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Solutions of the linearized geodetic boundary value problem for an ellipsoidal boundary to order e^3

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Abstract. The geodetic boundary value problem (GBVP) was originally formulated for the topographic surface of the Earth. It degenerates to an ellipsoidal problem, for example when topographic and downward continuation reductions have been applied. Although these ellipsoidal GBVPs possess a simpler structure than the original ones, they cannot be solved analytically, since the boundary condition still contains disturbing terms due to anisotropy, ellipticity and centrifugal components in the reference potential. Solutions of the so-called scalar-free version of the GBVP, upon which most recent practical calculations of geoidal and quasi-geoidal heights are based, are considered. Starting at the linearized boundary condition and presupposing a normal field of Somigliana–Pizzetti type, the boundary condition described in spherical coordinates is expanded into a series with respect to the flattening f of the Earth. This series is truncated after the linear terms in f, and first-order solutions of the corresponding GBVP are developed in closed form on the basis of spherical integral formulae, modified by suitable reduction terms. Three alternative representations of the solution are discussed, implying corrections by adding a first-order non-spherical term to the solution, by reducing the boundary data, or by modifying the integration kernel. A numerically efficient procedure for the evaluation of ellipsoidal effects, in the case of the linearized scalar-free version of the GBVP, involving first-order ellipsoidal terms in the boundary condition, is derived, utilizing geopotential models such as EGM96.

Keywords: Geoid – Quasigeoid – Geodetic boundary value problem – Stokes' formula – Ellipsoidal correction

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1 Introduction

Considering the presently achievable precision of GPS positioning, the need for highly precise local and regional geoid and quasigeoid determination has strongly increased. As a consequence, precise modelling of the terrestrial gravity field in the framework of the geodetic boundary value problem (GBVP) has become a central subject in geodesy. Facing these new challenges, the classical theory of the GBVP, originating from G.G. Stokes (Stokes 1849) and M.S. Molodenskii (Molodenskii et al. 1962) and extended first of all by T. Krarup (Krarup 1973) and H. Moritz in the 1970s (Moritz 1980), is no longer adequate. In particular, the approximations used in the individual solution steps (Rummel 1988) have to be reflected upon, and the respective approximation errors have to be analysed. As a second step, suitable reduction and evaluation procedures have to be developed which remove the approximation errors with sufficient precision and low numerical effort.

Comparisons of gravimetrically determined (quasi-) geoidal heights and differences between GPS and levelled heights at discrete points have shown that short- and long-wavelength errors of several centimetres exist in the resulting (quasi-) geoidal heights (Torge and Denker 1999), which may partly be due to the neglection of ellipsoidal effects. For this reason considerable effort has been put into the analysis of the first-order solution of the so-called 'ellipsoidal GBVP'. Based on the classical work by Sagrebin (1956), Bjerhammar (1962), Molodenskii et al. (1962) and Moritz (1980), formulae for the calculation of the ellipsoidal reduction – in first-order approximation with respect to the Earth's flattening – have been derived by different authors (see e.g. Koch 1968; Mather 1973; Zhu 1981; Martinec and Grafarend 1997; Martinec 1998a, b; Fei 2000; Fei and Sideris 2000; Brovar et al. 2001). In Huang et al. (2003) a numerical comparison between four main approaches (Molodenskii et al. 1962; Moritz 1974; Martinec and

Grafarend 1997; Fei and Sideris 2000) is carried out, indicating that the main differences are in the first-degree spherical harmonic terms, but otherwise the results are equivalent. Although the solution procedures are quite different, the formulae for the calculation of the ellipsoidal corrections in first-order approximation are mostly represented by spherical integral formulae, partly involving rather complicated integral kernels.

The main purpose of the present paper is the study of first-order solutions to the ellipsoidal GBVP, based on the linearized, scalar-free GBVP. As a preparation, a short review of the scalar-free GBVP at successive levels of approximation is provided in Sect. 2, followed by a discussion of first-order solutions of the ‘simple Molodenskii problem’ referring to an ellipsoidal boundary. The more general case of the linearized scalar-free GBVP involving first-order ellipsoidal terms is tackled in Sect. 4, resulting in a numerically efficient procedure for calculating the ellipsoidal effects, which is based on spherical harmonic expansions.

2 Review of the scalar-free GBVP at various levels of approximation

Starting at the highest level of complexity, the non-linear, scalar-free version of the GBVP for an arbitrary, star-shaped boundary surface S can be formulated as follows (Sacerdote and Sansò 1986; Heck 1989, 1997; Grafarend et al. 1999).

Let the observables g (modulus of gravity) and C (geopotential number with respect to a global fundamental point P_o) be given in continuous form on the star-shaped boundary surface S (representing the Earth’s topographical surface), where

$$g = |\nabla W||_S \tag{1}$$

$$C = W_o - W|_S \tag{2}$$

$$S : \mathbf{X}\big(\varphi_g, \lambda; h(\varphi_g, \lambda)\big) \tag{3}$$

The gravity potential W is composed of a known reference (normal) potential U and the unknown disturbing potential T, harmonic outside S and regular at infinity

$$W = U + T \tag{4}$$

$$\nabla^2 T = 0 \text{ outside } S \tag{5}$$

$$T \sim 0\left(\frac{1}{r}\right), \quad r \to \infty \tag{6}$$

The surface S is described by the position vector $\mathbf{X}$, parameterized in terms of geodetic coordinates (geodetic latitude φ_g, longitude λ, ellipsoidal height h, referring to an ellipsoid of revolution S_E with given size and orientation) where the unknown ellipsoidal height $h(\varphi_g, \lambda)$ is a function of the horizontal coordinates φ_g and λ as well as the boundary data $g(\varphi_g, \lambda)$ and $C(\varphi_g, \lambda)$, assuming that φ_g and λ are known at any point $P \in S$.

