J Geod (2006) 79: 641–651
DOI 10.1007/s00190-005-0017-5

ORIGINAL ARTICLE

W. Freeden · M. Schreiner

Local multiscale modelling of geoid undulations from deflections of the vertical

Received: 8 March 2005 / Accepted: 17 November 2005 / Published online: 6 January 2006

Abstract This paper deals with the problem of determining a scalar spherical field from its surface gradient, i.e., the modelling of geoid undulations from deflections of the vertical. Essential tools are integral formulae on the sphere based on Green's function of the Beltrami operator. The determination of geoid undulations from deflections of the vertical is formulated as multiscale procedure involving scale-dependent regularized versions of the surface gradient of Green's function. An advantage of the presented approach is that the multiscale method is based on locally supported wavelets. In consequence, local modelling of geoid undulations are calculable from locally available deflections of the vertical.

Keywords Deflections of the vertical · Regularization of Green's function · Multiscale modelling · Geoid undulations · Local approximation

1 The problem

The gravity potential (W) of the Earth is the sum of the gravitational potential (V) and the centrifugal potential (Φ), i.e., $W = V + \Phi$. In an Earth-fixed coordinate system, Φ is explicitly known. Hence, the determination of equipotential surfaces of the potential W is strongly related to the knowledge of the potential V. The gravity vector (g), introduced by $g(x) = \nabla_x W(x)$, where the point $x \in \mathbb{R}^3$ is located outside and on a sphere around the origin with Earth's mean radius R, is normal to the equipotential surfaces passing through the same point [for the definition of the radius R, the reader is referred, e.g., to Groten (1979), Heiskanen and Moritz (1967), Torge (1991)]. Thus, equipotential surfaces intuitively express the notion of tangential surfaces, as they are normal to the plumblines given by the direction of the gravity vector.

Equipotential surfaces of the gravity potential allow, in general, no simple representation. This is the reason why a reference surface, usually an ellipsoid of revolution, is chosen for the (approximate) construction of the geoid. As a matter of fact, the deviations of the gravity field of the Earth from the normal field of such an ellipsoid are small, by typically five orders of magnitude. The remaining parts of the gravity field are gathered in a so-called disturbing gravity field ∇T corresponding to the disturbing potential T (for more details see, e.g., Groten 1979; Heiskanen and Moritz 1967; Rummel 1992; Torge 1991).

The aim of physical geodesy can, therefore, be seen as the determination of equipotential surfaces of the Earth's gravity field or, equivalently, the determination of the gravity potential W normally (via a linearisation process) involving the disturbing potential T. Knowing the gravity potential, all equipotential surfaces, including the geoid, are given by an equation of the form $W(x) = \text{const}$. By introducing U as the normal gravity potential corresponding to the ellipsoidal field and T as the disturbing potential (in the usual Pizetti–Somigliana concept), we are led to a decomposition of the gravity potential in the form $W = U + T$ with zero- and first-order vanishing moments of T in terms of spherical harmonics (for details see, e.g., Heiskanen and Moritz 1967).

A point x on the geoid can be projected onto a point y on the reference ellipsoid by means of the ellipsoidal normal. The gravity anomaly vector is defined as the difference between the gravity vector $g(x)$ and the normal gravity vector $\gamma(y)$, $\gamma = \nabla U$, i.e., $g(x) - \gamma(y)$. It is also possible to

W. Freeden (✉)
Geomathematics Group,
University of Kaiserslautern,
67653 Kaiserslautern,
P.O. Box 3049,
Germany
E-mail: freeden@mathematik.uni-kl.de
Tel.: +49-31-2052852
Fax.: +49-631-2054736

M. Schreiner
Laboratory for Industrial Mathematics,
University of Buchs,
Werdenbergstrasse 4,
Buchs 9471 Switzerland
E-mail: schreiner@ntb.ch
Tel.: +41-81-7553463
Fax: +41-71-7565434

difference the vectors g and γ at the same point x to get the gravity disturbance vector $g(x) - \gamma(x)$ (see Fig. 1).

Several basic mathematical relations are known between the quantities just mentioned. In what follows, we only illustrate heuristically the relation of the deflections of the vertical to the surface gradient of the geoid undulation (in spherical approximation). We start by observing that the gravity disturbance vector at the point x can be written as

$$g(x) - \gamma(x) = \nabla_x (W(x) - U(x)) = \nabla_x T(x). \tag{1}$$

Expanding the potential U at x according to Taylor's theorem and truncating the series at the linear term, we get (see Fig. 1)

$$U(x) \doteq U(y) + \frac{\partial U}{\partial \nu'}(y) N(x). \tag{2}$$

Here, $\nu'(y)$ is the ellipsoidal normal at y, i.e., $\nu'(y) = -\gamma(y) \backslash |\gamma(y)|$, and the geoid undulation $N(x)$ is the distance between x and y, i.e., between the geoid and the reference ellipsoid (cf. Fig. 1). Using

$$\begin{aligned} |\gamma(y)| &= -\nu'(y) \cdot \gamma(y) \\ &= -\nu'(y) \cdot \nabla_y U(y) = -\frac{\partial U}{\partial \nu'}(y) \end{aligned} \tag{3}$$

we arrive at

$$N(x) = \frac{T(x) - (W(x) - U(y))}{|\gamma(y)|}. \tag{4}$$

Letting $U(y) = W(x) = \text{const.} = W_0$ [see, e.g., Torge 1991, Eq. (5.38)], we obtain Bruns (1878) formula

$$N(x) = \frac{T(x)}{|\gamma(y)|}. \tag{5}$$

Equation (5) relates the physical quantity T to the geometric quantity N.

Letting $\nu(x) = -g(x) \backslash |g(x)|$, we find

$$g(x) = \nabla_x W(x) = -|g(x)| \nu(x). \tag{6}$$

Furthermore, we have

$$\gamma(x) = \nabla_x U(x) = -|\gamma(x)| \nu'(x). \tag{7}$$

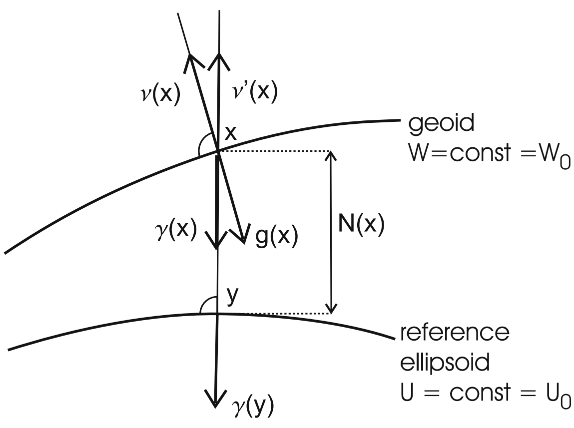

Fig. 1 Illustration of the definition of the gravity anomaly vector $g(x) - \gamma(y)$

Now, the total deflection of the vertical $\Theta(x)$ at the point x is defined to be the angular (i.e., tangential) difference between the directions $\nu(x)$ and $\nu'(x)$, i.e., the plumbline and the ellipsoidal normal through the same point:

$$\Theta(x) = \nu(x) - \nu'(x) - \big((\nu(x) - \nu'(x)) \cdot \nu'(x)\big)\, \nu'(x). \tag{8}$$

Clearly, because of its definition (Eq. 8), $\Theta(x)$ is orthogonal to $\nu'(x)$:

$$\Theta(x) \cdot \nu'(x) = 0. \tag{9}$$

Since the plumblines are orthogonal to the level surfaces of the geoid and the ellipsoid, respectively, the deflections of the vertical give a measure of the horizontal gradient of the level surfaces. This aspect will be described in more detail below. From Eq. (6) we obtain, in connection with Eq. (8),

$$\begin{aligned} g(x) &= \nabla_x W(x) \\ &= -|g(x)| \big(\Theta(x) + \nu'(x) \\ &\quad + ((\nu(x) - \nu'(x)) \cdot \nu'(x)) \nu'(x)\big). \end{aligned} \tag{10}$$

