7\%$. It was investigated whether these approximation errors are small enough to guarantee a successful iterative gravity field recovery. Initially only gravity field coefficient differences up to degree and order 10 were taken into account. Two different approaches were adopted. In the first approach the homogeneous solution and resonance effects [Eqs.(7)–(9)] were eliminated from the orbit perturbations. In the second approach this was not done, and the orbit perturbations were used directly in the computation of the normal equations. In this way the effect of errors in the initial conditions and of resonance effects on the analytical block-diagonal approach can be investigated.

As a quality measure of the gravity recovery, the global root mean square (RMS) of geoid differences between the true model and recovered model were used. The total geoid difference between JGM-3 and OSU91A up to degree and order 10 is equal to 11.93 cm. The accuracy of the recovery may depend on which (combination of) orbit difference component(s) is used. The results indicate that the homogeneous solution and resonance effects have to be taken into account: when using radial orbit differences the accuracy improves from 0.92 to 0.52 cm in terms of geoid error (Table 2). The improvement is from 19.02 to 1.79 cm when using

Table 1. Comparison of orbit differences obtained with numerical integration and with the analytical perturbation theory (LPT)

Truncation degree	Radial (cm)	Along-track (cm)	Cross-track (cm)
10[a]	71.60	274.59	109.27
10[b]	10.35	24.67	90.34
10[c]	1.21	2.60	0.46
30[a]	858.57	13794.88	739.87
30[b]	684.86	13076.60	612.86
30[c]	82.15	172.95	88.61
50[a]	1245.00	18341.01	1443.40
50[b]	825.88	17583.36	1001.22
50[c]	114.89	250.81	117.47

[a] Total signal: numerical integration
[b] After subtracting particular LPT solution [Eqs. (1)–(6)]
[c] Plus homogeneous/resonance solution [Eqs. (7)–(9)]

Table 2. Gravity field recovery from orbit perturbations using the block-diagonal approach: global RMS of geoid error (cm)

Maximum degree	Geoid signal	Orbit component			
		Radial	Along-track	Cross-track	Three directions
10	11.93	0.52	1.09	1.79	1.49
10[a]	11.93	0.92	1.46	19.02	5.78
10[b]	11.93	0.01	0.01	4.09	0.85
10[c]	11.93	0.01	0.01	0.02	0.04
13	14.38	0.54	1.42	13.01	3.07
13[a]	14.38	0.60	1.49	102.12	28.52
16	17.94	10.10	79.54	4.08	51.92
16[a]	17.94	12.31	11.26	123.60	32.03
30	37.15	52.07	5190.15	432.36	1587.57
30[a]	37.15	236.85	5268.51	834.27	1636.59

[a] Homogeneous solution/resonances not taken into account
[b] Second iteration
[c] Second iteration: zonals excluded

cross-track orbit differences. It can be seen that the approximation errors lead to different gravity field recovery errors when using orbit differences in different directions: the geoid error is equal to 0.52, 1.09, and 1.79 cm when using radial, along-track, and cross-track orbit differences, respectively. In addition, differences may be caused by the elimination of the homogeneous solution and resonance effects. By separately estimating the relevant amplitudes [Eqs. (7)–(9)], a part of the gravity signal represented by the particular solution of the LPT may be absorbed as well. Especially for the cross-track direction, the signal is reduced significantly by this estimation and elimination (Table 1).

Best results are obtained when using only the radial orbit differences. Adding normal equations based on along-track and cross-track orbit differences with the same weight leads to a larger gravity recovery error. It may thus be concluded that by using the particular solution of the LPT, which gives an approximation of the partial derivatives of the orbit differences to the gravity unknowns and excludes resonance terms that are eliminated separately, relatively large errors occur for the along-track and cross-track directions, although the recovery error is in all cases much smaller than the signal. The analytical block-diagonal approach is in principle an approach that should be iterated until convergence. Therefore, the four gravity field solutions of the first iteration, i.e. based on only the radial, only the along-track, only the cross-track orbit differences, and also the combination of these differences, were used in a second iteration. The remaining gravity field recovery error after the first iteration is almost entirely eliminated after this second iteration, except for the solution based on the cross-track orbit differences and the combined solution which included the cross-track orbit differences as well. It was found that the error is almost completely contained in the zonal coefficients, which are correlated most with the homogeneous solution and the resonance terms. By excluding the zonal coefficients, the recovery error becomes around 0.04 cm for the combined solution.

