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ORIGINAL ARTICLE

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Degradation of geopotential recovery from short repeat-cycle orbits: application to GRACE monthly fields

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Abstract Throughout 2004 the GRACE (Gravity Recovery And Climate Experiment) orbit contracted slowly to yield a sparse repeat track of 61 revolutions every 4 days on 19 September 2004. As a result, we show from linear perturbation theory that geopotential information previously available to fully resolve a gravity field every month of 120×120 (degree by order) in spherical harmonics was compressed then into about one-fourth of the necessary observation space. We estimate from this theory that the ideal gravity field resolution in September 2004 was only about 30×30. More generally, we show that any repeat-cycle mission for geopotential recovery with full resolution $L \times L$ requires the number of orbit-revolutions-to-repeat to be greater than $2L$.

Keywords Geopotential resolution · Repeat orbits · Resonances · GRACE · Satellite gravimetry

1 Introduction

Since April 2002, the Gravity Recovery And Climate Experiment or GRACE (Tapley et al. 2004) has been monitoring the Earth's geopotential field using precise microwave K-band ranging (KBR) between two co-planar satellites about 2° apart at about 480 km altitude. These measurements–along with GPS positions of centimeter-level precision – have been inverted to yield a series of unconstrained independent 'monthly' high-degree gravitational field models band limited at $L \times L$ (degree by order) with L generally being 120. (Some months with restricted coverage resulted in 70×70 models.)

These monthly models can and have been used to track time-varying mass changes in the Earth's surface fluids (e.g., Wahr et al. 2004). However, starting in July 2004 the unconstrained monthly fields have (1) not been released (Bettadpur 2004), while (2) at the same time the GRACE orbit was slowly contracting towards its repeat cycle of 61 revolutions in 4 days in September 2004. These two events may be related to the degradation of high-degree geopotential recovery in such a short repeat orbit.

Subsequent to the original issue of GRACE *unconstrained* monthly fields (Tapley et al. 2004), as foreseen by Bettadpur (2004), a second release is underway from mid-2005 of *constrained* monthly fields to address these and other concerns (http://www.podaac.jpl.nasa.gov/grace/data.access.html).

In Sects. 2 – 4, we use linear perturbation theory to gain an insight into the resolution problem with repeat-cycle GRACE orbits. In Sect. 5, we identify past and future critical GRACE orbits (in its continued contraction) with regard to 120×120 gravitational field resolution and summarize our findings and recommendations in Sect. 6.

2 Geopotential orbit frequencies and signal on the GRACE tracker

We start with the standard expression for the geopotential in solid spherical harmonics:

$$V_e = \mu_e / r \sum_{l=2}^{\infty} \sum_{m=0}^{l} (r_e/r)^l \overline{P}_{lm}(\phi, \lambda)[\overline{C}_{lm} \cos(m\lambda) + \overline{S}_{lm} \sin(m\lambda)], \quad (1)$$

where r, ϕ, λ are the radius to the satellite, its geocentric latitude and longitude; r_e and μ_e the Earth's mean equatorial radius and geocentric gravitational constant, respectively; and $\overline{P}_{lm}$ the fully normalized associated Legendre functions of degree l and order m. The $\overline{C}_{lm}$, $\overline{S}_{lm}$ are the fully normalized geopotential harmonic coefficients (e.g., Kaula 1966, p. 7).

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Considering a band-limited (degree and order $L \times L$) gravitational field acting on a nearly circular satellite orbit (a standard for all Earth observation satellites since 1980), the geopotential effects on it have frequencies (in integers q, k and m):

$$\dot{\alpha} = -q\dot{\omega} + k(\dot{\omega} + \dot{M}) + m(\dot{\Omega} - \dot{\theta}_e), \tag{2}$$

where $-\infty \leq q \leq \infty$, but only $q = 0$ covers the dominant effects, $-L \leq k \leq L$, $0 \leq m \leq L$ and ω, M, Ω are the orbit arguments of perigee, mean anomaly and right ascension of the ascending node, respectively, and θ_e is the Greenwich hour angle (e.g., Kaula 1966). Further, since $k = l - 2p + q$ and $0 \leq p(integer) \leq l$, each dominant m, k effect involves only degrees l of the same parity as k. Thus, the total number of dominant geopotential frequencies in perturbations on a 'circular orbit' for $L = 120$ is $121 \times 241 = 29,161$.

Wagner (1987) showed from linear perturbations of the orbit that the range-rate signal between two close co-planar subsatellites in near-circular polar orbits is given in terms of geopotential harmonics through these dominant frequencies as

$$\Delta\dot{R} = \sum_{\dot{\alpha}} [C_{\dot{\alpha}} \cos(\dot{\alpha}t) + S_{\dot{\alpha}} \sin(\dot{\alpha}t)], \tag{3}$$

$$C_{\dot{\alpha}},\ S_{\dot{\alpha}} = \sum_l H_{lmk} \begin{pmatrix} \overline{C}_{lm}\ , -\overline{S}_{lm} \\ \overline{S}_{lm}\ ,\ \overline{C}_{lm} \end{pmatrix}_{l-m \text{ odd}}^{l-m \text{ even}}, \tag{4}$$

$$\begin{aligned} H_{lmk} = {} & \dot{M} a (r_e/a)^l \bar{F}_{l,m,(l-k)/2}(\pi/2) \\ & \times \left\{ \delta u \cos(-k\,\delta u/2)[\beta(l+1) - 2k](\beta^2 - 1)^{-1} \right. \\ & +2\sin(-k\,\delta u/2)[2(l+1+2k\beta)(\beta^2-1)^{-1} \\ & \left. +(3k\beta^{-1})] \right\}, \end{aligned} \tag{5}$$

for $l \geq \max(m, |k|)$, with the same parity as k, where δu is the separation of the two subsatellites, $\bar{F}(\pi/2)$ the fully normalized inclination functions, a the semi-major axis of the orbits of the two subsatellites, and for simplicity (with no loss of generality) the track starts (at $t = 0$) with the mean position of the subsatellites at Greenwich on the equator. The term β, the normalized frequency of each wave, is $\dot{\alpha}/\dot{M}$.

