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ORIGINAL ARTICLE

An analytical study on the carrier-phase linear combinations for triple-frequency GNSS

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Abstract The linear combinations of multi-frequency carrier-phase measurements for Global Navigation Satellite System (GNSS) are greatly beneficial to improving the performance of ambiguity resolution (AR), cycle slip correction as well as precise positioning. In this contribution, the existing definitions of the carrier-phase linear combination are reviewed and the integer property of the resulting ambiguity of the phase linear combinations is examined. The general analytical method for solving the optimal integer linear combinations for all triple-frequency GNSS is presented. Three refined triple-frequency integer combinations solely determined by the frequency values are introduced, which are the ionosphere-free (IF) combination that the Sum of its integer coefficients equal to 0 (IFS0), the geometry-free (GF) combination that the Sum of its integer coefficients equal to 0 (GFS0) and the geometry-free and ionosphere-free (GFIF) combination. Besides, the optimal GF, IF, extra-wide lane and ionosphere-reduced integer combinations for GPS and BDS are solved exhaustively by the presented method. Their potential applications in cycle slip detection, AR as well as precise positioning are discussed. At last, a more straightforward GF and IF AR scheme than the existing method is presented based on the GFIF integer combination.

Keywords GPS · BDS · Triple-frequency · Optimal linear combination · Cycle slip · Ambiguity resolution · Geometry-free and ionosphere-free combination

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1 Introduction

Global Navigation Satellite System (GNSS) precise positioning at the centimeter level can be attained when the carrier-phase integer-cycle ambiguity is resolved correctly. If multi-frequency carrier-phase measurements are available, some linear combinations of these measurements can be formed to improve positioning performance and to achieve ambiguity resolution (AR) as well as cycle slip correction more efficiently.

The linear combinations of GPS dual-frequency carrier-phase measurements were studied firstly, such as wide-lane (WL) combination, the ionosphere-free (IF) combination (Blewitt 1989; Dong and Bock 1989) and the geometer-free (GF) combination (Blewitt 1990). Afterward, the systematic research on the theory of linear combination arose. The systematic search for all possible WL combination of the dual-frequency GPS was presented by Cocard and Geiger (1992). The general definition of linear combination in cycles was given by Han (1995), Han and Rizos (1996). The comprehensive study of the inter-frequency combinations was presented by Collins (1999), in which the WL, reduced-ionosphere and noise-reduction combinations were examined in detail.

For the three or four frequency carrier-phase measurements case, such as the modernized GPS, Beidou navigation satellite system (BDS) and GALILEO, the linear combinations are much more complicated. Based on the pre-defined extra-wide lane (EWL) and WL linear combinations, Forssell et al. (1997) and Jung et al. (2000) presented the three-carrier ambiguity resolution (TCAR) method and the cascade integer resolution (CIR) method for GALILEO and GPS, respectively. Han and Rizos (1999) presented the definition of the carrier-phase linear combination for the triple-frequency case and discussed the AR strategies without and with distance constraints by applying the LAMBDA method (Teunissen 1995) to the GF GNSS model. Richert and El-Sheimy (2007) studied the optimal GPS and GALILEO linear combinations for differential positioning over medium–long baselines. Feng (2008) introduced the optimal ionosphere-reduced linear combinations for the geometry-based TCAR. Cocard et al. (2008) systematically investigated the GPS triple-frequency integer phase combinations with an analytical method firstly and found that the sum of the integer coefficients of the combinations was an important indicator for systematic classification of sets of combinations, and this method was extended to BDS by Li et al. (2012a). Zhang and He (2015) examined the BDS triple-frequency linear combinations based on the relevant methods of Richert and El-Sheimy (2007) and Cocard et al. (2008). Besides, Odijk (2003) examined the IF combinations for the modernized triple-frequency GPS. Simsky (2006) presented a triple-frequency GF and IF combination for extracting the carrier-phase multipath information. Hatch (2006) presented a GF and refraction-corrected method for long baseline AR by using some refined linear combinations. Li et al. (2010) also studied the GF and IF combinations for estimating the narrow lane (NL) ambiguity without distance constraints. Li et al. (2012b) presented the optimal triple-frequency IF combination and the GF and IF combination for long baseline AR and precise positioning and shown that the GALILEO (E1, E6, E5a) has the best performance of long baselines AR, and the similar study was presented by Wang and Rothacher (2013).

A review of the existing study on linear combination for triple-frequency GNSS reveals that at least the following four problems have not been dealt with completely: (1) some phase combinations expressed in units of meter are still considered to be suffering from a loss of the integer nature of ambiguities, for example the GF and IF combination or the phase multipath-combination (Henkel 2009) ; (2) though the analytical method presented by Cocard et al. (2008) for solving optimal integer combinations for GPS frequencies can be applied to other GNSS system, some tiny modification may be still needed when it is applied to the GNSS system with especial frequency distribution, for instance BDS; (3) a systematic search for the GF integer combinations is still absent; (4) though several GF and IF schemes for solving the third or NL ambiguity without distance constraints have been presented, such as Forssell et al. (1997), Bonillo-Martínez et al. (1999), Hatch (2006), Li et al. (2010, 2012b), Wang and Rothacher (2013), it is still ambiguous that whether they are equivalent or which scheme should be the best choice.

In this contribution, the existing several definitions of the carrier-phase linear combination are reviewed and the integer property of the resulting ambiguity of the phase linear combinations is examined in Sect. 2. Extending to the approach described in Cocard et al. (2008), the optimal integer linear combinations are solved rigorously by a generalized analytical method in Sect. 3. Then three refined triple-frequency integer linear combinations solely determined by the frequency values are introduced. In Sect. 4, the optimal GF, IF, EWL and ionosphere-reduced integer linear combinations are presented for GPS and BDS, followed by the analysis on their potential application in cycle slip detection, short or long baseline AR as well as precise positioning. In Sect. 5, a more straightforward scheme for the third or NL ambiguity is developed based on the presented GF and IF integer combination. The summaries are given in Sect. 6.

We make use of the following notation: the integer and nonzero integer sets are denoted as $\mathbb{Z}$ and $\mathbb{Z}^*$, gcd() is the greatest common divisor operator, det() is the determinate operator. Three frequency values f_1, f_2 and f_3 of GNSS can be expressed as the product of the virtual fundamental frequency f_0 and three prime integer multiplier l_1, l_2 and l_3, with $f_0 = \gcd(f_1, f_2, f_3)$ and $l_1 > l_2 > l_3$. $\lambda_0 = \frac{c}{f_0}$ is the virtual fundamental wavelength respecting to the virtual fundamental frequency f_0. The overview of different GNSS frequency triplets is given in Table 1.

2 Definition of the carrier-phase linear combination

The original phase measurements in cycles and meters can be, respectively, expressed as:

$$\varphi_j = \frac{f_j}{c}\rho - \kappa_j d_{\text{ion}} + N_j + \upsilon_j \tag{1a}$$

$$\phi_j = \lambda_j \varphi_j = \rho - \mu_j d_{\text{ion}} + B_j + \varepsilon_j, \tag{1b}$$

where φ_j is the original phase measurement in units of cycle; f_j is the jth frequency; c is the velocity of light in vacuum; ρ is the frequency-independent term containing the geometrical distance between receiver and satellite antenna phase center, the receiver and satellite clock biases and the troposphere delay; $\kappa_j = \frac{f_1^2}{cf_j}$ is a frequency-dependent amplification factor; d_{ion} is the first-order ionosphere delay on the first frequency f_1 in meters; N_j is the sum of the initial phase, the phase ambiguity and the instrumental phase delay; υ_j is the unmodeled errors in units of cycle, such as the measurement noise and the multipath error; $\lambda_j = \frac{c}{f_j}$ is the wavelength of the jth frequency; $\phi_j = \lambda_j \varphi_j$ is the original phase measurement in meters; $\mu_j = \lambda_j \kappa_j = \frac{f_1^2}{f_j^2}$, $B_j = \lambda_j N_j$, $\varepsilon_j = \lambda_j \upsilon_j$ are the counterparts in meters of κ_j, N_j, υ_j, respectively.

From Eq. (1a), the triple-frequency phase combination expressed in units of cycle can be written as (Han 1995; Han and Rizos 1996, 1999; Cocard et al. 2008):

