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A comparison between the Airy/Heiskanen and the Pratt/Hayford isostatic models for the computation of potential harmonic coefficients

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1 Introduction

There are many brilliant contributions to spherical harmonic analysis of a function given on a sphere as well as to spherical harmonic synthesis which is the forward computation of retrieving the function from its Fourier coefficients (see e.g. Colombo 1981; Sneeuw 1994; Sneeuw and Bun 1996). An important application of this theory is when the function is a global terrain model. The theory of reduction of a global elevation model to spherical harmonic coefficients taking isostasy into consideration can be found in many bibliographic sources (see e.g. Balmino et al. 1973; Rapp 1982; Sünkel 1985, 1986; Rummel et al. 1988; Pavlis and Rapp 1990). All of these contributions consider the Airy/Heiskanen isostatic model, apart from Sünkel who introduced a smoothing operator to the linearized Vening Meinesz model and determined both depth to the compensation level and the smoothing factor to account for a regional compensation. The scope of the present paper is to revisit the theory of spherical harmonic analysis of a global Digital Elevation Model (DEM) using the Airy/Heiskanen model and expand it for the Pratt/Hayford model as well. The spectra resulting from both models will be computed and compared to the observed gravity field of EGM96.

2 Expansion for the potential of mass distributions into spherical harmonics

For the inverse distance function the following series expression in spherical coordinates holds, or equally its spherical harmonic expansion

$$\frac{1}{l_{PQ}} = \frac{1}{r_P}\sum_{l=0}^{\infty}\left(\frac{r_Q}{r_P}\right)^l P_l(\cos\psi_{PQ}) \quad \text{for } r_Q < r_P \tag{1}$$

$$\frac{1}{l_{PQ}} = \frac{1}{r_Q}\sum_{l=0}^{\infty}\left(\frac{r_P}{r_Q}\right)^l P_l(\cos\psi_{PQ}) \quad \text{for } r_P < r_Q \tag{2}$$

where $P_l(\cos\psi_{PQ})$ are the Legendre polynomials of degree l and ψ_{PQ} the angle linking attracting point Q to the computation point P. A separation of the functions related to P from those related to Q can be made by means of the addition theorem of the spherical harmonic functions

$$P_l(\cos\psi_{PQ}) = \frac{1}{2l+1}\sum_{m=0}^{l}\bar{P}_{lm}(\cos\theta_P)\bar{P}_{lm}(\cos\theta_Q) \times (\cos m\lambda_P \cos m\lambda_Q + \sin m\lambda_P \sin m\lambda_Q) \tag{3}$$

where $\bar{P}_{lm}$ are the fully normalized associated Legendre functions and m denotes order. Equation (3) refers to normalized quantities and can be derived from the non-normalized expression given by Lense (1954, pp. 75–76). Using the abbreviation

$$Y_{lm}^{\alpha}(P) = \bar{P}_{lm}(\cos\theta_P)\begin{cases}\cos m\lambda_P & \text{for } \alpha = 0\\ \sin m\lambda_P & \text{for } \alpha = 1\end{cases} \tag{4}$$

Eq. (3) becomes

$$P_l(\cos\psi_{PQ}) = \frac{1}{2l+1}\sum_{m=0}^{l}\sum_{\alpha=0}^{1} Y_{lm}^{\alpha}(P)Y_{lm}^{\alpha}(Q) \tag{5}$$

The gravitational potential at an arbitrary point in space P due to the Earth's mass distribution is given by Newton's law of gravitation

$$V_P = G\iiint_{\Sigma}\frac{\rho(Q)}{l_{PQ}}\mathrm{d}\Sigma_Q \tag{6}$$

where G denotes the gravitational constant, ρ the density inside the Earth and l_{PQ} the distance between P and the infinitesimal volume element $\mathrm{d}\Sigma_Q$ at Q. Inserting Eqs. (1) and (5) into Eq. (6), we obtain

$$V_P = G\iiint_{\Sigma} \rho(Q)\frac{1}{r_P}\sum_{l=0}^{\infty}\left(\frac{r_Q}{r_P}\right)^l \frac{1}{2l+1} \times \sum_{m=0}^{l}\sum_{\alpha=0}^{1} Y_{lm}^{\alpha}(P) Y_{lm}^{\alpha}(Q)\ \mathrm{d}\Sigma_Q$$
$$= G\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{\alpha=0}^{1}\frac{1}{r_P^{l+1}}\frac{1}{2l+1}Y_{lm}^{\alpha}(P) \times \iiint_{\Sigma}\rho(Q) r_Q^l Y_{lm}^{\alpha}(Q)\ \mathrm{d}\Sigma_Q$$
$$= \frac{GM}{R}\sum_{l=0}^{\infty}\frac{R^{l+1}}{r_P^{l+1}}\sum_{m=0}^{l}\sum_{\alpha=0}^{1} Y_{lm}^{\alpha}(P) C_{lm}^{\alpha} \quad \text{for } r_P > r_Q \tag{7}$$

