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The gravitational attraction of any polygonally shaped vertical prism with inclined top and bottom faces

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Abstract. A closed formula for computing the gravitational attraction of a general vertical prism with $N+2$ faces (N faces are vertical planes, the other two are the inclined top and bottom planes) in Cartesian coordinates is presented. In addition, the special case of a triangular prism is discussed. Algebraic differences and overlooked singularity conditions of a previously published formula of this computation (which was only for the triangular special case) were identified and are also presented.

Key words: Numerical integration – Gravity – Terrain corrections

Introduction

Computation of gravity signals from 3-D masses of homogeneous density is a problem with a long history in gravimetric geodesy and geophysics (Rausenberger 1888; Mader 1951; Götze 1978; Petrović 1996). A method is described herein where an exact closed formula (using mostly standard intrinsic computer functions) for the gravitational attraction of a vertical prism can be computed in Cartesian coordinates. For the purposes of this paper, a vertical prism is defined as an $N+2$- faced polyhedron (called "Π") where N faces are perpendicular to the x–y plane (so that the projection of Π onto the x–y plane is an N-sided polygon), and the other two faces are inclined planes (see Fig. 1). Full details on the general case where $N \geq 3$ are given, with some discussion on the special case of a triangular prism ($N=3$). While it is true that three points define a plane, a prism with four or more vertices can exist where the four or more corners lie on a plane. This plane may just be an approximation for the true topography, but that is all any surface is, whether flat or curved – an approximation of the true topographic surface. And while the utility of a "flat, but inclined" planar top to the prism may be limited, it is more accurate than a prism with "flat and un-inclined" tops and bottoms (to clarify: "flat" means the surface has an infinite radius of curvature while "inclined" means the surface is not parallel to the x–y plane). No closed-form solution to this problem with curved prism tops has yet been found. The applications of $N>3$-sided prisms are few, but could include computing gravity signals in mostly flat areas near rivers and gorges, or the forward modeling of density anomaly features of odd shapes on the gravity signal. Since the applications for $N>3$ are limited, the emphasis of this paper will be on the $N=3$ problem, although a few brief notes on the $N>3$ problem are made near the end of the paper.

Solution of a five-term mixed logarithmic and square-root integral

In anticipation of its use later in this paper, a solution of the following equation is presented:

$$F(a,b,g,d,e,x) = \int_x \ln\left\lfloor ax + b + \sqrt{gx^2 + 2dx + e}\right\rfloor \mathrm{d}x \tag{1}$$

The solution of this integral is complicated, yet is recurrent in the computation of gravitational attraction over a vertical prism with sloping top and bottom faces. Its solution is generally *not* found in most mathematical handbooks and tables of integrals. Woodward (1975) presents one method of computation by substitution of variables, but that paper does not fully address all singularity conditions that occur, and has an algebraic difference from the solution presented in the present paper (this will be discussed later). With the advent of modern symbolic languages, such as Mathematica 3.0 (Wolfram 1996), faster solutions to such integrals are possible than through manual techniques by a human using paper and pencil (albeit with perhaps less aesthetically pleasing

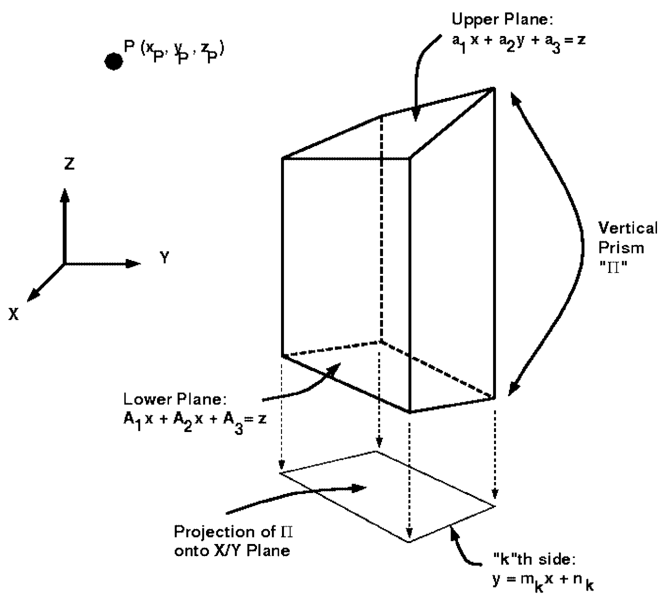

Fig. 1.

