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ORIGINAL ARTICLE

Adaptive collocation with application in height system transformation

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Abstract In collocation applications, the prior covariance matrices or weight matrices between the signals and the observations should be consistent to their uncertainties; otherwise, the solution of collocation will be distorted. To balance the covariance matrices of the signals and the observations, a new adaptive collocation estimator is thus derived in which the corresponding adaptive factor is constructed by the ratio of the variance components of the signals and the observations. A maximum likelihood estimator of the variance components is thus derived based on the collocation functional model and stochastic model. A simplified Helmert type estimator of the variance components for the collocation is also introduced and compared to the derived maximum likelihood type estimator. Reasonable and consistent covariance matrices of the signals and the observations are arrived through the adjustment of the adaptive factor. The new adaptive collocation with related adaptive factor constructed by the derived variance components is applied in a transformation between the geodetic height derived by GPS and orthometric height. It is shown that the adaptive collocation is not only simple in calculation but also effective in balancing the contribution of observations and the signals in the collocation model.

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1 Introduction

Collocation is usually applied in approximation in gravity field (Tscherning 1978). It can also be applied in coordinate transformation (You and Hwang 2006) and height datum transformation (Featherstone and Sproule 2006). To control the outlier influences, robust collocation is studied (Schaffrin 1986; Yang 1992). In the transformation of different height systems, the collocation method can also be applied to fit the errors that remained after the functional transformation of the two height systems or in a synthetic transformation process combining the functional transformation (trend-fitting) and stochastic fitting. The covariance function of the signals (stochastic part of the transformation) is, however, a key problem in the synthetic height transformation procedure (Yang 1992). Once the covariance function is chosen, the coefficients of the function are then determined by the observations and/or the known values of the parameters. The prior determined covariance elements are usually not changed in the collocation process.

Similar approach is kriging, which has also been widely researched and applied (Journel and Huijbregt 1989; Robeson 1997; Karniefi 1990; Oliver and Webster 1990; Felus et al. 2005). In theory, the kriging and collocation are equivalent from the expressions. The kriging evaluates the parameters and signals simultaneously; the dimension of the normal equation is larger than that of collocation, because the collocation estimates the trend parameters and signals separately. Furthermore, the collocation is only suitably applied in the smooth and stable stochastic fields to estimate the parameters and signals (Bian and Menz 2000).

The collocation results are sometimes distorted. This is due predominantly to the well-known distortions of the prior weight matrix of the measurements and the fitted covariance matrix of the signals. In theory, the weight matrices between the measurements and signals should be consistent to their actual contributions to the unknown parameters and signal estimates. In other words, the covariance matrices of the measurements and signals should reflect their actual uncertainties. Otherwise, the collocation results for the height transformation will be distorted.

Incomplete knowledge of variance matrix of the observations occurs in many geodetic application (Teunissen and Amiri-Simkooei 2008). In collocation, the incomplete knowledge of variance matrices of the measurements and signals will result in systematic errors similar to the influences of the functional model errors in the error effect point of view. Refitting of the residuals of the collocation is a way to reduce the systematic error influences (Yang and Liu 2002). In the view point of stochastic model error influences, the errors of the covariance elements of the signals can partly be adjusted through the variance factor. Thus, the variance component estimation can be employed in adjusting the ratio of the prior weights of the signals and the measurements (Koch and Kusche 2002; Shen and Liu 2002; Xu et al. 2006; Yang and Xu 2003). Following Koch and Kusche's idea to use the ratio of the variance components to overcome the ill-conditioned problem, we employ the ratio of the variance components of measurements and signals to adjust their weight matrices.

In geodetic applications, a lot of variance component estimation methods have been proposed and researched, such as minimum norm quadratic unbiased estimator (MINQUE) (Rao 1971), best invariant quadratic unbiased estimates (BIQUE) (Caspary 1987; Sjöeberg 1984), the maximum likelihood estimates (Koch 1986) or the restricted maximum likelihood estimation (Searle et al. 1992) and least squares (LS) variance component estimation (Teunissen and Amiri-Simkooei 2007). An often used variance component estimation of Helmert type is analyzed with respect to translation invariance and unbiasedness, which is also generalized into variance and covariance component estimation (Grafarend 1980). The best invariance component estimator is studied and its application to the generalized multivariate adjustment of heterogeneous deformation observations is proposed (Schaffrin 1981). Using the orthogonal complement likelihood function and rank factorization (Schaffrin 1983), an iterative procedure for the maximum likelihood estimates of the variance and covariance components is derived (Koch 1986). It is shown that BIQUE, MINQUE and LS variance component estimates are identical with Helmert's estimate under the Gaussian distribution (Amiri-Simkooei 2007; Ou 1989; Yu 1992). Bayesian inference for variance components is researched by Koch (1987). Furthermore, the variance component estimation is proposed to be applied in the determination of gravity field by different types of observations and solving the regularization problems (Koch and Kusche 2002; Xu et al. 2006), as well as in adaptive navigation (Yang and Xu 2003; Yang and Gao 2005).