where

$$C_{lm}^{\alpha} = \left\{\begin{matrix}\bar{C}_{lm}\\ \bar{S}_{lm}\end{matrix}\right\} = \frac{1}{M}\frac{1}{2l+1} \times \iiint_{\Sigma}\rho(Q)\left(\frac{r_Q}{R}\right)^l Y_{lm}^{\alpha}(Q)\mathrm{d}\Sigma_Q$$
$$= \frac{3}{2l+1}\frac{1}{\bar{\rho}R^3}\frac{1}{4\pi} \times \iint_{\sigma}\left(\int_r \rho(Q)\left(\frac{r_Q}{R}\right)^l r_Q^2 \mathrm{d}r_Q\right) Y_{lm}^{\alpha}(Q)\mathrm{d}\sigma_Q \tag{8}$$

In Eq. (8), M is the Earth's mass in a spherical approximation, $M = \frac{4}{3}\pi\bar{\rho}R^3$, with $\bar{\rho}$ denoting a mean density value of the Earth, e.g. 5500 kg m^{-3}, and R its radius. For the transition from a volume to a surface integral in Eq. (8), the volume element in spherical coordinates $\mathrm{d}\Sigma_Q = r_Q^2\,\mathrm{d}r_Q\,\mathrm{d}\sigma_Q$ was introduced. The objective of this paper is the computation of the dimensionless potential harmonic coefficients of Eq. (8) by taking the isostatic compensation of topography into account. For this purpose the idealized isostatic models of both Airy/Heiskanen and Pratt/Hayford are considered. It should be stressed that the computation refers to points situated outside or on a sphere including all masses (Brillouin sphere), i.e. throughout this paper only Eq. (1) is taken into consideration. A theoretically correct computation taking place on the Bjerhammar sphere should comprise both Eqs. (1) and (2), the former for $r_Q < R$ and the latter for those Q's for which $r_Q > R$.

3 Topographic/isostatic harmonic coefficients with the Airy/Heiskanen model

The crust in the Airy/Heiskanen model is considered to have constant density $\rho_{\mathrm{cr}} = 2670$ kg m^{-3} but variable thickness, where highly elevated terrain is compensated by thick crust and low terrain or oceans by thin crust. The density of the denser mantle layer on which the mountains float is considered also to have a constant value, namely $\rho_{\mathrm{m}} = 3270$ kg m^{-3}. Thus, the density contrast between crust and mantle becomes $\Delta\rho = \rho_{\mathrm{m}} - \rho_{\mathrm{cr}} = 600$ kg m^{-3}. A relation for the variable root (t) and anti-root (t') thickness can be obtained from the condition of floating equilibrium for the continents and the oceans, respectively. For flat columns one obtains, respectively

$$t = \frac{\rho_{\mathrm{cr}}}{\Delta\rho}h \tag{9}$$

and

$$t' = \frac{(\rho_{\mathrm{cr}} - \rho_{\mathrm{w}})}{\Delta\rho}h' \tag{10}$$

where h and h' denote the positive (heights) and negative elevations (depths) of a global elevation set and $\rho_{\mathrm{w}} = 1030$ kg m^{-3} the density of sea water. When the convergence of the verticals is taken into account one obtains in linear approximation (Lambeck 1988; Rummel et al. 1988)

$$t = \left(\frac{R}{R - D_{\mathrm{A}}}\right)^2\frac{\rho_{\mathrm{cr}}}{\Delta\rho}h \tag{11}$$

and

$$t' = \left(\frac{R}{R - D_{\mathrm{A}}}\right)^2\frac{(\rho_{\mathrm{cr}} - \rho_{\mathrm{w}})}{\Delta\rho}h' \tag{12}$$