results than might be achieved by a human). As an additional bonus, symbolic languages are generally not prone to algebraic errors. Although the solution for the indefinite integral will be shown, ultimately that solution will be used to solve for the definite integral, as follows:

$$\int_{x=x_1}^{x_2} \ln[ax + b + \sqrt{gx^2 + 2dx + e}]\,\mathrm{d}x = F(a,b,g,d,e,x_2) - F(a,b,g,d,e,x_1) \qquad (2)$$

The solution to Eq. (1) was computed using Mathematica 3.0 (Wolfram 1996), and can be written as the sum of seven terms

$$F = C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 \qquad (3)$$

where the first five are fairly simple to read and implement into computer programs. The 6th and 7th terms are more complex, but are very similar to one another. Symmetries and patterns re-occur, and the form in which all seven terms are presented attempts to exploit re-occurring terms

$$C_1 = -x \qquad (4)$$

$$C_2 = \left[\sqrt{2abd - d^2 - a^2e - b^2g + eg} \times \arctan\left[\frac{ab - d + a^2x - gx}{\sqrt{2abd - d^2 - a^2e - b^2g + eg}}\right]\right] \Big/ (a^2 - g) \qquad (5)$$

$$C_3 = \frac{(ab - d)\ln[b^2 - e + 2abx - 2dx + a^2x^2 - gx^2]}{2(a^2 - g)} \qquad (6)$$

$$C_4 = x\ln[ax + b + \sqrt{gx^2 + 2dx + e}] \qquad (7)$$

$$C_5 = \frac{(ad - bg)\ln\left[\frac{2(d+gx)}{\sqrt{g}} + 2\sqrt{gx^2 + 2dx + e}\right]}{(a^2 - g)\sqrt{g}} \qquad (8)$$

$$C_6 = \left(-\frac{1}{Q_1}\right)\sqrt{Q_2 - Q_3}(Q_4 - Q_5) \times \ln\left(\frac{Q_6 - Q_7}{\sqrt{Q_8 - Q_9}(Q_{10} - Q_{11})(Q_{17} - Q_{18})} + \frac{Q_{12}}{(Q_{13} - Q_{14})(Q_{15} - Q_{16})}\right) \qquad (9)$$

$$C_7 = \left(-\frac{1}{Q_1}\right)\sqrt{Q_2 + Q_3}(Q_4 + Q_5) \times \ln\left(\frac{Q_6 + Q_7}{\sqrt{Q_8 + Q_9}(Q_{10} + Q_{11})(Q_{17} + Q_{18})} + \frac{Q_{12}}{(Q_{13} + Q_{14})(Q_{15} + Q_{16})}\right) \qquad (10)$$

where

$$Q_1 = -2(a^2 - g)(-2abd + a^2e + b^2g)$$

$$Q_2 = -2a^3bd + a^4e - 2abdg + b^2g^2 + a^2(2d^2 + (b^2 - e)g)$$

$$Q_3 = 2a(ad - bg)\sqrt{-2abd + d^2 + a^2e + b^2g - eg}$$

$$Q_4 = bd - ae$$

$$Q_5 = b\sqrt{-2abd + d^2 + a^2e + b^2g - eg}$$

$$Q_8 = -2a^3bd + a^4e - 2abdg + b^2g^2 + a^2(2d^2 + (b^2 - e)g)$$

$$Q_9 = 2a(ad - bg)\sqrt{-2abd + d^2 + a^2e + b^2g - eg}$$

$$Q_{10} = (a^2b - 2ad + bg)(-2abd + d^2 + a^2e + (b^2 - e)g)$$

$$Q_{11} = (3a^2bd - a^3e + bdg + a(-2d^2 + (-2b^2 + e)g)) \times \sqrt{-2abd + d^2 + a^2e + b^2g - eg}$$

$$Q_{13} = -2(a^2b - 2ad + bg)(-2abd + d^2 + a^2e + (b^2 - e)g)$$

$$Q_{14} = 2(-3a^2bd + a^3e - bdg + a(2d^2 + 2b^2g - eg)) \times \sqrt{-2abd + d^2 + a^2e + b^2g - eg}$$

$$Q_{15} = \sqrt{-2abd + d^2 + a^2e + b^2g - eg}$$

$$Q_{18} = -\sqrt{-2abd + d^2 + a^2e + b^2g - eg}$$

The variables Q_6, Q_7, Q_{16}, and Q_{17} are linearly dependent on x. The variable Q_{12} is quadratically dependent on x. It is useful, for computational speed, to break Q_6, Q_7, Q_{16}, and Q_{17} down to their linear forms in x (so that $Q_6 = R_{60} + xR_{61}$), while Q_{12} remains in its original form