Altogether, we get for the gravity disturbance vector

$$\begin{aligned} g(x) - \gamma(x) = \nabla_x T(x) = -|g(x)| \,(\Theta(x) + ((\nu(x) \\ - \nu'(x)) \cdot \nu'(x)\big)\, \nu'(x)\big) \\ - (|g(x)| - |\gamma(x)|)\, \nu'(x). \end{aligned} \tag{11}$$

The magnitude $|g(x)| - |\gamma(x)|$ is called the gravity disturbance. Since the vector $\nu(x) - \nu'(x)$ is (almost) orthogonal to $\nu'(x)$, it can be neglected in Eq. (11). Hence, it follows that

$$\begin{aligned} g(x) - \gamma(x) &= \nabla_x T(x) \\ &\doteq -|g(x)| \Theta(x) \\ &\quad - (|g(x)| - |\gamma(x)|)\, \nu'(x). \end{aligned} \tag{12}$$

In spherical approximation

$$x = R\xi, \;\; R = |x|, \;\; |\xi| = 1 \tag{13}$$

the gradient $\nabla_x T(x)$ can be split into a normal part (pointing into the direction of $\xi = \nu'(x)$) and an angular (tangential) part (characterized by the surface gradient ∇^*) (see, e.g., Freeden et al. (1998) for more details on ∇^*). It follows that

$$\begin{aligned} \nabla_x T(x) &= \left(\frac{\partial T}{\partial r}(r\xi) \bigg|_{r=R} \right) \xi + \frac{1}{R} \nabla^*_\xi T(R\xi) \\ &= \frac{\partial T}{\partial \nu'}(x) \nu'(x) + \frac{1}{R} \nabla^*_\xi T(R\xi). \end{aligned} \tag{14}$$

By comparison of Eqs. (12) and (14), we obtain

$$|g(x)| - |\gamma(x)| = -\frac{\partial T}{\partial \nu'}(x), \tag{15}$$

i.e., the gravity disturbance, besides being the difference in magnitudes between the actual and the normal gravity vectors, is also the normal component of the gravity disturbance vector.

In addition, we are led to the angular, i.e., (tangential) differential equation

$$\frac{1}{R} \nabla^*_\xi T(R\xi) = -|g(x)| \Theta(R\xi). \tag{16}$$

Since $|\Theta(R\xi)|$ is a small quantity, it may (without loss of precision) multiplied either by $-|g(x)|$ or by $-|\gamma(x)|$. In spherical approximation [with $|\gamma(x)| = kM\backslash R^2$, see, e.g., Heiskanen and Moritz (1966)], this gives

$$\nabla^*_\xi T(R\xi) = -\frac{kM}{R}\Theta(R\xi), \tag{17}$$

where k is the gravitational constant and M is the Earth's mass. By virtue of the Bruns formula, we finally find

$$\frac{kM}{R^2}\nabla^*_\xi N(R\xi) = -\frac{kM}{R}\Theta(R\xi), \tag{18}$$

i.e.,

$$\nabla^*_\xi N(R\xi) = -R\Theta(R\xi). \tag{19}$$

In other words, the knowledge of the geoid undulations allows the determination of the deflections of the vertical by taking the surface gradient on the unit sphere.

In physical geodesy (see e.g., Groten 1981; Heiskanen and Moritz 1967; Rummel 1992; Torge 1991), the deflection of the vertical, which is a (tangential) vector field, is usually decomposed into mutually perpendicular scalar components. In fact, there are various distinctions in the introduction of the deflections of the vertical (e.g., Featherstone and Rüeger 2000; Jekeli 1999; Torge 1991). In our approach, we essentially follow the elaboration by Torge (1991) (see, e.g., Groten 1979; Heiskanen and Moritz 1997; Rummel 1992, for further details).

We assume that the minor axis of the reference ellipsoid is parallel to the mean rotational axis of the Earth, and that the zero meridian of the ellipsoidal system is parallel to the mean meridian plane of Greenwich. If the axes are brought into coincidence by parallel displacements and if a unit sphere is centered at the point of intersection of the coinciding rotational axes with the unit sphere, then $\Theta(x)$ enters as the spherical distance between the points on this sphere, which corresponds to the astronomic zenith Z_a and the geodetic zenith Z_g (Fig. 2).

The north–south component of the deflection of the vertical along the astronomic meridian is denoted by NSC, and the east–west component in the prime vertical is denoted by EWC. From spherical trigonometry

$$\begin{aligned}\sin\varphi &= \cos\eta\sin(\Phi - \text{NSC}),\\ \sin\text{EWC} &= \cos\varphi\sin(\Lambda - \lambda)\end{aligned} \tag{20}$$

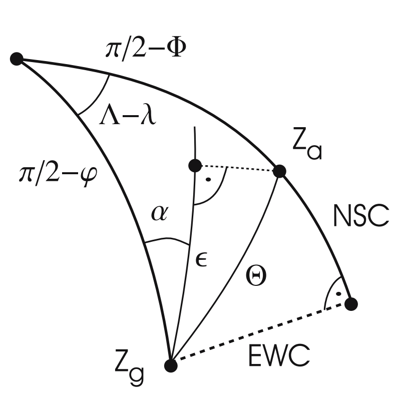

Fig. 2 Illustration of the vertical deflections

and using

$$\begin{aligned}&\cos\text{EWC} \approx 1, \quad \sin\text{EWC} \approx \text{EWC},\\ &\sin(\Lambda - \lambda) \approx \Lambda - \lambda,\end{aligned} \tag{21}$$

the components of the deflection of the vertical are given by

$$\text{NSC} = \Phi - \varphi, \text{EWC} = (\Lambda - \lambda)\cos\varphi. \tag{22}$$

The element

$$\varepsilon = \text{NSC}\cos\alpha + \text{EWC}\sin\alpha \tag{23}$$

is the component in azimuth α (see Torge 1991). $\Phi - \varphi$ and $\Lambda - \lambda$, respectively, are called the *latitude disturbance* and *the longitude disturbance* (in geodesy, NSC and EWC are usually denoted by ξ and η). Altogether, in connection with the (λ, φ)-component representation of the surface gradient, the deflections of the vertical read in (λ, φ) coordinates as

$$-\frac{1}{R}\frac{\partial N}{\partial\varphi}(\lambda, \varphi) = \text{NSC}(\lambda, \varphi) = \Phi - \varphi, \tag{24}$$

$$-\frac{1}{R}\frac{1}{\cos\varphi}\frac{\partial N}{\partial\lambda}(x, \varphi) = \text{EWC}(\lambda, \varphi) = (\Lambda - \lambda)\cos\varphi. \tag{25}$$

It should be noted that the deflections of the vertical play an important role when the deviations of the physical plumbline from the ellipsoidal normal can no longer be ignored in high-accuracy observations. For example, they are taken into account for the reduction of topographic masses, the computation of the ellipsoidal parameters of the reference ellipsoid (for more details see, e.g., Groten 1979; Heiskanen and Moritz 1967; Rummel 1992; Torge 1991 and the literature therein).

In this paper, we are concerned with the problem of determining the geoid undulations from given deflections of the vertical, i.e., we study the differential equation (cf. Eq. 19)

$$\nabla^* F = d \tag{26}$$

on the unit sphere $\Omega \subset \mathbb{R}^3$, thereby assuming that $F : \Omega \to \mathbb{R}$ (with $F(\xi) = N(R\xi)$) is a continuously differentiable scalar field and $d : \Omega \to \mathbb{R}^3$ (with $d(\xi) = -R\Theta(\xi)$) is a continuous vector field on Ω.

Seen from physical geodesy, at least in the opinion of the authors, two concepts are realized in the paper. First, an unknown method of determining the geoid undulation from deflections of the vertical is established by inversion of the surface gradient operator on the sphere using the Green function of the Beltrami operator. In other words, an alternative way of representing geoid undulations from deflections of the vertical is deduced by certain tools of spherical vector theory, thus avoiding the occurrence of the Stokes kernel function within the solution process (see Groten 1979; Heiskanen and Moritz 1967; Rummel 1992; Torge 1991, for this 'classical' approach).