Table 1 Overview of different GNSS frequency triplets

System	Nominal frequency (MHz)			Virtual fundamental frequency (MHz)	Integer multiplier			Virtual fundamental wavelength (m)
	f_1	f_2	f_3	f_0	l_1	l_2	l_3	λ_0
GPS (L1,L2,L5)	1575.42	1227.60	1176.45	10.23	154	120	115	29.3
BDS (B1,B3,B2)	1561.098	1268.52	1207.14	2.046	763	620	590	146.5
GAL-a (E1,E6,E5b)	1575.42	1278.75	1207.14	10.23	154	125	118	29.3
GAL-b (E1,E6,E5)	1575.42	1278.75	1191.795	5.115	308	250	233	58.6
GAL-c (E1,E6,E5a)	1575.42	1278.75	1176.45	10.23	154	125	115	29.3
GAL-d (E1,E5b,E5a)	1575.42	1207.14	1176.45	10.23	154	118	115	29.3

$$\varphi_{(i.j.k)} = i\cdot\varphi_1 + j\cdot\varphi_2 + k\cdot\varphi_3 = \frac{f_{(i.j.k)}}{c}\rho - \kappa_{(i.j.k)}d_{\text{ion}} + N_{(i.j.k)} + \upsilon_{(i.j.k)}, \tag{2}$$

where $()_{(i.j.k)} = i\cdot()_1 + j\cdot()_2 + k\cdot()_3$, for example, the resulting frequency of the phase combination $f_{(i.j.k)} = i\cdot f_1 + j\cdot f_2 + k\cdot f_3$. When $f_{(i.j.k)} \neq 0$, this combination can be expressed in meters again in the form of Eq. (1b) (Feng 2008; Li et al. 2010):

$$\phi_{\langle i.j.k\rangle} = \frac{c}{f_{(i.j.k)}}\varphi_{(i.j.k)} = \frac{i\cdot f_1\cdot\phi_1 + j\cdot f_2\cdot\phi_2 + k\cdot f_3\cdot\phi_3}{i\cdot f_1 + j\cdot f_2 + k\cdot f_3} = \rho - \mu_{\langle i.j.k\rangle}d_{\text{ion}} + B_{\langle i.j.k\rangle} + \varepsilon_{\langle i.j.k\rangle}, \tag{3}$$

where $()_{\langle i.j.k\rangle} = \frac{i\cdot f_1\cdot()_1 + j\cdot f_2\cdot()_2 + k\cdot f_3\cdot()_3}{i\cdot f_1 + j\cdot f_2 + k\cdot f_3}$. Though the linear coefficients i, j and k are not necessary to be integer in Eqs. (2) and (3), it is only necessary to consider the case of integer because any rational coefficients can be converted to integer coefficients by multiplying a common integer. In other words, the resulting ambiguities in Eqs. (2) and (3) still preserve the integer characteristic as long as the linear coefficients are the rational number.

From Eq. (1b), the triple-frequency phase combination in meters can be written as (Cocard and Geiger 1992; Collins 1999; Urquhart 2009):

$$\phi_{(x,y,z)} = x\cdot\phi_1 + y\cdot\phi_2 + z\cdot\phi_3 = (x+y+z)\cdot\rho - \mu_{(x,y,z)}d_{\text{ion}} + B_{(x,y,z)} + \varepsilon_{(x,y,z)}, \tag{4}$$

where $()_{(x,y,z)} = x\cdot()_1 + y\cdot()_2 + z\cdot()_3$. When the triple-frequency phase combinations are expressed as Eq. (4), the integer property of the resulting ambiguity is implicit or is even considered to be lost. Actually, for the any phase combination in the form of Eq. (4), it is easy to validate that the resulting ambiguity still retains the integer nature as long as the linear coefficients are the rational number. In other words, it can always find their counterparts in cycles with the form of Eq. (2).

3 Solving optimal phase combinations by the analytical method

From Eqs. (2) and (4), we know that the number of linear combinations is unlimited. Among the infinite number of linear combinations, however, only those that satisfy some important criteria are of interest (Seeber 2003): the resulting ambiguity retains the integer nature, reasonably large wavelength to help ambiguity fixing, low ionosphere influence and limited observation noise.

3.1 Optimal phase combination in cycles

The resulting frequency (or wavelength) $f_{(i,j,k)}$, ionosphere amplification factor $\kappa_{(i,j,k)}$ and the noise amplification factor are the key criteria for evaluating a linear combination. The frequency and the ionosphere amplification factor can be expressed by the so-called lane number l_n and ion number i_n with the definitions as follows (Cocard et al. 2008; Li et al. 2012a):

$$l_n = l_1\cdot i + l_2\cdot j + l_3\cdot k \tag{5a}$$

$$i_n = \frac{l_2 l_3}{g}\cdot i + \frac{l_1 l_3}{g}\cdot j + \frac{l_1 l_2}{g}\cdot k, \tag{5b}$$

where $g = \gcd(l_2 l_3, l_1 l_3, l_1 l_2)$. Then the wavelength and ionosphere amplification factor can be rewritten as:

$$\lambda_{[i,j,k]} = \frac{c}{f_{(i,j,k)}} = \frac{c}{l_n\cdot f_0} = \frac{\lambda_0}{l_n} \tag{6a}$$

$$\kappa_{(i,j,k)} = i\cdot\frac{f_1^2}{cf_1} + j\cdot\frac{f_1^2}{cf_2} + k\cdot\frac{f_1^2}{cf_3} = i_n\cdot\frac{g}{l_2 l_3}\cdot\frac{1}{\lambda_1} = q_n\cdot\frac{1}{\lambda_1} \tag{6b}$$

where, $\lambda_0 = \frac{c}{f_0}$, $q_n = i_n\cdot\frac{g}{l_2 l_3}$. Considering Eq. (5b), we know that for the combination $\varphi_{(1,0,0)}$ or φ_1, the value of q_n is equal to 1 because of $i_n = \frac{l_2 l_3}{g}$ when $(i,j,k) = (1,0,0)$.

Assuming that the noise of the original phase measurements expressed in cycles on all three frequencies are the same and statistically independent, then the standard deviation of the combined noise $\upsilon_{(i,j,k)}$ can be expressed as:

$$\sigma_{[i.j.k]} = \sqrt{i^2 + j^2 + k^2} \cdot \sigma_\upsilon = \eta \cdot \sigma_\upsilon, \tag{7}$$

where $\eta = \sqrt{i^2 + j^2 + k^2}$ is the noise amplification factor, σ_υ is the standard deviation of the phase noise in cycles.

Cocard et al. (2008) presented an analytical method for solving the optimal integer linear combination (i, j, k) with a given integer pair of l_n and i_n. The key of the method is that the integer linear coefficients (i, j, k) are expressed as the integer linear function of the lane number l_n, the ion number i_n and an arbitrary integer s_n by solving the Diophantine equation Eq. (5). Namely, there is an integer triplet (s_1, s_2, s_3) existing to make the determinant of the following integer matrix $\boldsymbol{Z}$ equal to 1, i.e., $\det(\boldsymbol{Z}) = \pm 1$:

$$\begin{bmatrix} l_n \\ i_n \\ s_n \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 \\ \frac{l_2 l_3}{g} & \frac{l_1 l_3}{g} & \frac{l_1 l_2}{g} \\ s_1 & s_2 & s_3 \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} = \boldsymbol{Z} \begin{bmatrix} i \\ j \\ k \end{bmatrix} \tag{8}$$

where $s_n = s_1 \cdot i + s_2 \cdot j + s_3 \cdot k$. For GPS ($l_1 = 154, l_2 = 120, l_3 = 115$), from Eq. (8), we get that

$$\det(\boldsymbol{Z})_{\mathrm{GPS}} = 18,095 s_1 - 125,892 s_2 + 107,134 s_3 \tag{9}$$

Because of gcd (18,095, −125,892, 107,134) = 1, there are integer triplets (s_1, s_2, s_3) existing to make the determinant $\det(\boldsymbol{Z})_{\mathrm{GPS}}$ equal to 1, for example (−121, 146, 192), then we have:

$$\begin{bmatrix} i \\ j \\ k \end{bmatrix}_{\mathrm{GPS}} = \boldsymbol{Z}^{-1} \begin{bmatrix} l_n \\ i_n \\ s_n \end{bmatrix}_{\mathrm{GPS}} = \begin{bmatrix} 70,224 & -6250 & 18,095 \\ -488,568 & 43,483 & -125,892 \\ 415,771 & -37,004 & 107,134 \end{bmatrix} \begin{bmatrix} l_n \\ i_n \\ s_n \end{bmatrix}_{\mathrm{GPS}} \tag{10}$$

For BDS (l_1=763, l_2=620, l_3=590), however

$$\det(\boldsymbol{Z})_{\mathrm{BDS}} = 2,769,690 s_1 - 14,512,278 s_2 + 1,166,8371 s_3 \tag{11}$$

Because of gcd (2,769,690, −14,512,278, 11,668,371)=33, there is no integer triplet (s_1, s_2, s_3) existing to make the determinant $\det(\boldsymbol{Z})_{\mathrm{BDS}}$ equal to 1. Namely, there are no integer linear combinations (i, j, k) existing for some given integer pairs (l_n, i_n). However, we can make the determinant $\det(\boldsymbol{Z})_{\mathrm{BDS}}$ equal to the greatest common divisor of the coefficients, for example $(s_1, s_2, s_3) = (-71, -143, -161)$, then:

$$\begin{bmatrix} i \\ j \\ k \end{bmatrix}_{\mathrm{BDS}} = \frac{1}{33} \begin{bmatrix} -482,979 & 15,450 \\ 2,530,654 & -80,953 \\ -2,034,733 & 65,089 \end{bmatrix} \begin{bmatrix} l_n \\ i_n \end{bmatrix} + \begin{bmatrix} 83,930 \\ -439,766 \\ 353,587 \end{bmatrix} s_n \tag{12}$$