Due to Eqs. (1) and (2) the boundary values $g(\varphi_g, \lambda)$ and $C(\varphi_g, \lambda)$ depend on the unknown functions T and h in a non-linear way. A second level of approximation of the scalar-free GBVP is achieved by linearization, referring to the normal potential U and an approximate surface, the telluroid Σ, which is constructed from the given boundary values by some telluroid mapping. Throughout this paper Molodenskii’s telluroid mapping $P \to Q,\ P \in S,\ Q \in \Sigma$ is applied (see Fig. 1).

$$\left.\begin{aligned} \varphi_g(Q) &\overset{!}{=} \varphi_g(P) \\ \lambda(Q) &\overset{!}{=} \lambda(P) \end{aligned}\right\} \sim \mathbf{n}(Q) = \mathbf{n}(P)$$

$$U(Q) - U_o \overset{!}{=} W(P) - W_o = -C(P) \tag{7}$$

where U_o = const. is the normal potential of Somigliana–Pizzetti type on the surface of the reference ellipsoid S_E, to which the horizontal coordinates φ_g, λ refer. The unit vector $\mathbf{n}$ denotes the direction of the ellipsoidal normal, parameterized by φ_g and λ. It should be noted that the deviations between the telluroid Σ and the ellipsoid S_E are small over oceanic regions, their size being comparable with the order of magnitude of sea surface topography.

Linearization of Eqs. (1) and (2) with respect to the normal potential U and the telluroid Σ, after elimination of the height unknown, yields the general linearized boundary condition, valid on Σ

$$T - \frac{\langle \boldsymbol{\gamma}, \mathbf{n}\rangle}{\langle \boldsymbol{\gamma}, M \cdot \mathbf{n}\rangle} \cdot \langle \boldsymbol{\gamma}, \nabla T\rangle = \Delta w_o - \frac{\gamma \cdot \langle \boldsymbol{\gamma}, \mathbf{n}\rangle}{\langle \boldsymbol{\gamma}, M \cdot \mathbf{n}\rangle} \cdot \Delta g \tag{8}$$

In Eq. (8), $\boldsymbol{\gamma} = \nabla U$ denotes the normal gravity vector, $\gamma = |\boldsymbol{\gamma}|$ the normal gravity, $M = \nabla\nabla U$ the second-order Marussi tensor of normal gravity gradients; these quantities, as well as T, have to be evaluated at the running point $Q \in \Sigma$. The gravity anomaly

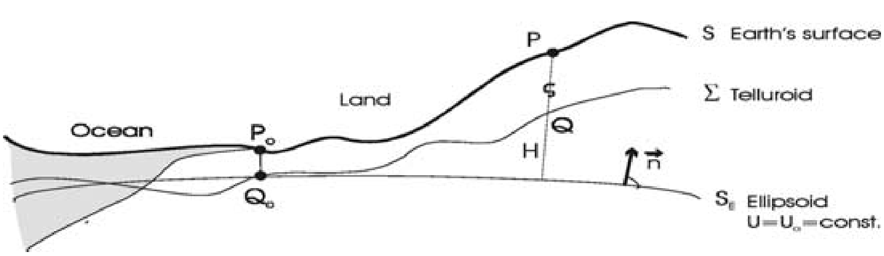

Fig. 1. Geometry of the scalar-free GBVP

$\Delta g = g(P) - \gamma(Q)$ is defined in the sense of Molodenskii's surface free-air anomaly. In the following, the unknown constant $\Delta w_o = W_o - U_o$ is set to zero for reasons of simplicity; for a discussion of the consequences of this assumption see e.g. Heck and Rummel (1990). The detailed form of the linearized boundary condition of Eq. (8) for various normal gravity fields has been derived by Grafarend et al. (1999), based on elliptical coordinates. It has been shown by Heck and Seitz (1993) that the errors due to linearization may amount to 2 cm globally and can easily be taken into account by reductions.

The next approximation step consists of expanding the coefficients in Eq. (8), depending on $\boldsymbol{\gamma}, \mathbf{n}$ and M, in binomial series with respect to the flattening f of the reference ellipsoid. In general, f is replaced by the square of the first numerical eccentricity e^2. In consistence with spherical harmonics, which will be applied in Sect. 3 for the description of harmonic functions, spherical coordinates (r, φ, λ) are introduced instead of the geodetic coordinates (h, φ_g, λ) for parameterizing the position vector of arbitrary points on and outside the telluroid Σ; φ is the geocentric latitude. Seitz (1997) has derived a complete second-order theory for the development of the linearized boundary condition of Eq. (8), based on a general reference field and specified for a Somigliana–Pizzetti field. A rough estimate of the second-order terms [see Seitz 1997, Eq. (3–22)] gives a maximum value of about $3 \cdot 10^{-8}$ m s^{-2}, proving that the series expansion of the coefficients in Eq. (8) can safely be truncated after the linear term. The linearized boundary condition in first-order approximation with respect to e^2 for a Somigliana–Pizzetti normal field takes the following form (Heck 1991; Seitz 1997):

$$\left[-\frac{2}{a}\left[1+\frac{1}{2}e^2\left(2\sin^2\varphi - 1\right)+\bar{m}\right]\cdot T - \frac{\partial T}{\partial r} - e^2 \sin\varphi\cos\varphi \cdot \frac{\partial T}{a\partial\varphi}\right]\Bigg|_{\Sigma} = \Delta g(\varphi, \lambda) \tag{9}$$

where a is the semi-major axis of the reference ellipsoid, and the parameter $\bar{m} = \omega^2 a^3/\mu$ depends on the angular velocity of the Earth's rotation and the geocentric gravitational constant μ. Terms of order $0(h/a)$ have been neglected in Eq. (9). It should be noted that the anisotropic term in Eq. (9) depending on the horizontal derivative of the disturbing potential is due to the difference between the radial direction and the direction of -$\boldsymbol{\gamma}$.