R denotes a mean Earth radius value ($R = 6370$ km) and D_{A} the thickness of the crust for zero elevation. A popular value for D_{A} in Airy's model is $D_{\mathrm{A}} = 30$ km. Equation (8) is written in the Airy/Heiskanen model as the difference between the coefficients generated by the potential of the surface topography and those generated by the compensation part. One writes

$$C_{lm}^{\alpha^{\mathrm{I}}} = \frac{3}{\bar{\rho}R(2l+1)}\frac{1}{4\pi}\iint_{\sigma}\left[A^{\mathrm{T}}(Q) - A^{\mathrm{C}}(Q)\right] \times Y_{lm}^{\alpha}(Q)\mathrm{d}\sigma_Q \tag{13}$$

where the surface topography part is

$$A^{\mathrm{T}}(Q) = \rho_{\mathrm{cr}}\int_{r=R}^{R+h}\left(\frac{r_Q}{R}\right)^{l+2}\mathrm{d}r_Q \tag{14}$$

and the compensation part

$$A^{\mathrm{C}}(Q) = \Delta\rho\int_{r=R-D_{\mathrm{A}}-t}^{R-D_{\mathrm{A}}}\left(\frac{r_Q}{R}\right)^{l+2}\mathrm{d}r_Q \tag{15}$$

Replacing ocean depths h' by equivalent rock topography and taking the convergence of the verticals into account, one obtains for the coefficients of the isostatically compensated topography (for derivations see Sünkel 1986; Rummel et al. 1988)

$$C_{lm}^{\alpha^{\mathrm{I}}} = C_{lm}^{\alpha^{\mathrm{T}}} - C_{lm}^{\alpha^{\mathrm{C}}} \tag{16}$$

where the coefficients from the uncompensated topography are

$$C_{lm}^{\alpha^{\mathrm{T}}} = \frac{3}{2l+1}\frac{\rho_{\mathrm{cr}}}{\bar{\rho}}\left\{h_{lm} + \frac{l+2}{2}h2_{lm} + \frac{(l+2)(l+1)}{6}h3_{lm}\right\} \tag{17}$$

and those corresponding to the isostatic compensation

$$\begin{aligned} C_{lm}^{\alpha^{\mathrm{C}}} = \frac{3}{2l+1}\frac{\rho_{\mathrm{cr}}}{\bar{\rho}}\Bigg\{ & \left(\frac{R-D_{\mathrm{A}}}{R}\right)^{l} h_{lm} \\ & - \frac{l+2}{2}\frac{\rho_{\mathrm{cr}}}{\Delta\rho}\left(\frac{R-D_{\mathrm{A}}}{R}\right)^{l-3} h2_{lm} \\ & + \frac{(l+2)(l+1)}{6}\frac{\rho_{\mathrm{cr}}^2}{\Delta\rho^2}\left(\frac{R-D_{\mathrm{A}}}{R}\right)^{l-6} h3_{lm}\Bigg\} \end{aligned} \tag{18}$$

For a shorter notation the following surface harmonic expansions were introduced in Eqs. (17) and (18) (Rummel et al. 1988, Eq. 23):

$$h_{lm} = \frac{1}{4\pi}\iint_{\sigma} \frac{h(Q)}{R} Y_{lm}^{\alpha}(Q)\mathrm{d}\sigma_Q \tag{19}$$

$$h2_{lm} = \frac{1}{4\pi}\iint_{\sigma} \frac{h^2(Q)}{R^2} Y_{lm}^{\alpha}(Q)\mathrm{d}\sigma_Q \tag{20}$$

and

$$h3_{lm} = \frac{1}{4\pi}\iint_{\sigma} \frac{h^3(Q)}{R^3} Y_{lm}^{\alpha}(Q)\mathrm{d}\sigma_Q \tag{21}$$

For initial presentation of the numerical results the potential coefficient spectrum was computed. The degree variance (spectral power or simply spectrum) of a set of coefficients for degree l is defined by

$$\sigma_l^2 = \sum_{m=0}^{l}\left(\bar{C}_{lm}^2 + \bar{S}_{lm}^2\right) \tag{22}$$

Figure 1 displays the separate contributions of h_{lm}, $h2_{lm}$ and $h3_{lm}$ to the computation of the uncompensated potential coefficients according to Eq. (17). The elevations and depths for these computations were taken from a 1° × 1° mean elevation file, known as TUG87 (Wieser 1987). The h_{lm}'s are properly multiplied by the respective factors given in Eqs. (17), so that the curves in Fig. 1 have dimensions of $\sigma_l^2 = \sum_{m=0}^{l}((\bar{C}_{lm}^{\mathrm{T}})^2 + (\bar{S}_{lm}^{\mathrm{T}})^2)$. The harmonic coefficients of Eqs. (19)–(21) were computed to degree 180 using the program 'gsha.m' written in MATLAB (Tsoulis and Sneeuw 1998) using integrated associated Legendre functions (Gerstl 1980). The heights/depths were treated as mean values and this was properly taken into consideration in the spherical harmonic analysis algorithm (Albertella and Sacerdote 1995; Tsoulis 1999). The computations shown in Fig. 1 agree with those reported in Rummel et al. (1988): the power spectra of $(h/R)^2$ and $(h/R)^3$ are approximately 10^{-6} and 10^{-13}, respectively, of the power of (h/R). Figure 2 compares the uncompensated, the Airy-compensated and the truncated up to maximum degree and order 180 potential coefficient spectrum of EGM96 (Lemoine et al. 1998). In all representations, the degree variances σ_l^2 for degree $l \geq 2$ are displayed; the zeroth and first degree of the spectrum refer to the deviation between the geocentre and the centre of mass of the topography and its isostatic compensation and a remark on their magnitude can be found in Pavlis and Rapp (1990, p. 374).