Without loss of universality, a group of the integer linear combinations with the given integer pair (l_n, i_n) can be expressed as:

$$\boldsymbol{c} = \begin{bmatrix} i \\ j \\ k \end{bmatrix} = \frac{\boldsymbol{Z}_{\mathrm{adj}}}{\det(\boldsymbol{Z})} \begin{bmatrix} l_n \\ i_n \\ s_n \end{bmatrix} = \frac{\boldsymbol{l} \cdot l_n + \boldsymbol{i} \cdot i_n + \boldsymbol{s} \cdot s_n}{\det(\boldsymbol{Z})} = \frac{\boldsymbol{h} + \boldsymbol{s} \cdot s_n}{\det(\boldsymbol{Z})} \qquad s_n \in \mathbb{Z}, \tag{13}$$

where $\boldsymbol{Z}_{\mathrm{adj}}$ denotes the adjoint of the matrix $\boldsymbol{Z}$; $\boldsymbol{l}$, $\boldsymbol{i}$ and $\boldsymbol{s}$ are the first, second and third column of the matrix $\boldsymbol{Z}_{\mathrm{adj}}$; $\boldsymbol{h} = \boldsymbol{l} \cdot l_n + \boldsymbol{i} \cdot i_n$. Then, the noise amplification factor can be expressed as:

$$\eta = \sqrt{\boldsymbol{c}^{\mathrm{T}} \boldsymbol{c}} = \frac{\sqrt{(\boldsymbol{h} + \boldsymbol{s} \cdot s_n)^{\mathrm{T}} (\boldsymbol{h} + \boldsymbol{s} \cdot s_n)}}{\det(\boldsymbol{Z})} = \frac{\sqrt{\boldsymbol{s}^{\mathrm{T}} \boldsymbol{s} \cdot s_n^2 + 2 \boldsymbol{s}^{\mathrm{T}} \boldsymbol{h} \cdot s_n + \boldsymbol{h}^{\mathrm{T}} \boldsymbol{h}}}{\det(\boldsymbol{Z})} \tag{14}$$

The noise amplification factor can be minimized when

$$s_n = \mathrm{round}\left(-\frac{\boldsymbol{s}^{\mathrm{T}} \boldsymbol{h}}{\boldsymbol{s}^{\mathrm{T}} \boldsymbol{s}}\right) = \mathrm{round}\left(-\frac{\boldsymbol{s}^{\mathrm{T}} \boldsymbol{l} \cdot l_n + \boldsymbol{s}^{\mathrm{T}} \boldsymbol{i} \cdot i_n}{\boldsymbol{s}^{\mathrm{T}} \boldsymbol{s}}\right) = \mathrm{round}\left(-\frac{\boldsymbol{s}^{\mathrm{T}} \boldsymbol{l}}{\boldsymbol{s}^{\mathrm{T}} \boldsymbol{s}} l_n - \frac{\boldsymbol{s}^{\mathrm{T}} \boldsymbol{i}}{\boldsymbol{s}^{\mathrm{T}} \boldsymbol{s}} \frac{l_2 l_3}{g} q_n\right) \tag{15}$$

Because the $\det(\boldsymbol{Z})$ is equal to the greatest common divisor of the elements of the column vector $\boldsymbol{s}$, if the column vector $\boldsymbol{h}$ can be divided exactly by the $\det(\boldsymbol{Z})$, then the optimal integer coefficients (i, j, k) with the given integer pair (l_n, i_n) can be solved by Eqs. 13 and 15. Consequently, with given lane number l_n and ion number i_n, the process for solving the optimal integer combination (i, j, k) in terms of the noise amplification factor is as follows:

- From Eq. (8), the elements of the first and second row of the matrix $\boldsymbol{Z}$ can be determined based on the given triple-frequency values of GNSS. Then the determinant of the matrix $\boldsymbol{Z}$ can be expressed as $\det(\boldsymbol{Z}) = A_{31} \cdot s_1 + A_{32} \cdot s_3 + A_{33} \cdot s_3$, where A_{31}, A_{32} and A_{33} are the algebraic complements corresponding to the elements s_1, s_2 and s_3 of the third row, respectively.
- Solving a particular solution of the integer linear equation $A_{31} \cdot s_1 + A_{32} \cdot s_3 + A_{33} \cdot s_3 = \gcd(A_{31}, A_{32}, A_{33})$ in order

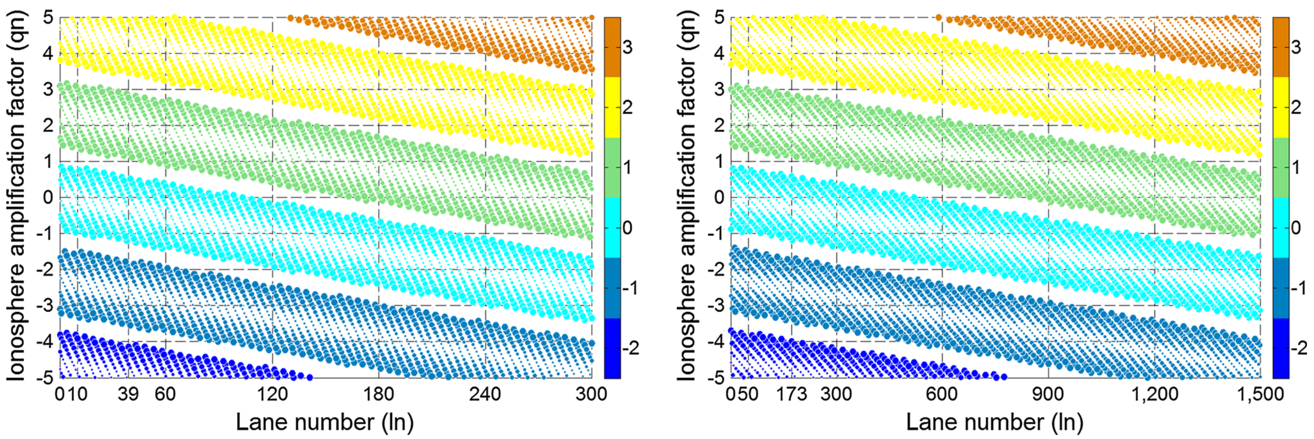

Fig. 1 GPS (*left*) and BDS (*right*) optimal integer phase combinations in the l_n–q_n plane

to determine the elements of the third row of the matrix $\boldsymbol{Z}$. As a result, we have $\det(\boldsymbol{Z}) = \gcd(A_{31}, A_{32}, A_{33})$.

- Computing the adjoint $\boldsymbol{Z}_{\text{adj}}$ of the matrix $\boldsymbol{Z}$; $\begin{bmatrix} \boldsymbol{l} & \boldsymbol{i} & \boldsymbol{s} \end{bmatrix} = \boldsymbol{Z}_{\text{adj}}$, computing $\boldsymbol{h} = \boldsymbol{l} \cdot l_n + \boldsymbol{i} \cdot i_n$ with the given integer pair (l_n, i_n). If $\boldsymbol{h}$ can be divided exactly by the $\det(\boldsymbol{Z})$, then the optimal integer linear coefficients (i, j, k) is solved by $\boldsymbol{c} = \frac{\boldsymbol{h} + \boldsymbol{s} \cdot s_n}{\det \boldsymbol{Z}}$ with $s_n = \text{round}\left(-\frac{\boldsymbol{s}^{\text{T}}\boldsymbol{h}}{\boldsymbol{s}^{\text{T}}\boldsymbol{s}}\right)$; otherwise, there will be no relevant integer linear coefficients for the given lane number l_n and ion number i_n.

From the above algorithms, for GPS: $0 \le l_n \le 300$ and $|i_n| \le 5 \times 1380$ ($|q_n| \le 5$), for BDS: $0 \le l_n \le 1500$ and $|i_n| \le 5 \times 36{,}580$ ($|q_n| \le 5$), $\eta \le 100$, we can find 4994 and 5829 optimal integer combinations for GPS and BDS, respectively. The results are shown in the l_n–q_n plane (see Fig. 1). In Fig. 1, the color of the point denotes the sum of the integer coefficients; the smaller the point, the smaller the noise is.

It is found from Fig. 1 that these optimal integer combinations are reclassified by the sum of their integer coefficients for both GPS and BDS. For each group of combinations distinguished by the sum of their coefficients, there is a lowest noise axis in the l_n–q_n plane: the nearer to this axis the lower the noise amplification factor is. These conclusions are the same as that given by Cocard et al. (2008).

3.2 Sum of the integer coefficients and three refined integer combinations

If we make all elements of the third row of the matrix $\boldsymbol{Z}$ equal to 1, namely, (s_1, s_2, s_3)=(1, 1, 1), the integer number s_n denotes the sum of the integer coefficients. Considering Eq. (13), for GPS and BDS, we have

$$\begin{bmatrix} i \\ j \\ k \end{bmatrix}_{\text{GPS}} = \frac{1}{-663} \begin{bmatrix} -77 & -5 & 18{,}095 \\ 468 & 39 & -125{,}892 \\ -391 & -34 & 107{,}134 \end{bmatrix} \begin{bmatrix} l_n \\ i_n \\ s_n \end{bmatrix}_{\text{GPS}} \tag{16a}$$

$$\begin{bmatrix} i \\ j \\ k \end{bmatrix}_{\text{BDS}} = \frac{1}{-74{,}217} \begin{bmatrix} -2289 & -30 & 2{,}769{,}690 \\ 10726 & 173 & -14{,}512{,}278 \\ -8437 & -143 & 11{,}668{,}371 \end{bmatrix} \begin{bmatrix} l_n \\ i_n \\ s_n \end{bmatrix}_{\text{BDS}} \tag{16b}$$

From Eqs. (16a,b) if we make any two elements of the triplet (l_n, i_n, s_n) equal to zero, we can find three special integer combinations, namely: IFS0 (IF combination that the Sum of its integer coefficients equal to 0), GFS0 (GF combination that the Sum of its integer coefficients equal to 0) and GFIF (GF and IF combination), as shown in Table 2. For any triple-frequency GNSS, there exist similar three integer combinations which are solely determined by the triple-frequency values of GNSS.

The GPS IFS0 integer combination (77, −468, 391) is first used to assist AR over long baseline by Han and Rizos (1999) and also given in the Cocard et al. (2008). For BDS, the relevant IFS0 combination is (2289, −10,726, 8437), which also can be applied in the long baseline AR similar to that of GPS. The GFS0 combination is useful in the cycle slip detection and the GFIF combination is crucial for AR over long baseline. The practicality of these three combinations will be introduced in detail in Sect. 4.