For reasons of simplicity the further approximation $\bar{m} \approx e^2/2$ (Moritz 1980) can be introduced, resulting in the boundary equation

$$\left[-\frac{2}{a}(1+e^2\cos^2\varphi)\cdot T - \frac{\partial T}{\partial r} - e^2\sin\varphi\cos\varphi\cdot\frac{\partial T}{a\partial\varphi}\right]\Bigg|_{\Sigma} = \Delta g(\varphi,\lambda) \tag{10}$$

a rough estimate based on the parameters of the GRS80 yields a maximum approximation error of about $4 \cdot 10^{-8}\,\mathrm{m\,s^{-2}}$.

Neglecting the terms of order $0(e^2)$ in Eq. (10) provides the next level in the approximation scheme, often denoted as 'spherical' or 'isotropic' approximation. In this way the boundary condition of the 'simple' Molodenskii problem (Krarup, 1973) is generated

$$\left[-\frac{2}{a}\cdot T - \frac{\partial T}{\partial r}\right]\Bigg|_{\Sigma} = \Delta g(\varphi,\lambda) \tag{11}$$

which still refers to the telluroid as boundary. Considering the order of magnitude of the terms omitted in Eq. (11), amounting to about $0.6 \cdot 10^{-5}$ m s^{-2} (Cruz 1986; Seitz 1997), it becomes obvious that they have to be balanced by some ellipsoidal corrections in order to fulfil the actual precision requirements.

A further level of approximation is attained by considering $\Delta g(\varphi, \lambda)$ as boundary values on the reference ellipsoid S_E rather than on the telluroid Σ. The effect of this approximation is generally taken into account by Molodenskii's series expansion or by application of Moritz's L-operators (Moritz 1980; Sideris 1990) in the context of the harmonic downward continuation to sea level. In the boundary condition the respective effects may amount to some tens of milliGals, propagating into a 1–2 m effect in the (quasi-)geoidal heights (Seitz 1997, p. 57 and 93). It should be mentioned that no closed solution of the GBVP exists at this level of simplification.

Finally the boundary surface can formally be replaced by a sphere S_a of radius a, resulting in the boundary condition

$$\left[-\frac{2}{a}\cdot T - \frac{\partial T}{\partial r}\right]\Bigg|_{S_a} = \Delta g(\varphi,\lambda) \tag{12}$$

Associated with the Laplace equation $\nabla^2 T = 0$ holding outside the sphere S_a, this corresponds formally to a third boundary value problem of potential theory for a spherical boundary which is solvable in closed form by Stokes' integral (Stokes 1849)

$$T(r,\varphi,\lambda) = \frac{a}{4\pi}\iint_{\sigma} \Delta g(\varphi',\lambda')\cdot\left[S(r,\psi)-\frac{a}{r}\right]\mathrm{d}\sigma + T_1(r,\varphi,\lambda) \tag{13}$$

$S(r, \psi)$ is the Stokes–Pizzetti function (Heiskanen and Moritz 1967), σ the domain of the unit sphere with the surface element $\mathrm{d}\sigma = \cos\varphi' \cdot \mathrm{d}\varphi' \cdot \mathrm{d}\lambda' = \sin\psi \cdot \mathrm{d}\psi \cdot \mathrm{d}\alpha$ (ψ spherical distance, α azimuth), and (φ', λ') denote the position of the variable integration point. It is well-known that the first-degree term T_1 (r, φ, λ) in Eq. (13) cannot be determined at this level of approximation, often denoted as 'spherical and constant radius approximation'.

The sequence of approximations described above is summarized in Table 1. It becomes clear that the com-

Table 1. GBVP: sequence of approximations

Level	Denotation	Equation number of the boundary condition
0	Non-linear GBVP	(1), (2)
	← Linearization (normal field, telluroid mapping)	
1	Linearized GBVP, boundary surface = telluroid	(8)
	← First-order approximation of the boundary operator	
2	Linearized GBVP, including first-order 'ellipsoidal effects'	(9) or (10)
	← Zero-order approximation of the boundary operator ('spherical approximation')	
3	'Simple' GBVP involving only radial components in the boundary operator; boundary = telluroid	(11)
	← Downward continuation to 'sea level'	
4	'Simple' GBVP for an ellipsoidal surface (oblique derivative)	(11)
	← Constant-radius approximation	
5	'Simple' GBVP for a spherical surface (normal derivative)	(12)

plexity decreases with the approximation level, starting at level 0 which is related to the full non-linear GBVP, up to level 5 which is analytically solvable. The solution of the GBVP at the respective upper levels can be obtained by an iterative scheme, using the solution available from the lower level (Rummel 1988). It has been shown by Seitz (1997) that such an iterative procedure between the levels 0 and 4 provides a fast numerical convergence, while the iteration between the levels 0 and 5 proves to be divergent. It follows that the convergence behaviour depends on both the initial solution and the form of the boundary surface. For this reason, the use of ellipsoidal corrections should provide a significant improvement. It is worth mentioning that this procedure is strongly related to the 'change of boundary' method proposed by Sansò and Sona (1995).