This paper generalizes the maximum likelihood estimation of the variance component to the collocation model and employs the variance component ratio to set up an adaptive factor to balance the contribution of the measurements and prior signal information. Thus, a new adaptive collocation procedure is derived by introducing the adaptive factor.

2 Least square collocation estimators

Assume that we have an $n \times 1$ vector L of measurements, the corresponding error vector e with the expectation $E(e) = 0$ and covariance matrix $\Sigma_e = \sigma_e^2 P_e^{-1}$, an $m \times 1$ vector X of trend parameters, and a $u \times 1$ signal vector S with prior covariance matrix $\Sigma_S = \sigma_s^2 P_S^{-1}$, which has functional relation with measurements, where σ_e^2 and σ_s^2 are variance components, P_e and P_S are the weight matrices of e and S, respectively. The observational equation is

$$L = AX + BS + e \tag{1}$$

where A denotes an $n \times m$ design matrix with rank m and B is an $n \times u$ coefficient matrix.

The corresponding error equation is

$$V = A\hat{X} + B\hat{S} - L \tag{2}$$

where $\hat{X}$ is the estimated vector of the trend parameters and $\hat{S}$ is the estimated signal vector.

If the signal vector S', which has no relation with the measurements, is correlated with the measured signal vector S, that is, $\Sigma_{SS'} = \Sigma_{S'S}^{\mathrm{T}} \neq 0$, and if the unmeasured signal vector needs to be estimated, then Eq. 2 can be changed into

$$V = A\hat{X} + [B\ \mathbf{0}] \begin{bmatrix} \hat{S} \\ \hat{S}' \end{bmatrix} - L$$

where $\mathbf{0}$ is a zero matrix. The covariance matrix between the measured signals S and unmeasured signals S' must be evaluated at first, which is the task of covariance function fitting.

Assume that the signals and the measurement noises are uncorrelated, by using the least squares collocation principle (Koch 1977; Krarup 1978; Moritz 1980)

$$V^{\mathrm{T}} P_e V + \hat{S}^{\mathrm{T}} P_S \hat{S} = \min \tag{3}$$

we get

$$\hat{X} = (A^{\mathrm{T}} P_L A)^{-1} A^{\mathrm{T}} P_L L \tag{4}$$

and

$$\hat{S} = \Sigma_S P_L (L - A\hat{X}) \tag{5}$$

where

$$P_L = (B\Sigma_S B^{\mathrm{T}} + \Sigma_e)^{-1} \tag{6}$$

The signal estimates at unmeasured stations are

$$\hat{S}' = \Sigma_{S'S} P_L (L - A\hat{X}) \tag{7}$$

3 Adaptive collocation estimators

It is different from the standard collocation that the object function of the adaptive collocation is similar to the adaptive Kalman filter (Yang et al. 2001; Yang and Gao 2006)

$$\Omega = V^{\mathrm{T}} P_e V + \alpha \hat{S}^{\mathrm{T}} P_S \hat{S} = \min \tag{8}$$

where α is an adaptive factor, which balances the contribution of the measurements and signals to the parameter estimates. By derivation we have

$$\begin{bmatrix} A^{\mathrm{T}} P_e A & A^{\mathrm{T}} P_e B \\ B^{\mathrm{T}} P_e A & B^{\mathrm{T}} P_e B + \alpha P_S \end{bmatrix} \begin{bmatrix} \hat{X} \\ \hat{S} \end{bmatrix} = \begin{bmatrix} A^{\mathrm{T}} P_e L \\ B^{\mathrm{T}} P_e L \end{bmatrix} \tag{9}$$