The dominant perturbing frequencies on the orbit [expressed in Eq. (2) and characterized by m, k] may not change substantially for long periods of time so that, provided their periods do not all repeat in that time, the resulting aperiodic signal series will continue to yield new information. However, we will see immediately that the orbit condition (mean motions of perigee, node and mean anomaly) resulting in a resonance provides just such a repeating periodic cycle for this series. It further defines the band limit L that can yield the maximum amount of information in this cycle.

3 Repeat (geostationary) or resonant orbits

The condition for a repeating trajectory (in geographic space), given a circular orbit, is

$$D(\dot{M} + \dot{\omega}) = R(\dot{\theta}_e - \dot{\Omega}), \tag{6}$$

for R and D co-prime integers (R/D irreducible). This condition results in a stationary ground track of R nodal revolutions in D synodic days (revolutions of Greenwich with respect to the satellite's node). By comparison with Eq. (2) it is also seen that in the same repeat cycle (of D synodic days) all the near-circular geopotential perturbations (m, k) of frequency $\dot{\alpha}$ also repeat with each frequency having wave number with respect to that cycle time:

$$N = kR - mD. \tag{7}$$

We should point out, as Colombo (1984) first discussed, that an actual trajectory is never exactly circular and so cannot repeat exactly in this sense, due to other external perturbations. However, as we shall see, the broad characteristics of geopotential resolution on a nearly circular orbit such as GRACE can be quickly found by analysis of a nearby R, D 'circular' resonant case.

For this purpose we imagine the range-rate signal in Eqs. (3)–(5) to be generated over a repeat cycle at a certain data rate and then its spectrum taken, with each phase of each line considered as an observation. We assume a data rate sufficient to assure resolution of at least as many lines as geopotential frequencies in L. For example, for a repeat in 4 days, $L \times L$ resolution requires a data rate of at least one point every 12 s, which is less than half the rate (0.2 Hz) actually used in the GRACE project's analysis, whose tracker has an estimated 1 μm/s precision (Tapley et al. 2004).

In this context we define the 'ideal' resolution of a band-limited geopotential $L \times L$ in a repeat cycle as occurring when each frequency (m, k) in L has a unique (exclusive-to-it) wave number $|N|$. (We do not include the zonal harmonics $m = 0$ in this definition because for any orbit, they always occur in pairs with opposite k signs but the same perturbation.) In the general case, we also take the absolute wave number because an observation of a perturbation with a negative wave number is the same as that of its 'positive' with only a reversed sine phase and so can be lumped with another perturbation of the same wave number (positive or negative).

4 Ideal and degraded geopotential resolution

Let non-zonal (m, k, N) and (m_1, k_1, N_1) represent two geopotential frequencies on the repeat orbit (R, D). When are their frequencies and wave numbers $|N|$ unique within L? Clearly uniqueness is guaranteed only when both (1) $N_1 \neq N$ and (2) $N_1 \neq -N$ are satisfied for all possible pairs of frequencies. To exclude same-sign wave numbers (1) note the condition for equality is $N = N_1$ or

$$kR - mD = k_1R - m_1D, \tag{8}$$

or, for integers $k' = k - k_1$ and $m' = m - m_1$:

$$k' = Dm'/R, \tag{9}$$

a Diophantine equation (i.e., with only integer solutions), which can only be satisfied for

$$m_1 = m + iR, \quad (10)$$

$$k_1 = k + iD, \quad (11)$$

where $i = 0, \pm 1, \pm 2, \ldots$, with the band limits $0 \leq (m, m_1) \leq L$ and $(|k|, |k_1|) \leq L$.

For sub-synchronous repeat orbits ($R > D$) the m equation (Eq. 10) will govern, where it is seen that uniqueness holds for $R > L$. [For the extreme counter-examples, if $R = L$, the frequency pairs (m, k): $(0, k)$ and $(L, k + D)$ for $i = 1$ have a common wave number $N = kL$, and the pairs (L, k) and $(0, k - D)$ for $i = -1$ have another common wave number $N = L(k - D)$].

However, to exclude opposite-sign wave numbers (2), note that the condition for equality is $N = -N_1$ or

$$kR - mD = -[k_1 R - m_1 D], \quad (12)$$

leading to the same Diophantine equation as Eq. (9) but with $k' = k + k_1$ and $m' = m + m_1$, whose solution is the set of parametric equations:

$$m_1 = iR - m, \quad (13)$$

$$k_1 = iD - k, \quad (14)$$

where $i = 0, 1, 2, \ldots$, with the same band limitations on m, m_1 and k, k_1 as above. (Note there are no negative integers i permitted in this set since negative m_1s are excluded.)

Thus, for the opposite-sign wave numbers to be unique (in subsynchronous orbits) all that is necessary is for $R > 2L$, since then for $i = 1$ the lowest m_1 correlating with a wave involving an m within L is always greater than L. [For the extreme counter-example if $R = 2L$, for $i = 1$ the (m, k) frequency pairs (L, k) and $(L, D - k)$ would have equal and opposite wave numbers and so not have unique wave representations for an $L \times L$ field.]

To summarize, considering both same- and opposite-sign wave numbers, ideal resolution from a repeat-cycle orbit (R, D) can only occur when $R > 2L$. Note that since there are $(L + 1)^2$ geopotential coefficients in an $L \times L$ field and $2(2L + 1)(L + 1)$ phase observations of the lumped harmonics $C_{\dot{\alpha}}$ and $S_{\dot{\alpha}}$, the average ratio of observations to coefficients in any ideal repeat cycle approaches 4/1 for large L and is 3.98 for $L = 120$.