When (s_1, s_2, s_3)=(1, 1, 1), from Eq. (15), we can depict the value of the sum of optimal integer combinations in the l_n–q_n plane. For example, $|q_n| \le 5$, for GPS: $0 \le l_n \le 300$, for BDS: $0 \le l_n \le 1500$, the sum values of the optimal integer combinations for GPS and BDS in the l_n–q_n plane are shown in Fig. 2.

From Fig. 2, it is demonstrated again that the sum of the integer coefficients of the combinations is an important indicator for grouping the optimal integer combinations. From Figs. 1 and 2, we find that the sums of the integer coefficients of the optimal combinations with larger wavelength and lower ionosphere amplification factor have small absolute value ($\le$2) for GPS and BDS. For the EWL and WL combinations with long wavelength (GPS, $1 \le l_n \le 39$;

Table 2 Three refined integer combinations for GPS and BDS

	l_n	i_n	s_n	i	j	k	$\lambda_{[i,j,k]}$ (m)
GPS							
IFS0	663	0	0	77	−468	391	0.044
GFS0	0	663	0	5	−39	34	–
GFIF	0	0	−663	18,095	−125,892	107,134	–
BDS							
IFS0	74,217	0	0	2289	−10,726	8437	0.002
GFS0	0	74,217	0	30	−173	143	–
GFIF	0	0	−2249	83,930	−439, 766	353,587	–

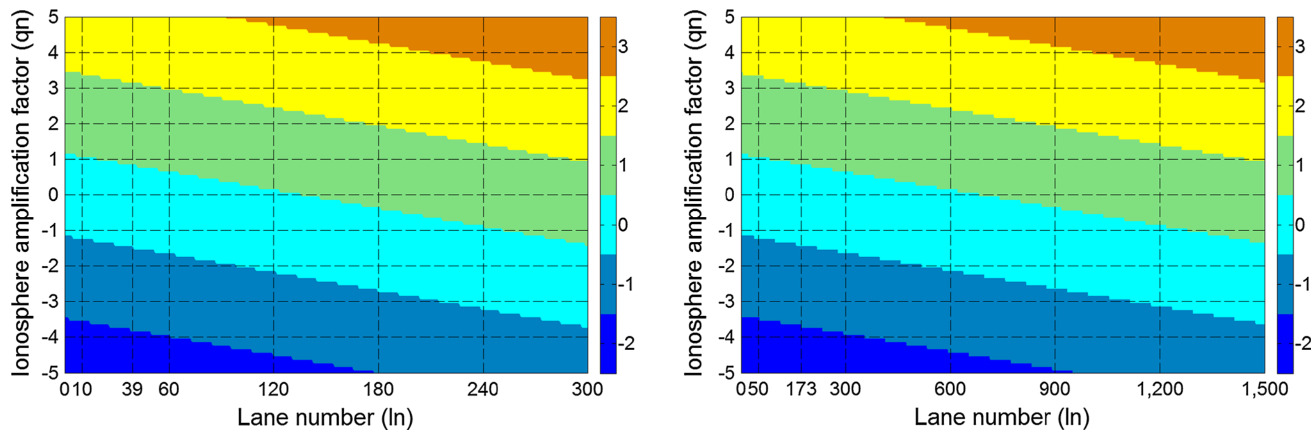

Fig. 2 The sum of the integer coefficients of the optimal combinations in the l_n–q_n plane for GPS (*left*) and BDS (*right*)

BDS, $1 \leq l_n \leq 173$), their ionosphere amplification factor increases with the absolute value of the sum of their coefficients. Therefore, the sum of the integer coefficients for the optimal EWL and WL combinations with lowest ionosphere influence should be equal to zero, namely, the area: $|q_n| \leq 1$, GPS: $1 \leq l_n \leq 39$, BDS: $1 \leq l_n \leq 173$, shown in Figs. 1 and 2. Besides, the sum of the integer coefficients for the optimal NL combinations with lowest ionosphere influence should be equal to 1, namely, the area: $|q_n| \leq 1$, GPS: $240 \leq l_n \leq 300$, BDS: $1200 \leq l_n \leq 1500$, shown in Figs. 1 and 2.

3.3 Some especial phase combinations in meters and their counterparts in cycles

For the phase combinations in meters, their linear coefficients (x, y, z) can also be expressed as the linear function of three characteristic parameters with their definitions as follows:

$$\begin{bmatrix} s_r \\ \mu_r \\ \lambda_r \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{17}$$

where s_r is the sum of the real coefficients, μ_r is the ionosphere amplification factor, λ_r is the weighted sum of the real coefficients by their corresponding wavelength. From Eq. (17), we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{f_1^2}{(f_1-f_2)(f_1-f_3)} & \frac{f_2 f_3}{(f_1-f_2)(f_1-f_3)} & \frac{-f_1^2(f_2+f_3)}{(f_1-f_2)(f_1-f_3)}\frac{1}{c} \\ \frac{f_2^2}{(f_2-f_1)(f_2-f_3)} & \frac{f_2 f_3}{(f_2-f_1)(f_2-f_3)}\frac{f_2}{f_1} & \frac{-f_2^2(f_1+f_3)}{(f_2-f_1)(f_2-f_3)}\frac{1}{c} \\ \frac{f_3^2}{(f_3-f_1)(f_3-f_2)} & \frac{f_2 f_3}{(f_3-f_1)(f_3-f_2)}\frac{f_3}{f_1} & \frac{-f_3^2(f_1+f_2)}{(f_3-f_1)(f_3-f_2)}\frac{1}{c} \end{bmatrix} \times \begin{bmatrix} s_r \\ \mu_r \\ \lambda_r \end{bmatrix} \tag{18}$$

From Eq. (18), we can get the generic expression of the linear coefficients for the GF combinations if we make the s_r equal to zero; we can also get the generic expression of the linear coefficients for the IF combinations if we make the μ_r equal to zero. A special GF combination can be obtained if we make both s_r and λ_r equal to zero, with its coefficients as follows:

$$x_{\mathrm{GF}} = \frac{f_2 f_3 \mu_r}{(f_1-f_2)(f_1-f_3)} \quad y_{\mathrm{GF}} = \frac{f_2 f_3 \mu_r}{(f_2-f_1)(f_2-f_3)}\frac{f_2}{f_1}$$
$$z_{\mathrm{GF}} = \frac{f_2 f_3 \mu_r}{(f_3-f_1)(f_3-f_2)}\frac{f_3}{f_1} \tag{19}$$

If we make both μ_r and λ_r equal to zero, an especial IF combination can be derived with its coefficients as follows:

$$x_{\mathrm{IF}} = \frac{f_1^2}{(f_1 - f_2)(f_1 - f_3)} s_r \quad y_{\mathrm{IF}} = \frac{f_2^2}{(f_2 - f_1)(f_2 - f_3)} s_r$$
$$z_{\mathrm{IF}} = \frac{f_3^2}{(f_3 - f_1)(f_3 - f_2)} s_r \tag{20}$$

If we make both s_r and μ_r equal to zero, a GFIF combination can be is obtained with its coefficients as follows:

$$x_{\mathrm{GFIF}} = \frac{-f_1^2(f_2+f_3)}{(f_1 - f_2)(f_1 - f_3)} \frac{\lambda_r}{c} \quad y_{\mathrm{GFIF}} = \frac{-f_2^2(f_1+f_3)}{(f_2 - f_1)(f_2 - f_3)} \frac{\lambda_r}{c}$$
$$z_{\mathrm{GFIF}} = \frac{-f_3^2(f_1+f_2)}{(f_3 - f_1)(f_3 - f_2)} \frac{\lambda_r}{c} \tag{21}$$

It should be noted that the variables μ_r, s_r and λ_r in Eqs. (19)–(21) only perform a scale function. As a result, the Eqs. (19)–(21) can be simplified as follows, respectively:

$$x_{\mathrm{GF}} = \frac{f_1}{(f_1 - f_2)(f_1 - f_3)} \quad y_{\mathrm{GF}} = \frac{f_2}{(f_2 - f_1)(f_2 - f_3)}$$
$$z_{\mathrm{GF}} = \frac{f_3}{(f_3 - f_1)(f_3 - f_2)} \tag{22a}$$

$$x_{\mathrm{IF}} = \frac{f_1^2}{(f_1 - f_2)(f_1 - f_3)} \quad y_{\mathrm{IF}} = \frac{f_2^2}{(f_2 - f_1)(f_2 - f_3)}$$
$$z_{\mathrm{IF}} = \frac{f_3^2}{(f_3 - f_1)(f_3 - f_2)} \tag{22b}$$

$$x_{\mathrm{GFIF}} = f_1^2\left(f_2^2 - f_3^2\right) \quad y_{\mathrm{GFIF}} = f_2^2\left(f_3^2 - f_1^2\right)$$
$$z_{\mathrm{GFIF}} = f_3^2\left(f_1^2 - f_2^2\right) \tag{22c}$$

It is easy to find that the Eq. (22b) is the same as the Eq. (12) in Hatch (2006) and the Eq. (22c) is a scaled version of the Eq. (2) in Simsky (2006), the Eqs. (29), (30) and (34) in Hatch (2006) and the Eq. (7) in Li et al. (2012b). Besides, it is not difficult to validate that the integer combinations GFS0, IFS0 and GFIF in Sect. 3.2 are namely the counterparts in cycles of the real combinations Eq. (22a)–(22c), respectively.