Second, a method is guaranteed establishing the transition from global to local modelling of geoid undulations from deflections of the vertical within a scale-dependent approximation procedure. Surprisingly, our approach enables us to obtain a locally reflected improvement of the fine structure of the geoid by (surface curl-free) vector wavelets. To be more specific, for more detailed information about the geoid, only smaller local amounts of the vectorial signal, i.e., the deflections of the vertical, are needed within the multiscale process proposed in this paper.

The structure of this paper is as follows: In the next section, we give a concise review on the notations to be used throughout the paper. Then, we recapitulate Green's function with respect to the Beltrami operator (see Freeden 1979). From Freeden et al. (1998) we borrow the fundamental theorem for ∇^* on the unit sphere. This enables us to invert the differential equation under canonical constraints. Then, we introduce a regularized version of Green's function. The integral formulae associated with regularized Green's functions are formulated in detail. Based on these results, we develop certain reconstruction formulas for the deflections of the vertical, where we make use of locally supported wavelets. These wavelets form the keystones for multiscale modelling of the geoid undulations. As a matter of fact, our approach enables us to derive local corrections to lower scaled global results.

2 Basic notation

$\mathbb{R}^3$ denotes the three-dimensional Euclidean space. The inner product of $x, y \in \mathbb{R}^3$, $x = (x_1, x_2, x_3)^{\mathrm{T}}$, $y = (y_1, y_2, y_3)^{\mathrm{T}}$, is defined, as usual, by $x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3$. The corresponding norm in $\mathbb{R}^3$ is given by $|x| = \sqrt{x \cdot x}$. As usual, the vector product is given by $x \wedge y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)^{\mathrm{T}}$. Each $x \in \mathbb{R}^3 \setminus \{0\}$ has a unique representation of the form $x = r\xi$, $r = |x|$, $\xi = (\xi_1, \xi_2, \xi_3)^{\mathrm{T}}$, where $\xi \in \mathbb{R}^3$ with $|\xi| = 1$ is the uniquely determined directional unit vector of x. The unit sphere in $\mathbb{R}^3$ will be denoted by Ω. As is well known, the total surface $\|\Omega\| = \int_\Omega d\omega$ is equal to 4π ($d\omega$ denotes the surface element). Each element $\xi \in \Omega$ can be represented by spherical coordinates as follows:

$$\xi = \sin\varphi \varepsilon^3 + \cos\varphi(\cos\lambda\varepsilon^1 + \sin\lambda\varepsilon^2), \tag{27}$$

$$0 \le \lambda < 2\pi, -\pi/2 \le \varphi \le \pi/2$$

(λ: spherical latitude, φ: spherical longitude), where ε^1, ε^2, ε^3, respectively, form the (canonical) orthonormal basis in $\mathbb{R}^3$

$$\varepsilon^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \varepsilon^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \varepsilon^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{28}$$

As is well known, the representation in polar coordinates leads to a moving orthonormal triad on the unit sphere Ω given in the form

$$\varepsilon^r = \begin{pmatrix} \cos\lambda\cos\varphi \\ \sin\lambda\cos\varphi \\ \sin\varphi \end{pmatrix}, \quad \varepsilon^\lambda = \begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix},$$

$$\varepsilon^\varphi = \begin{pmatrix} -\cos\lambda\sin\varphi \\ \sin\lambda\sin\varphi \\ \cos\varphi \end{pmatrix}. \tag{29}$$

Throughout this paper, we need a number of differential operators which are listed in Table 1 (for more details see, e.g., Freeden et al. 1998; Freeden and Michel 2004).

For the convenience of the reader, we also give their representation in local polar coordinates:

$$\nabla_x = \xi \frac{\partial}{\partial r} + \frac{1}{r}\nabla^*_\xi, \tag{30}$$

$$\Delta_x = \left(\frac{\partial}{\partial r}\right)^2 + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta^*_\xi, \tag{31}$$

$$\nabla^*_\xi = \varepsilon^\lambda \frac{1}{\cos\varphi}\frac{\partial}{\partial\lambda} + \varepsilon^\varphi \frac{\partial}{\partial\varphi}, \tag{32}$$

$$\Delta^*_\xi = \frac{1}{\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\varphi}\right), \tag{33}$$

$$L^*_\xi = -\varepsilon^\lambda \frac{\partial}{\partial\varphi} + \varepsilon^\varphi \frac{1}{\cos\varphi}\frac{\partial}{\partial\lambda}. \tag{34}$$

It should be mentioned that the operators ∇^*, L^*, Δ^* will be always used here in coordinate-free representation, thereby avoiding any singularity at the poles.

Let $\eta \in \Omega$ be fixed; then it is not difficult to see that, for $\xi, \eta \in \Omega$,

$$\nabla^*_\xi(\xi \cdot \eta) = \eta - (\xi \cdot \eta)\xi, \tag{35}$$

$$L^*_\xi(\xi \cdot \eta) = \xi \wedge \nabla^*_\xi(\xi \cdot \eta) = \xi \wedge \eta, \tag{36}$$

and

$$\Delta^*_\xi(\xi \cdot \eta) = -2(\xi \cdot \eta). \tag{37}$$

Furthermore, if F is of class $C^{(1)}[-1, +1]$ and $F' \in C[-1, +1]$ is its derivative, then

$$\nabla^*_\xi F(\xi \cdot \eta) = F'(\xi \cdot \eta)(\eta - (\xi \cdot \eta)\xi), \tag{38}$$

$$L^*_\xi F(\xi \cdot \eta) = F'(\xi \cdot \eta)(\xi \wedge \eta), \tag{39}$$

whereas for $F \in C^{(2)}[-1, +1]$

$$\Delta^*_\xi F(\xi \cdot \eta) = -2(\xi \cdot \eta)F'(\xi \cdot \eta) + (1 - (\xi \cdot \eta)^2)F''(\xi \cdot \eta). \tag{40}$$

Following the nomenclature of Freeden et al. (1998) and Freeden and Michel (2004), we denote the surface divergence on Ω by $\nabla^*\cdot$, and the surface curl on Ω by $L^*\cdot$ (cf. Table 1).

Since the operators ∇^*, L^* and $\nabla^*\cdot$, $L^*\cdot$ are of particular interest throughout this work, we list some of their properties (for more details, see Freeden et al. 1998):

Table 1 Differential operators

Symbol	Differential Operator
∇_x	gradient at x
$\Delta_x = \nabla_x \cdot \nabla_x$	Laplace operator at x
∇^*_ξ	surface gradient at ξ
$\Delta^*_\xi = \nabla^*_\xi \cdot \nabla^*_\xi$	Beltrami operator at ξ
$L^*_\xi = \xi \wedge \nabla^*_\xi$	surface curl gradient at ξ
$\nabla^*_\xi\cdot$	surface divergence at ξ
$L^*_\xi\cdot$	surface curl at ξ

$$\nabla^*_\xi \cdot \nabla^*_\xi F(\xi) = \Delta^*_\xi F(\xi), \qquad \xi \in \Omega, \tag{41}$$
$$L^*_\xi \cdot L^*_\xi F(\xi) = \Delta^*_\xi F(\xi), \qquad \xi \in \Omega, \tag{42}$$
$$\nabla^*_\xi \cdot L^*_\xi F(\xi) = 0, \qquad \xi \in \Omega, \tag{43}$$
$$L^*_\xi \cdot \nabla^*_\xi F(\xi) = 0 \qquad \xi \in \Omega, \tag{44}$$
$$\nabla^*_\xi F(\xi) \cdot L^*_\xi F(\xi) = 0, \qquad \xi \in \Omega, \tag{45}$$
$$\nabla^*_\xi \cdot (F(\xi) f(\xi)) = (\nabla^*_\xi F(\xi)) \cdot f(\xi) + F(\xi) \nabla^*_\xi \cdot f(\xi), \quad \xi \in \Omega. \tag{46}$$

3 Green's theorems on the (unit) sphere

An important tool for our considerations is Green's function on Ω with respect to the Beltrami operator Δ^* (see Freeden 1979).