4.1 Ideal geopotential resolution in a repeat orbit ($R > 2L$)

Following Colombo (1984), each m, k perturbation is represented by a unique wave N in the repeat time D so that each phase observation of this perturbation [Eqs. (4) and (5) above for a given resonant orbit] represents a lumped harmonic of a given order m of either odd or even degree l, depending on the parity of k and of the same kind ($\overline{C}_{lm}$ or $\overline{S}_{lm}$).

What are the powers (amplitudes) of the signals in these (m, k) perturbation waves in a GRACE orbit? For simplicity (and order of magnitude), we used Eqs. (4) and (5) in the GRACE orbit from May 2002 with a 'Kaula field' (120×120) consisting of normalized geopotential harmonics of $10^{-5}/l^2$ for all terms beyond $\overline{C}_{20}$. As Fig. 1 shows, the signals are dominated by low-order m terms with small k (which are the frequencies of the zonal terms in cycles per revolution). Notice that the effects are somewhat stronger for positive k compared to negative k because the absolute wave numbers $|N|$ for these are smaller (thus of longer wavelength and lower frequency).

Aggregating all such observations of the same order, degree parity and kind into a block of condition equations for these common geopotential coefficients results in minimum block-diagonal normal matrices for their precisions (on inversion). Roughly, for a band limit L, the maximum size (or order) of these minimum (ideal) matrices will be $(L - m)/2$ or at most 60 for $L = 120$. To resolve an $L \times L$ field in this ideal repeat orbit will require the inversion of $2(L + 1)$ of these 'small' matrices, two for each order (of odd and even degree parities). Since the time to invert a matrix is proportional to the cube of the matrix size (e.g., Press et al. 1987, p. 37) and the size of a full geopotential matrix is roughly $(L+1)^2$, the inversion of all the 'small' blocks will take only about $1/16L^2$ of the time for a full inverse ($L >> 1$), which for a repeat cycle is mostly filled with zeros.

Figures 2 (left panel) and 3 (bottom curve) illustrate the ideal precision obtained in a polar repeat orbit near GRACE's altitude in 2002 (480 km, $R, D = 243, 16$) using 1μm/s precision inter-satellite range-rate observations every second for 1 month. They were derived by inverting the minimum block diagonal normal matrices described above, each one using a selected set of the 29,161 independent perturbation waves within the repeat cycle D. For example, for the matrix of order $m = 1$ and odd degree l, the cosine phase observations applying would be all $m = 1, k = odd$ or 120 in total (negative as well as positive k) to determine 59 geopotential coefficients $\overline{C}_{(l\, odd \neq 1,1)}$.

The ratio of observations to unknowns in this 'ideal' case is roughly 2/1, but for all the matrices in this gravity recovery, the average ratio is close to 4/1 (as mentioned previously).

For the ideal precisions (Fig. 2, left panel), note the general fall off for each degree from zonals (best) to sectorials (worst), which follow from the polar orbit whose ground track is best suited to discriminating zonal harmonics. Visualizing the way a polar orbit tracks a spherical harmonic, it is seen that sectorials ($l = m$) would generate mainly strong low-frequency terms (<1 cyles/revolution), while the tesserals ($l > m$) for the same degree would have a much richer spectrum of effects. We will also see a close similarity of the ideal precisions in these cases and more realistic ones generated by the GRACE project.

We also assess the condition of these normal matrices by noting the inverse of their diagonal terms (the 'ideal' variance if the normals were diagonal matrices) and comparing them to the diagonals of the inverse ('actual' variances). This variance ratio 'ideal'/'actual' for each l, m term, which we call its condition number, is always less than 1.0 in a correlated normal matrix and $\ll 1$ in a poorly conditioned one which may need a priori information (or stabilization/regularization) to produce a sensible result.