The stepwise estimators are

$$\hat{X} = (A^{\mathrm{T}} \bar{P}_L A)^{-1} A^{\mathrm{T}} \bar{P}_L L \tag{10}$$

$$\hat{S} = \bar{\Sigma}_S B^{\mathrm{T}} \bar{P}_L (L - A\hat{X}) \tag{11}$$

The signal estimator at the unmeasured stations is

$$\hat{S}' = \bar{\Sigma}_{S'S} \bar{\Sigma}_S^{-1} \hat{S} \tag{12}$$

where $\bar{P}_L$, $\bar{\Sigma}_S$ and $\bar{\Sigma}_{S'S}$ denote the evaluated matrices by using the adaptive factor, they are respectively expressed as

$$\bar{P}_L = (B\Sigma_S B^{\mathrm{T}}/\alpha + \Sigma_e)^{-1} = \alpha (B\Sigma_S B^{\mathrm{T}} + \alpha \Sigma_e)^{-1} \tag{13}$$

$$\bar{\Sigma}_S = \frac{1}{\alpha} \sum_S \tag{14}$$

$$\bar{\Sigma}_{S'S} = \frac{1}{\alpha} \sum_{S'S} \tag{15}$$

Let

$$\tilde{P}_L = (B\Sigma_S B^{\mathrm{T}} + \alpha \Sigma_e)^{-1} \tag{16}$$

then we have

$$\bar{P}_L = \alpha \tilde{P}_L \tag{17}$$

Substituting Eqs. (14)–(16) into Eqs. (10)–(12), we have

$$\hat{X} = (A^{\mathrm{T}} \tilde{P}_L A)^{-1} A^{\mathrm{T}} \tilde{P}_L L \tag{18}$$

$$\hat{S} = \Sigma_S B^{\mathrm{T}} \tilde{P}_L (L - A\hat{X}) \tag{19}$$

$$\hat{S}' = \frac{1}{\alpha} \Sigma_{S'S} (\frac{1}{\alpha} \Sigma_S)^{-1} \hat{S} = \Sigma_{SS} \Sigma_S^{-1} \hat{S} \tag{20}$$

Equations 7 and 20 have the same form, but the estimated values are not equivalent, because the estimated signals $\hat{S}$ of the measured stations is changed by the adaptive factor. If the adaptive factor $\alpha = 1$, the solutions provided by the two formulae will be equal.

If the measurement outliers are taken into account, the object function may further be expressed as

$$\Omega = \sum_{i=1}^{n} P_i \rho(V_i) + \alpha \hat{S}^{\mathrm{T}} P_S \hat{S} = \min \tag{21}$$

where $\rho(V_i)$ is convex, continuous and not decreased function of the residual V_i, P_i is the weight element of the weight matrix P_e. Equation 21 is called an adaptively robust collocation object function, in which the robust object function is employed for dealing with the measurement outliers and adaptive factor α is introduced for balancing the prior weight matrices of the measurements and the signals.

Similar to the robust collocation (Yang 1992) and adaptive Kalman filter (Yang and Gao 2006), after taking the derivatives of the object function with respect to the unknown parameters, we have

$$\begin{bmatrix} A^{\mathrm{T}} \bar{P}_e A & A^{\mathrm{T}} \bar{P}_e B \\ B^{\mathrm{T}} \bar{P}_e A & B^{\mathrm{T}} \bar{P}_e B + \alpha P_S \end{bmatrix} \begin{bmatrix} \hat{X} \\ \hat{S} \end{bmatrix} = \begin{bmatrix} A^{\mathrm{T}} \bar{P}_e L \\ B^{\mathrm{T}} \bar{P}_e L \end{bmatrix} \tag{22}$$

where $\bar{P}_e$ denotes the equivalent weight matrix of the measurement vector which is determined by a weight function (Yang 1991, 1992).

4 Adaptive factor determined by variance components

It has been explained in Sect. 1 that the weight matrices of the signals and measurements should adapt to their proper uncertainties. In actual computation, however, the covariance matrix of the signals evaluated by the chosen covariance function and the known data points and the priori covariance matrix of the measurements may be inaccurate, which result in the weight matrices of the measurements and the signals unsuitable. Thus, we use an adaptive factor α to adjust the weight matrices and balance the contribution of the measurements and signals to the parameter estimates. Different statistical principles have different variance component estimators. In this section, two types of variance component estimators are derived based on the maximum likelihood estimation and Helmert estimation, respectively. An adaptive factor is presented.