$$Q_i = R_{i,0} + xR_{i,1} \quad i \in (6, 7, 16, 17)$$

$$R_{6,0} = 2d(a^2 - g)^2(-2abd + d^2 + a^2e + (b^2 - e)g)$$

$$R_{6,1} = 2g(a^2 - g)^2(-2abd + d^2 + a^2e + (b^2 - e)g)$$

$$R_{7,0} = 2(a^2 - g)^2(-abd + d^2 + a^2e - eg) \times \sqrt{-2abd + d^2 + a^2e + b^2g - eg}$$

$$R_{7,1} = 2a(a^2 - g)^2(ad - bg) \times \sqrt{-2abd + d^2 + a^2e + b^2g - eg}$$

$$R_{16,0} = d - ab$$

$$R_{16,1} = g - a^2$$

$$R_{17,0} = ab - d$$

$$R_{17,1} = a^2 - g$$

$$Q_{12} = 4(a^2 - g)^2\sqrt{-2abd + d^2 + a^2e + b^2g - eg} \times \sqrt{gx^2 + 2dx + e}$$

While this solution to Eq. (1) appears lengthy, but easy enough to put into computer code, there are two complications. The first complication is that geometry will occasionally cause singularities in these formulas. (For example, when the bottom plane of the prism is parallel to the x–y plane, then $a_1 = a_2 = 0$, and if at the same time one of the side faces, k, of the prism is parallel to the x–z plane, then $m_k = 0$, and this combination causes a singularity in the C_6 and C_7 terms which will cause a computer failure if the above solution is coded without caution.) Using limits, these singularities may all be eliminated with appropriate coding in the integration subroutine. This is covered in a later section. The second complication is that complex numbers are used in the above equations. In FORTRAN 77 and C, no standard function exists for taking square roots of complex numbers, although coding one is a fairly simple task (such a function does, formally, exist in C++ using the *complex* template). However, whether one writes one's own subroutine, or uses an existing one, means that two equally possible roots of a complex number could be computed, which was shown in initial tests to cause occasional incorrect computations. In order to remove this source of error, the following steps were taken. Note in Eq. (2) that there is a difference of two functions. Of specific interest, the following two terms will occur if Eq. (2) is expanded:

$$\begin{aligned} \Delta C_6 &= C_6(x_2) - C_6(x_1) \\ \Delta C_7 &= C_7(x_2) - C_7(x_1) \end{aligned} \tag{11}$$

If Eq. (11) is used as written, there is an instability in the computer code which depends on the choice of which complex square root is computed. However, because C_6 and C_7 are logarithmic functions, and also because the Q_1 through Q_5 terms have no dependence on x one may re-write Eq. (11) as

$$\begin{aligned} \Delta C_6 &= D_6 \ln[E_6(x_2)] - D_6 \ln[E_6(x_1)] = D_6 \ln\left[\frac{E_6(x_2)}{E_6(x_1)}\right] \\ \Delta C_7 &= D_7 \ln[E_7(x_2)] - D_7 \ln[E_7(x_1)] = D_7 \ln\left[\frac{E_7(x_2)}{E_7(x_1)}\right] \end{aligned} \tag{12}$$

where D_6, E_6, D_7 and E_7 are functions of the Q variables and can be inferred from Eqs. (9), (10) and (11). Using Eq. (12) for ΔC_6 and ΔC_7 [rather than Eq. (11)] will yield stability and correct results, so long as consistency is used in computing the square root of complex numbers.

Gravitational attraction of a polygonally shaped vertical prism

The gravitational attraction generated at a point P by an homogeneously dense (ρ=constant) mass distributed over a volume, Π, in (x, y, z) space, is

$$g(x_p, y_p, z_p) = -G\rho \int_{\Pi} \frac{(z - z_p)}{l^3} \mathrm{d}\Pi \tag{13}$$

where l is the distance from point (x_p, y_p, z_p) to point (x, y, z). For the purpose of this paper, the volume (Π) is assumed to be a vertical prism, with sloping top and bottom. (A "vertical prism" will be defined as any polyhedron, of $N + 2$ faces, where N of those faces are contained in planes perpendicular to the x–y plane.). The nature of the "sides" (the N vertical faces) of a vertical prism allows any volumetric (3-D) integral over such a prism to be broken up into an integral over an area A (where A is the projection of Π onto the x–y plane) and the vertical coordinate, z

$$\int_{\Pi} f(x, y, z)\mathrm{d}\Pi = \int_{A} \int_{z} f(x, y, z)\mathrm{d}z\, \mathrm{d}A \tag{14}$$