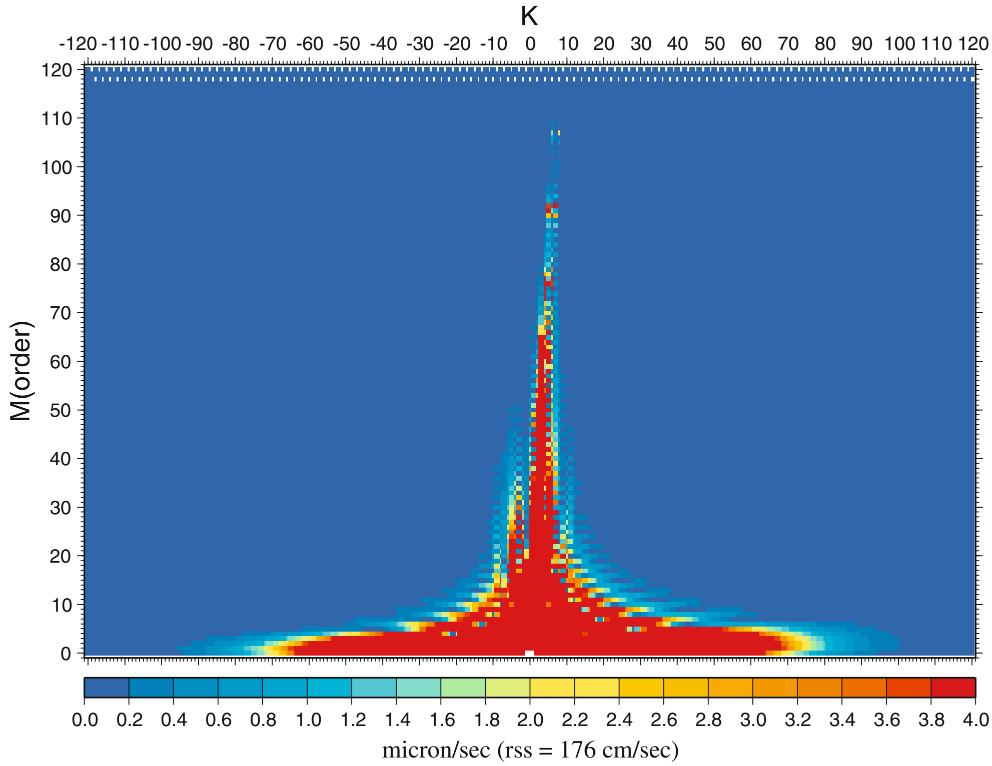

Fig. 1 GRACE geopotential range-rate signal from a Kaula 120 × 120 field. Orbits 243 revs/16 days, 2° separation (May 2002). The observation space (M, K), where each frequency is (roughly) K cycles/revolution–M cycles/day, is dominated by high-frequency terms of low-order M and (shallow resonance) terms of small positive K yielding low frequencies. Because each frequency is seen distinctly, the inverse of this signal for a complete 120 × 120 field is well-conditioned

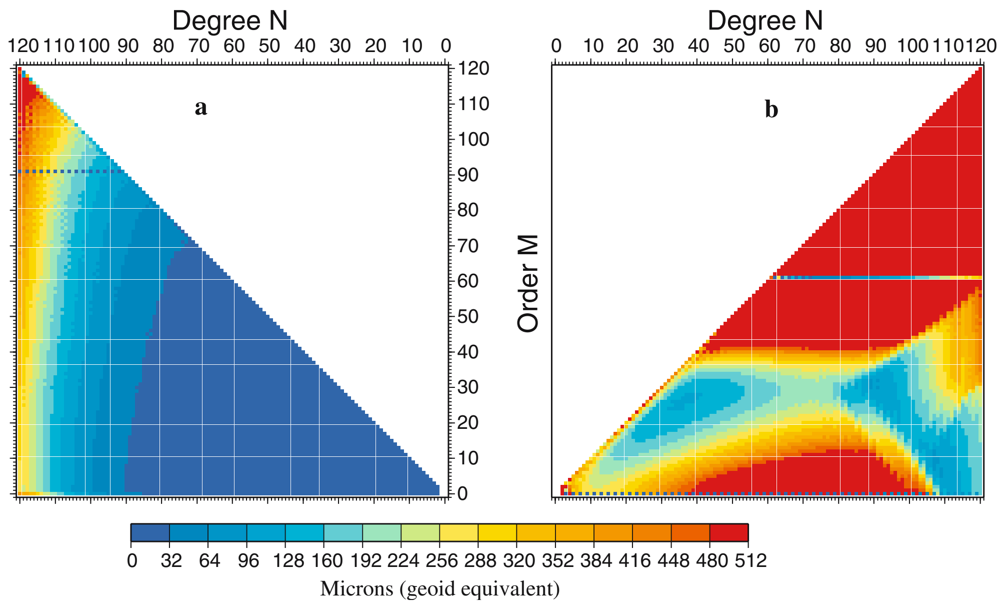

Fig. 2 Theoretical geoid precisions for $\overline{C}_{N,M}$ in GRACE 120 × 120 monthly solutions: **a** in an ideal orbit for 120 × 120 resolution (Rev/Day = 243/16), unconstrained solution, total error = 17 mm; **b** for the September 2004 orbit (Rev/Day = 61/4), uses Kaula constraint, total error = 410 mm. In spite of the constraint, the deficient 61/4 orbit recovers the gravitational field poorly compared to the 243/16 orbit at almost all degrees and orders

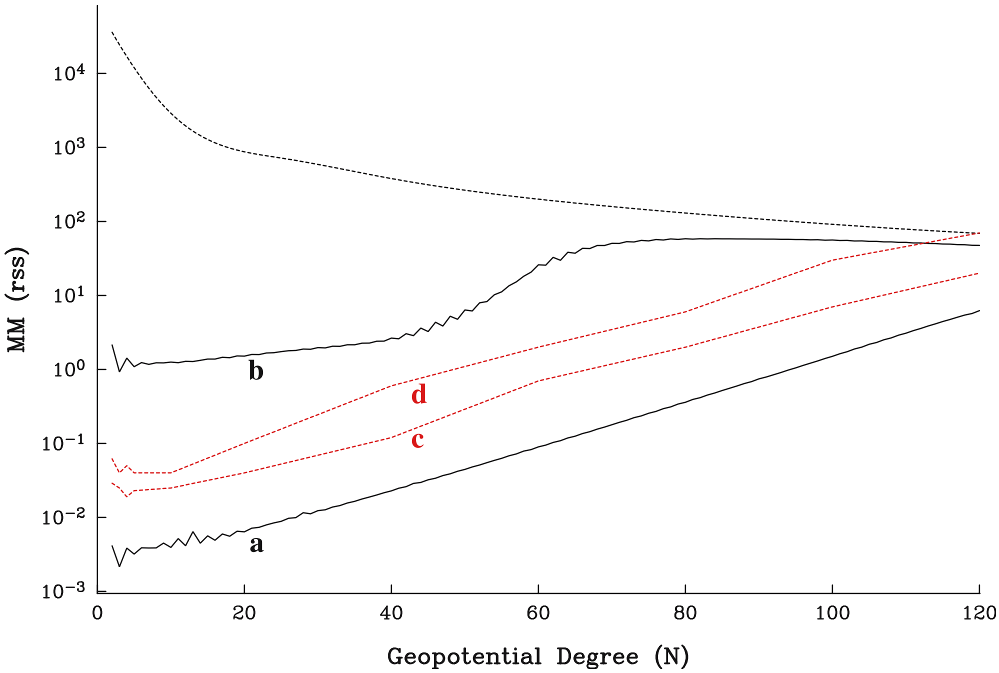

Fig. 3 Geoid error degree variances in GRACE 120 × 120 monthly recoveries (lower curves). *Solid* from linear perturbation theory using $1\mu m/s$ data (every 12 seconds) for 30 days for orbits: *a* 243 revs/16 synodic days (15.25 revs/day), (no stabilization used); *b* 61 revs/4 synodic days (15.31 revs/day), uses Kaula's rule to stabilize inverse. *Dash* from GRACE project solutions in 2004 for: *c* February (15.30 revs/day), *d* August (15.31 revs/day). *Top dash* curve from Kaula's rule for full field coefficients ($10^{-5}N^{-2}$)