Definition 3.1 *$G(\Delta^*; \cdot, \cdot)\colon (\xi, \eta) \mapsto G(\Delta^*; \xi, \eta)$, $-1 \leq \xi \cdot \eta < 1$, is called Green's function on Ω with respect to the operator Δ^*, if it satisfies the following properties:*

1. *(differential equation) for every point $\xi \in \Omega$, $\eta \mapsto G(\Delta^*; \xi, \eta)$ is infinitely often differentiable on $\{\eta \in \Omega : -1 \leq \xi \cdot \eta < 1\}$, and we have*
$$\Delta^*_\eta G(\Delta^*; \xi, \eta) = -\frac{1}{4\pi}, \quad -1 \leq \xi \cdot \eta < 1, \tag{47}$$
2. *(characteristic singularity) for every $\xi \in \Omega$, the function*
$$\eta \mapsto G(\Delta^*; \xi, \eta) - \frac{1}{4\pi} \ln(1 - \xi \cdot \eta) \tag{48}$$
is continuously differentiable on Ω,
3. *(rotational symmetry) for all orthogonal transformations* **t**
$$G(\Delta^*; \boldsymbol{t}\xi, \boldsymbol{t}\eta) = G(\Delta^*; \xi, \eta) \tag{49}$$
4. *(normalisation) for every $\xi \in \Omega$,*
$$\frac{1}{4\pi} \int_\Omega G(\Delta^*; \xi, \eta)\, \mathrm{d}\omega(\eta) = 0. \tag{50}$$

The uniqueness of Green's function with respect to Δ^* is guaranteed. In terms of a (maximal) $L^2(\Omega)$-orthonormal system $\{Y_{n,m}\}$ of spherical harmonics of degree n and order m, we obtain a spectral representation of the bilinear expansion (see Freeden 1979; Freeden et al. 1998)

$$G(\Delta^*; \xi, \eta) = \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{n(n+1)} Y_{n,m}(\xi) Y_{n,m}(\eta) \tag{51}$$

Observing the addition theorem of spherical harmonics (e.g., Freeden et al. 1998), we find the following series representation in terms of Legendre polynomials $\{P_n\}$

$$G(\Delta^*; \xi, \eta) = \sum_{n=1}^{\infty} \frac{2n+1}{4\pi} \frac{1}{n(n+1)} P_n(\xi \cdot \eta). \tag{52}$$

In connection with the bilinear expansion, we are able (cf. Freeden 1979) to verify that

$$G(\Delta^*; \xi, \eta) = \frac{1}{4\pi} \ln(1 - \xi \cdot \eta) - \frac{1}{4\pi} - \frac{1}{4\pi} \ln 2 \tag{53}$$

is an explicit representation of Green's function with respect to the Beltrami operator Δ^*.

An easy calculation using Eq.(38) shows that applying the differential operator ∇^* to the Green's function with respect to the Beltrami operator yields

$$\nabla^*_\xi G(\Delta^*; \xi, \eta) = -\frac{1}{4\pi} \frac{\eta - (\xi \cdot \eta)\xi}{1 - \xi \cdot \eta}, \quad -1 \leq \xi \cdot \eta < 1. \tag{54}$$

From Freeden et al. (1998), we know the fundamental theorem for ∇^* on the unit sphere Ω.

Theorem 3.2 (Fundamental theorem for ∇^* on Ω) *Suppose that F is of class $C^{(1)}(\Omega)$. Then, for all $\xi \in \Omega$,*

$$F(\xi) = \frac{1}{4\pi} \int_\Omega F(\eta)\, \mathrm{d}\omega(\eta) - \int_\Omega \nabla^*_\eta F(\eta) \cdot \nabla^*_\eta G(\Delta^*; \xi, \eta)\, \mathrm{d}\omega(\eta). \tag{55}$$

This theorem can be used to establish the remainder term for an approximate integration formula on the sphere Ω (see Freeden 1979; Freeden et al. 1998) by comparing $F(\xi)$ for each $\xi \in \Omega$ with the mean integral value

$$F^\wedge(0, 1) = \frac{1}{4\pi} \int_\Omega F(\eta)\, \mathrm{d}\omega(\eta). \tag{56}$$

Combining Theorem 3.2 and Eq. (54), we obtain the following theorem.

Theorem 3.3 (Differential equation for ∇^* on Ω) *Let $v\colon \Omega \to \mathbb{R}^3$ be a continuously differentiable vector field on Ω with $\xi \cdot v(\xi) = 0$, $L^*_\xi \cdot v(\xi) = 0$, $\xi \in \Omega$. Then*

$$F(\xi) = \frac{1}{4\pi} \int_\Omega \frac{1}{1 - \xi \cdot \eta} (\xi - (\xi \cdot \eta)\eta) \cdot v(\eta)\, \mathrm{d}\omega(\eta) \tag{57}$$

is the uniquely determined solution of the differential equation

$$\nabla^*_\xi F(\xi) = v(\xi), \quad \xi \in \Omega, \tag{58}$$

satisfying

$$F^\wedge(0, 1) = \frac{1}{4\pi} \int_\Omega F(\xi)\, \mathrm{d}\omega(\xi) = 0. \tag{59}$$

4 Integral formulae associated with regularized Green's functions

In what follows, for real values $\rho > 0$, we consider – as an auxiliary function – the so-called ρ-regularized Green's kernel function with respect to Δ^*

$$G^\rho(\Delta^*; \xi, \eta) = \begin{cases} \frac{1}{4\pi} \ln(1 - \xi \cdot \eta) + \frac{1}{4\pi}(1 - \ln 2) & \text{for } 1 - \xi \cdot \eta > \rho \\ \frac{1 - \xi \cdot \eta}{4\pi\rho} + \frac{1}{4\pi}(\ln \rho - \ln 2) & \text{for } 1 - \xi \cdot \eta \leq \rho. \end{cases} \tag{60}$$

The kernel function $(\xi, \eta) \mapsto G^\rho(\Delta^*; \xi, \eta)$ only depends on the inner product of ξ and η, hence, $G^\rho(\Delta^*; \xi, \eta)$ is (as in the case of $G(\Delta^*; \xi, \eta)$) a radial basis function such that

$$G^\rho(\Delta^*; \mathbf{t}\xi, \mathbf{t}\eta) = G^\rho(\Delta^*; \xi, \eta) \tag{61}$$

for all orthogonal transformations $\mathbf{t}$. The surface gradient of the ρ-regularized Green's kernel is given as

$$\nabla^*_\xi G^\rho(\Delta^*; \xi, \eta) = \begin{cases} -\frac{1}{4\pi}\frac{1}{1-\xi\cdot\eta}(\eta - (\xi\cdot\eta)\xi) & \text{for } 1-\xi\cdot\eta > \rho \\ -\frac{1}{4\pi\rho}(\eta - (\xi\cdot\eta)\xi) & \text{for } 1-\xi\cdot\eta \le \rho. \end{cases} \tag{62}$$

Consequently, $G^\rho(\Delta^*; \cdot, \eta)$ is a continuously differentiable function on Ω for every (fixed) $\eta \in \Omega$, $G^\rho(\Delta^*; \xi, \cdot)$ is a continuously differentiable function on Ω for every (fixed) $\xi \in \Omega$, and

$$t \mapsto G^\rho(\Delta^*; t) = \begin{cases} \dfrac{1}{4\pi}\ln(1-t) + \dfrac{1}{4\pi}(1-\ln 2) & \text{for } 1-t > \rho \\ \dfrac{1-t}{4\pi\rho} + \dfrac{1}{4\pi}(\ln\rho - \ln 2) & \text{for } 1-t \le \rho \end{cases} \tag{63}$$

is a (one-dimensional) continuously differentiable function on the interval $[-1, +1]$.

For $F \in C^{(0)}(\Omega)$, we consider the "regularized potential"

$$P^\rho(F)(\xi) = \int_\Omega G^\rho(\Delta^*; \xi, \eta)F(\eta)\, d\omega(\eta) \tag{64}$$

as counterpart to the potential

$$P(F)(\xi) = \int_\Omega G(\Delta^*; \xi, \eta)F(\eta)\, d\omega(\eta). \tag{65}$$

We are now able to formulate the following theorem.