Note that A is a polygon, and can be of any general shape (i.e. convex or concave). Applying Eq. (13) to Eq. (14) yields

$$g(x_p, y_p, z_p) = -G\rho \int_{A} \int_{z=z_1(x,y)}^{Z_2(x,y)} \frac{z - z_p}{\left[(x - x_p)^2 + (y - y_p)^2 + (z - z_p)^2\right]^{(3/2)}} \mathrm{d}z\mathrm{d}A \tag{15}$$

where $z_1(x, y)$ and $z_2(x, y)$ are the equations of the bottom and top bounding planes of the vertical prism. The outer integral (over A) is shorthand for a double integration in x and y, over the area bounded by polygon A. Attempting to evaluate this integral over an irregularly shaped polygon, A, can involve very complicated upper and lower limits for both the x and y integrals. In order to avoid this complication, one can use a trick of calculus (used also in Woodward 1975), breaking down 2-D integration over an irregularly shaped polygon into a sum of integrals, where each component integral is "quasi-trapezoidal"

$$\int_{A} f(x, y)\mathrm{d}A = \sum_{i=1}^{N} \int_{x=x_i}^{x_j} \int_{y=y_0}^{m_k x + n_K} f(x, y)\mathrm{d}x\ \mathrm{d}y \tag{16}$$

In Eq. (16), N is the number of sides of polygon A and as i equals $(1, 2 \ldots N-2, N-1, N)$, the variable j equals $(2, 3 \ldots N-1, N, 1)$ and k equals (3, 4 ... $N, 1, 2$). The lower limit y_0 is arbitrary, but for simplicity can be taken as the y coordinate of the centroid of the polygon. The values m_k and n_k are the coefficients of line k, where the kth line connects the ith and jth points of the polygon (see Fig. 1). Caution should be used when applying Eq. (16). When $x_i = x_j, m_k$ goes to infinity (that is, the kth side of the polygon is parallel to the y axis). Thankfully, when this occurs, the integral in Eq. (16) goes to zero, and should therefore be separately coded in programs as such

$$\int_{x=x_i}^{x_j} \int_{y=y_0}^{m_k x+n_k} f(x,y)\mathrm{d}x\,\mathrm{d}y = 0 \quad \text{if } m_k = \infty \ (x_i = x_j) \tag{17}$$

This was not mentioned in Woodward (1975), even though the occurrence of north/south lines (where m_k goes to infinity) is very prevalent in most DEM (Digital Elevation Model) applications. Inserting Eq. (16) into Eq. (15), and centering the coordinate frame at the point of interest, the following equation arises:

$$g(x_p=0, y_p=0, z_p=0) = -G\rho \sum_{i=1}^{N} \int_{x=x_i}^{x_j} \int_{y=y_0}^{m_k x+n_k} \int_{z=A_1x+A_2y+A_3}^{a_1x+a_2y+a_3} \frac{z}{\sqrt{x^2+y^2+z^2}} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{18}$$

where the equations of the top and bottom planes (z_1 and z_2) have been written out as

$$\begin{aligned} z_1(x,y) &= A_1x + A_2y + A_3 \\ z_2(x,y) &= a_1x + a_2y + a_3 \end{aligned} \tag{19}$$

Performing the inner two integrals (over y and z) in Eq. (18) is fairly straightforward, and leads to the following:

$$g = -G\rho \sum_{i=1}^{N} \int_{x=x_i}^{x_j} \sum_{l=1}^{4} s_l \ln\left[\alpha_l x + \beta_l + \sqrt{\gamma_l x^2 + 2\delta_l x + \eta_l}\right] \tag{20}$$

where $s_l, \alpha_1, \beta_l, \gamma_l, \delta_l$ and η_l are as follows:
For $l=1$:

$$\begin{aligned} s_l &= +\frac{1}{\sqrt{1+a_2^2}} \\ \alpha_l &= \frac{2}{\sqrt{1+a_2^2}}\left(a_1a_2 + m_k + m_ka_2^2\right) \\ \beta_l &= \frac{2}{\sqrt{1+a_2^2}}\left(a_2a_3 + n_k + n_ka_2^2\right) \\ \gamma_l &= 4\left(1 + a_1^2 + 2a_1a_2m_k + m_k^2 + a_2^2m_k^2\right) \\ \delta_l &= 4\left(a_1a_3 + a_2a_3m_k + a_1a_2n_k + m_kn_k + a_2^2m_kn_k\right) \\ \eta_l &= 4\left(a_3^2 + 2a_2a_3n_k + n_k^2 + a_2^2n_k^2\right) \end{aligned} \tag{21}$$