Figure 4 (left panel) shows the condition numbers for all geopotential coefficients in a 120 × 120 gravity recovery from the range-rate perturbations in a 243/16 GRACE orbit. No a priori information was required to stabilize the minimum block matrices here yet, except for orders m near shallow resonances (15, 30, 46, 61, 91 and 106), which are dominated by a few large perturbations of fairly long period (1–6 days), the condition numbers of the inverse are excellent (≈ 1.0). [Note that for order 91, however, we removed an especially large shallow resonant effect which, while improving the condition numbers for these terms, degraded their precisions compared to their neighbors (Fig. 2, left panel).]

As an overall result, the error degree variances for high-resolution gravity recovery in this 'ideal' repeat orbit are uniformly small (Fig. 3, bottom curve). Also shown in Fig. 3 are two error degree variances for two-monthly recoveries in early and mid-2004 from the GRACE project (Flechtner, personal communication, 2005). These already show evidence of degradation with the approach of the 61-revs/4-day repeat orbit in September 2004. In contrast, these GRACE project estimates are based on a full inverse including many other empirical parameters (e.g., accelerometer biases) with unknown matrix stabilization and calibration and may also include GPS observations of centimeter-level precision, which when taken together account for their more pessimistic estimates.

Figure 5 shows the individual geopotential standard error estimates (geoid equivalents) in the March 2004 monthly recovery, which also has characteristics similar to the theoretical recovery in the 'ideal' 243/16 repeat orbit (Fig. 2, left panel). Note the full inverse result (Fig. 5) shows an even steeper decline for each degree from zonals to sectorials, probably due to correlations with empirical non-geopotential parameters having low-frequency effects, which are especially needed to resolve non-zonal terms.

4.2 Geopotential resolution in a degraded repeat orbit ($R < 2L$)

What happens to the resolution space when $R < 2L$? For $L = 120$, $R = 61$ and $D = 4$, as in the GRACE orbit for September 2004, we find from the evaluation of Eqs. (10), (11) and (13), (14) for each order m that for each wave number N, there are about three other orders m_1, with generally distinct k and k_1 that yield the same $|N|$ and thus are correlated to the same lumped harmonic observation $C_{\dot\alpha}$ or $S_{\dot\alpha}$ (Table 1). The observation space has shrunk by about one-fourth so that the average ratio of observations to coefficients in a 120 × 120 field is roughly 1/1 with a severe degradation of precision.

While in an ideal repeat orbit the string length (the number of degrees l in each condition equation for a given wave phase observation) is never more than $(L - m)/2$, or 60 for 120 × 120 resolution, in a degraded repeat orbit the presence of more orders m in each 'observation' keeps the average length elevated for each block. More importantly, we note certain features of each block and of the ensemble of blocks due to the parametric solutions of the Diophantine equations (10), (11) and (13), (14):

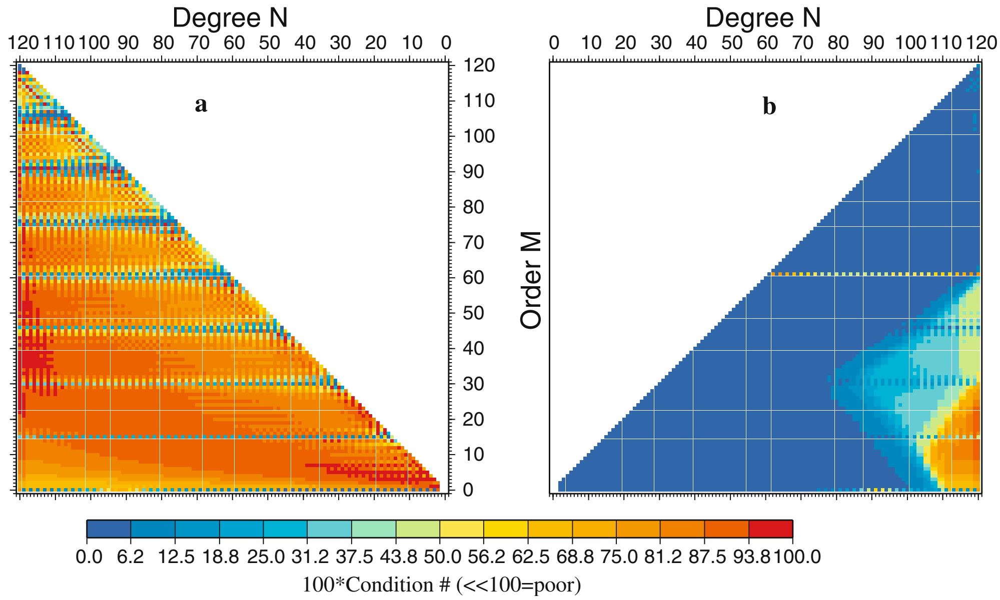

Fig. 4 Theoretical condition numbers in GRACE 120 × 120 monthly solutions. **a** $\overline{C}_{N,M}$: ideal orbit (Rev/Day = 243/16), unconstrained condition # = $[ideal/actual\ variances]^{1/2}$. Note conditions degrade only at resonant orders overly dominated by the sensitivity to its frequencies. **b** $\overline{C}_{N,M}$ September 2004 orbit (Rev/Day = 61/4), uses Kaula constraint. Poor conditions hold for the majority of the coefficients recovered from this degraded orbit except notably order 61 which is the only one resolved from observations free from other orders (see text)