Theorem 4.1 *For (sufficiently small) values $\rho > 0$ and $F \in C^{(0)}(\Omega)$, the potential $P^\rho(F)$ is of class $C^{(1)}(\Omega)$, and we have*

$$\lim_{\rho\to 0}\sup_{\xi\in\Omega}\left|\int_\Omega G(\Delta^*; \xi, \eta)F(\eta)\, d\omega(\eta) - \int_\Omega G^\rho(\Delta^*; \xi, \eta)F(\eta)\, d\omega(\eta)\right| = 0 \tag{66}$$

and

$$\lim_{\rho\to 0}\sup_{\xi\in\Omega}\left|\int_\Omega \nabla^*_\xi G(\Delta^*; \xi, \eta)F(\eta)\, d\omega(\eta) - \nabla^*_\xi\int_\Omega G^\rho(\Delta^*; \xi, \eta)F(\eta)\, d\omega(\eta)\right| = 0. \tag{67}$$

Furthermore,

$$\sup_{\xi\in\Omega}\left|\int_\Omega \nabla^*_\xi G(\Delta^*; \xi, \eta)F(\xi)\, d\omega(\eta) - \nabla^*_\xi\int_\Omega G(\Delta^*; \xi, \eta)F(\eta)\, d\omega(\eta)\right| = 0. \tag{68}$$

Proof For $F \in C^{(0)}(\Omega)$, $P^\rho(F)$ being of class $C^{(1)}(\Omega)$. Moreover,

$$\nabla^*_\xi P^\rho(F)(\xi) = \nabla^*_\xi \int_\Omega G^\rho(\Delta^*; \xi, \eta)F(\eta)\, d\omega(\eta) = \int_\Omega \nabla^*_\xi G^\rho(\Delta^*; \xi, \eta)F(\eta)\, d\omega(\eta) \tag{69}$$

with $G^\rho(\Delta^*; \xi, \eta)$ and $G(\Delta^*; \xi, \eta)$ differing only on the cap around ξ given by $\{\eta \in \Omega | 1-\xi\cdot\eta \le \rho\}$. Thus, we obtain (for sufficiently small values $\rho > 0$)

$$\begin{aligned} &\left|P^\rho(F)(\xi) - P(F)(\xi)\right| \\ &\le \left(\sup_{\xi\in\Omega}|F(\xi)|\right) \int_{1-\xi\cdot\eta<\rho}\left(|\ln(1-\xi\cdot\eta)| + \ln\rho + \left|\frac{1-\xi\cdot\eta}{\rho}\right|\right) d\omega(\eta) \\ &\le 2\pi\left(\sup_{\xi\in\Omega}|F(\xi)|\right)\int_{1-\rho}^{1}(|\ln(1-t)| + 2 + |\ln\rho|)\, dt. \end{aligned} \tag{70}$$

In other words, for all $\xi \in \Omega$ and $F \in C^{(0)}(\Omega)$,

$$\left|P^\rho(F)(\xi) - P(F)(\xi)\right| = O(\rho\ln\rho), \quad \rho \to 0. \tag{71}$$

This proves the first assertion (Eq. 66) of our theorem. Furthermore, we are able to verify that

$$\begin{aligned} &\left|\nabla^*_\xi P(F)(\xi) - \nabla^*_\xi P^\rho(F)(\xi)\right| \\ &\le \left(\sup_{\xi\in\Omega}|F(\xi)|\right)\int_{1-\xi\cdot\eta<\rho}\left(\left|\frac{\nabla^*_\xi(\xi\cdot\eta)}{1-\xi\cdot\eta}\right| + \frac{1}{\rho}|\nabla^*_\xi(\xi\cdot\eta)|\right) d\omega(\eta) \\ &\le 2\left(\sup_{\xi\in\Omega}|F(\xi)|\right)\int_{1-\xi\cdot\eta<\rho}\left|\frac{\eta-(\xi\cdot\eta)\xi}{1-\xi\cdot\eta}\right| d\omega(\eta). \end{aligned} \tag{72}$$

Therefore, we find for all $\xi \in \Omega$ and $F \in C^{(0)}(\Omega)$,

$$\left|\nabla^*_\xi P^\rho(F)(\xi) - \nabla^*_\xi P(F)(\xi)\right| = O(\rho^{1/2}), \quad \rho \to 0. \tag{73}$$

Altogether this yields the desired results stated in Theorem 4.1. □

The regularized Green's kernels now enable us to reformulate the integral formula (Theorem 3.2) developed in Sect. 3.

Theorem 4.2 (Regularized fundamental theorem for ∇^*) *Suppose that F is a continuously differentiable function on Ω. Then*

$$\lim_{\rho\to 0}\sup_{\xi\in\Omega}\left|F(\xi)-(ST)(F)(\rho;\xi)-F^\wedge(0,1)\right|=0, \tag{74}$$

where we have used the abbreviations

$$(ST)(F)(\rho;\xi)=-\int\limits_\Omega \nabla^*_\eta F(\eta)\cdot s_\rho(\xi,\eta)\,\mathrm{d}\omega(\eta) \tag{75}$$

and

$$s_\rho(\xi,\eta)=\nabla^*_\eta G^\rho(\Delta^*;\xi,\eta) = \begin{cases} -\frac{1}{4\pi}\frac{1}{1-\xi\cdot\eta}(\xi-(\xi\cdot\eta)\eta) & \text{for } 1-\xi\cdot\eta>\rho \\ -\frac{1}{4\pi\rho}(\xi-(\xi\cdot\eta)\eta) & \text{for } 1-\xi\cdot\eta\le\rho. \end{cases} \tag{76}$$

5 Reconstruction formulae and wavelet transform

For all (sufficiently small) values $\rho>0$, the family $\{s_\rho\}_{\rho>0}$ of kernels $s_\rho\colon\Omega\times\Omega\to\mathbb{R}^3$, $(\xi,\eta)\mapsto s_\rho(\xi,\eta)$, $(\xi,\eta)\in\Omega\times\Omega$, is called a *scaling vector field*. Moreover, $s_1\colon\Omega\times\Omega\to\mathbb{R}^3$ (i.e., $\rho=1$) is called the *mother kernel of the scaling vector field*. Correspondingly, for $\rho>0$, the family $\{w_\rho\}_{\rho>0}$ of kernels $w_\rho\colon\Omega\times\Omega\to\mathbb{R}^3$ given by

$$w_\rho(\xi,\eta)=-w(\rho)^{-1}\frac{\mathrm{d}}{\mathrm{d}\rho}s_\rho(\xi,\eta),\quad \xi,\eta\in\Omega, \tag{77}$$

is called a wavelet vector field. The kernel $w_1\colon\Omega\times\Omega\to\mathbb{R}^3$ (i.e., $\rho=1$) is called the *mother kernel of the wavelet function*. Equation (77) is called the *scale-continuous scaling equation*.

In the remainder of this paper, we particularly choose $w(\rho)=\rho^{-1}$ in analogy to Euclidean wavelet theory (of course, other weight functions than $w(\rho)=\rho^{-1}$ can be chosen in our approach).

For the scaling function $\{s_\rho\}_{\rho>0}$, the associated wavelet transform is defined by

$$(WT)(F)(\rho;\xi)=\int\limits_\Omega \nabla^*_\eta F(\eta)\cdot w_\rho(\xi,\eta)\,\mathrm{d}\omega(\eta) \tag{78}$$

where

$$w_\rho(\xi,\eta)=\begin{cases} 0 & \text{for } 1-\xi\cdot\eta>\rho \\ \frac{1}{4\pi\rho}(\xi-(\xi\cdot\eta)\eta) & \text{for } 1-\xi\cdot\eta\le\rho. \end{cases} \tag{79}$$

It should be pointed out that the kernels constituting the wavelet vector field possess local support. This is of great significance for computational purposes, since approximate integration procedures have only to observe the contributions inside the local support of w_ρ.

It is obvious that the wavelets behave like $O(\rho^{-1})$; hence, the convergence of the following integrals in the reconstruction theorem is guaranteed.