4.1 Maximum likelihood estimator of variance components

Assume that the measurement vector L follows normal distribution

$$L \sim N(AX, \Sigma_L) \tag{23}$$

where Σ_L includes the variance components σ_s^2 and σ_e^2 of the stochastic signals and observational errors, that is

$$\Sigma_L = \sigma_s^2 B Q_s B^{\mathrm{T}} + \sigma_e^2 Q_e \tag{24}$$

where Q_s and Q_e are the cofactor matrices of the signals and measurements, respectively. Then the likelihood function of the unknown parameter vector X and the variance components σ_s^2 and σ_e^2 is (Koch 1986; Ou 1989)

$$\begin{aligned} l(X, \sigma_s^2, \sigma_e^2|L) &= (2\pi)^{-\frac{n}{2}} |\Sigma_L|^{-\frac{1}{2}} \\ &\times \exp\{-\frac{1}{2}(L - AX)^{\mathrm{T}} \Sigma_L^{-1} (L - AX)\} \end{aligned} \tag{25}$$

Taking the logarithm of the Eq. 25 and neglecting the constant terms, we have

$$\begin{aligned} \ln l(x, \sigma_s^2, \sigma_e^2|L) &= -\ln|\Sigma_L| - (L - AX)^{\mathrm{T}} \Sigma_L^{-1} (L - AX) \\ &= -\ln|\Sigma_L| - \mathrm{tr}[\Sigma_L^{-1} \\ &\quad \times (L - AX)(L - AX)^{\mathrm{T}}] \end{aligned} \tag{26}$$

Taking the derivatives of Eq. 26 with respect to σ_s^2 and σ_e^2, respectively, we obtain

$$\begin{aligned} \frac{\partial \ln l}{\partial \sigma_s^2} &= -\mathrm{tr}(B Q_S B^{\mathrm{T}} \Sigma_L^{-1}) + \mathrm{tr}[\Sigma_L^{-1} B Q_S B^{\mathrm{T}} \Sigma_L^{-1} \\ &\quad \times (L - AX)(L - AX)^{\mathrm{T}}] \end{aligned} \tag{27}$$

$$\begin{aligned} \frac{\partial \ln l}{\partial \sigma_e^2} &= -\mathrm{tr}(Q_e \Sigma_L^{-1}) + \mathrm{tr}[\Sigma_L^{-1} Q_e \Sigma_L^{-1} \\ &\quad \times (L - AX)(L - AX)^{\mathrm{T}}] \end{aligned} \tag{28}$$

Let the derivatives be equal to zeros and considering $\Sigma_L^{-1} = P_L$, we get

$$\begin{aligned} &\mathrm{tr}[P_L B Q_S B^{\mathrm{T}} P_L (L - AX)(L - AX)^{\mathrm{T}}] \\ &\quad = \mathrm{tr}(B Q_S B^{\mathrm{T}} P_L) \end{aligned} \tag{29}$$

$$\mathrm{tr}[P_L Q_e P_L (L - AX)(L - AX)^{\mathrm{T}}] = \mathrm{tr}(Q_e P_L) \tag{30}$$

The right-hand term of Eq. 29 can be expressed as

$$\begin{aligned} \mathrm{tr}(B Q_S B^{\mathrm{T}} P_L) &= \mathrm{tr}(B Q_S B^{\mathrm{T}} P_L \Sigma_L P_L) \\ &= \mathrm{tr}[B Q_S B^{\mathrm{T}} P_L (B Q_S B^{\mathrm{T}} \sigma_s^2 \\ &\quad + Q_e \sigma_e^2) P_L] \\ &= \mathrm{tr}(B Q_S B^{\mathrm{T}} P_L B Q_S B^{\mathrm{T}} P_L) \sigma_s^2 \\ &\quad + \mathrm{tr}(B Q_S B^{\mathrm{T}} P_L Q_e P_L) \sigma_e^2 \end{aligned} \tag{31}$$

The right-hand term of the Eq. 30 can be expressed as

$$\begin{aligned} \mathrm{tr}(Q_e P_L) &= \mathrm{tr}(Q_e P_L B Q_S B^{\mathrm{T}} P_L) \sigma_s^2 \\ &\quad + \mathrm{tr}(Q_e P_L Q_e P_L) \sigma_e^2 \end{aligned} \tag{32}$$

Combining the Eqs. 29–32 and expressing $L - AX$ as the residual vector V, we arrive at

$$\begin{aligned} &\begin{pmatrix} \mathrm{tr}(B Q_S B^{\mathrm{T}} P_L B Q_S B^{\mathrm{T}} P_L) & \mathrm{tr}(B Q_S B^{\mathrm{T}} P_L Q_e P_L) \\ \mathrm{tr}(B Q_S B^{\mathrm{T}} P_L Q_e P_L) & \mathrm{tr}(Q_e P_L Q_e P_L) \end{pmatrix} \\ &\times \begin{pmatrix} \hat{\sigma}_s^2 \\ \hat{\sigma}_e^2 \end{pmatrix} = \begin{pmatrix} V^{\mathrm{T}} P_L B Q_S B^{\mathrm{T}} P_L V \\ V^{\mathrm{T}} P_L Q_e P_L V \end{pmatrix} \end{aligned} \tag{33}$$