Based on these promising results, the truncation degree was increased to 30, unfortunately resulting in an unsuccessful gravity field recovery: the error in the recovered gravity field coefficients is for most coefficients larger than the coefficient differences between the a priori and true gravity field models. For example, the geoid error is equal to 52.07 cm compared to a signal of 37.15 cm when using radial orbit differences. The explanation for this failure may be the occurrence of the first orbit resonance order, which is at order 16, because GOCE completes about 16 orbit revolutions per day. As stated before, the LPT is less accurate for (near-) resonances. It may thus be concluded that for a low-flying satellite like GOCE, the approximation errors induced by the analytical model used in the block-diagonal approach become too large when the maximum degree is larger than the first resonance order, leading to a true normal matrix that is not block-diagonal. A few more cases were investigated where the truncation degree was equal to 13 and 16 (Table 2). It was found that up to degree 13 the block-diagonal approach still seems to recover the gravity field signal quite well: the geoid error is less than 1 cm when using radial orbit differences compared to about 14.38 cm for the signal. However, when the maximum degree is increased to 16 or higher, the recovery error is much above the signal.

This result seems to be in conflict with successful application of analytical perturbation methods for modeling gravity-field-induced orbit perturbations including resonances and recovery experiments reported in the literature, (see e.g. King-Hele and Winterbottom 1994; Visser 1995). However, it can be shown that this apparent conflict is due to the block-diagonal approximation approach and not to the analytical perturbation theory and its solution. To show this, one additional experiment was conducted in which a full set of normal equations was computed and solved using orbit differences in the radial direction. In this experiment, the gravity field coefficients were estimated simultaneously with the initial conditions and the resonance solution for the zonal coefficients was included in the computation of the observation equations. This solution reads (Visser 1992)

$$l = \text{even} : \Delta r = a \sum_{l=2,4,6\ldots}^{l_{\max}} \Delta\bar{C}_{l0} \left(\frac{a_e}{a}\right)^l \times (l+1) F_{l0l/2} (\cos\omega_0 t - 1)$$

$$l = \text{odd} : \Delta r = a \sum_{l=3,5,7\ldots}^{l_{\max}} \Delta\bar{C}_{l0} \left(\frac{a_e}{a}\right)^l \{F_{l0(l-1)/2} - F_{l0(l+1)/2}\} \times (l-1)(\omega_0 t \cos\omega_0 t - \sin\omega_0 t) \quad (20)$$

In this case, a full normal matrix was obtained, although with a block-diagonal dominant structure. However, clear side bands could be observed close to the near-resonance orders. The matrix looks similar to the matrix obtained with numerical integration to be discussed in the next section (Fig. 3). The resulting geoid error is equal to 35.65 cm, compared to 37.15 cm for the JGM-3/OSU91A geoid difference to degree and order 30. It was found that the error is contained for the larger part in the low-degree zonal coefficients. Excluding the zonal terms, the geoid error is equal to 8.02 cm after one iteration compared to 36.74 cm for the JGM-3/OSU91A geoid difference.

In summary, the analytical block-diagonal approach based on the particular solution of the LPT was unsuccessful for gravity field recovery for truncation degrees larger than 16. A full matrix approach seems to lead to satisfactory results. However, in this case the required computer time and memory requirements are equivalent to those for the full matrix approach with numerical integration. The latter approach leads to smaller approximation errors. This will be corroborated by the results presented in the next section.