3 First-order solutions of the simple Molodenskii problem on a reference ellipsoid

Before the solution of the more realistic 'ellipsoidal' GBVP at level 2 is investigated in Sect. 4, first the simpler problem at level 4 is studied in detail, reflecting on the behaviour of the harmonic continuation from a spherical to an ellipsoidal surface and vice versa. Starting from Eq. (11), specified for an ellipsoidal boundary surface, the respective basic equations for the 'simple' Molodensky problem on the ellipsoid S_E are as follows:

$$\left[-\frac{2}{a}\cdot T-\frac{\partial T}{\partial r}\right]\Bigg|_{r=r_E(\varphi)}=\Delta g(\varphi,\lambda) \tag{14}$$

$$\nabla^2 T=0 \text{ outside } S_E$$

$$T=0\left(\frac{1}{r}\right) \text{ for } r\to\infty \tag{15}$$

where $r_E(\varphi)$ represents the geocentric radius of the ellipsoidal surface which can be developed into a Taylor series

$$\begin{aligned} r_E(\varphi)&=a\sqrt{\frac{1-e^2}{1-e^2\cos^2\varphi}}\\ &=a\left(1-\frac{1}{2}e^2\sin^2\varphi-\frac{1}{8}e^4\sin^2\varphi\left(4-3\sin^2\varphi\right)+\cdots\right) \end{aligned} \tag{16}$$

neglecting terms of order $0(e^6)$.

The harmonic disturbing potential T can be represented by a series of solid spherical harmonics

$$T(r,\varphi,\lambda)=\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^{n+1}\sum_{m=-n}^{+n}T_{nm}\cdot Y_{nm}(\varphi,\lambda) \tag{17}$$

where $Y_{nm}(\varphi,\lambda)$ denote the Laplacian surface spherical harmonics of degree n and order m. The coefficients T_{nm} can be interpreted as the spherical harmonic coefficients of the function T restricted to the sphere of radius a. Equation (17) is generally used in satellite geodesy, but cannot be applied directly in the present context, since it might diverge between the sphere S_a and the ellipsoidal boundary surface S_E. Due to the validity of the Runge–Krarup theorem (Moritz 1980) the disturbing potential outside S_E can be approximated arbitrarily well by the representation

$$T(r,\varphi,\lambda)=\sum_{n=0}^{\infty}\left(\frac{b}{r}\right)^{n+1}\sum_{m=-n}^{+n}\bar{T}_{nm}\cdot Y_{nm}(\varphi,\lambda) \tag{18}$$

where b is the semi-minor axis of the reference ellipsoid. Equation (18) converges in the domain outside the Bjerhammar sphere S_b with radius b and represents the harmonic downward continued potential between S_b and S_E (Fig. 2). The coefficients $\bar{T}_{nm}$ correspond to the spherical harmonic coefficients of the downward continued function on the sphere S_b.

Due to the well-known relationship

$$b=a\cdot\sqrt{1-e^2} \tag{19}$$

the coefficients $\bar{T}_{nm}$ can be transformed into T_{nm}

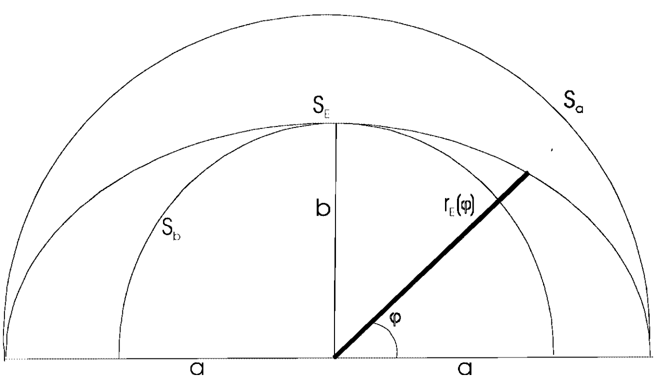

Fig. 2. Ellipsoidal boundary S_E and spheres S_a and S_b

$$T_{nm} = \left(1 - e^2\right)^{\frac{n+1}{2}} \cdot \bar{T}_{nm} \tag{20}$$

The gravity anomaly $\Delta g(\varphi, \lambda)$, given on the ellipsoid S_E as a function of (φ, λ), can be expanded into a series of surface spherical harmonics

$$\Delta g(\varphi, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \Delta g_{nm} \cdot Y_{nm}(\varphi, \lambda) \tag{21}$$

Inserting Eqs. (18), (16) and (21) into the boundary condition of Eq. (14) results in

$$\sum_{n=0}^{\infty} \frac{\left(1-e^2\right)^{(n+2)/2}}{b} \left[(n-1) + \frac{n(n+1)}{2} \cdot e^2 \sin^2\varphi + \cdots\right] \times \sum_{m=-n}^{+n} \bar{T}_{nm} \cdot Y_{nm}(\varphi, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \Delta g_{nm} \cdot Y_{nm}(\varphi, \lambda) \tag{22}$$

where the terms of order $0\left(n^2 \cdot e^2\right)$ can be attributed to the procedure of downward continuation. From Moritz [1980, Eq. (39–76)] it follows that the product $\sin^2\varphi \cdot Y_{nm}(\varphi, \lambda)$ can be expressed on the basis of surface spherical harmonics

$$\sin^2\varphi \cdot Y_{nm}(\varphi, \lambda) = \alpha_{nm} \cdot Y_{n+2,m}(\varphi, \lambda) + \beta_{nm} \times Y_{nm}(\varphi, \lambda) + \gamma_{nm} \cdot Y_{n-2,m}(\varphi, \lambda) \tag{23}$$

involving the coefficients

$$\begin{aligned} \alpha_{nm} &= \frac{(n-k+1)(n-k+2)}{(2n+1)(2n+3)} \\ \beta_{nm} &= \frac{2n^2 - 2k^2 + 2n - 1}{(2n-1)(2n+3)} \\ \gamma_{nm} &= \frac{(n+k)(n+k-1)}{(2n-1)(2n+1)}, \quad k = |m| \end{aligned} \tag{24}$$

These coefficients tend to finite constants for $n \to \infty$, i.e. they have the order $0(n^0)$. Inserting Eq. (23) into Eq. (22) and rearranging the left-hand side in terms of $Y_{nm}(\varphi, \lambda)$ provides the basis for a comparison of coefficients of the same degree and order. This procedure results in the system of algebraic equations

$$\frac{\left(1-e^2\right)^{(n+2)/2}}{b} \cdot \left[(n-1)\bar{T}_{nm} + \frac{e^2}{2}\left((n-2)(n-1) \times \alpha_{n-2,m}\bar{T}_{n-2,m} + n(n+1)\beta_{nm}\bar{T}_{nm} + (n+2)(n+3) \times \gamma_{n+2,m}\bar{T}_{n+2,m}\right) + 0\left(e^4\right)\right] = \Delta g_{nm}, \quad \forall n \geq 0 \tag{25}$$

(For $n \leq 1$ the coefficients $\bar{T}_{n-2,m}$ have to be set equal to zero).