If $B = I$, then we get the estimator of the variance components as

$$\begin{aligned} \begin{pmatrix} \hat{\sigma}_s^2 \\ \hat{\sigma}_e^2 \end{pmatrix} &= \begin{pmatrix} \mathrm{tr}(Q_S P_L Q_S P_L) & \mathrm{tr}(Q_S P_L Q_e P_L) \\ \mathrm{tr}(Q_S P_L Q_e P_L) & \mathrm{tr}(Q_e P_L Q_e P_L) \end{pmatrix}^{-1} \\ &\quad \times \begin{pmatrix} V^{\mathrm{T}} P_L Q_S P_L V \\ V^{\mathrm{T}} P_L Q_e P_L V \end{pmatrix} \end{aligned} \tag{34}$$

It is interesting that the new estimators of variance components make the calculation procedure simpler than the Helmert type estimates, because they do not have the need to calculate the normal matrix and its inverse.

After solving $\hat{\sigma}_s^2$ and $\hat{\sigma}_e^2$, the adaptive factor α can be set up like (Yang et al. 2001; Yang and Xu 2003; Yang and Gao 2005)

$$\alpha = \hat{\sigma}_e^2 / \hat{\sigma}_s^2 \tag{35}$$

in which $\hat{\sigma}_s^2 \neq 0$ must be satisfied. If the variance component of the signals is large, that is, their weight matrix should be reduced. The adaptive factor expressed by Eq. 35 just adapts the weight matrix of the signals to its proper one and also adapts the signals to their proper contributions to the parameter estimates.

Having the adaptive factor α, we re-estimate the trend parameter vector $\hat{X}$ and the signal vectors $\hat{S}$ and $\hat{S}'$ by using Eqs. 16 and 18–20.

4.2 Simplified estimator of variance components

If the signals are viewed as pseudo observations, then the variance components with respect to the two kinds of observations can be obtained (Koch and Kusche 2002). We start from the error equations

$$\begin{cases} V = A\hat{X} + B\hat{S} - L & \text{with} \quad P_e = \Sigma_e^{-1} \\ V_S = \hat{S} - 0 & \text{with} \quad P_S = \Sigma_S^{-1} \end{cases} \tag{36}$$

where "0" denotes the prior expectation vector of the signals. Then the corresponding Helmert type of variance component estimator is like (Koch and Kusche 2002)

$$\hat{\sigma}_e^2 = \frac{V^{\mathrm{T}} P_e V}{r_e} \tag{37}$$

$$\hat{\sigma}_s^2 = \frac{\hat{S}^{\mathrm{T}} P_s \hat{S}}{r_s} \tag{38}$$

where r_e and r_s are the partial redundancies, i.e., the contributions of the observations L and the prior signal information S to the overall redundancy $r = n + u - m$ of the model of Eq. 1. The partial redundancies are computed from

(Koch and Kusche 2002)

$$r_e = n - \mathrm{tr}\left(\frac{1}{\sigma_e^2} N^{-1} N_e\right) \tag{39}$$

$$r_s = u - \mathrm{tr}\left(\frac{1}{\sigma_s^2} N^{-1} N_s\right) \tag{40}$$

with

$$N = \begin{bmatrix} \frac{1}{\sigma_e^2} A^{\mathrm{T}} P_e A & \frac{1}{\sigma_e^2} A^{\mathrm{T}} P_e B \\ \frac{1}{\sigma_e^2} B^{\mathrm{T}} P_e A & \frac{1}{\sigma_e^2} B^{\mathrm{T}} P_e B + \frac{1}{\sigma_s^2} P_S \end{bmatrix},$$

$$N_e = \begin{bmatrix} \frac{1}{\sigma_e^2} A^{\mathrm{T}} P_e A & \frac{1}{\sigma_e^2} A^{\mathrm{T}} P_e B \\ \frac{1}{\sigma_e^2} B^{\mathrm{T}} P_e A & \frac{1}{\sigma_e^2} B^{\mathrm{T}} P_e B \end{bmatrix}, \quad N_s = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\sigma_e^2} P_S \end{bmatrix}$$

After solving the estimates of variance components $\hat{\sigma}_s^2$ and $\hat{\sigma}_e^2$ as well as the adaptive factor α, we can re-estimate the trend parameter vector $\hat{X}$ and the signal vectors $\hat{S}$ and $\hat{S}'$ in the same way as listed in Sect. 4.1.