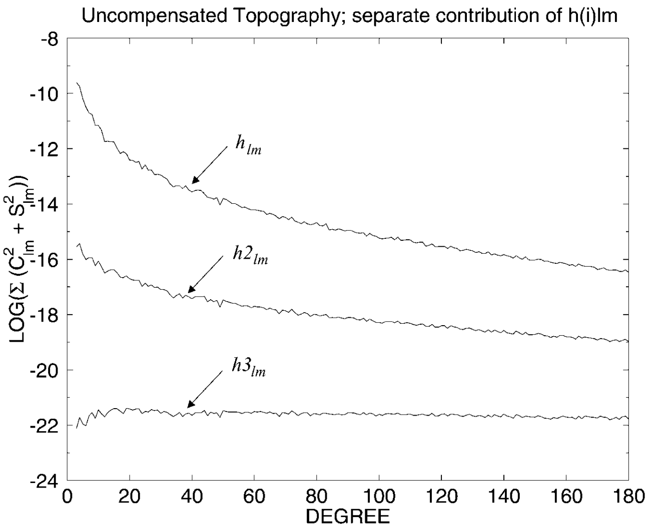

Fig. 1. Separate contributions of h_{lm}, $h2_{lm}$ and $h3_{lm}$ [Eqs. (19)–(21)] to the computation of the uncompensated potential coefficients [Eq. (17)]

4 Topographic/isostatic harmonic coefficients with the Pratt/Hayford model

According to the isostatic model of Pratt/Hayford there exists uniform density ρ_m below the level of compensa-

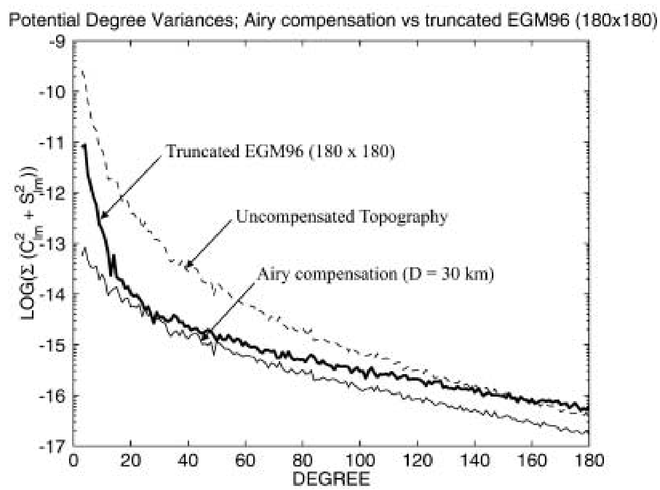

Fig. 2. A comparison between the following potential coefficient spectra: uncompensated topography [Eq. (17)], isostatically compensated topography after Airy [Eq. (18)] and the truncated (180 × 180) EGM96 model

tion ($D_P = 100$ km). Above floats a layer of thickness $D_P + h$ or $D_P - h'$ relative to sea level and of variable density, so that the mass of each column of the same cross-section remains constant. Consequently, mountains are underlain by low-density crust and oceans by high-density material. For a flat-Earth approximation the density ρ of a column of height $D_P + h$, h representing the height of the topography, should satisfy the equation $(D_P + h)\rho = D_P\rho_{cr}$, assuming that the density of a column of thickness D_P equals the mean density of the crust ρ_{cr}. The variable density underlying the continents in a flat and a spherical column representation is given respectively by (Lambeck 1988)

$$\rho_L = \frac{D_P}{D_P + h}\rho_{cr} \tag{23}$$

and

$$\rho_L = \left(\frac{R}{R - D_P}\right)^2 \frac{D_P}{D_P + h}\rho_{cr} \tag{24}$$

For the ocean part it holds respectively that

$$\rho_O = \frac{D_P\rho_{cr} - h'\rho_w}{D_P - h'} \tag{25}$$

and

$$\rho_O = \left(\frac{R}{R - D_P}\right)^2 \frac{D_P\rho_{cr} - h'\rho_w}{D_P - h'} \tag{26}$$