4 Optimal phase combinations for cycle slip detection and AR

4.1 GF combinations for cycle slip detection

For cycle slip detection, the GF combinations in cycles with low noise amplification factor and low ionosphere amplification factor are helpful. The lower the noise amplification factor is, the fewer the number of insensitive cycle slip groups is. The low ionosphere amplification factor can ease the ionosphere influence on cycle slip detection and make them still efficient when dealing with those data with low sampling rate or under high ionosphere activity. When $f_{(i.j.k)} = 0$, the epoch-difference GF phase combinations from Eq. (2) can be expressed as:

$$\Delta\varphi_{(i.j.k)} = -\kappa_{(i.j.k)}\Delta d_{\mathrm{ion}} + \Delta N_{(i.j.k)} + \Delta\upsilon_{(i.j.k)}$$
$$f_{(i.j.k)} = 0, \tag{23}$$

where Δd_{ion} and $\Delta N_{(i.j.k)}$ are, respectively, the ionospheric delay variation and the possible cycle slip value between consecutive epoch. The cycle slip detector based on the GF phase combinations can be defined as (Li 2014)

$$T = \frac{\Delta\varphi_{(i.j.k)}}{\sqrt{2}\sqrt{i^2+j^2+k^2}} = \frac{\Delta\varphi_{(i.j.k)}}{\sqrt{2}\cdot\eta} > \iota\cdot\sigma_{\upsilon} \quad f_{(i.j.k)} = 0, \tag{24}$$

where $\iota = 3$ (with confidence level of 99.7 %). The influence of the ignored ionosphere delay on the cycle slip detection by the Eq. (24) is evaluated by the following factor:

$$\gamma = \frac{\kappa_{(i,j,k)}}{\sqrt{2}\cdot\sqrt{i^2+j^2+k^2}} = \frac{\kappa_{(i,j,k)}}{\sqrt{2}\cdot\eta} \tag{25}$$

Considering $l_n = 0$, for GPS: $0 \le i_n \le 7 \times 1380$ $(0 \le q_n \le 7)$, $\eta \le 300$, for BDS: $0 \le i_n \le 7 \times 36{,}580$ $(0 \le q_n \le 12)$, $\eta \le 600$, we get the optimal GF integer combinations by the algorithms of Sect. 3.1, which are listed in the Tables 3 and 4 for GPS and BDS, respectively. For comparison, the most frequently used dual-frequency GF combinations are also shown in the second and third row.

From Tables 3 and 4, we can know that both the ionosphere and noise amplification factor of the triple-frequency GF combinations are evidently smaller than those of the dual-frequency GF combinations. It indicates that the better performance of cycle slip detection can be achieved in the triple-frequency case. If the ionospheric delay variation between consecutive epoch can be ignored (<0.001 m), the best two GF phase combinations for cycle slip detection are (−10, 9, 4) and (−5, −7, 14) for GPS, (−20, 17, 8) and (−10, −21, 35) for BDS because of their lowest noise amplification factor. If the ionospheric delay variation is the mm-level (<0.01 m), the GF combinations (−5, 16, −10) and (0, −23, 24) for GPS and (−10, 38, −27) and (0, −59, 62) for BDS are the best two selections. If the ionospheric delay variation reach the cm-level (<0.1 m), the GF combinations (5, −39, 34) and (−15, 94, −78) for GPS, (30, −173, 143) and (40, −211, 170) for BDS should be the best two GF phase combinations. Moreover, the GF combinations (−25, 172, −146) and (30, −211, 180) for GPS, (40, −211, 170) and (−90, 460, −367) for BDS are still applicable for the dm-level (<1 m) ionospheric delay variation due to their very small ionosphere amplification factor. Besides, these GF phase combinations shown in Tables 3 and 4 can be also used for GF phase-only AR using multi-epoch observations.

Table 3 GPS optimal GF combinations

i_n	s_n	i	j	k	Sum	$\kappa_{(i.j.k)}$	η	q_n	γ
53,567	18,502	−60	77	0	17	204.0	97.62	38.82	1.4776
125,892	43,483	−115	0	154	39	479.4	192.20	91.23	1.7637
9531	3292	−10	9	4	3	36.3	14.04	6.91	1.8285
6575	2271	−5	−7	14	2	25.0	16.43	4.76	1.0774
2956	1021	−5	16	−10	1	11.3	19.52	2.14	0.4078
3619	1250	0	−23	24	1	13.8	33.24	2.62	0.2931
663	229	5	−39	34	0	2.5	51.98	0.48	0.0343
2293	792	−10	55	−44	1	8.7	71.14	1.66	0.0868
4282	1479	5	−62	58	1	16.3	85.05	3.10	0.1356
5249	1813	−15	71	−54	2	20.0	90.45	3.80	0.1563
8205	2834	−20	87	−64	3	31.2	109.84	5.95	0.2011
7901	2729	5	−85	82	2	30.1	118.21	5.73	0.1800
1630	563	−15	94	−78	1	6.2	123.07	1.18	0.0357
4945	1708	10	−101	92	1	18.8	136.99	3.58	0.0972
7542	2605	−25	126	−98	3	28.7	161.57	5.47	0.1257
967	334	−20	133	−112	1	3.7	175.02	0.70	0.0149
5608	1937	15	−140	126	1	21.4	188.95	4.06	0.0799
3923	1355	−25	149	−122	2	14.9	194.19	2.84	0.0544
9227	3187	15	−163	150	2	35.1	222.02	6.69	0.1119
304	105	−25	172	−146	1	1.2	226.99	0.22	0.0036
6271	2166	20	−179	160	1	23.9	240.92	4.54	0.0701
6216	2147	−35	204	−166	3	23.7	265.32	4.50	0.0631
359	124	30	−211	180	−1	1.4	278.96	0.26	0.0035
6934	2395	25	−218	194	1	26.4	292.89	5.02	0.0637
2597	897	−35	227	−190	2	9.9	298.08	1.88	0.0235

4.2 IF combinations for AR over long baselines

From the algorithms of Sect. 3.1, considering i_n=0, for GPS: $0 \leq l_n \leq 10{,}491$, $\eta \leq 200$, for BDS: $0 \leq l_n \leq 234{,}069$, $\eta \leq 1000$, we find the optimal IF integer combinations, which are shown in the Tables 5 and 6 for GPS and BDS, respectively. The IFS0 combinations presented in Sect. 3.2 are also shown in the first row. In Tables 5 and 6, σ_ε is the standard deviation of the linear combinations noise in meters and λ_e is the effective wavelength when the EWL ambiguity $N_{(0,1,-1)}$ and/or the WL ambiguity $N_{(1,-1,0)}$ have been fixed to their integer values in advance.

From Tables 5 and 6, we know that except for the combinations (0, 24, −23), (0, 62, −59) and IFS0, the wavelength of the optimal IF combinations are smaller than 1 cm for GPS and 1 mm for BDS. Furthermore, considering their large noise standard deviation in meters, we can infer that it is almost impossible to fix their ambiguities directly. However, when the EWL ambiguity $N_{(0,1,-1)}$ and/or $N_{(1,-1,0)}$ are fixed in advance, the effective wavelength of all optimal IF combinations are larger than 10 cm. For the IFS0 combinations, the effective wavelengths even reach to 3.4 m for GPS and 4.5 m for BDS. As a result, it becomes possible to solve their ambiguities to their integer values.

The minimal noise standard deviation in meters for the IF combinations is desired for precise positioning application. The most optimal triple-frequency IF combinations have smaller noises than those of the dual-frequency IF combinations. For example, the standard deviations in meters of the triple-frequency IF combinations (77, −36, −23), (77, −12, −46) and (154, −48, −49) for GPS, (763, −248, −354), (763, −186, −413) and (763, −124, −472) for BDS, are very close to the minimum value and at the same time have relatively small integer coefficients. Hence, better precise baseline or coordinates estimation will be achieved in the triple-frequency case.

As a whole, it can be concluded that the improvement in position estimation and AR for the triple-frequency IF combinations is insignificant for GPS and BDS compared to the dual-frequency IF combinations, considering the slight differences in noise standard deviations σ_ε and the effective wavelength λ_e.