3.2 Full matrix approach with numerical integration

The analytical block-diagonal method was shown to fail for gravity field recovery for spherical harmonic expansions above degree 16. Therefore, use was made of a full matrix approach with numerical integration to assess gravity field recovery for higher maximum degrees and orders. Due to the data period of 10 days, the maximum resolvable degree is 80, about half the number of orbital revolutions. This can be explained by the fact that the ground-track pattern of the satellite orbit divides the Earth's circumference into 160 parts at ground-track crossings, leading to a Nyquist cut-off degree of 80. In addition, it may be expected that the polar caps not covered by the GOCE ground track due to the orbital inclination of 96.6° start to play a role (these gaps, however, represent less than 1% of the Earth's surface). The observations that were used in setting up the observation equations were the inertial Cartesian x, y, and z coordinates with 10-s time interval. All coordinates were assigned the same weight. Gravity field recovery experiments were conducted with the same true (OSU91A) and a priori (JGM-3) gravity field models used in the analytical block-diagonal approach, this time truncated at degrees 30, 50, and 80 (the last one leading to a normal matrix with a size of 6557×6557). First, the linearization error of the observation equations was assessed. This was done by inserting the known gravity field differences between the true and a priori gravity field models and the known initial condition differences between the true and reference orbits in the observation equations and analyzing the misfit. It was found that, as expected, the linearization error is much smaller than the approximation errors when using the analytical perturbation theory (Table 3). The linearization errors are much smaller than 1 cm, except for the along-track direction, compared to signal magnitudes for the orbit differences of several meters. It was found that the linearization error grows with the arc length; for example, the linearization error in the along-track direction is equal to 6.35 cm or a 10-day arc compared to 0.01 cm for a 1-day arc for a truncation at degree 50.

The gravity field recovery accuracy in terms of the RMS of coefficient errors as a function of the degree is displayed in Fig. 1 (top) for truncations at degrees 30 and 50. Up to degree 25 the recovery errors are an order of magnitude smaller than the signal (OSU91-JGM-3), which is quite acceptable for a first iteration with a relatively short data period of 10 days. It was found that no regularization was required for solutions up to degree and order 50. In fact, the regularization is effectively realized by the truncation at a certain degree.

It was found that truncation at degree 80 resulted in an unstable normal matrix that could not be inverted due to high correlations between the initial conditions and (near-) resonance gravity field terms. Therefore, gravity field recovery experiments were conducted where the 10-day data period was split into 10 1-day orbital arcs. As for the 10-day orbital arc, for each 1-day arc a dynamic GOCE reference orbit was estimated from the true x, y, and z coordinates and the normal equations established. The resulting 10 sets of normal equations were accumulated using the principle of partitioning [Eqs. (18) and (19)]. Using this approach, 10 sets of initial conditions have to be accounted for instead of one for the 10-day orbital arc length. By splitting up the data period into shorter orbital arcs, the correlations have less chance to build up. In addition, linearization errors will be reduced (Table 3). This approach proved to be successful and no regularization was required in solving the normal equations. A remarkable result is the much smaller gravity field recovery error for the truncation at degree 50 due to the smaller linearization errors (Fig. 1, bottom). In terms of geoid recovery accuracy, the error is reduced from 16.83 to 0.04 cm (Table 4). For a truncation degree equal to 80, this error is equal to 19.55 cm compared to a signal of 66.42 cm. However, it should be mentioned that the normal equations complete to degree and order 80 could be solved only by making use of enhanced computer precision (16-byte instead of 8-byte words), because of instability problems that can be explained by the earlier-mentioned Nyquist cutt-off degree and the polar gaps. In addition, it is well

Table 3. Linearization error of observation equations obtained by numerical integration

Truncation degree	Arc length	Radial (cm)	Along-track (cm)	Cross-track (cm)
10[a]	10 days[b]	71.60	274.59	109.27
10[a]	10 days[c]	0.00	0.07	0.00
30	10 days[b]	858.57	13794.88	739.87
30	10 days[c]	0.22	7.27	0.07
50	10 days[b]	1245.00	18341.01	1443.40
50	10 days[c]	0.40	6.35	0.09
50	1 day[b]	333.21	1054.26	240.00
50	1 day[c]	0.00	0.01	0.01
80	1 day[b]	329.10	1058.90	272.66
80	1 day[c]	0.01	0.02	0.01

[a] Included as a reference case (cf. Table 1)
[b] Total signal: numerical integration
[c] Linearization error