For $l=2$:

$$\begin{aligned} s_l &= -\frac{1}{\sqrt{1+A_2^2}} \\ \alpha_l &= \frac{2}{\sqrt{1+A_2^2}}\left(A_1A_2 + m_k + m_kA_2^2\right) \\ \beta_l &= \frac{2}{\sqrt{1+A_2^2}}\left(A_2A_3 + n_k + n_kA_2^2\right) \\ \gamma_l &= 4\left(1 + A_1^2 + 2A_1A_2m_k + m_k^2 + A_2^2m_k^2\right) \\ \delta_l &= 4\left(A_1A_3 + A_2A_3m_k + A_1A_2n_k + m_kn_k + A_2^2m_kn_k\right) \\ \eta_l &= 4\left(A_3^2 + 2A_2A_3n_k + n_k^2 + A_2^2n_k^2\right) \end{aligned} \tag{22}$$

For $l=3$:

$$\begin{aligned} s_l &= -\frac{1}{\sqrt{1+a_2^2}} \\ \alpha_l &= \frac{2}{\sqrt{1+a_2^2}}(a_1a_2) \\ \beta_l &= \frac{2}{\sqrt{1+a_2^2}}(a_2a_3 + y_0 + y_0a_2^2) \\ \gamma_l &= 4(1 + a_1^2) \\ \delta_l &= 4(a_1a_3 + a_1a_2y_0) \\ \eta_l &= 4(a_3^2 + 2a_2a_3y_0 + y_0^2 + a_2^2y_0^2) \end{aligned} \tag{23}$$

For $l=4$:

$$\begin{aligned} s_l &= +\frac{1}{\sqrt{1+A_2^2}} \\ \alpha_l &= \frac{2}{\sqrt{1+A_2^2}}(A_1A_2) \\ \beta_l &= \frac{2}{\sqrt{1+A_2^2}}(A_2A_3 + y_0 + y_0A_2^2) \\ \gamma_l &= 4(1 + A_1^2) \\ \delta_l &= 4(A_1A_3 + A_1A_2y_0) \\ \eta_l &= 4(A_3^2 + 2A_2A_3y_0 + y_0^2 + A_2^2y_0^2) \end{aligned} \tag{24}$$

Up to this point, the calculus involved has already been documented in previous papers. Specifically, Woodward (1975) arrives at an equation similar to Eq. (20), but that paper has an algebraic difference with regard to the α, β, γ, δ and η values (and has unmentioned singularity conditions, discussed later). Woodward's values for α, β, γ, δ and η are scaled relative to the values listed in Eqs. (21)–(24) as follows:

$$[\alpha_l, \beta_l] = \left[\frac{2}{\sqrt{1+a_2^2}} \text{or} \frac{2}{\sqrt{1+A_2^2}}\right] \times \left[\alpha_l^W, \beta_l^W\right]$$

$$[\gamma_l, \delta_l, \eta_l] = \left[\frac{4}{1+a_2^2} \text{or} \frac{4}{1+A_2^2}\right] \times \left[\gamma_l^W, \delta_l^W, \eta_l^W\right] \quad (25)$$

where the superscript W refers to Woodward's values, and the "or" indicates that a_2 values be used for $l = 1, 3$ and A_2 values be used for $l = 2, 4$. Although this would at first indicate an error in Woodward (1975), there is not. Specifically, if one uses the α, β, γ, δ and η values presented in Eqs. (21)–(24), or those of Woodward (1975), the exact same value for the gravitational attraction is computed. Thus a word of warning about symbolic packages like Mathematica: do not assume that the solution which comes out of such a package is either the simplest, or the only, solution to any given problem.

The final step in computing the gravitational attraction is to perform the integration over x which remains in Eq. (20). Unlike the method of Woodward (1975), the method used here will be the one presented at the beginning of this paper. Therefore, Eq. (20) becomes

$$g = -G\rho \sum_{i=1}^{N} \sum_{l=1}^{4} s_l \big(F(\alpha_l, \beta_l, \gamma_l, \delta_l, \eta_l, x_j) - F(\alpha_l, \beta_l, \gamma_l, \delta_l, \eta_l, x_i)\big) \quad (26)$$