Table 4 BDS optimal GF combinations

i_n	s_n	i	j	k	Sum	$\kappa_{(i.j.k)}$	η	q_n	γ
11,668,371	−65,089	−620	763	0	143	1661.02	983.14	318.98	1.1947
14,512,278	−80,953	−590	0	763	173	2065.86	964.50	396.73	1.5145
412,137	−2299	−20	17	8	5	58.67	27.44	11.27	1.5118
344,553	−1922	−10	−21	35	4	49.05	42.02	9.42	0.8253
67,584	−377	−10	38	−27	1	9.62	47.68	1.85	0.1427
276,969	−1545	0	−59	62	3	39.43	85.59	7.57	0.3257
209,385	−1168	10	−97	89	2	29.81	132.02	5.72	0.1596
141,801	−791	20	−135	116	1	20.19	179.11	3.88	0.0797
74,217	−414	30	−173	143	0	10.56	226.45	2.03	0.0330
6633	− 37	40	−211	170	−1	0.94	273.90	0.18	0.0024
351,186	−1959	30	−232	205	3	49.99	311.05	9.60	0.1136
60,951	−340	−50	249	−197	2	8.68	321.42	1.67	0.0191
128535	−717	−60	287	−224	3	18.30	368.98	3.51	0.0351
216,018	−1205	50	−308	259	1	30.75	405.52	5.91	0.0536
196,119	−1094	−70	325	−251	4	27.92	416.56	5.36	0.0474
263,703	−1471	−80	363	−278	5	37.54	464.17	7.21	0.0572
80,850	−451	70	−384	313	−1	11.51	500.32	2.21	0.0163
331,287	−1848	−90	401	−305	6	47.16	511.79	9.06	0.0652
398,871	−2225	−100	439	−332	7	56.78	559.41	10.90	0.0718
357,819	−1996	70	−443	375	2	50.94	584.61	9.78	0.0616
54,318	−303	−90	460	−367	3	7.73	595.31	1.48	0.0092

Table 5 GPS optimal IF combinations

l_n	s_n	i	j	k	Sum	λ (cm)	η	σ_ε (cm)	λ_e (cm)
663	−2573	77	−468	391	0	4.42	614.68	27.17	340.35
235	−912	0	24	−23	1	12.47	33.24	4.15	12.47
3718	−14,429	77	−156	92	13	0.79	196.80	1.55	10.25
3953	−15,341	77	−132	69	14	0.74	167.67	1.24	10.38
4188	−16,253	77	−108	46	15	0.70	140.39	0.98	10.50
4423	−17,165	77	−84	23	16	0.66	116.25	0.77	10.60
4658	−18,077	77	−60	0	17	0.63	97.62	0.61	10.70
4893	−18,989	77	−36	−23	18	0.60	88.06	0.53	10.78
5128	−19,901	77	−12	−46	19	0.57	90.49	0.52	10.86
5363	−20,813	77	12	−69	20	0.55	104.09	0.57	10.93
5598	−21,725	77	36	−92	21	0.52	125.26	0.66	10.99
5833	−22,637	77	60	−115	22	0.50	150.84	0.76	11.05
6068	−23,549	77	84	−138	23	0.48	178.97	0.86	11.11
9551	−37,066	154	−96	−23	35	0.31	182.92	0.56	10.74
10,021	−38,890	154	−48	−69	37	0.29	175.45	0.51	10.82
10,491	−40,714	154	0	−115	39	0.28	192.20	0.54	10.89

4.3 EWL combinations for AR over short baselines

For geometry-free AR over short baseline, these combinations with large wavelength and small noise are helpful. From the algorithms of Sect. 3.1, for GPS: $0 < l_n \leq 10$, $|i_n| \leq 5 \times 1380$ ($|q_n| \leq 5$), $\eta \leq 15$, for BDS: $0 < l_n \leq 50$, $|i_n| \leq 5 \times 36{,}580$ ($|q_n| \leq 5$), $\eta \leq 15$, the optimal EWL integer combinations are obtained and shown in the Tables 7 and 8 for GPS and BDS, respectively.

From Tables 7 and 8, we can know that the ionosphere amplification factor of all optimal EWL combinations for GPS and BDS is proportionally increasing with the sum of their integer linear coefficients. The proportion of the factor

Table 6 BDS optimal IF combinations

l_n	s_n	i	j	k	Sum	λ (cm)	η	σ_ε (cm)	λ_e(cm)
74,217	12,942	2289	−10,726	8437	0	0.197	13,837.25	27.319	451.92
3630	633	0	62	−59	3	4.037	85.59	3.455	12.11
197,769	34,487	763	−620	0	143	0.074	983.14	0.728	10.59
201,399	35,120	763	−558	−59	146	0.073	947.11	0.689	10.62
205,029	35,753	763	−496	−118	149	0.071	917.66	0.656	10.65
208,659	36,386	763	−434	−177	152	0.070	895.46	0.629	10.67
212,289	37,019	763	−372	−236	155	0.069	881.05	0.608	10.70
215,919	37,652	763	−310	−295	158	0.068	874.81	0.594	10.72
219,549	38,285	763	−248	−354	161	0.067	876.92	0.585	10.75
223,179	38,918	763	−186	−413	164	0.066	887.32	0.583	10.77
226,809	39,551	763	−124	−472	167	0.065	905.72	0.585	10.79
230,439	40,184	763	−62	−531	170	0.064	931.65	0.592	10.81
234,069	40,817	763	0	−590	173	0.063	964.50	0.604	10.83

Table 7 GPS EWL combinations

l_n	i_n	s_n	i	j	k	Sum	λ (m)	$\kappa_{(i.j.k)}$	$\mu_{\langle i.j.k\rangle}$	η	q_n
1	−3104	−1076	4	−8	3	−1	29.31	−11.82	−346.39	9.43	−2.25
1	−148	−55	−1	8	−7	0	29.31	−0.56	−16.52	10.68	−0.11
1	6427	2216	−6	1	7	2	29.31	24.47	717.22	9.27	4.66
2	−3252	−1131	3	0	−4	−1	14.65	−12.38	−181.45	5.00	−2.36
2	3323	1140	−2	−7	10	1	14.65	12.65	185.41	12.37	2.41
2	6279	2161	−7	9	0	2	14.65	23.91	350.35	11.40	4.55
3	−6356	−2207	7	−8	−1	−2	9.77	−24.20	−236.43	10.68	−4.61
3	−3400	−1186	2	8	−11	−1	9.77	−12.95	−126.47	13.75	−2.46
3	3175	1085	−3	1	3	1	9.77	12.09	118.10	4.36	2.30
4	71	9	1	−7	6	0	7.33	0.27	1.98	9.27	0.05
4	3027	1030	−4	9	−4	1	7.33	11.53	84.45	10.63	2.19
5	−77	−46	0	1	−1	0	5.86	−0.29	−1.72	1.41	−0.06
6	−3181	−1122	4	−7	2	−1	4.88	−12.11	−59.16	8.31	−2.31
6	−225	−101	−1	9	−8	0	4.88	−0.86	−4.18	12.08	−0.16
7	−3329	−1177	3	1	−5	−1	4.19	−12.68	−53.07	5.92	−2.41
7	3246	1094	−2	−6	9	1	4.19	12.36	51.75	11.00	2.35
7	6202	2115	−7	10	−1	2	4.19	23.62	98.87	12.25	4.49
8	−6433	−2253	7	−7	−2	−2	3.66	−24.50	−89.74	10.10	−4.66
8	3098	1039	−3	2	2	1	3.66	11.80	43.21	4.12	2.24
9	−6581	−2308	6	1	−9	−2	3.26	−25.06	−81.60	10.86	−4.77
9	−6	−37	1	−6	5	0	3.26	−0.02	−0.07	7.87	0.00
9	2950	984	−4	10	−5	1	3.26	11.23	36.58	11.87	2.14
10	6421	2179	−5	−5	12	2	2.93	24.45	71.65	13.93	4.65

and the sum is about 12. Therefore, the optimal EWL combinations with Sum = 0 are the best selection for AR since they suffer from the lowest ionosphere influence. However, it is impossible to find three linear independent EWL combinations with low noise and ionosphere amplification factors simultaneously since there are only two linear independent combinations with Sum = 0. It means that the third EWL combination will inevitably suffer from about 12 times larger ionosphere influence than that of the other two combinations with Sum = 0.

For short baselines, however, all the optimal EWL combinations shown in Tables 7 and 8 are helpful to reduce the influence of large pseudorange multipath error and noise on AR. For example, the GPS EWL combinations (−6, 1, 7), (−1, 8, −7) and (4, −8, 3) can be used at one time to reduce the influence of pseudorange multipath error on geometry-

Table 8 BDS EWL combinations

l_n	i_n	s_n	i	j	k	Sum	λ[m]	$\kappa_{(i.j.k)}$	$\mu_{\langle i.j.k\rangle}$	η	q_n
1	−167,405	934	7	−1	−8	−2	146.53	−23.83	−3491.80	10.68	−4.58
4	87,070	−485	−2	−8	11	1	36.63	12.39	454.04	13.75	2.38
5	−80,335	449	5	−9	3	−1	29.31	−11.44	−335.13	10.72	−2.20
6	164,397	−916	−8	7	3	2	24.42	23.40	571.51	11.05	4.49
7	−3008	18	−1	6	−5	0	20.93	−0.43	−8.96	7.87	−0.08
11	84,062	−467	−3	−2	6	1	13.32	11.97	159.40	7.00	2.30
12	−83,343	467	4	−3	−2	−1	12.21	−11.86	−144.87	5.39	−2.28
16	3727	−18	2	−11	9	0	9.16	0.53	4.86	14.35	0.10
18	81,054	−449	−4	4	1	1	8.14	11.54	93.93	5.74	2.22
19	−86,351	485	3	3	−7	−1	7.71	−12.29	−94.80	8.19	−2.36
23	719	0	1	−5	4	0	6.37	0.10	0.65	6.48	0.02
25	78,046	−431	−5	10	−4	1	5.86	11.11	65.12	11.87	2.13
29	165,116	−916	−7	2	7	2	5.05	23.50	118.76	10.10	4.51
30	−2289	18	0	1	−1	0	4.88	−0.33	−1.59	1.41	−0.06
31	−169,694	952	7	0	−9	−2	4.73	−24.16	−114.18	11.40	−4.64
34	84,781	−467	−2	−7	10	1	4.31	12.07	52.01	12.37	2.32
35	−82,624	467	5	−8	2	−1	4.19	−11.76	−49.24	9.64	−2.26
37	−5297	36	−1	7	−6	0	3.96	−0.75	−2.99	9.27	−0.14
41	81,773	−449	−3	−1	5	1	3.57	11.64	41.60	5.92	2.24
42	−85,632	485	4	−2	−3	−1	3.49	−12.19	−42.53	5.39	−2.34
47	−165,967	934	9	−11	0	−2	3.12	−23.63	−73.66	14.21	−4.54
48	78,765	−431	−4	5	0	1	3.05	11.21	34.23	6.40	2.15
49	−88,640	503	3	4	−8	−1	2.99	−12.62	−37.73	9.43	−2.42

free AR to the greatest extent since all their wavelengths reach to the maximum 29.3 m.