The formulas above may be derived by the maximum likelihood method, by the best invariant quadratic unbiased estimation or Helmert's method (see Koch 2000, p. 146; Koch and Kusche 2002).

5 An actual computation and analysis

Three hundred and sixty two GPS/leveling stations in the area of Southern China with two different heights are chosen for the height transformation (see Fig. 1), in which the geodetic heights are referenced to the geocentric reference frame ITRF97 and the normal heights are referenced to the Chinese height datum 1985 defined by the mean sea surface of Huanghai Sea determined by tide gauge observations at Qingdao. To check the accuracy of the applied transformation methods, 332 stations are employed for computing the transformation parameters and the signals, and the rest 30 stations are served as checking stations, that is, the geodetic heights of the 30 stations are transformed into the orthometric heights, then compared with the observed orthometric ones and the discrepancies as well as the RMS are obtained.

The polynomial model is chosen as the functional model to express the trend of the height differences. Considering the stochastic part, we employ the following error model

$$V_i = \sum_{m=1}^{K} \sum_{n=0}^{m} \hat{\alpha}_{mn} \mathrm{d}B_i^{m-n} \mathrm{d}L_i^n + \hat{S}_i - (h_i - H_i) \tag{41}$$

where V_i is the residual of ith pseudo-observation $(h_i - H_i)$, which is also defined in Eq. 2; $\hat{\alpha}_{mn}$ denotes the estimated unknown coefficients, which construct the unknown parameter vector in Eq. 2; $\mathrm{d}B_i$ and $\mathrm{d}L_i$ are the centered latitude and longitude coordinates, which construct the design matrix in Eq. 2; h_i and H_i denote the ellipsoidal height and

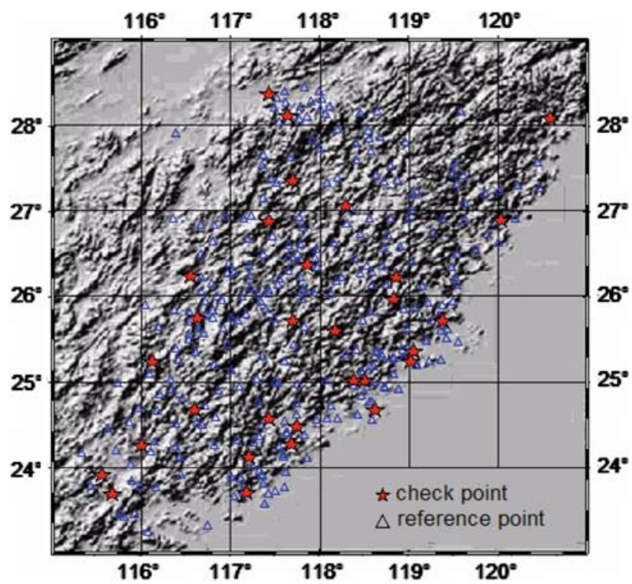

Fig. 1 Distribution of common stations with the heights of two systems (The *blue triangles* are the reference data stations, the *red stars* denote the checking stations, which do not take part in the collocation)

orthometric height of the ith GPS/leveling station; and K denotes the order of the polynomial model. The Gaussian covariance function of the signals is chosen (Moritz 1980; You and Hwang 2006)

$$C(d) = C_0 e^{-k^2 d^2} \tag{42}$$

where C_0 and k are unknown parameters to be estimated and $C(d)$ denotes the covariance between the ith point and the jth point with distance d. A method proposed by Mikhail and Ackermann (1976) to fit the empirical covariance function is employed. Firstly, the data stations (332 reference stations) are divided into m groups, each of which with nearly equal distance $d_i (i = 1, \ldots, m)$. Secondly, the initial covariance is obtained by the mean of all possible products $l_i l_j$ with the distance d_i

$$C(d_i) = \frac{1}{n_i} \sum_{i,j}^{n_i} l_i l_j \tag{43}$$

where n_i denotes the number of pseudo-observations in the ith group; $C(d_i)$ is regarded as the covariance at the distance d_i, and the pseudo-observations are expressed as

$$l_i = (h_i - H_i) - \sum_{m=1}^{K=2} \sum_{n=0}^{m} \hat{\alpha}_{mn} \mathrm{d}B_i^{m-n} \mathrm{d}L_i^n \tag{44}$$

Thirdly, we get the unknown parameter estimates in the chosen Gaussian function, $C_0 = 0.0123$, $k = 0.0162$, based on the m covariance elements by using least squares estimation.