Hence, the density deficiency in continental regions for a flat column approximation is

$$\Delta\rho^L = \rho_L - \rho_{cr} = -\frac{h}{D_P + h}\rho_{cr} \tag{27}$$

For the spherical column representation one obtains, after some reordering

$$\Delta\rho^L = \rho_L - \rho_{cr} = -\frac{h}{D_P + h}\rho_{cr} - \frac{D_P^2(D_P - 2R)}{(R - D_P)^2(D_P + h)}\rho_{cr} \tag{28}$$

Similarly, the density surplus for the sub-oceanic columns in the flat and the spherical column approximation is given respectively by

$$\Delta\rho^O = \rho_O - \rho_{cr} = \frac{h'}{D_P - h'}(\rho_{cr} - \rho_w) \tag{29}$$

and

$$\Delta\rho^O = \rho_O - \rho_{cr} = \left(\frac{R}{R - D_P}\right)^2 \frac{h'}{D_P - h'}(\rho_{cr} - \rho_w) - \frac{(D_P^2 - 2RD_P)}{(R - D_P)^2}\rho_{cr} \tag{30}$$

Equation (13) holds for the Pratt/Hayford model as well. The distinction to a surface topography part and a compensation part is made also for Pratt/Hayford. The former consists of the contribution of the model-defined densities over the continental regions and of the density contrasts $\rho_w - \rho_{cr}$ over the oceans. The compensation part corresponds to the potential generated by the variable density anomalies Eqs. (27)–(30). Thus, the topographic/isostatic coefficients using the Pratt/Hayford model will be given by Eq. (13) with

$$A^T(Q) = \begin{cases} \int_{r=R}^{R+h} \left(\frac{r_Q}{R}\right)^{l+2} \rho_L(Q)\mathrm{d}r_Q & \text{Land part} \\ \int_{r=R-h'}^{R} \left(\frac{r_Q}{R}\right)^{l+2} [\rho_w - \rho_{cr}]\mathrm{d}r_Q & \text{Ocean part} \end{cases} \tag{31}$$

and

$$A^C(Q) = \begin{cases} \int_{r=R-D_P}^{R} \left(\frac{r_Q}{R}\right)^{l+2} \Delta^L\rho(Q)\mathrm{d}r_Q & \text{Land part} \\ \int_{r=R-D_P}^{R-h'} \left(\frac{r_Q}{R}\right)^{l+2} \Delta^O\rho(Q)\mathrm{d}r_Q & \text{Ocean part} \end{cases} \tag{32}$$

The integral for r_Q in the first of Eqs. (31) is identical to the one appearing in Eq. (14). Thus, the only difference for the "topography" contribution over continental regions between the two isostatic models arises from the difference between ρ_{cr} and ρ_L. For the ocean part one obtains, by integrating the second of Eqs. (31) with respect to r_Q

$$A^T_{ocean}(Q) = [\rho_w - \rho_{cr}]\frac{R}{l+3}\left[\left(\frac{r_Q}{R}\right)^{l+3}\right]_{R-h'}^{R} = \rho_w\frac{R}{l+3}\left[1 - \left(\frac{R - h'}{R}\right)^{l+3}\right] \tag{33}$$

Expanding the second term in the bracket into a binomial series up to third order in h'/R gives

$$A^T_{ocean}(Q) = [\rho_w - \rho_{cr}]R\left[\frac{h'}{R} - \frac{l+2}{2}\left(\frac{h'}{R}\right)^2 + \frac{(l+2)(l+1)}{6}\left(\frac{h'}{R}\right)^3\right] \tag{34}$$

For the compensation part one neglects at first the effect of the convergence of the verticals. Thus, for flat columns, inserting Eq. (27) into the first equation of Eq. (32) and integrating over r_Q yields, after a few steps

$$A^C_{land}(Q) = \frac{h}{D_P + h}\rho_{cr}\frac{R}{l+3}\left[1 - \left(\frac{R - D_P}{R}\right)^{l+3}\right] \tag{35}$$