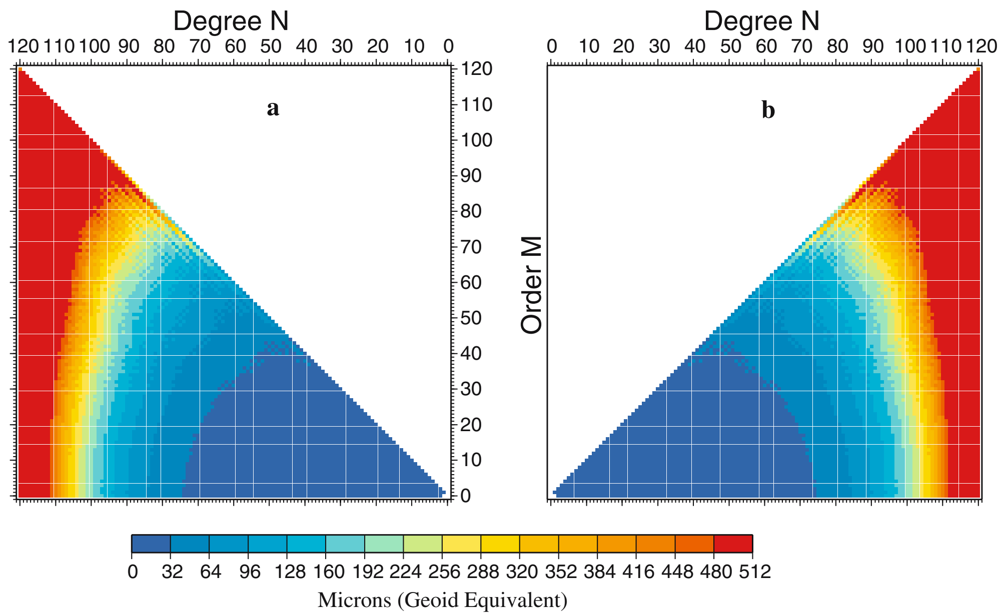

Fig. 5 Formal errors for a GRACE monthly 120 × 120 solution (March 2004): **a** for $\overline{C}_{N,M}$, **b** for $\overline{S}_{N,M}$. The characteristics from this early 2004 recovery is similar to that in an 'ideal' (theoretical) recovery shown in Fig. 2 (see text)

1. The blocks generally have four orders present with constant order-sum, two orders being odd and two even.
2. The ks and thus the degrees l for each observation are all either odd or even.

Therefore, since each phase observation is associated with a distinct geopotential coefficient depending on the parity of $l - m$, each phase observation of each wave in a block (characterized in Table 1 by the m initially listed) can be assigned to one of two sub-block matrices (a total of four for each block): for cosine phases, one for k or l odd waves, combining the $\overline{S}_{l,m}$ coefficients if $l - m$ is even and/or the $\overline{C}_{l,m}$ if $l - m$ is odd, and one for k or l even waves combining the $\overline{S}_{l,m}$, $l - m$ even and $\overline{C}_{l,m}$, $l - m$ odd coefficients. Similarly for the observation sine phase, one sub-block matrix covers k or l even waves, combining $-\overline{C}_{l,m}$, $l - m$ even with $\overline{S}_{l,m}$, $l - m$ odd coefficients, and another covers k or l odd waves combining the same kinds of coefficients.

Table 1 Content of block matrices for 120 × 120 recovery from a 61-revolution/4-day orbit

Waves	Observations (wave phases)	Frequencies	Geopotential coefficients	Average string length	Observations/ coefficients
Summary: all blocks					
7711	15421	29161	14637	77.548	1.054

Block	Waves (number)	m (number)	Observations (number)	Geopotential coefficients (number)	Average string length	Observations/ coefficients	Orders (m) in block
1	125	2	249	239	102.944	1.042	0 61
2	249	3	498	478	73.723	1.042	1 62 60
3	253	4	506	480	73.498	1.054	2 63 59 120
4	253	4	506	480	73.466	1.054	3 64 58 119
5	253	4	506	480	73.893	1.054	4 65 57 118
.....	.....	.....	.....	.....	.....	......	....
29	253	4	506	480	79.395	1.054	28 89 33 94
30	253	4	506	480	79.427	1.054	29 90 32 93
31	253	4	506	480	79.443	1.054	30 91 31 92

Finally, we note that for the observations of the sine phases the sensitivities of geopotential coefficients associated with frequencies (m, k) yielding negative wave numbers [see Eq. (7)] get an additional -1 multiplier since the spectrum of observations is defined only for positive waves.

With the above rules for the sub-blocks, we cover all the geopotential coefficients for each block (of generally four orders m) inverting four sub-block matrices with a size no greater than 120 (for a total of 124 such 'minimum' matrices). However, unlike the 'ideal' case where the *maximum* size of the blocks was 60 (for $L = 120$) here, because of the compressed space with roughly equal m sum for each block, the size of *each* sub-block matrix is roughly the same, about 120. Table 2 shows an example of the content of the four sub-block matrices for block # 1 (Table 1 above) involving $m = 0$ and 61.

Notice that the two sub-blocks involving the $\overline{C}_{l,61}$ geopotential coefficients in Table 2 have more than twice as many 'observations' as unknowns (because the $\overline{S}_{l,0}$ coefficients are not present), while the other two sub-blocks are deficient in 'observations'. These deficient matrices for block 1 are the only ones of the 124 that demand a priori information for inversion, while the other two of this block are the only ones that do not require it. For the other blocks, all sub-block matrices involve more than one order m but also more observations (see Table 1), with the result that the ratio of observations/equations is always slightly greater than 1/1. However, since each matrix tends to be dominated by only a few strong perturbations (of low $|k|$, recall Fig. 1), they all are found to need some a priori constraint for stabilization.