Theorem 5.1 *Let $\{s_\rho\}$ be a scaling vector field [as introduced by Eq. (76)]. Suppose that F is of class $C^{(1)}(\Omega)$. Then the reconstruction formula*

$$\int_0^2 (WT)(F)(\rho;\cdot)\frac{\mathrm{d}\rho}{\rho}=F-F^\wedge(0,1) -\frac{1}{3}\sum_{m=1}^{3}F^\wedge(1,m)Y_{1,m} \tag{80}$$

holds in the sense of $\|\cdot\|_{C^{(0)}(\Omega)}$, where

$$F^\wedge(0,1)=\int\limits_\Omega F(\eta)Y_{0,1}(\eta)\,\mathrm{d}\omega(\eta) \tag{81}$$

and

$$F^\wedge(1,m)=\int\limits_\Omega F(\eta)Y_{1,m}(\eta)\mathrm{d}\omega(\eta),\qquad m=1,2,3. \tag{82}$$

Proof Let $\rho\in(0,2]$ be a sufficiently small quantity. By observing Fubini's theorem (Fubini 1958) of real analysis and the identity

$$s_R(\xi,\eta)-s_2(\xi,\eta) =\int_R^2 w_\rho(\xi,\eta)\frac{\mathrm{d}\rho}{\rho},\quad \xi,\eta\in\Omega\times\Omega \tag{83}$$

we have

$$\int_R^2 (WT)(F)(\rho;\xi)\frac{\mathrm{d}\rho}{\rho}=(ST)(F)(R;\xi) -(ST)(F)(2;\xi),\quad \xi\in\Omega. \tag{84}$$

Taking the limit $R\to 0$, we obtain with Theorem 4.2

$$\int_0^2 (WT)(F)(\rho;\xi)\frac{\mathrm{d}\rho}{\rho}=F(\xi)-F^\wedge(0,1) -(ST)(F)(2;\xi),\quad \xi\in\Omega. \tag{85}$$

Using the definition of s_ρ (Eq. 76), Green's formula, and the addition theorem, we have for all $\xi\in\Omega$,

$$\begin{aligned}(ST)(F)(2,\xi)&=\int_\Omega \nabla^*_\eta F(\eta)\cdot s_2(\xi,\eta)\,\mathrm{d}\omega(\eta)\\ &=-\int_\Omega F(\eta)\Delta^*_\eta s_2(\xi,\eta)\,\mathrm{d}\omega(\eta)\\ &=\int_\Omega F(\eta)\frac{1}{4\pi}(\xi\cdot\eta)\,\mathrm{d}\omega(\eta)\\ &=\frac{1}{4\pi}\int_\Omega F(\eta)P_1(\xi\cdot\eta)\,\mathrm{d}\omega(\eta)\end{aligned} \tag{86}$$

(note that $s_2(\xi,\cdot)$ is twice continuously differentiable on Ω). Obviously, the addition theorem for spherical harmonics of degree one shows us that

$$(ST)(F)(2;\xi)=\frac{1}{3}\sum_{m=1}^{3}F^\wedge(1,m)Y_{1,m}(\xi). \tag{87}$$

This yields the desired result formulated in Theorem 5.1 □

An immediate consequence of Theorem 5.1 is the following corollary.

Corollary 5.2 *Suppose that $F \in C^{(1)}(\Omega)$ fulfills $F^\wedge(0,1) = 0$ and $F^\wedge(1,m) = 0$, $m = 1,2,3$. Then*

$$\int_0^2 (WT)(F)(\rho;\cdot)\frac{d\rho}{\rho} = F. \tag{88}$$

6 Multiscale modelling

Until now, we have been concerned with a scale-continuous approach to wavelets. In what follows, scale-discrete scaling vector fields and wavelets will be introduced to establish a numerical solution process. We start with the choice of a sequence that divides the continuous scale interval (0, 2] into discrete pieces. More explicitly, $(\rho_j)_{j\in\mathbb{N}_0}$ denotes a monotonically decreasing sequence of real numbers satisfying

$$\rho_0 = 2, \quad \lim_{j\to\infty}\rho_j = 0. \tag{89}$$

Remark 6.1 For example, one may choose $\rho_j = 2^{1-j}$, $j \in \mathbb{N}_0$ (note that in this case $2\rho_{j+1} = \rho_j$, $j \in \mathbb{N}_0$). Another possibility, which is related to what is developed in Freeden and Schreiner (2005a,b), is to set $\varphi_j = 2\pi 2^{-j}$ and $\rho_j = 1 - \cos\varphi_j$, $j \in \mathbb{N}_0$. In this case, the diameter of the support of the wavelets doubles when going from j to $j-1$.

Given a scaling vector field $\{s_\rho\}$, then we clearly define the (scale) discretized scaling vector field $\{s_j^{\mathrm{D}}\}_{j\in\mathbb{N}_0}$ as follows:

$$s_j^{\mathrm{D}} = s_{\rho_j}, \quad j \in \mathbb{N}_0. \tag{90}$$

An illustration of $|s_j^{\mathrm{D}}|$ can be found in Fig. 3. We immediately get the following result.

Theorem 6.2 *For $F \in C^{(1)}(\Omega)$*

$$\lim_{j\to\infty}\sup_{\xi\in\Omega}\left|F(\xi) - F^\wedge(0,1) - (ST)^{\mathrm{D}}(F)(j;\xi)\right| = 0, \tag{91}$$

where

$$(ST)^{\mathrm{D}}(F)(j;\xi) = (ST)(F)(\rho_j;\xi) \tag{92}$$

with

$$(ST)^{\mathrm{D}}(F)(j;\xi) = \int_\Omega \nabla_\eta^* F(\xi)\cdot s_j^{\mathrm{D}}(\xi,\eta)\,\mathrm{d}\omega(\eta). \tag{93}$$

Our procedure canonically leads us to the following type of scale-discretized wavelets.

Definition 6.3 *Let $\{s_j^{\mathrm{D}}\}_{j\in\mathbb{N}_0}$ be a scale-discretized scaling vector field. Then the (scale) discretized wavelet vector field is defined by*

$$w_j^{\mathrm{D}}(\cdot,\cdot) = \int_{\rho_{j+1}}^{\rho_j} w_\rho(\cdot,\cdot)\frac{\mathrm{d}\rho}{\rho}, \quad j \in \mathbb{N}_0. \tag{94}$$

A graphical impression of the scale-discretised wavelets can be found in Fig. 4.
It follows from Eq. (94) that

$$\begin{aligned} w_j^{\mathrm{D}}(\cdot,\cdot) &= -\int_{\rho_{j+1}}^{\rho_j} \rho\frac{\mathrm{d}}{\mathrm{d}\rho}s_\rho(\cdot,\cdot)\frac{\mathrm{d}\rho}{\rho} \\ &= s_{j+1}^{\mathrm{D}}(\cdot,\cdot) - s_j^{\mathrm{D}}(\cdot,\cdot). \end{aligned} \tag{95}$$

The last formulation is called the *(scale) discretized scaling equation*. It is not hard to see that, for $\xi,\eta \in \Omega$,

$$w_j^{\mathrm{D}}(\xi,\eta) = \begin{cases} 0 & \text{for } \rho_{j+1} < 1-\xi\cdot\eta \\ -\frac{1}{4\pi}\left(\frac{1}{\rho_{j+1}} - \frac{1}{1-\xi\cdot\eta}\right)(\xi-(\eta\cdot\xi)\eta) & \text{for } \rho_j < 1-\xi\cdot\eta \le \rho_{j+1} \\ -\frac{1}{4\pi}\left(\frac{1}{\rho_{j+1}} - \frac{1}{\rho_j}\right)(\xi-(\eta\cdot\xi)\eta) & \text{for } 1-\xi\cdot\eta \le \rho_j. \end{cases} \tag{96}$$