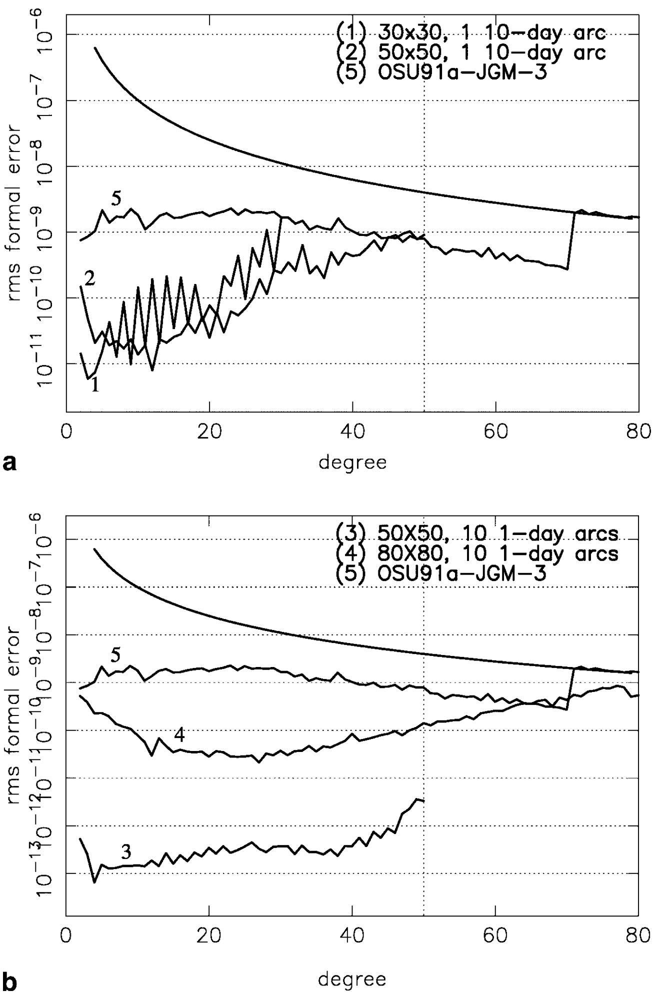

Fig. 1a, b. Gravity field recovery based on a full matrix approach with numerical integration. The observations for the gravity field estimation consisted of error-free orbit perturbations. The a priori gravity field model used was JGM-3. The coefficient differences between JGM-3 and OSU91A (*5*) were truncated at different degrees: 30 (*1*), 50 (*2*/*3*) and 80 (*4*)

known that at satellite altitude the orbit becomes less sensitive to gravity field perturbations with increasing degree. The geoid recovery error for a truncation degree equal to 30 is 10.44 cm. The results indicate that the orbital arc length plays an important role in gravity field recovery and can be seen as an optimization parameter. This optimization is, however, beyond the scope of this paper and is left for future research.

Table 4. Gravity field recovery from orbit perturbations using the full matrix approach with numerical integration

Maximum degree	Arc length	Geoid signal (cm)	Geoid recovery error (cm)
30	10 days	37.15	10.44
50	10 days	46.28	16.83
50	10 × 1 day	46.28	0.04
80	10 days	66.42	failed
80	10 × 1 day	66.42	19.55

In principle, the full matrix approach with numerical integration should be repeated until convergence. However, due to the computationally demanding nature of the recovery experiments, only one iteration was applied. The results indicate, however, that this approach is an appropriate method to recover the gravity field from orbit perturbations to relatively high degree and order. It is interesting to assess the limiting gravity field recovery accuracy in the presence of realistic orbit errors. This has been done by using OSU91A as both true and a priori model, but synthetic orbit errors are added to the x, y, and z coordinates. The orbit errors were obtained by conducting a kinematic orbit determination experiment making use of double-differenced GPS carrier phase observations (Visser and van den IJssel 2000). The following error sources were taken into account: GPS carrier phase measurement noise, errors of GPS station coordinates, uncertainties in GPS ephemeris, and errors of atmospheric path length correction due to troposphere. The orbit errors were at the few-centimeter level. A detailed description can be found in SID (2000). It can be seen in Fig. 2 that the gravity field recovery errors are larger than for the case where JGM-3 was used as a priori model with zero orbits errors. It may thus be concluded that the full matrix approach with numerical integration itself introduces already after the first iteration smaller errors in the gravity field solution when using error-free orbit perturbations rather than orbit perturbations that are affected by orbit errors at the few-centimeter level. In order to verify and corroborate this conclusion, the orbit error spectrum was propagated to gravity field coefficient errors by means of an independent analytical covariance analysis (Colombo 1984; Schrama 1991; Visser 1992). The covariance analysis is based on the same particular solution of the LPT as described in Sect. 2.1 [Eqs. (1)–(6)]. It can be seen that the gravity field recovery error is in close