We tried two different inversion routines on these normal matrices [Gauss-Jackson and singular value decomposition

Table 2 Geopotential coefficients resolved in Block #1 (sub-block matrix size observation equations)

	Wave phase	
k, l	Cosine	Sine
Odd	$\overline{C}_{l,0} + \overline{S}_{l,61}$ (89 − 62)	$\overline{C}_{l,61}$ (30 − 62)
Even	$\overline{C}_{l,61}$ (30 − 63)	$\overline{C}_{l,0} + \overline{S}_{l,61}$ (90 − 63)

(Press et al. 1987)]. Yet even with the addition of a constraint of $10^{-5}/l^2$ for each coefficient (roughly the expected size of the term), the conditioning of the matrices was very poor for the great majority of terms (Fig. 4, right panel). Only for a band of orders m below about 40 do the theoretical precisions from these inverses approach that in an 'ideal' recovery (recall Fig. 2). An exception is for $\overline{C}_{l,61}$ (see the right panels of Figs. 2, 4), where we saw in Table 2 that these coefficients are the only ones not requiring stabilization.

Summarizing, as Fig. 3 shows, the overall degradation judged by the error degree variances with respect to the 'ideal' ranges from more than two orders of magnitude for low-degree l to about one order of magnitude for high-degree l. Indeed, increasingly for degrees l above about 50 the influence of the a priori constraint is felt to such an extent that by degree 120 there is almost no improvement in the field errors over what is already assumed.

5 Past and future encounters of the GRACE satellites with critical repeat orbits

In view of the severe degradation of precision for GRACE geopotential recovery during the (61, 4) repeat orbit, we surveyed the past and possible future GRACE orbits for similar occurrences (Figs. 6, 7). Figure 6 shows the decline of the altitude of the GRACE satellites (due mainly to atmospheric drag) from the start of the mission in the spring of 2002 to the end of 2004. The data come from the two-line element sets from the GeoforschungsZentrum, Potsdam (see Acknowledgments). The decline contains clear evidence of a slowing rate due to the weakening of the 11-year solar cycle since the last 'high' in 2000. There is also evidence of seasonal effects (especially strong in 2002) and the presence of magnetic storms in the fall (northern autumn) of 2003. At the end of 2004, the altitude (semi-major axis – mean Earth equatorial radius) was declining at the rate of about 17 m/day which we used (as an estimate of the minimum decline during the low of the solar cycle) to predict the altitude in 'free fall' for a number of years from January 2005.

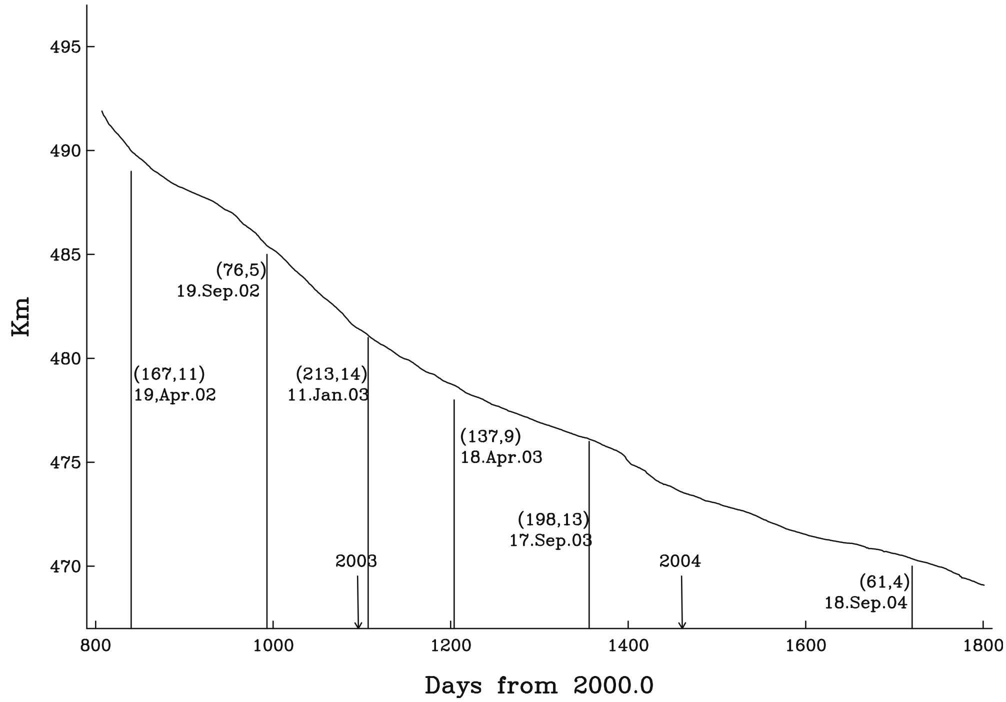

Fig. 6 Altitude of GRACE A in 2002–2004. The *vertical bars* show times of exact repeat orbits (R, D) since launch in the spring of 2002, which degrade a 120×120 geopotential recovery: R revolutions, D synodic days. The more rapid passages through the critical repeat orbits prior to 2004 reduces the impact of their effect on 120×120 recoveries in these months