4.4 Ionosphere-reduced combinations for AR over medium–long baselines

From Sect. 3.1, we know that there are two optimal ionosphere-reduced areas in the l_n–q_n plane and the sum of the corresponding linear coefficients are 0 and 1, respectively.

From the algorithms of Sect. 3.1, when $\eta \le 10$, if $0 < l_n \le 39$ and $|i_n| < 1380$ ($|q_n| < 1$) for GPS and $0 < l_n \le 50$ and $|i_n| < 36{,}580$ ($|q_n| < 1$) for BDS, then we get the optimal ionosphere-reduced combinations that the sum of their coefficients equal to 0. The results are shown in the Tables 9 and 10 for GPS and BDS, respectively.

From Tables 9 and 10, we know that the optimal ionosphere-reduced combinations with the sum of their coefficients being 0 are EWL or WL combinations. The ionosphere amplification factor for the ionosphere-reduced combinations (1, −6, 5), (1, −7, 6), (0, 1, −1) and (1, −5, 4) for GPS, (1, −5, 4), (1, −4, 3), (0, 1, −1) and (−1, 6, −5) for BDS are very small. Especially for the GPS combination (1, −6, 5) and the BDS combination (1, −5, 4), the influence of 1 m ionosphere delay in the first frequency on AR are smaller 0.02 and 0.10 cycles, respectively.

Table 9 GPS ionosphere-reduced integer combinations (Sum = 0)

l_n	i_n	s_n	i	j	k	λ (m)	$\kappa_{(i.j.k)}$	$\mu_{\langle i.j.k\rangle}$	η	q_n
4	71	9	1	−7	6	7.33	0.27	1.98	9.27	0.05
5	−77	−46	0	1	−1	5.86	−0.29	−1.72	1.41	−0.06
9	−6	−37	1	−6	5	3.26	−0.02	−0.07	7.87	−0.004
14	−83	−83	1	−5	4	2.09	−0.32	−0.66	6.48	−0.06
19	−160	−129	1	−4	3	1.54	−0.61	−0.94	5.10	−0.12
24	−237	−175	1	−3	2	1.22	−0.90	−1.10	3.74	−0.17
29	−314	−221	1	−2	1	1.01	−1.20	−1.21	2.45	−0.23
34	−391	−267	1	−1	0	0.86	−1.49	−1.28	1.41	−0.28
39	−468	−313	1	0	−1	0.75	−1.78	−1.34	1.41	−0.34

Considering the ionosphere amplification factor and noise amplification factor synthetically, the EWL combinations (0, 1, −1) and (1, −6, 5) for GPS, (0, 1, −1) and (1, −5, 4) for BDS are the first and second selection for AR in the triple case. Besides, compared to the dual-frequency ionosphere-reduced WL combination (1, −1, 0), the triple frequency ones have larger wavelength and smaller ionosphere amplification factor in cycles.

From the algorithms of Sect. 3.1, when $\eta \le 10$, if $240 < l_n \le 300$ and $|i_n| \le 1380$ ($|q_n| \le 1$) for GPS

Table 10 BDS ionosphere-reduced integer combinations (Sum = 0)

l_n	i_n	s_n	i	j	k	λ (m)	$\kappa_{(i.j.k)}$	$\mu_{\langle i.j.k\rangle}$	η	q_n
7	−3008	18	−1	6	−5	20.93	−0.43	−8.96	7.87	−0.08
23	719	0	1	−5	4	6.37	0.10	0.65	6.48	0.02
30	−2289	18	0	1	−1	4.88	−0.33	−1.59	1.41	−0.06
37	−5297	36	−1	7	−6	3.96	−0.75	−2.99	9.27	−0.14
53	−1570	18	1	−4	3	2.76	−0.22	−0.62	5.10	−0.04
83	−3859	36	1	−3	2	1.77	−0.55	−0.97	3.74	−0.11
113	−6148	54	1	−2	1	1.30	−0.88	−1.13	2.45	−0.17
136	−5429	54	2	−7	5	1.08	−0.77	−0.83	8.83	−0.15
143	−8437	72	1	−1	0	1.02	−1.20	−1.23	1.41	−0.23
173	−10,726	90	1	0	−1	0.85	−1.53	−1.29	1.41	−0.29

Table 11 GPS ionosphere-reduced integer combinations (Sum = 1)

l_n	i_n	s_n	i	j	k	λ (cm)	$\kappa_{(i.j.k)}$	$\mu_{\langle i.j.k\rangle}$	η	σ_ε (cm)	q_n
154	1380	−121	1	0	0	19.03	5.255	1.000	1.00	0.19	1.00
241	438	−784	4	−6	3	12.16	1.668	0.203	7.81	0.95	0.32
242	290	−839	3	2	−4	12.11	1.104	0.134	5.39	0.65	0.21
246	361	−830	4	−5	2	11.91	1.375	0.164	6.71	0.80	0.26
247	213	−885	3	3	−5	11.86	0.811	0.096	6.56	0.78	0.15
251	284	−876	4	−4	1	11.68	1.081	0.126	5.74	0.67	0.21
252	136	−931	3	4	−6	11.63	0.518	0.060	7.81	0.91	0.10
256	207	−922	4	−3	0	11.45	0.788	0.090	5.00	0.57	0.15
257	59	−977	3	5	−7	11.40	0.225	0.026	9.11	1.04	0.04
261	130	−968	4	−2	−1	11.2	0.495	0.056	4.58	0.51	0.09
266	53	−1014	4	−1	−2	11.02	0.202	0.022	4.58	0.50	0.04
271	−24	−1060	4	0	−3	10.81	−0.091	−0.010	5.00	0.54	−0.02
275	47	−1051	5	−7	3	10.66	0.179	0.019	9.11	0.97	0.03
276	−101	−1106	4	1	−4	10.62	−0.385	−0.041	5.74	0.61	−0.07
280	−30	−1097	5	−6	2	10.47	−0.114	−0.012	8.06	0.84	−0.02
281	−178	−1152	4	2	−5	10.43	−0.678	−0.071	6.71	0.70	−0.13
285	−107	−1143	5	−5	1	10.28	−0.407	−0.042	7.14	0.73	−0.08
286	−255	−1198	4	3	−6	10.25	−0.971	−0.099	7.81	0.80	−0.18
290	−184	−1189	5	−4	0	10.11	−0.701	−0.071	6.40	0.65	−0.13
291	−332	−1244	4	4	−7	10.07	−1.264	−0.127	9.00	0.91	−0.24
295	−261	−1235	5	−3	−1	09.93	−0.994	−0.099	5.92	0.59	−0.19
300	−338	−1281	5	−2	−2	09.77	−1.287	−0.126	5.74	0.56	−0.24

and $1200 < l_n \leq 1500$ and $|i_n| \leq 36{,}580$ ($|q_n| \leq 1$) for BDS, then we obtain the optimal ionosphere-reduced combinations with the sum of their coefficients being 1, which are shown in the Tables 11 and 12 for GPS and BDS, respectively.

From Tables 11 and 12, we know that the optimal ionosphere-reduced combinations that the sum of their coefficients equal to 1 are the NL combinations with a wavelength of about 11 cm and a noise standard deviation of 5–10 mm. Unlike the IF combinations presented in Sect. 4.2, it is possible to solve their ambiguities to the integer values directly. The ionosphere influence of the combinations (4, 0, −3) for GPS, (4, 2, −5) and (5, −3, −1) for BDS can be still ignored in the process of AR and position estimation even if the ionosphere delay on the first frequency reaches to the level of 1 m. The ionosphere-reduced combinations (4, −1, −2) and (4, −2, −1) for GPS, (4, 0, −3), (4, 1, −4) and (5, −2, −2) for BDS may be the appropriate choices for the ionosphere delay level of 0.1 m. Besides, considering the fact that the noise standard deviation in meters for the optimal ionosphere-reduced combinations is comparative with that of the optimal IF combinations presented in Sect. 4.2, they can be used for precise positioning directly over medium–long baselines.