Equation (26) represents a closed analytical form for the gravitational attraction of any polygonally shaped vertical prism with sloping top and bottom faces. A single subroutine (called *intlog5*, not presented in this manuscript) was written in FORTRAN 77 which computes, for any α_l, β_l, γ_l, δ_l, η_l, x_i and x_j values, the difference seen in Eq. (26). For actual implementation, however, a second subroutine, *intlog5a* (available at the National Geodetic Survey web page at http://www.ngs.noaa.gov/GEOID), was written which specifically exploits the interactions between the values of α_l, β_l, γ_l, δ_l and η_l that occur in this particular problem, and has special code to avoid some singularities that occur due to geometry (mentioned in the next section). Prior to calling *intlog5a*, the individual α_l, β_l, γ_l, δ_l and η_l values are computed for a given i, l combination (remember that α_l, β_l, γ_l, δ_l and η_l are dependent on m_k and n_k, and k is dependent on i). The subroutine is then called for a given x_i and x_j with the appropriate α_l, β_l, γ_l, δ_l and η_l values.

Coding in FORTRAN 77 and avoiding singularities

Specific cases where singularities occur do exist, and should be coded appropriately to avoid computational overflows. It must be stressed that the method of Woodward (1975) fails to mention many singularities of geometry, and coding that method without proper care can cause the code to crash or, more dangerously, run and yield false results. The case of a face parallel to the y axis has been mentioned in Eq. (17). In addition, the geometry of the solution (i.e., station of interest sits on a face or a vertex of the triangular prism) will occasionally cause terms either to equal zero exactly, or to approach zero in the limit. Special code should be used to prevent attempts to compute these values, and they should instead be identified and set to zero. Most of the "singularity" conditions will be easily identified by one of α, β, γ, δ or η (or some combination of them) being zero. One exception is the case where a_3 (or A_3) is equal to zero (i.e., the plane of the top or bottom face of the prism intersects point P). In that case, the singularity is identified through a polynomial equation in α, β, γ, δ and η (see footnote "a" in Table 1). The special cases are summarized in Table 1. A "0" entry indicates that when the subroutine identifies the appropriate combination of input variables being equal to zero, certain $\Delta C_I = C_I(x_j) - C_I(x_i))$ values should be set to zero (where $I = 1, 2, \ldots 7$). For ease of reading, the "l" subscripts on α, β, γ, δ and η have been dropped.

In addition to the special cases in Table 1 which can cause ΔC_I to go to zero, there are a few special cases where the individual components of the ΔC_I term [either $C_I(x_j)$ or $C_I(x_i)$] can go to zero. Specifically, these cases occur when x_i or x_j (but not both at the same time) equals zero. In general, when $x_i = 0$ or $x_j = 0$, then $C_1(x_i$ or $x_j) = 0$ and $C_4(x_i$ or $x_j) = 0$, and for a few special cases, $C_2(x_i$ or $x_j) = 0$ (see Table 2).

Application to a triangular prism

Although Eq. (26) works for any polygonally shaped vertical prism, the case of greatest interest would be for a triangular prism (rectangular prisms have been studied and documented with simpler solutions in many other papers). For the case of a triangular prism, the number of vertical faces, N, is 3, so that

$$g = -G\rho \sum_{i=1}^{3} \sum_{l=1}^{4} s_l (F(\alpha_l, \beta_l, \gamma_l, \delta_l, \eta_l, x_j) - F(\alpha_l, \beta_l, \gamma_l, \delta_l, \eta_l, x_i)) \quad (27)$$

The stability of the software was tested in a number of ways. First, gravitational attraction of triangular prisms was computed using Eq. (27) as well as using numerical integration. In every case, Eq. (27) was correct to sub-microGal accuracy, with a significant increase in speed (milliseconds, versus seconds to days) over the numerical integration. Specifically, a right-triangular prism was built, 1000 m on a side, with the following initial coordinates, relative to a station at coordinates (0,0,0):

$$\begin{aligned} &x(1) = 1000 \;\; y(1) = 1000 \;\; z_{\text{lower}}(1) = -4000 \;\; z_{\text{upper}}(1) = 0 \\ &x(2) = 1000 \;\; y(2) = 2000 \;\; z_{\text{lower}}(2) = -3900 \;\; z_{\text{upper}}(2) = -100 \\ &x(3) = 2000 \;\; y(3) = 1000 \;\; z_{\text{lower}}(3) = -4500 \;\; z_{\text{upper}}(3) = -200 \end{aligned} \quad (28)$$

Table 1. Values to set to zero when singularity conditions occur. For any values x_i and x_j