On the other hand, the potential function on S_E can be expressed in the spectral domain, involving the coefficients $\tilde{T}_{nm}$

$$T(r_E(\varphi), \varphi, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \tilde{T}_{nm} \cdot Y_{nm}(\varphi, \lambda) \tag{26}$$

This expression can be compared with Eq. (18), specified for $r = r_E(\varphi)$

$$T(r_E(\varphi), \varphi, \lambda) = \sum_{n=0}^{\infty} \left(\frac{b}{r_E(\varphi)}\right)^{n+1} \sum_{m=-n}^{+n} \bar{T}_{nm} \cdot Y_{nm}(\varphi, \lambda) \tag{27}$$

where

$$\left(\frac{b}{r_E(\varphi)}\right)^{n+1} = \left(\frac{b}{a}\right)^{n+1} \cdot \left(\frac{a}{r_E}\right)^{n+1} = \left(1-e^2\right)^{(n+1)/2} \times \left(1 + \frac{n+1}{2} e^2 \sin^2\varphi + \cdots\right) \tag{28}$$

This term can be inserted in Eq. (27). After rearranging the term $\sin^2\varphi \cdot Y_{nm}(\varphi, \lambda)$ with the aid of Eq. (23) and performing a comparison of the respective coefficients in Eq. (26), a relationship between the coefficients $\bar{T}_{nm}$ of the spatial function T in terms of solid spherical harmonics and the coefficients $\tilde{T}_{nm}$ of the surface function $T(r_E(\varphi), \varphi, \lambda)$ is obtained

$$\left(1-e^2\right)^{(n+1)/2} \cdot \left[\bar{T}_{nm} + \frac{e^2}{2}\left((n-1) \cdot \alpha_{n-2,m} \times \bar{T}_{n-2,m} + (n+1) \cdot \beta_{nm} \cdot \bar{T}_{nm} + (n+3) \cdot \gamma_{n+2,m} \times \bar{T}_{n+2,m}\right) + \cdots\right] = \tilde{T}_{nm} \tag{29}$$

Combining Eqs. (29) and (25) results in the spectral relationship between the surface functions $T(r_E(\varphi), \varphi, \lambda)$ and $\Delta g(\varphi, \lambda)$, neglecting any terms of order $0(e^4)$

$$\frac{(n-1)}{a} \cdot \left[\tilde{T}_{nm} + \frac{e^2}{2}\left(-\alpha_{n-2,m}\tilde{T}_{n-2,m} + \frac{n+1}{n-1}\beta_{nm}\tilde{T}_{nm} + 3\frac{n+3}{n-1}\gamma_{n+2,m}\tilde{T}_{n+2,m}\right) + \cdots\right] = \Delta g_{nm} \tag{30}$$

The ellipsoidal correction terms in Eq. (30) have an impact of order $0\left(e^2 \cdot n^0\right)$, while in Eq. (25) the corresponding expressions increase with n, showing the order $0\left(e^2 \cdot n^1\right)$.

Equation (30) provides a system of equations which can be solved iteratively for $n \neq 1$, starting with the 'spherical' solution

$$\tilde{T}_{nm}^{(0)} = \frac{a}{n-1} \Delta g_{nm} \tag{31}$$

If a suitable global geopotential model with coefficients T_{nm} is available, this prior information can also be used for evaluating the terms of order $0(e^2)$ in Eq. (30), which yields

$$\begin{aligned}\tilde{T}_{nm} = &\frac{a}{n-1} \Delta g_{nm} + \frac{e^2}{2} \left(\alpha_{n-2,m} T_{n-2,m} - \frac{n+1}{n-1} \beta_{nm} T_{nm} \right. \\ &\left. -3 \frac{n+3}{n-1} \gamma_{n+2,m} T_{n+2,m} \right) + 0\left(e^4\right)\end{aligned} \tag{32}$$

Due to the factor $(n-1)$ in Eq. (30), the solution for $n=1$ has to be considered separately. Applying Eq. (25) for $n=1$ yields

$$\frac{1-e^2}{a} \cdot e^2 (\beta_{1m} \bar{T}_{1m} + 6 \gamma_{3m} \bar{T}_{3m}) = \Delta g_{1m} \tag{33}$$

where, by further application of Eq. (25) for $n=3$ and neglecting terms of order $0(e^2)$, $\bar{T}_{3m}$ can be replaced by

$$\bar{T}_{3m} = \frac{a}{2} \Delta g_{3m} + 0(e^2) \tag{34}$$

This results in the expressions

$$\bar{T}_{10} = \frac{5 \cdot a \cdot \Delta g_{10}}{3e^2(1-e^2)} - \frac{6}{7} \cdot a \cdot \Delta g_{30} \tag{35a}$$

$$\bar{T}_{1,\pm 1} = \frac{5a}{e^2(1-e^2)} \Delta g_{1,\pm 1} - \frac{36}{7} a \cdot \Delta g_{3,\pm 1} \tag{35b}$$