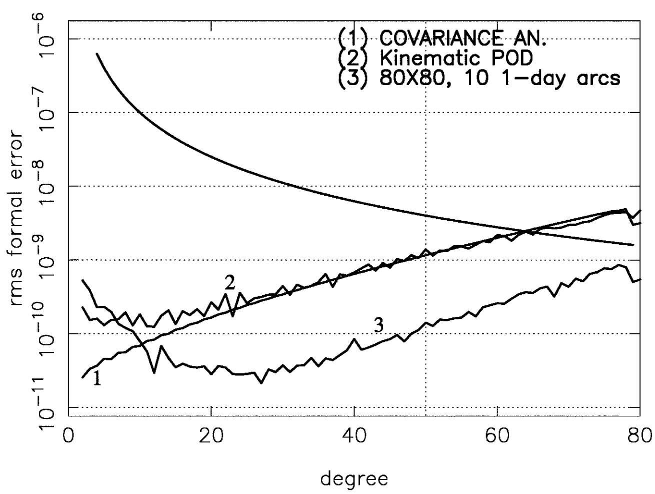

Fig. 2. Propagation of orbit error to gravity field recovery error: covariance analysis (*1*) and numerical integration approach for orbit errors based on kinematic precise orbit determination (*2*). The gravity field recovery error from orbit perturbations based on the JGM-3/OSU91A gravity field model differences is included as a reference (*3*)

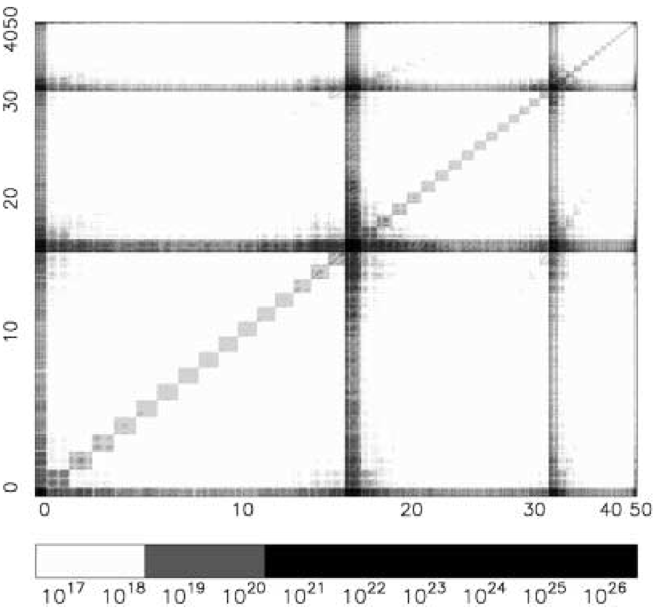

Fig. 3. Magnitude of normal matrix for 50 × 50 gravity field estimation from GOCE inertial position coordinates. The matrix is organized per order (indicated along the axes)

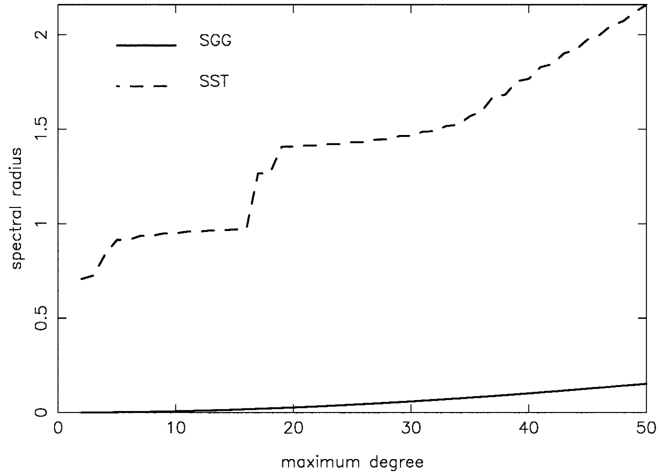

Fig. 4. Spectral radius for block-diagonal approach for gravity field recovery from SST and SGG observations

agreement with the predicted error (Fig. 2) except at low degrees. The difference at the lower degrees can be explained by the fact that the 10-day data period was split up into 10 1-day orbital arcs, resulting in additional initial condition unknowns and thereby reducing observability of long-period gravity field orbit perturbations.