Variables(s) which equal zero	ΔC_1	ΔC_2	ΔC_3	ΔC_4	ΔC_5	ΔC_6	ΔC_7
a_3 (or A_3)[a]	–	0	0	–	–	0	0
None of α, β, γ, δ, η	–	–	–	–	–	–	–
α	–	–	–	–	–	–	–
β	–	–	–	–	–	–	–
γ	Imp[b]	Imp[b]	Imp[b]	Imp[b]	Imp[b]	Imp[b]	Imp[b]
δ	–	–	–	–	–	–	–
η	Imp[c]	Imp[c]	Imp[c]	Imp[c]	Imp[c]	Imp[c]	Imp[c]
α, β	–	–	–	–	0	0	0
α, δ[d]	–	–	0	–	–	–	–
β, δ[e]	–	–	0	–	0	–	–
α, β, δ	–	–	0	–	0	0	0
β, δ, η[f]	–	0	0	–	0	0	0
α, β, δ, η[g]	–	0	0	–	0	0	0

[a] Check for a_3 or $A_3 = 0$ by checking if $2\alpha\beta\delta - \delta^2 - \alpha^2\eta - \beta^2\gamma + \eta\gamma = 0$
[b] It is impossible for $\gamma = 0$. All further combinations of variables with γ will be omitted
[c] It is impossible for $\eta = 0$ without the additional requirements of $\beta = 0$ and $\delta = 0$ at the same time
[d] This combination occurs when a_1, a_2 and $m_k = 0$ or when A_1, A_2 and $m_k = 0$
[e] This combination occurs when a_1, a_2 and $n_k = 0$ or when A_1, A_2 and $n_k = 0$
[f] This combination occurs when a_3 and $n_k = 0$ or when A_3 and $n_k = 0$
[g] This combination occurs when m_k, n_k, a_3 and either a_1 or $a_2 = 0$ or when m_k, n_k, A_3 and either A_1 or $A_2 = 0$

Table 2. Additional values to set to zero when singularity conditions occur, and $x_i = 0$ or $x_j = 0$

Variable(s) which equal zero	$C_1(x_i$ or $x_j)$	$C_4(x_i$ or $x_j)$	$C_2(x_i$ or $x_j)$
x_i or x_j	0	0	–
x_i or x_j and α, δ	0	0	0
x_i or x_j and β, δ	0	0	0
x_i or x_j and α, β, δ	0	0	0

The entire prism was moved progressively closer to the station, by moving the position of $x(1), y(1)$ along the line $x = y$ in steps of $dx = dy = 200$ m (see Fig. 2), and the gravitational attraction of the prism computed to an accuracy of 1 μGal. Table 3 summarizes the results. For this example, Woodward's formulas either fail to yield the correct result or fail entirely because the geometry of this situation allows $a_3 = 0$ without his two "special cases" of $n = 0$ or $a_1 = a_2 = m_k = 0$.

Table 3. Computational time for closed form, and numerical integration for a test prism

$x(1)$, $y(1)$	Time for Woodward closed form (s)	Time for Smith closed form (s)	Time for numerical quadrature (s)	Number of iterations for quadrature
0, 0	0.075[a]	0.058	22.476	6
200, 200	0.082[a]	0.059	177.662	7
400, 400	0.083[a]	0.059	177.617	7
600, 600	0.082[a]	0.061	1412.510	8
800, 800	0.082[a]	0.061	1412.510	8
1000, 1000	fails[b]	0.061	578 564 096[c]	12[c]

[a] Although speed is comparable, the Woodword scheme arrives at the wrong value by 0.01 to 0.02 mGal
[b] In this geometry, situations occur where $a_3 = 0$ while neither $n_k = 0$ nor $a_1 = a_2 = m_k = 0$, which are the only two situation which Woodward has discussed
[c] Extrapolated after 11 iterations (the 12th iteration would have taken 2 months)

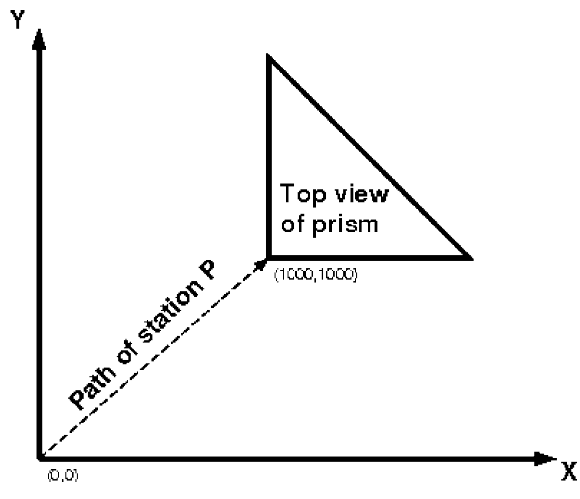

Fig. 2.