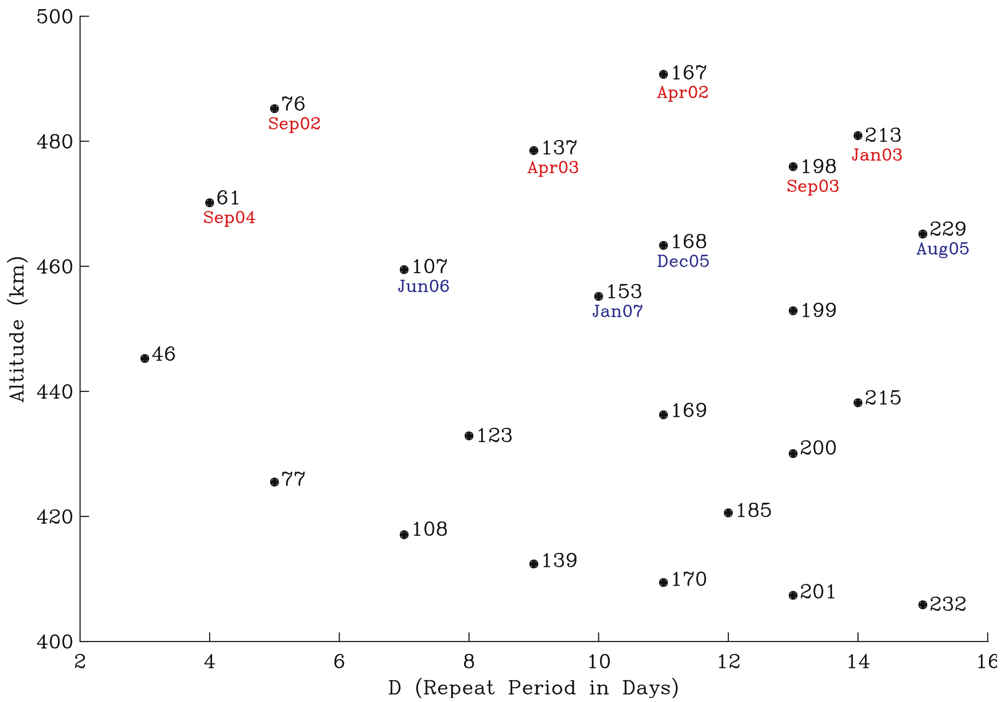

Fig. 7 Degrading resonances (R/D) for GRACE 120×120 fields: 2002–2007. *Upper number* Revolutions R in D days. *Lower number* month. Year of exact repeat orbit, predicted after January 2005 assuming 17 m/day decline of the orbit. The most serious critical repeat orbit in 'free fall' will not be reached until 2008 $(R/D = 46/3)$

To estimate the time-averaged semi-major axes in (R, D) repeat orbits, we used the formula in Wagner (1991) (accurate to within ~100 m):

$$a = a_0\left\{1 - C'_{2,0}(r_e/a_0)^2[4\cos^2(I) - (R/D)\cos(I) - 1]\right\}, \quad (15)$$

$$a_0 = \mu_e^{1/3}(\dot{\theta}_e R/D)^{-2/3}, \quad (16)$$

where I is the mean inclination of the orbit and $C'_{2,0}$, the Earth's dominant oblateness geopotential coefficient, is -1.08263×10^{-3} (here unnormalized).

Figure 6 shows there were five periods since launch (prior to September 2004) when GRACE passed through a critical (degrading) resonance with respect to a resolution of 120 × 120. However, none were as severe [had as low a ratio R/(2 × 120)] as the 61/4 repeat orbit. In addition, the previous resonance passages occurred faster than during the slower orbit decline in late 2004 lessening the impact that a prolonged passage with a 'stationary' track would have had.

Nevertheless there is some evidence of degraded precision in attempted 120 × 120 monthly recoveries during these periods as well. For example, a gravitational field released by the project (Tapley et al. (2004)) for September 2002 during the (76/5) resonance was reduced to 70 × 70 and still shows elevated errors compared to its companion 'monthlies' earlier and later in the year.

Finally in Fig. 7, we show the full panoply of the GRACE altitude in 'critical repeat-orbit space' from launch to altitudes above 400 km. Predictions beyond January 2007 are increasingly uncertain, but it seems clear that the very severe 46/3 repeat orbit would not be reached in 'free fall' until at least 2008. Yet as early as 2006, the slow approach of the (107/7) repeat orbit during the low in the solar cycle is likely to cause precision problems similar to the ones with (61/4).

6 Conclusions, discussion and recommendations

The attempt to extract a complete high-degree geopotential field during a repeat-orbit period of insufficient coverage results in degraded precision at all wavelengths. Monthly GRACE fields presented prior to 2005 show this degradation for at least solutions in early to mid-2004. There is also some evidence of similar degradation of the monthly solutions near previous repeat orbits.

As mentioned in Sect. 1, the GRACE project team is addressing this problem (started in mid-2005) by releasing new monthly 120 × 120 fields from 2002, but strongly constrained to a multi-year mean GRACE geopotential above degree 40. This approach sacrifices more complete (unrestrained) information for a good part of the mission to its necessary constraint at other times. For example, a reduced resolution 30 × 30 field for monthly variations (which could be accomplished without constraint until about 2008) has been found to have essentially the same precision at its band limit as the 'ideal' unconstrained 120 × 120 field in non-critical orbits.

As it happens 30 × 30 also appears to be near the practical resolution for sensing geopotential time-variations from current GRACE data analysis (Bettadpur 2004; Wahr et al. 2004). However, some significant variable GRACE signals have been found from the unconstrained 120 × 120 solutions at considerably higher degree (e.g. Wagner and McAdoo 2004). Thus, the project's new continuously constrained monthly solutions to high degree may well represent the best that can be achieved from the GRACE orbit in 'free fall'.

Our position, however, is to avoid the necessity for accepting reduced precision or loss of information in future monthly GRACE gravitational field models by maneuvering the GRACE pair (high or low) to avoid the short repeat cycles shown in Fig. 7. Furthermore, we believe that avoiding the sparse ground tracks in short-repeat cycles should also benefit recoveries of both mean and variable fields using other approaches (besides global harmonics), which transform the tracking data directly into anomalous potential differences and downward-continue these to the Earth's surface (e.g., Han et al. 2005). However, likely deficiencies in these methods, also during periods of sparse tracks, will require further analysis.

Summarizing, future geopotential recovery missions should seriously consider either 'station-keeping' their orbits in favorable long repeat cycles or maneuvering around the short ones when encountered in 'free fall'.

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