Inserting Eq. (29) into the second of Eqs. (32), one obtains similarly for the ocean part

$$A^{\mathrm{C}}_{\mathrm{ocean}}(Q) = \frac{h'}{D_{\mathrm{P}} - h'}(\rho_{\mathrm{cr}} - \rho_{\mathrm{w}})h' \left[-1 + \frac{l+2}{2}\frac{h'}{R} - \frac{(l+2)(l+1)}{6}\left(\frac{h'}{R}\right)^2\right] + \frac{h'}{D_{\mathrm{P}} - h'}(\rho_{\mathrm{cr}} - \rho_{\mathrm{w}})\frac{R}{l+3} \times \left[1 - \left(\frac{R - D_{\mathrm{P}}}{R}\right)^{l+3}\right] \tag{36}$$

h' denotes here the bathymetry information, i.e. the original depths of the global elevation set taken as absolute value; the concept of equivalent rock topography is absent in the present analysis of the Pratt model. Inserting expressions (34), (35) and (36) into Eq. (13), one obtains a separate contribution to the potential coefficients from the ocean topography part, the land and the ocean isostatic parts respectively. The land topography part will be given by Eq. (17) using another harmonic expansion hi_{lm}. It holds that

$$C^{\alpha\mathrm{I}}_{lm} = C^{\alpha\mathrm{T/land}}_{lm} + C^{\alpha\mathrm{T/ocean}}_{lm} - C^{\alpha\mathrm{C/land}}_{lm} - C^{\alpha\mathrm{C/ocean}}_{lm} \tag{37}$$

$C^{\alpha\mathrm{T/land}}_{lm}$ is given from Eq. (17) with $hi_{lm}(i = 1, 2, 3)$ defined as follows:

$$h_{lm} = \frac{1}{4\pi}\iint_{\sigma}\left(\frac{D_{\mathrm{P}}}{D_{\mathrm{P}} + h}\right)\frac{h}{R}Y^{\alpha}_{lm}(Q)\mathrm{d}\sigma_Q \tag{38}$$

$$h2_{lm} = \frac{1}{4\pi}\iint_{\sigma}\left(\frac{D_{\mathrm{P}}}{D_{\mathrm{P}} + h}\right)\left(\frac{h}{R}\right)^2 Y^{\alpha}_{lm}(Q)\mathrm{d}\sigma_Q \tag{39}$$

$$h3_{lm} = \frac{1}{4\pi}\iint_{\sigma}\left(\frac{D_{\mathrm{P}}}{D_{\mathrm{P}} + h}\right)\left(\frac{h}{R}\right)^3 Y^{\alpha}_{lm}(Q)\mathrm{d}\sigma_Q \tag{40}$$

For the other three contributions to the potential coefficients after Pratt/Hayford compensation, one has

$$C^{\alpha\mathrm{T/ocean}}_{lm} = \frac{(\rho_{\mathrm{w}} - \rho_{\mathrm{cr}})}{\overline{\rho}}\frac{3}{2l+1}\left\{h'_{lm} - \frac{l+2}{2}h2'_{lm} + \frac{(l+2)(l+1)}{6}h3'_{lm}\right\} \tag{41}$$

$$C^{\alpha\mathrm{C/land}}_{lm} = \frac{\rho_{\mathrm{cr}}}{\overline{\rho}}\frac{3}{(2l+1)(l+3)}\left[1 - \left(\frac{R - D_{\mathrm{P}}}{R}\right)^{l+3}\right]h_{dh} \tag{42}$$

and

$$C^{\alpha\mathrm{C/ocean}}_{lm} = \frac{(\rho_{\mathrm{cr}} - \rho_{\mathrm{w}})}{\overline{\rho}}\frac{3}{2l+1} \times \left\{-h''_{lm} + \frac{l+2}{2}h2''_{lm} - \frac{(l+2)(l+1)}{6}h3''_{lm}\right\} + \frac{(\rho_{\mathrm{cr}} - \rho_{\mathrm{w}})}{\overline{\rho}}\frac{3}{(2l+1)(l+3)} \times \left[1 - \left(\frac{R - D_{\mathrm{P}}}{R}\right)^{l+3}\right]h''_{dh} \tag{43}$$

In Eqs. (41)–(43) enter the following spherical harmonic expansions are entered:

$$h_{dh} = \frac{1}{4\pi}\iint_{\sigma}\left(\frac{h}{D_{\mathrm{P}} + h}\right)Y^{\alpha}_{lm}(Q)\mathrm{d}\sigma_Q \tag{44}$$

$$h'_{lm} = \frac{1}{4\pi}\iint_{\sigma}\frac{h'}{R}Y^{\alpha}_{lm}(Q)\mathrm{d}\sigma_Q \tag{45}$$

$$h2'_{lm} = \frac{1}{4\pi}\iint_{\sigma}\left(\frac{h'}{R}\right)^2 Y^{\alpha}_{lm}(Q)\mathrm{d}\sigma_Q \tag{46}$$

$$h3'_{lm} = \frac{1}{4\pi}\iint_{\sigma}\left(\frac{h'}{R}\right)^3 Y^{\alpha}_{lm}(Q)\mathrm{d}\sigma_Q \tag{47}$$