Note that as the prism comes closer and closer to the station, the number of iterations increases. Each iteration breaks down the prism into smaller and smaller cubes. When the prism is too close to the station, the near-field effects of quadrature approximation are too large, requiring smaller cubes for a more accurate computation. This makes the quadrature solution take longer and longer, eventually making it quite impractical to compute, through numerical integration, the gravitational attraction of nearby prisms. (One can reduce this time, but not to the level of a closed form, by using multiple-sized cubes in the quadrature solution.) However, the closed form shows no distance-dependent time to compute, and even for the prism located at $x(1), y(1) = (0, 0)$, the time advantage is a factor of 388 times faster. Clearly there is good reason to use the closed form.

Next, as a simple example, various flat-topped rectangular prisms were broken up into two right-triangular prisms, and gravitational attraction computed using Eq. (27), as well as closed forms for rectangular prisms (Nagy 1966). Agreement was always to the nanoGal or better.

Finally, an equilateral triangularly shaped prism, 1 m high, was centered 10 m below a test station. The side of the prism was increased from 1 to 1 000 000 m, and the attraction of the prism approached 0.1119 mGal. This is to be expected, as the triangular prism is approaching a Bouguer plate (where the attraction of the plate, no matter how far it is below the station, is 0.1119 mGal, for a plate 1 m thick).

New method versus Woodward

There are distinct similarities between the computational method outlined here and that of Woodward (1975). However, numerous differences make the present method superior, even discounting the non-essential algebraic difference in computing α, β, γ, δ and η. The most important difference between the two methods is in the treatment of singularities. Woodward identifies only two special cases ($a_3 = 0$ with $n_k = 0$ and $a_3 = 0$ with $a_1 = a_2 = m_k = 0$). The present paper addresses seven special cases. It was seen that Woodward's formulas fail if singularities occur which he has not covered. Additionally, Woodward's method occasionally suffers from rounding errors, yielding 0.01- to 0.02-mGal errors. If these errors are systematic, they can significantly affect the total computation of the terrain signal on gravity. Finally, Woodward's method addresses only the special case of an $N = 3$-sided prism, whereas this paper addresses the general case for all $N \geq 3$. While the utility of prisms with four or more faces is limited, it is nonetheless an advantage over Woodward. This is briefly discussed in the next section.

The $N > 3$ problem

There are few cases where a DEM is broken up into prisms that are neither triangular ($N = 3$) nor rectangular ($N = 4$), and therefore the $N > 3$ solution has very limited applications (the $N = 4$ solution, as already mentioned, has already seen a great deal of historical treatment). However, the solution presented in this paper is good for all $N \geq 3$, so a few words must be said here for completeness.

The use of a flat plane to model the topography defined by $N > 3$ points means that some approximate surface must be used, if the N points are not co-planar. A simple best-fit plane is the easiest solution to this problem. Equation (29) shows the formulas for computing A_1, A_2, and A_3 for an $N > 3$ prism top

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i y_i & \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i y_i & \sum_{i=1}^{N} y_i^2 & \sum_{i=1}^{N} y_i \\ \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} y_i & N \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{N} x_i z_i \\ \sum_{i=1}^{N} x_i z_i \\ \sum_{i=1}^{N} z_i \end{bmatrix} \quad (29)$$

where x_i, y_i, and z_i are the Cartesian coordinates of the prism top corners. Equation (29) can be used in a similar way to compute a_1, a_2, and a_3 for the bottom of the prism. Equation (29) should only be used in cases where the topography is sufficiently smooth to allow for a flat approximation (such as in large flood plains). In all other cases, the topography should be modeled with triangular prisms which allow for a diverse number of slopes in a local region.

Conclusions

A closed, stable, fast formula for computing the gravitational attraction of any $N+2$-faced vertical prism, with sloping top and bottom faces, has been shown. The algebra is lengthy, but is easily coded. Occasionally geometry can cause the formula to become unstable, but these instabilities have been identified, and special logic for the subroutines has been presented. In deriving this formula there was a heavy dependence on the symbolic language of Mathematica 3.0. Through the use of Mathematica, a correct, but different formula than has been previously published (Woodward 1975) was found. The advantages over the method of Woodward include better treatment of singularities, closer agreement with numerical integration, and allowance for $N > 3$-sided prisms. It was found that the closed formula provided a more efficient method for computing triangularly shaped prisms, particularly in the region near a station of interest.

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