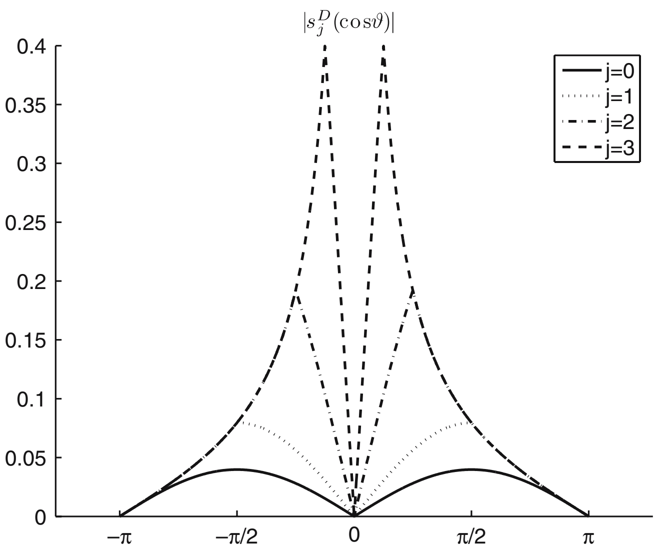

Fig. 3 Illustration of the norm of the scaling function $|s_j^{\mathrm{D}}(\cos\vartheta)|$

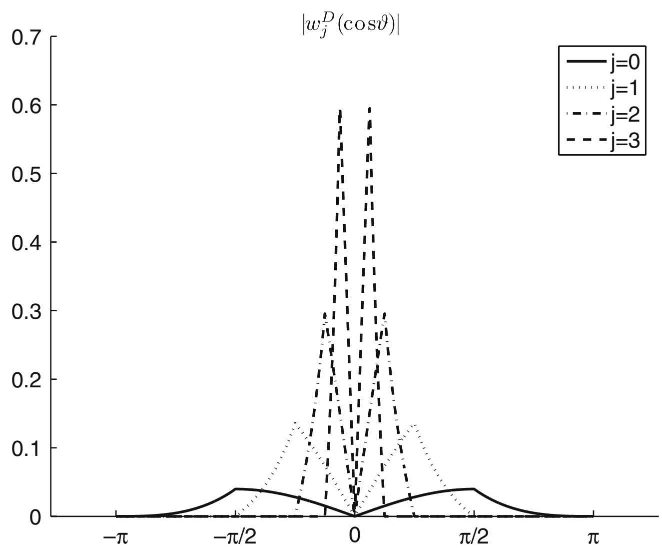

Fig. 4 Illustration of the norm of the wavelets $|w_j^{\mathrm{D}}(\cos\vartheta)|$

Assume now that F is a function of class $C^{(1)}(\Omega)$. Observing the discretized scaling equation (Eq. 95), we get for $J \in \mathbb{N}_0$ and $K \in \mathbb{N}$

$$(ST)^{\mathrm{D}}(F)(J+K;\xi) = (ST)^{\mathrm{D}}(F)(J;\xi) + \sum_{j=J}^{J+K-1} (WT)^{\mathrm{D}}(F)(j;\xi), \quad \xi \in \Omega. \tag{97}$$

with

$$(WT)^{\mathrm{D}}(F)(j;\xi) = (WT)(F)(\rho_j;\xi), \quad j = J, \ldots, J+K-1. \tag{98}$$

Therefore, we are able to formulate the following corollary.

Corollary 6.4 *Let $\{s_J^{\mathrm{D}}\}_{j\in\mathbb{N}_0}$ be a (scale) discretized scaling vector field. Then the multiscale representation of a function $F \in C^{(1)}(\Omega)$ is given by*

$$\sup_{\xi\in\Omega} \left| F(\xi) - \left(\sum_{j=0}^{\infty} (WT)^{\mathrm{D}}(F)(j;\xi) + F^\wedge(0,1) + \frac{1}{3}\sum_{m=1}^{3} F^\wedge(1,m) Y_{1,m}(\xi) \right) \right| = 0. \tag{99}$$

The aforementioned corollary admits the following reformulation.

Corollary 6.5 *Under the assumptions of Corollary 6.4*

$$(ST)^{\mathrm{D}}(F)(J;\cdot) + \sum_{j=J}^{\infty} (WT)^{\mathrm{D}}(F)(j;\cdot) = F - F^\wedge(0,1) - \frac{1}{3}\sum_{m=1}^{3} F^\wedge(1,m) Y_{1,m} \tag{100}$$

for every $J \in \mathbb{N}_0$ (in the sense of $\|\cdot\|_{C(\Omega)}$).

As in the classical theory of wavelets (e.g., Freeden et al. 1998), the integrals $(ST)^{\mathrm{D}}(F)(j;\cdot)$, $(WT)^{\mathrm{D}}(F)(j;\cdot)$ may be understood as low-pass and band-pass filters, respectively. As such,

$$(ST)^{\mathrm{D}}(F)(j+1;\cdot) = (ST)^{\mathrm{D}}(F)(j;\cdot) + (WT)^{\mathrm{D}}(F)(j;\cdot). \tag{101}$$

Equation (101) may be interpreted in the following way. The $(j+1)$-scale low-pass filtered version of F is obtained by the j-scale low-pass filtered version of F added by the j-scale band-pass filtered version of F.

7 Multiscale modelling of geoid undulations

The above considerations lead us to the following results in the determination of geoid undulations $N\colon \Omega \to \mathbb{R}$ from deflections of the vertical $\Theta\colon \Omega \to \mathbb{R}^3$ via Eq. (19), i.e.,

$$\nabla_\xi^* N(R\xi) = -R\Theta(R\xi), \quad \xi \in \Omega. \tag{102}$$

As in the classical process of gravimetric determination of the geoid (e.g., Heiskanen and Moritz 1967), we assume that the zero-degree term of the disturbing potential vanishes at the (spherical) surface of the Earth (i.e., the difference between the mass of the Earth and the mass of the reference ellipsoid is supposed to be zero). Moreover, we assume that the centre of the reference ellipsoid coincides with the center of gravity of the Earth so that the first-degree term is zero. In other words, the quantities

$$N^\wedge(n,m) = \int_\Omega N(R\xi) Y_{n,m}(\xi)\, \mathrm{d}\omega(\xi); \ n = 0,1; \ m = 1,\ldots,2n+1,$$

are all allowed to be zero and do not enter into our consideration.

To be more specific, we find the following result from Theorem 6.2.

Corollary 7.1 *Let $\Theta(R\cdot)\colon \xi \mapsto \Theta(R\xi)$, $\xi \in \Omega$ be a continuous field on Ω. Then the geoid undulations can be determined by the formula*

$$N(R\xi) = \frac{R}{4\pi} \int_\Omega \frac{1}{1-\xi\cdot\eta} (\xi - (\xi\cdot\eta)\eta) \cdot \Theta(R\xi)\, \mathrm{d}\omega(\eta). \tag{103}$$

According to our approach, the integral on the right-hand side of Eq. (103) can be written approximately by replacing the improper integral by proper parameter-integrals involving the "regularized Green function kernel" with respect to the Beltrami operator.

In our notation, for $\xi \in \Omega$ and $J \in \mathbb{N}_0$, we obtain

$$N(R\xi) = -R\int_\Omega \Theta(R\eta) \cdot s_J^{\mathrm{D}}(\xi,\eta)\, \mathrm{d}\omega(\eta) - R\sum_{j=J}^{\infty} \int_\Omega \Theta(R\eta) \cdot w_j^{\mathrm{D}}(\xi,\eta)\, \mathrm{d}\omega(\eta). \tag{104}$$

Consequently we find with $d = -R\Theta(R\cdot)$

$$N(R\xi) = \int_\Omega d(\eta) \cdot s_J^{\mathrm{D}}(\xi,\eta)\, \mathrm{d}\omega(\eta) - R\sum_{j=J}^{\infty} \int_{\substack{1-\xi\cdot\eta\le\rho_j+1 \\ \eta\in\Omega}} \Theta(R\eta) \cdot w_j^{\mathrm{D}}(\xi,\eta)\, \mathrm{d}\omega(\eta). \tag{105}$$