Table 1 Root mean square residuals of heights at data stations (m)

σ_0^2	Scheme	Minimum	Maximum	RMS
$\sigma_0^2 = 1$	X^{LS}	−0.385	0.304	0.105
	X^{C}	−0.375	0.307	0.100
	X^{MAC}	−0.329	0.296	0.085
	X^{HAC}	−0.326	0.295	0.085
$\sigma_0^2 = 0.1$	X^{C}	−0.339	0.296	0.088
	X^{MAC}	−0.329	0.296	0.085
	X^{HAC}	−0.326	0.295	0.085

The root mean square residuals calculated by the differences of the fitted height values by collocation methods and the measurements

Table 2 RMS of the heights at checking stations (m)

σ_0^2	Scheme	Minimum	Maximum	RMS
$\sigma_0^2 = 1$	X^{LS}	−0.209	0.197	0.102
	X^{C}	−0.197	0.171	0.096
	X^{MAC}	−0.194	0.111	0.070
	X^{HAC}	−0.192	0.105	0.069
$\sigma_0^2 = 0.1$	X^{C}	−0.195	0.128	0.076
	X^{MAC}	−0.194	0.111	0.070
	X^{HAC}	−0.192	0.105	0.069

The root mean square errors calculated from the differences of the fitted height values by collocation methods and the known values which do not take part in the collocation, by using Eq. (45)

The RMS of the transformed orthometric heights from the checking stations is computed by

$$\mathrm{RMS} = \left[\frac{1}{n}\sum_{i=1}^{n}(\Delta H_i)^2\right]^{1/2} \tag{45}$$

where $n = 30$ is the number of checking stations and ΔH_i is the difference between the computed and known orthometric height at the ith checking station. The smaller the RMS is, the better the transformation scheme is.

In the computation, following four schemes are performed:

Scheme 1: Least squares estimation based on a polynomial model, expressed as X^{LS}
Scheme 2: Collocation, expressed as X^{C}
Scheme 3: Adaptive collocation based on maximum likelihood estimates of variance components, expressed as X^{MAC}
Scheme 4: Adaptive collocation based on Helmert type estimates of variance components, expressed as X^{HAC}.

To show how the adaptive collocation proposed in this paper adapts to the weight ratios of the measurements and signals, the initial scale variance is supposed as 1.0 and 0.1, respectively. Table 1 shows the minimum, maximum

Table 3 The error region of the fitted heights at check stations

σ_0^2	Scheme	<5 cm	[5 cm,10 cm]	>10 cm
$\sigma_0^2 = 1$	X^{LS}	9 (30%)	12 (40%)	9 (30%)
	X^{C}	10 (33%)	12 (40%)	8 (27%)
	X^{MAC}	17 (57%)	9 (30%)	4 (13%)
	X^{HAC}	17 (57%)	10 (33%)	3 (10%)
$\sigma_0^2 = 0.1$	X^{C}	16 (53%)	8 (27%)	6 (20%)
	X^{MAC}	17 (57%)	9 (30%)	4 (13%)
	X^{HAC}	17 (57%)	10 (33%)	3 (10%)

In table 3, the percentages reflect the absolute error region of the fitted height values by collocation methods

values and root mean squares of residuals of height measurements at reference stations for different schemes, respectively. Figure 2 presents distribution of residuals of the heights at the reference stations. The minimum and maximum values and RMS of the transformed heights at the checking stations are shown in Table 2, and the intervals of the corresponding discrepancies between the transformed orthometric heights and the original ones are shown in Table 3. If the scale factor $\sigma_0^2 = 1$ is chosen, then the adaptive factor $\alpha = 50.61$ after three iterations, by using Scheme 3; while $\alpha = 50.60$ after four iterations, by using Scheme 4. If $\sigma_0^2 = 0.1$ is chosen, then the adaptive factor $\alpha = 5.04$ after three iterations,