$$h''_{lm} = \frac{1}{4\pi}\iint_{\sigma}\left(\frac{h'}{D_{\mathrm{P}} - h'}\right)\frac{h'}{R}Y^{\alpha}_{lm}(Q)\mathrm{d}\sigma_Q \tag{48}$$

$$h2''_{lm} = \frac{1}{4\pi}\iint_{\sigma}\left(\frac{h'}{D_{\mathrm{P}} - h'}\right)\left(\frac{h'}{R}\right)^2 Y^{\alpha}_{lm}(Q)\mathrm{d}\sigma_Q \tag{49}$$

$$h3''_{lm} = \frac{1}{4\pi}\iint_{\sigma}\left(\frac{h'}{D_{\mathrm{P}} - h'}\right)\left(\frac{h'}{R}\right)^3 Y^{\alpha}_{lm}(Q)\mathrm{d}\sigma_Q \tag{50}$$

and

$$h''_{dh} = \frac{1}{4\pi}\iint_{\sigma}\left(\frac{h'}{D_{\mathrm{P}} - h'}\right)Y^{\alpha}_{lm}(Q)\mathrm{d}\sigma_Q \tag{51}$$

In order to take the convergence effect of spherical columns into consideration, one has to insert Eqs. (28) and (30) into the first and second integral of Eq. (32), respectively. Following the same procedure one is once more led for the land part of the topography to Eq. (17) with the expansions of Eqs. (38)–(40) now modified as follows:

$$h_{lm} = \frac{1}{4\pi}\iint_{\sigma}\left(\frac{D_{\mathrm{P}}}{D_{\mathrm{P}} + h}\right)\left(\frac{R}{R - D_{\mathrm{P}}}\right)\frac{h}{R}Y^{\alpha}_{lm}(Q)\mathrm{d}\sigma_Q \tag{52}$$

the remaining two terms being written analogously. For the continental part of the compensation, $C^{\alpha\mathrm{C/land}}_{lm}$, Eq. (42) holds with the expansion of Eq. (44) written as

$$h_{lm} = \frac{1}{4\pi}\iint_\sigma \left(\frac{h}{D_P + h} + \frac{D_P^2(D_P - 2R)}{(R - D_P)^2(D_P + h)}\right) \times \frac{h}{R} Y_{lm}^\alpha(Q)\mathrm{d}\sigma_Q \quad (53)$$

Thus, in order to account for the convergence of the verticals in the computation of the land compensation part, one has to replace the factor $h/(D_P + h)$ in up expansion of Eq. (44) with Eq. (28) and then apply Eq. (42). For the ocean part of the compensation one obtains the following slightly modified version of Eq. (43):

$$C_{lm}^{\alpha\mathrm{C/ocean}} = \frac{3}{\bar{\rho}(2l+1)}\left\{-h''_{lm} + \frac{l+2}{2}h2''_{lm} - \frac{(l+2)(l+1)}{6}h3''_{lm}\right\} + \frac{3}{\bar{\rho}(2l+1)(l+3)} \times \left[1 - \left(\frac{R - D_P}{R}\right)^{l+3}\right]h''_{dh} \quad (54)$$

with the expansions of Eqs. (48)–(51) modified to

$$h''_{lm} = \frac{1}{4\pi}\iint_\sigma \Delta^{\mathrm{O}}\rho(Q)\frac{h'}{R}Y_{lm}^\alpha(Q)\mathrm{d}\sigma_Q \quad (55)$$

$\Delta^{\mathrm{O}}\rho(Q)$ is given by Eq. (30) and the remaining [Eqs. (49)–(51)] are treated in an analogous manner. Thus, for the computation of the ocean part of the isostatic compensation of a Pratt-compensated elevation model when the convergence of verticals is taken into account, Eq. (54) should be used. The expansions h''_{lm}, $h2''_{lm}$, $h3''_{lm}$ and h''_{dh} are then taken from Eqs. (48)–(51), replacing the factor $h'/(D_P - h')$ by the density anomaly $\Delta^{\mathrm{O}}\rho(Q)$ given by Eq. (30).