proving that for the non-spherical GBVP the first-degree coefficients of the external potential can be determined from the boundary data Δg, at least in principle (Heck 1991); obviously the first-degree coefficients cannot be calculated very precisely due to the small denominators in Eqs. (35a) and (35b). On the other hand, even if the first-degree terms $\bar{T}_{1m}$ (or equivalently T_{1m}) of the spatial, harmonic function T vanish, the first-degree surface harmonics Δg_{1m} of the surface function $\Delta g(\varphi, \lambda)$ will not disappear, as is obvious from Eq. (33). Furthermore, from Eq. (29) it follows that when $\bar{T}_{1m} \equiv 0$ (or $T_{1m} \equiv 0$), the first-degree terms of the boundary values of T on S_E will not vanish either

$$\tilde{T}_{1m} = \frac{2(3+|m|)(2+|m|)}{35} e^2 \cdot \bar{T}_{3m}, \quad -1 \le m \le 1 \tag{36}$$

Imposing the condition $T_{1m} \equiv 0$ on the external potential will produce three consistency conditions for the boundary data Δg, resulting from Eqs. (35a) and (35b)

$$\Delta g_{10} = \frac{18}{35} e^2 \cdot \Delta g_{30}, \quad \Delta g_{1,\pm 1} = \frac{36}{35} e^2 \cdot \Delta g_{3,\pm 1} \tag{37}$$

or in integral form

$$\iint_\sigma \Delta g(\varphi', \lambda') \cdot \left[P_1(\varphi') - \frac{6}{5} e^2 \cdot P_3(\varphi') \right] \mathrm{d}\sigma = 0 \tag{38a}$$

$$\iint_\sigma \Delta g(\varphi', \lambda') \cdot \left[P_{11}(\varphi') - \frac{2}{5} e^2 \cdot P_{31}(\varphi') \right] \left\{ \begin{matrix} \cos \lambda' \\ \sin \lambda' \end{matrix} \right\} \mathrm{d}\sigma = 0 \tag{38b}$$

In general, the data will not fulfil these conditions, but it is reasonable to assume that the bias generated in the solution will be negligibly small.

As a final step, Eq. (26) involving the coefficients $\tilde{T}_{nm}$ can be summed up by using Eq. (32). A 'near-closed' solution on the ellipsoidal surface is constructed by separating a dominant 'spherical' part T_S and an 'ellipsoidal correction' δT_E

$$\begin{aligned}T(r_E(\varphi), \varphi, \lambda) &= \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \tilde{T}_{nm} \cdot Y_{nm}(\varphi, \lambda) \\ &= T_S(\varphi, \lambda) + \delta T_E(\varphi, \lambda)\end{aligned} \tag{39}$$

Assuming that the usually applied conditions $T_{00} \equiv 0, T_{1m} \equiv 0$ (or equivalently $\bar{T}_{00} \equiv 0, \bar{T}_{1m} \equiv 0$) hold, the 'spherical' part can be expressed by the well-known Stokes integral

$$\begin{aligned}T_S(r_E(\varphi), \varphi, \lambda) &= \sum_{n=2}^{\infty} \sum_{m=-n}^{+n} \frac{a}{n-1} \Delta g_{nm} \cdot Y_{nm}(\varphi, \lambda) \\ &= \frac{a}{4\pi} \iint_\sigma \Delta g(\varphi', \lambda') \cdot S(\psi) \cdot \mathrm{d}\sigma\end{aligned} \tag{40}$$

where $S(\psi)$ denotes the original Stokes function. Then the 'ellipsoidal correction', based on Eq. (32), is given by

$$\begin{aligned}&\delta T_E(r_E(\varphi), \varphi, \lambda) \\ &= \sum_{n=0}^{N} \sum_{m=-n}^{+n} \left(\tilde{T}_{nm} - \frac{a}{n-1} \Delta g_{nm} \right) \cdot Y_{nm}(\varphi, \lambda) \\ &= \frac{e^2}{5} \left[T_{20} + \frac{12}{7} T_{30} \cdot \sin \varphi \right. \\ &\quad \left. + \frac{24}{7} \left(T_{31} \cdot \cos \lambda + T_{3,-1} \cdot \sin \lambda \right) \cdot \cos \varphi \right] \\ &\quad + \frac{e^2}{2} \cdot \sum_{n=2}^{N} \sum_{m=-n}^{+n} \left(\alpha_{n-2,m} \cdot T_{n-2,m} - \tfrac{n+1}{n-1} \cdot \beta_{nm} \cdot T_{nm} \right. \\ &\quad \left. - 3 \frac{n+3}{n-1} \cdot \gamma_{n+2,m} \cdot T_{n+2,m} \right) \cdot Y_{nm}(\varphi, \lambda) + 0\left(e^4\right)\end{aligned} \tag{41}$$

truncating the infinite series at the upper limit $n = N$. This form of the ellipsoidal correction can easily be evaluated on the surface of the reference ellipsoid by means of a global geopotential model containing spherical harmonic coefficients up to degree $n = N$, e.g. EGM96 (Lemoine et al. 1998), where $N = 360$. Since the coefficients of Eq. (41) behave like $0\left(e^2 \cdot n^0\right)$ for large n, the resulting function is practically as smooth as the potential function itself. For this reason, the series shows a fast numerical convergence, justifying a truncation at $N = 360$ or less.