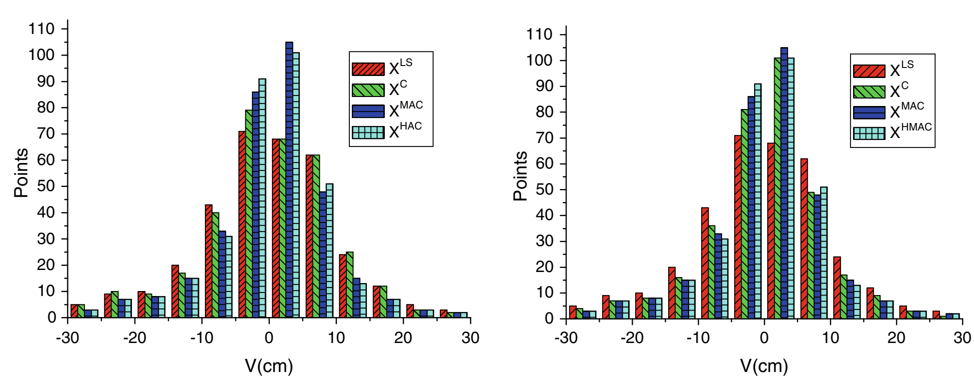

Fig. 2 Residual distribution of the measured stations (σ_0^2 equals 1.0 and 0.1, respectively)

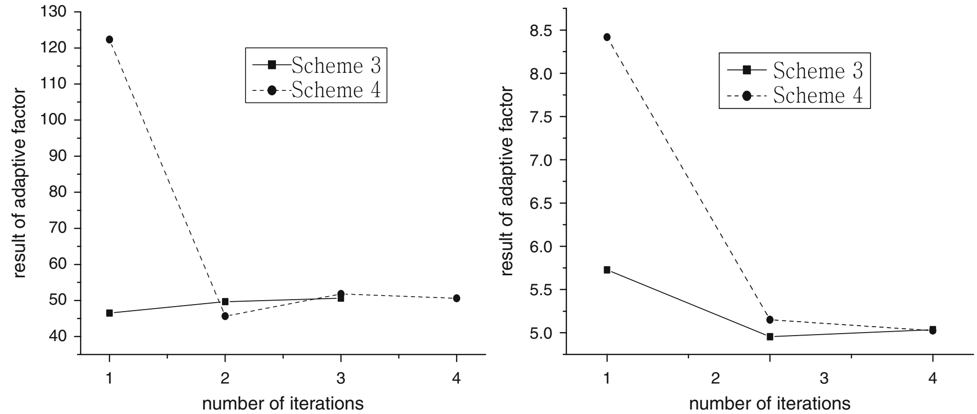

Fig. 3 Iterative adaptive factors (σ_0^2 equals 1.0 and 0.1, respectively)

by using Scheme 3; while $\alpha = 5.03$ after three iterations, by using Scheme 4. The adaptive factor α corresponding to different schemes are shown in Fig. 3. It shows that the priori variance scale will not influence the adaptive collocation results, but it will affect the nonadaptive collocation results.

From the computation results, the following facts have been drawn:

1. The initial value of scale factor σ_0^2 has significant effects on the transformed heights for both of the collocation method and least squares polynomial fitting method, which has been reflected by the residuals of the measured stations and the discrepancies of the transformed heights of the checking stations. It means that the inconsistent variance components between the measurements and the signals will result in systematic errors of the transformed heights.

2. In theory and practice, the weight matrices of the measurements and signals should reflect their uncertainties when applying collocation. In actual computation, however, it may have errors in the fitted variance–covariance matrix of the signals, which may lead to a biased variance scale of the signals, thus, leading to inconsistent weight matrices for the measurements and the signals. Thus, it is reasonable to employ the variance components to construct an adaptive factor, to adapt the ratio of the weight matrices of the signals and measurements to their proper value, and in turn to adjust their contributions to the parameter estimates.

3. For the two kinds of adaptive collocation, the different initial values of σ_0^2 do not influence transformed heights, since the adaptive factor plays a role in balancing the contributions of the stochastic models between the measurements and signals according to their actual uncertainties, which keeps the weight matrices between the measurements and signals in a reasonable ratio, and keeps the collocation results stable.

4. The RMS of the checking stations shows that the adaptive collocation based on the ratio of the estimated variance components is superior to the standard collocation and the polynomial fitting in transforming the geodetic heights to the orthometric heights.

6 Conclusions

The determination of the covariance function is a key problem in collocation. If the prior weight matrices of the signals and the measurements inversed from the related covariance matrices are consistent, in other words, the two weight matrices reflect their uncertainties of the measurements and signals, the collocation results are stable and reliable. Otherwise, the standard collocation results will be distorted. Usually, it is difficult to make the prior weight matrices of the signals and the measurements consistent in practice; thus, the standard collocation may be influenced in some cases.

The adaptive collocation by introducing an adaptive factor α, which makes the weight matrices of the signals and measurements consistent, balances the contributions of the measurements and signals to the estimated parameters.

It should be pointed that the adaptive factor α determined by variance component estimates may be influenced by the preliminary values of the prior covariance matrices. Thus, the preliminary covariance matrices should be as precise as possible.

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