It is interesting to look at the structure of the normal matrix, which is displayed in Fig. 3 for a maximum degree equal to 50 containing only the gravity field unknowns for a 10-day orbital arc. The effect of the initial conditions was taken into account by partitioning [Eq. (19)]. It can be seen that although the matrix is dominantly block-diagonal, clear horizontal and vertical side bands can be distinguished. These bands occur close to the resonance orders 0, 16, and 32, approximately integer multiples of the number of orbital revolutions per day. In addition, less pronounced oblique bands occur for matrix elements representing two coefficient terms for which the sum or the difference of the respective harmonic orders is close to the resonance orders. The side bands explain why a block-diagonal approach applied to orbit perturbations, such as that used in Sect. 3.1 did not lead to an accurate gravity field recovery. Based on this normal matrix, the spectral radius [Eq.(17)] was computed as a function of the maximum degree. For each maximum degree an appropriate subset of the normal matrix was used. It can be seen in Fig. 4 that the spectral radius becomes larger than 1 for degree 17, around degree 16, which is in agreement with the occurrence of the first resonance order and also reflects the strong side bands at this order in the normal matrix. For comparison, the spectral radius for iterative block-diagonal gravity field recovery from radial diagonal SGG observations (Klees et al. 2000) for a 10-day repeat period is included. Based on the sparse structure of the matrix, it will be interesting to investigate, for example, fast conjugate gradient methods with preconditioners based on the dominant structures of the normal equations, for example as implemented for space-wise gravity field recovery from SGG observations (ESA 2000).

4 Discussion and conclusions

A block-diagonal approach was implemented and tested for gravity field recovery from orbit perturbations of GOCE. This approach is based on the particular solution of an analytical LPT. The approach was found to be successful as long as the maximum degree is less than the first resonance order for orbit perturbations, in the case of GOCE at order 16. For higher degrees, at least up to degree 30, it was found that a successful recovery is possible when resonance effects are included in setting up the observation equations for the gravity field zonal coefficients and initial conditions are estimated simultaneously with the gravity field unknowns. However, in this case a full, although sparse and block-diagonal dominant, normal matrix was obtained, resulting in a significant increase in required computer central processing unit (CPU) time and memory. By analyzing the shape of the normal equations obtained by numerical integration, it was found that clear resonance bands show up in the normal matrix around multiples of order 16, giving a strong indication of the failure of the block diagonal approach.

The full matrix approach with numerical integration was found to be successful and viable for gravity field recovery up to degree and order 80 with existing software tools and computer facilities, although for a recovery complete to degree and order 80 the data period had to be split into shorter orbital arcs. It was found that the orbital arc length plays an important role and is an important factor in the gravity field recovery error. Increased orbital arc lengths lead to larger linearization errors of the observation equations, but also result in better observability of the long-period orbit perturbations. For a maximum degree equal to 50, the gravity field recovery was almost perfect after one iteration provided the orbital arc length was equal to 1 day. It is fair to assume that the maximum resolvable degree will increase with longer GOCE data observation and repeat periods. The projected observation period for GOCE is 2×6 months and the repeat period will be longer than 2 months, resulting in larger maximum degrees for the observable part of the gravity field from GOCE orbit perturbations. Due to the sparse structure of the normal matrix, which is strongly block-diagonal dominant with side bands at the resonance orders, it is interesting to investigate fast iterative methods, for example a conjugate gradient solver with preconditioners based on this dominant structure. Finally, gravity field recovery errors obtained with the full matrix approach from simulated GPS-based orbit errors were found to be in close agreement with error predictions based on an independent covariance analysis tool.

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