The solution of the simple Moldenskii problem on an ellipsoidal surface, as derived above, is presented in the form

$$T(r_E(\varphi), \varphi, \lambda) = \frac{a}{4\pi} \iint_\sigma \Delta g(\varphi', \lambda') \cdot S(\psi) \mathrm{d}\sigma + \delta T_E(\varphi, \lambda) \tag{42}$$

consisting of a 'spherical' part and an 'ellipsoidal correction' acting on the solution. Different equivalent solution formulae, also based on 'ellipsoidal corrections' with respect to Stokes integral, have been derived and proposed in the geodetic literature. Using Green's identities, Fei (2000) and Fei and Sideris (2000) derived a form similar to Eq. (42), but representing $\delta T_E(\varphi, \lambda)$ by a spherical integral, i.e. (in the present notation)

$$\begin{aligned} \delta T_E(\varphi, \lambda) = \frac{e^2}{5} \Big[& T_{20} + \frac{12}{7} T_{30} \cdot \sin\varphi \\ & + \frac{24}{7} \left(T_{31} \cdot \cos\lambda + T_{3,-1} \cdot \sin\lambda \right) \cdot \cos\varphi \Big] \\ & + \frac{e^2}{4\pi} \iint_\sigma T(r_E(\varphi'), \varphi', \lambda') \cdot f_o(\psi, \varphi, \varphi') \, \mathrm{d}\sigma \end{aligned} \tag{43}$$

involving the anisotropic kernel function $f_o(\psi, \varphi, \varphi')$. It can be proved by manipulating Eq. (41) that for $N \to \infty$ the alternative expressions of Eqs. (41) and (43) will provide the same results, if terms of order $0(e^4)$ are omitted.

A second alternative is generated by modifying the gravity anomaly data instead of correcting the disturbing potential, resulting in the following form of the solution:

$$\begin{aligned} & T(r_E(\varphi), \varphi, \lambda) \\ & = \frac{a}{4\pi} \iint_\sigma [\Delta g(\varphi', \lambda') + \delta g_E(\varphi', \lambda')] S(\psi) \, \mathrm{d}\sigma \end{aligned} \tag{44}$$

Here the form of the solution as a spherical integral of Stokes type has been retained, and the impact of the ellipsoidal boundary has been integrated in the modified boundary data. As Moritz (1980) has shown, this result can be interpreted as a continuation of the boundary data, originally given on the ellipsoid S_E, to the enclosing sphere S_a, followed by the solution of the spherical problem on S_a and downward continuation of the potential from S_a to S_E. The correction term δg_E can easily be represented in series form, using Eqs. (32) and (39).

$$\begin{aligned} \delta g_E(\varphi, \lambda) = -\frac{e^2}{2a} \sum_n \sum_m & \left[(n-1) \alpha_{n-2,m} \cdot T_{n-2,m} \right. \\ & - (n+1) \beta_{nm} \cdot T_{nm} \\ & \left. - 3(n+3) \cdot \gamma_{n+2,m} \cdot T_{n+2,m} \right] Y_{nm}(\varphi, \lambda) \end{aligned} \tag{45}$$

As a third alternative, the spherical Stokes kernel can be modified to allow for the consideration of the ellipticity of the boundary surface

$$\begin{aligned} & T(r_E(\varphi), \varphi, \lambda) \\ & = \frac{a}{4\pi} \iint_\sigma \Delta g(\varphi', \lambda') \cdot [S(\psi) + \delta S_E(\psi, \varphi, \varphi')] \, \mathrm{d}\sigma \end{aligned} \tag{46}$$

The ellipsoidal correction kernel $\delta S_E(\psi, \varphi, \varphi')$ is anisotropic, depending on the latitudes φ, φ' (besides the spherical distance ψ) or, equivalently, on the azimuth of the integration point. This approach to the solution of the ellipsoidal GBVP has been elaborated by Martinec (1998a, b) and Martinec and Grafarend (1997), based on the use of ellipsoidal harmonics. It can be proved (but is not worked out here in more detail) that the same result can be derived by manipulation of Eq. (41) for $N \to \infty$.

In addition to the three basic types of solution of Eqs. (42), (44) and (46), mixed forms are also possible, including the formula given by Molodenskii et al. (1962). Despite of the equivalence of these approaches from the analytical point of view, the numerical properties of the ellipsoidal terms to be evaluated are quite different. A preliminary inspection of the numerical properties of the procedures described above reveals that the approach based on Eqs. (39) and (41) may have some advantages over the other methods, since the ellipsoidal term δT_E is a comparatively smooth function, represented by a rapidly converging spherical harmonic series, which can easily be evaluated on the basis of a given geopotential model up to degree N.

4 'Near-closed' solutions of the linearized scalar-free GBVP involving first-order ellipsoidal terms

The procedure applied in Sect. 3 for the simple Molodenskii problem will now be applied to the linearized scalar-free GBVP including first-order ellipsoidal terms, defined in Sect. 2 as approximation level 2, which models terrestrial gravity anomaly data more closely to reality. It is assumed that the boundary surface has an ellipsoidal shape, approximating the telluroid surface Σ. This implies that the boundary data Δg have already been properly reduced for topographic effects. This level is based on the boundary equation [Eq. (10)], completed by the field equation [Eq. (15)] for the disturbing potential, i.e.

$$\begin{aligned} & \left. \left[-\frac{2}{a} \left(1 + e^2 \cos^2\varphi \right) \cdot T - \frac{\partial T}{\partial r} - e^2 \sin\varphi \cos\varphi \cdot \frac{\partial T}{a \partial \varphi} \right] \right|_{r = r_E(\varphi)} \\ & = \Delta g(\varphi, \lambda) \end{aligned} \tag{47}$$

$$\nabla^2 T = 0 \quad \text{outside } S_E \tag{48}$$

$$T = 0\left(\frac{1}{r}\right) \quad \text{for } r \to \infty$$

Again the representations of Eqs. (17) and (18) of the disturbing potential T, Eq. (16) for $r_E(\varphi)$, and Eq. (21) for the function $\Delta g(\varphi, \lambda)$ referring to the ellipsoidal surface will be used, resulting in