Table 12 BDS ionosphere-reduced integer combinations (Sum = 1)

l_n	i_n	s_n	i	j	k	λ (cm)	$\kappa_{(i.j.k)}$	$\mu_{\langle i.j.k\rangle}$	η	σ_ε (cm)	q_n
763	36,580	−71	1	0	0	19.20	5.207	1.000	1.00	0.19	1.00
1222	8980	163	4	−2	−1	11.99	1.278	0.153	4.58	0.55	0.25
1229	5972	181	3	4	−6	11.92	0.850	0.101	7.81	0.93	0.16
1245	9699	163	5	−7	3	11.77	1.381	0.162	9.11	1.07	0.27
1252	6691	181	4	−1	−2	11.70	0.952	0.111	4.58	0.54	0.18
1259	3683	199	3	5	−7	11.64	0.524	0.061	9.11	1.06	0.10
1275	7410	181	5	−6	2	11.49	1.055	0.121	8.06	0.93	0.20
1282	4402	199	4	0	−3	11.43	0.627	0.072	5.00	0.57	0.12
1305	5121	199	5	−5	1	11.23	0.729	0.082	7.14	0.80	0.14
1312	2113	217	4	1	−4	11.17	0.301	0.034	5.74	0.64	0.06
1335	2832	217	5	−4	0	10.98	0.403	0.044	6.40	0.70	0.08
1342	−176	235	4	2	−5	10.92	−0.025	−0.003	6.71	0.73	−0.005
1365	543	235	5	−3	−1	10.73	0.077	0.008	5.92	0.64	0.01
1372	−2465	253	4	3	−6	10.68	−0.351	−0.037	7.81	0.83	−0.07
1395	−1746	253	5	−2	−2	10.50	−0.249	−0.026	5.74	0.60	−0.05
1402	−4754	271	4	4	−7	10.45	−0.677	−0.071	9.00	0.94	−0.13
1418	−1027	253	6	−7	2	10.33	−0.146	−0.015	9.43	0.97	−0.03
1425	−4035	271	5	−1	−3	10.28	−0.574	−0.059	5.92	0.61	−0.11
1448	−3316	271	6	−6	1	10.12	−0.472	−0.048	8.54	0.86	−0.09
1455	−6324	289	5	0	−4	10.07	−0.900	−0.091	6.40	0.64	−0.17
1478	−5605	289	6	−5	0	9.91	−0.798	−0.079	7.81	0.77	−0.15
1485	−8613	307	5	1	−5	9.87	−1.226	−0.121	7.14	0.70	−0.24

5 GF and IF AR over long baselines

From Sect. 3.2 and Eq. (2), we know that

$$(\varphi_{\mathrm{GFIF}})_{\mathrm{GPS}} = \varphi_{(18{,}095,-125{,}892,107{,}134)} = N_{(18{,}095,-125{,}892,107{,}134)} + \upsilon_{(18{,}095,-125{,}892,107{,}134)} \quad (26a)$$

$$(\varphi_{\mathrm{GFIF}})_{\mathrm{BDS}} = \varphi_{(83{,}930,-439{,}766,353{,}587)} = N_{(83{,}930,-439{,}766,353{,}587)} + \upsilon_{(83{,}930,-439{,}766,353{,}587)} \quad (26b)$$

On the assumption that the combined ambiguity $N_{(i,j,k)}$ can be decomposed as the linear combination of three ambiguities $N_{(0,1,-1)}$, $N_{(1,-1,0)}$ and $N_{(1,0,0)}$, namely:

$$N_{(i,j,k)} = c_1 \cdot N_{(0,1,-1)} + c_2 \cdot N_{(1,-1,0)} + c_3 \cdot N_{(1,0,0)} \quad (27)$$

Then the decomposition coefficients can be obtained by the following formula:

$$\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} i & j & k \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \quad (28)$$

As a result, we can get

$$N_{(18{,}095,-125{,}892,107{,}134)} = -107{,}134 \cdot N_{(0,1,-1)} + 18{,}758 \cdot N_{(1,-1,0)} - 663 \cdot N_{(1,0,0)} \quad (29a)$$

$$N_{(83{,}930,-439{,}766,353{,}587)} = -353{,}587 \cdot N_{(0,1,-1)} + 86{,}179 \cdot N_{(1,-1,0)} - 2249 \cdot N_{(1,0,0)} \quad (29b)$$

It is well known that the EWL ambiguity $N_{(0,1,-1)}$ and the WL ambiguity $N_{(1,-1,0)}$ can be solved to their integer values effortlessly by the GF and IF method (Hatch et al. 2000). Consequently, the ambiguity $N_{(1,0,0)}$ for GPS and BDS can be solved by the following estimator:

$$\left(\hat{N}_{(1,0,0)}\right)_{GPS} = \frac{-107134 \cdot N_{(0,1,-1)} + 18758 \cdot N_{(1,-1,0)} - \varphi_{(18,095,-125,892,107,134)}}{663}$$
$$= N_{(1,0,0)} - \frac{1}{663}\upsilon_{(18,095,-125,892,107,134)} \tag{30a}$$

$$\left(\hat{N}_{(1,0,0)}\right)_{BDS} = \frac{86,179 \cdot N_{(1,-1,0)} - 353,587 \cdot N_{(0,1,-1)} - \varphi_{(83,930,-439,766,353,587)}}{2249}$$
$$= N_{(1,0,0)} - \frac{1}{2249}\upsilon_{(83,930,-439,766,353,587)} \tag{30b}$$

The performance of the estimator Eq. (30) is not affected by the orbit error, the troposphere delay error and first-order ionosphere delay error, etc., and only suffers from the carrier-phase measurement noise and multipath error. As a result, it is applicable to AR over long baselines without distance constraint. However, the standard deviation of the estimate $\hat{N}_{(1,0,0)}$ reach to 5.016 and 5.073 cycles for GPS and BDS, respectively, with the assumption that the raw phase measurement noise is 0.01 cycles on each of the three frequencies. A multi-epoch averaging process has to be implemented to reduce the noise of the ambiguity estimates. Regardless of the correlation between epochs, theoretically 1119 and 1145 epochs are required for GPS and BDS, respectively, to reduce the standard deviation to 0.15 cycles or improve the rounding success rate to 99.9 %. It should be noted that the standard deviation is independent of the integer coefficients of the three ambiguities in the right hand of Eq. 30a, 30b (Li 2011, 2014; Li et al. 2012b).

The GFIF-based AR schemes for different GNSS frequency triplets are shown in Table 13. It is shown that the GALILEO frequency triplet (E1, E6, E5a) has the best outcome among all frequency triplets, which agrees with the results of Li et al. (2012b), Wang and Rothacher (2013). Besides, the standard deviations σ_{NL} are the same as those presented in Li et al. (2010, 2012b), Wang and Rothacher (2013) when the different assumptions for the raw phase measurement noise are taken into account. Consequently, it is evident that these GF and IF schemes for solving the third or NL ambiguity are equivalent. However, the estimator Eq. (30) for the third or NL ambiguity is more straightforward than those presented in Hatch (2006), Li et al. (2010, 2012b), Wang and Rothacher (2013).

6 Summary and conclusions

This paper presented a systematical analysis on the carrier-phase linear combination for triple-frequency GNSS. The integer property of the resulting ambiguity of the phase linear combinations is examined. It is concluded that the resulting ambiguity of all linear combinations still retains the integer nature as long as their linear coefficients are the rational number. The present analytical method for solving the optimal integer linear combinations is generalized for all triple-frequency GNSS. Three refined integer phase combinations, namely, IFS0, GFS0, GFIF, are introduced for triple-frequency GNSS and they are (77, −468, 391), (5, −39, 34) and (18,095, −125,892, 107,134) for GPS, (2289, −10,726, 8437), (30, −173, 143) and (83, 930, −439,766, 353,587) for BDS.

The optimal GF, IF, EWL and ionosphere-reduced integer combinations for GPS and BDS are solved by the presented method. It is shown that when the ionospheric delay variation can be ignored between consecutive epoch, the best two GF phase combinations for cycle slip detection are (−10, 9, 4) and (−5, −7, 14) for GPS, (−20, 17, 8) and (−10, −21, 35)

Table 13 GFIF-based AR schemes for different GNSS frequency triplets

System	GFIF integer coefficients			Decomposition coefficients			σ_{NL}	Number of epochs
	i	j	k	c_1	c_2	c_3	(Cycles)	
GPS (L1, L2, L5)	18,095	−125,892	107,134	−107,134	18,758	−663	5.016	1119
BDS (B1, B3, B2)	83,930	−439,766	353,587	−353,587	86,179	−2249	5.073	1145
GAL-a (E1, E6, E5b)	14,553	−68,000	53,041	−53,041	14,959	−406	4.308	826
GAL-b (E1, E6, E5)	421,498	−1,690,625	1,256,802	−1,256,802	433,823	−12,325	3.486	541
GAL-c (E1, E6, E5a)	24,640	−87,425	62,031	−62,031	25,394	−754	2.917	379
GAL-d (E1, E5b, E5a)	17,941	−206,323	187,680	−187,680	18,643	−702	7.962	2819

for BDS. The GF combinations (−25, 172, −146) and (30, −211, 180) for GPS, (40, −211, 170) and (−90, 460, −367) for BDS are still valid for cycle slip detection in the presence of high ionosphere activity of dm-level. The optimal triple-frequency IF combinations have smaller noise than those of the dual-frequency IF combinations. The triple-frequency IF combinations (77, −36, −23), (77, −12, −46) and (154, −48, −49) for GPS, (763, −248, −354), (763, −186, −413) and (763, −124, −472) for BDS, have the minimum standard deviations in meters and have relatively small integer coefficients at the same time. However, compared to the dual-frequency IF combinations, the improvement of the triple-frequency IF combinations in position estimation and AR should be trivial for GPS and BDS. The ionosphere amplification factor of all optimal EWL combinations for GPS and BDS is proportionally increasing with the sum of their integer linear coefficients and the proportion of the factor and the sum is about 12 for GPS and BDS. As a result, it cannot find three linear independent optimal EWL combinations with low noise and low ionosphere influence at the same time. The optimal ionosphere-reduced combinations that the sum of their coefficients equal to 1 are the NL combinations with a wavelength of about 11 cm and a noise standard deviation of 5–10 mm. The ionosphere-reduced combinations (4, 0, −3) for GPS, (4, 2, −5) and (5, −3, −1) for BDS can be used for precise positioning over medium–long baselines.

All existing GF and IF AR schemes for solving the third or NL ambiguity without distance constraints are equivalent and it can be achieved by a more straightforward manner based on the GF–IF integer combination.

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