Similarly to the spectra computations carried out in the previous section, we proceed to respective computations for the Pratt/Hayford model. Figure 3 displays the effect of neglecting the convergence of verticals on the power spectra of the potential coefficients. Illustrated are the power spectra of the set of coefficients complying to flat columns for the land part and the isostatic part of the topography [Eqs. (38)– (40), (44)–(51)], the spectrum for spherical convergent columns [Eqs. (39)–(41), (52)–(55)] and the spectrum corresponding to the set of potential coefficients generated by subtracting the previous two. The spectrum of this residual set of coefficients is 10^{-2} to 10^{-1} of the power of either the flat or the spherical column coefficients. Figure 4 shows power spectra of both Airy and Pratt compensation in a single graph. One observes that the Pratt model seems to fail as a compensating mechanism, at least up to degree 60. From that point up to $l = 180$ its spectrum converges slowly to that of Airy. Consequently, the difference between the two up to $l \approx 60$ is approximately equal to the difference between Airy compensation and uncompensated topography. A first attempt to apply the set of coefficients that emerged from the difference between the two sets (Airy minus Pratt) to obtain a global field of gravity anomalies and geoid undulation differences produced fields that – apart from the identification of continents – could hardly be interpreted. Further investigation is necessary to examine the limitations of using the harmonic coefficients resulting from both models to the prediction of gravity anomalies in gravimetrically deficient areas. Such studies should also consider the effect of ice which Pavlis and Rapp (1990) demonstrated to be non-negligible (at least in the Airy/Heiskanen case). Extension of the maximum degree of expansion beyond 180 should also be pursued. The inability of the Pratt model to remove the effect of topography at the degree range 0–60 is also a matter that deserves further research. Regional harmonic synthesis applications of the two sets should reveal different local characteristics of the gravity field and thus set the one of the two isostatic models which is more appropriate to the specific region.

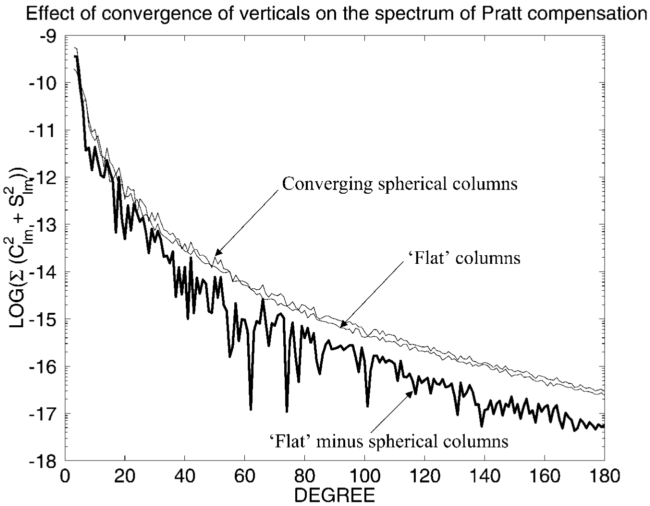

Fig. 3. The effect of the convergence of the vertical columns in Pratt's isostatic model

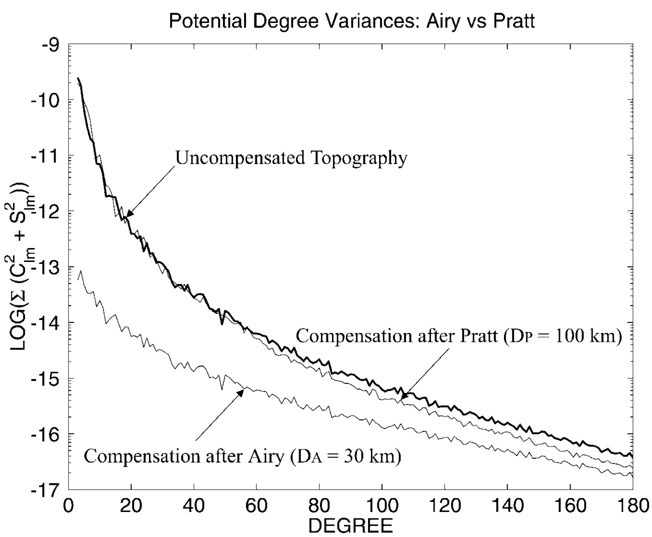

Fig. 4. Airy vs Pratt compensation

5 Concluding remarks

The theory on spherical harmonic analysis of a global elevation model accounting for the effect of isostasy according to Airy/Heiskanen was revisited and expanded to the Pratt/Hayford model. The obtained sets of harmonic coefficients can be used as an alternative technique for estimating gravity anomalies in areas where poor or no data are available. The present contribution shows that the Pratt isostatic model removes the effect of topography, i.e. acts as a compensating mechanism to the uncompensated topography spectrum only in the degree range 60–180. It can be used as an alternative to the Airy-resulting spectrum, which was used solely until now. However, further research is necessary for the optimal application of the present theory, for example to determine which degree range of the Pratt-resulting spectrum produces the most satisfactory results when used in combination with a satellite-derived model for the prediction of gravity anomalies or geoid undulations. With the dawn of the first gravity and gradiometry satellite missions these applications become more relevant than ever, particularly in areas where lack of data will continue to exist, such as at the polar gap regions.

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