More explicitly,

$$
\begin{aligned}
N(R\xi) &= -R \int_{\substack{1-\xi\cdot\eta>\rho_J \\ \eta\in\Omega}} \Theta(R\eta)\cdot\left(-\frac{1}{4\pi}\frac{1}{1-\xi\cdot\eta}(\xi-(\xi\cdot\eta)\eta)\right)\times \mathrm{d}\omega(\eta) \\
&\quad -R \int_{\substack{1-\xi\cdot\eta\le\rho_J \\ \eta\in\Omega}} \Theta(R\eta)\cdot\left(-\frac{1}{4\pi\rho_J}(\xi-(\xi\cdot\eta)\eta)\right)\times \mathrm{d}\omega(\eta) \\
&\quad -R \sum_{j=J}^{\infty} \left\{ \int_{\substack{\rho_j<1-\xi\cdot\eta\le\rho_j+1 \\ \eta\in\Omega}} \Theta(R\eta) \right. \\
&\quad \cdot\left(-\frac{1}{4\pi}\left(\frac{1}{\rho_{j+1}}-\frac{1}{1-\xi\cdot\eta}\right)(\xi-(\xi\cdot\eta)\eta)\right) d\omega(\eta) \\
&\quad - \int_{\substack{1-\xi\cdot\eta\le\rho_j \\ \eta\in\Omega}} \Theta(R\eta)\cdot\left(-\frac{1}{4\pi}\left(\frac{1}{\rho_{j+1}}-\frac{1}{\rho_j}\right)\right. \\
&\quad \left.\left.\times(\xi-(\xi\cdot\eta)\eta)\,\mathrm{d}\omega(\eta)\right)\right\}. \qquad (106)
\end{aligned}
$$

In other words, finer and finer detail information about the geoid is obtained by wavelets with smaller and smaller local support.

8 Discussion and conclusions

Over the last two centuries, geodesists and mathematicians have tried to formulate and solve gravimetric boundary-value problems (BVPs), mainly in terms of functionals of the gravitational potential on the Earth's topographic surface. The international reference sphere (IRS) and the internal reference ellipsoid of revolution (IRE) had been chosen as international reference figures of the Earth.

Stokes (1849) initiated the study of BVPs in physical geodesy, succeeding in transforming gravity anomalies on the IRS to gravitational potential disturbances. Pizzetti (1910) formulated the Stokes solution in terms of a BVP: given gravity anomalies on the IRS, find the harmonic gravitational potential (incremental potential with a reference (normal) potential) in the space external to the IRS. As we discussed in Sect. 1, Bruns (1878) converted the incremental potential to radial displacement such that the Stokes–Pizetti solution of the gravity anomaly BVP in spherical approximation produced undulations of the geoid, as well as other external equipotential surfaces.

While a large variety of papers concerned with the determination of the harmonic gravitational disturbing potential in the outer space of the IRS is available (see, e.g., Freeden and Michel 2004 for a survey), fewer attempts have been made to represent the disturbing potential and its functionals intrinsically from suitable quantities on the IRS itself. First, qualitative studies of various intrinsic approaches to determine the disturbing potential are due to Meissl (1971a,b); important results were continued by Rummel and van Gelderen (1992), Grafarend (2001) and Nutz (2001) in their papers about the completed Meissl diagram (for further information, the reader is also referred to the literature cited therein).

This paper addresses the intrinsic problem of computing a scalar field F from its surface gradient $\nabla^* F$. It is shown, that the unknown scalar field F is essentially the surface integral over the scalar product of the given surface gradient field $\nabla^* F$ and the surface gradient of Green's function with respect the Beltrami operator on the sphere. Since the resulting integral is singular, it can be replaced by an integral involving a regularized vector kernel s_ρ showing a local support. In this context, it is of significance that the kernel is explicitly available as elementary function in closed form. As a result, the singular integral (i.e., the convolution of $\nabla^* F$ against the kernels s_ρ) can be replaced by a family of integrals with regular kernels. In consequence, an economical and efficient multiscale procedure [including a vectorial variant of a tree algorithm (pyramid scheme)] can be established by use of the concept proposed by Freeden et al. (1998), Bayer et al. (1998) and Beth (2000). In addition, wavelet variances (illustrating the space evolution in the frequencies) and multiscale denoising (as known from Euclidean wavelet theory) can be introduced in close similarity to the ideas presented in Freeden and Maier (2002). Moreover, spectral as well as multiscale signal-to-noise thresholding can be performed as proposed by Freeden and Maier (2003).

It should be mentioned in closing that the local determination of differences in N from deflections of the vertical Θ is well known as astro-gravimetric levelling. However, to the knowledge of the authors, the problem of determining N globally from data Θ has not been discussed in the geodetic literature. Moreover, a local approximation of N can be deduced from locally given Θ data, because of the local character of the kernel fields involved in the multiscale approximation (see Freeden and Schreiner (2005a,b) for an alternate wavelet approach involving the up-function).

9 Acknowledgements

The authors wish to thank W. Featherstone, P. Holota, C. Kotsakis, I.N. Tziavos, and an unknown reviewer for their valuable comments.

References

Bayer M, Beth S, Freeden W (1998) Geophysical field modelling by multiresolution analysis. Acta Geod et Geophys Hungarica 33:289–319

Beth S (2000) Multiscale approximation by vector radial basis functions on the sphere. PhD thesis, Geomathematics Group, University of Kaiserslautern, Shaker, Aachen

Bruns EH (1878) Die Figur der Erde. Publikation Königl. Preussisch. Geodätisches Institut. P. Stankiewicz Buchdruckerei, Berlin
Featherstone WE, Rüeger JM (2000) The importance of using deviations of the vertical for the reduction of survey data to a geocentric datum. Aust Surveyor 45(2):46–61 (Erratum: Austr Surveyor 47(1):7)
Freeden W (1979) Über eine Klasse von Integralformeln der Mathematischen Geodäsie. Habilitation Thesis, Veröffentlichung des Geodätischen Instituts der RWTH Aachen, No. 27
Freeden W, Michel V (2004) Multiscale potential theory (with applications to geoscience). Birkhäuser, Basel
Freeden W, Gervens T, Schreiner M (1998) Constructive approximation on the sphere (with applications to geomathematics). Oxford Science Publications, Clarendon Press, Oxford
Freeden W, Maier T (2002) On multiscale denoising of spherical functions: basic theory and numerical aspects. Electr Trans Numer Analysis (ETNA) 14:40–62
Freeden W, Maier T (2003) Spectral and multiscale signal-to-noise thresholding of spherical vector fields. Comput Geosci 7(3):215–250
Freeden W, Schreiner M (2005a) Multiresolution analysis by spherical up functions. Constr Approx (in press)
Freeden W, Schreiner M (2005b) Spaceborne gravitational field determination by means of locally supported wavelets. J Geod 79:431–446
Fubini G (1958) Sugli integrali multipli. Opere scelte 2:243–249
Grafarend EW (2001) The spherical horizontal and spherical vertical boundary value problem – vertical deflections and geoidal undulations – the completed Meissl diagram. J Geod 75:363–390
Groten E (1979) Geodesy and the Earth's gravity field I, II. Dümmler, Bonn
Groten E (1981) Local and global gravity field representation. Rev Geophys Space Phys 19:407–414
Heiskanen WA, Moritz H (1967) Physical geodesy. Freeman, San Francisco
Jekeli C (1999) An analysis of vertical deflections derived from high-degree spherical harmonic models. J Geod 73(1):10–22
Meissl P (1971a) A study of covariance functions related to the Earth's disturbing potential. Report 151, Department of Geodetic Science, The Ohio State University Columbus
Meissl, P (1971b) On the linearization of the geodetic boundary value problem. Report 152, Department of Geodetic Science, The Ohio State University Columbus
Nutz H (2001) A unified setup of gravitational field observables. PhD Thesis, Geomathematics Group, University of Kaiserslautern, Shaker, Aachen
Pizzetti P (1910) Sopra il calcoba tesrico delle deviazioni del geoide dall' ellissoide. Att R Acad Sci Torino 46:331–350
Rummel R (1992) Fysische Geodesie I, II. Collegediktaat, Faculty of Geodesy, Delft University of Technology, Delft
Rummel R, van Gelderen M (1992) Spectral analysis of the full gravity tensor. Geophys J Int 111:159–169
Torge W (1991) Geodesy. de Gruyter, Berlin
Stokes GG (1849) On the variation of gravity at the surface of the Earth. Trans Cambridge Phil Soc 8:672–712. In: Mathematical and physical papers by George Gabriel Stokes, vol II. Johnson Reprint Corporation, New York, pp 131–171