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{+n}\frac{\left(1-e^2\right)^{(n+2)/2}}{b}\left[\left((n-1)+e^2\left(\frac{n^2+n+4}{2}\right.\right.\right.$$
$$\left.\left.\left.\times\sin^2\varphi-2\right)\right)Y_{nm}(\varphi,\lambda)-e^2\sin\varphi\cdot\cos\varphi\frac{\partial Y_{nm}(\varphi,\lambda)}{\partial\varphi}\right]$$
$$\times\bar{T}_{nm}=\sum_{n=0}^{\infty}\sum_{m=-n}^{+n}\Delta g_{nm}Y_{nm}(\varphi,\lambda) \tag{49}$$

Similar to the product $\sin^2\varphi\cdot Y_{nm}(\varphi,\lambda)$ [see Eqs. (23) and (24)], the derivative of $Y_{nm}(\varphi,\lambda)$ with respect to φ, multiplied by $\sin\varphi\cdot\cos\varphi$, can be represented on the basis of the surface spherical harmonics (Moritz 1980)

$$\sin\varphi\cdot\cos\varphi\cdot\frac{\partial Y_{nm}(\varphi,\lambda)}{\partial\varphi}=-n\cdot\alpha_{nm}Y_{n+2,m}(\varphi,\lambda)$$
$$+\frac{3\beta_{nm}-1}{2}Y_{nm}(\varphi,\lambda)+(n+1)\gamma_{nm}Y_{n-2,m}(\varphi,\lambda) \tag{50}$$

where the coefficients α_{nm},β_{nm} and γ_{nm} are given in Eq. (24). Inserting Eqs. (23) and (50) into Eq. (49) and comparing the coefficients of the surface spherical harmonics $Y_{nm}(\varphi,\lambda)$ results in the system of algebraic equations

$$\frac{\left(1-e^2\right)^{(n+2)/2}}{b}\cdot\left[(n-1)\bar{T}_{nm}+\frac{e^2}{2}\left(\left(n^2-n+2\right)\right.\right.$$
$$\times\alpha_{n-2,m}\bar{T}_{n-2,m}+\left(\left(n^2+n+1\right)\beta_{nm}-3\right)\bar{T}_{nm}$$
$$\left.\left.+\left(n^2+3n+4\right)\gamma_{n+2,m}\bar{T}_{n+2,m}\right)+0\left(e^4\right)\right]$$
$$=\Delta g_{nm},\quad\forall n\geq 0 \tag{51}$$

(again, for $n\leq 1$ the coefficients $\bar{T}_{n-2,m}$ have to be set equal to zero). Combining Eqs. (51) and (29) yields the spectral relationship between the surface functions $T(r_E(\varphi),\varphi,\lambda)$ and $\Delta g(\varphi,\lambda)$, neglecting terms of order $0(e^4)$

$$\frac{n-1}{a}\left[\tilde{T}_{nm}+\frac{e^2}{2}\left(\frac{n+1}{n-1}\alpha_{n-2,m}\tilde{T}_{n-2,m}\right.\right.$$
$$+\left(\beta_{nm}\frac{n+2}{n-1}-\frac{3}{n-1}\right)\tilde{T}_{nm}$$
$$\left.\left.+\frac{n+7}{n-1}\gamma_{n+2,m}\tilde{T}_{n+2,m}\right)+\cdots\right]=\Delta g_{nm} \tag{52}$$

Just as in Sect. 3 [Eq. (30)], the ellipsoidal correction terms have an impact of order $0\left(e^2\cdot n^0\right)$, while in Eq. (51) the corresponding expressions increase with increasing n, showing the order $0\left(e^2\cdot n^1\right)$.

Prior information about the coefficients T_{nm} from a global geopotential model can be used for evaluating the terms of order $0(e^2)$ in Eq. (52), resulting in

$$\tilde{T}_{nm}=\frac{a}{n-1}\Delta g_{nm}-\frac{e^2}{2}\left(\frac{n+1}{n-1}\alpha_{n-2,m}T_{n-2,m}\right.$$
$$+\left(\beta_{nm}\frac{n+2}{n-1}-\frac{3}{n-1}\right)T_{nm}$$
$$\left.+\frac{n+7}{n-1}\gamma_{n+2,m}T_{n+2,m}\right)+0\left(e^4\right),\quad n\neq 1 \tag{53}$$

Again the solution of Eq. (52) for the case $n=1$ has to be considered separately. From Eq. (51), in combination with Eq. (34), a formula for calculating $\bar{T}_{1m}$ can be derived

$$T_{1m}=-\frac{5a}{3(1+m^2)}\left(\frac{\Delta g_{1m}}{e^2(1-e^2)}\right.$$
$$\left.-\frac{2(3+|m|)(2+|m|)}{35}\Delta g_{3m}\right),\quad -1\leq m\leq 1 \tag{54}$$

which proves that, in principle, the first-degree coefficients of the external potential can be determined from the gravity anomalies given on S_E. On the other hand, due to the small denominator, these coefficients cannot be calculated very precisely. Therefore it is preferable to fix the first-degree coefficients to $\bar{T}_{1m}\equiv 0$; the first-degree coefficients $\tilde{T}_{1m}$ of the surface potential $T(r_E(\varphi),\varphi,\lambda)$ are then given by Eq. (36). Similarly, it is often postulated to fix the zero-degree coefficient $\bar{T}_{00}$ to zero, resulting in the condition

$$\tilde{T}_{00}=\frac{e^2}{5}T_{20}+0(e^4) \tag{55}$$

Forcing the zero and first-degree coefficients T_{00}, T_{1m} to be zero produces a (small) bias in the solution, since the consistency conditions

$$\Delta g_{00}=\frac{4}{15}e^2\Delta g_{20} \tag{56a}$$

$$\Delta g_{1m}=e^2\cdot\frac{2(3+|m|)(2+|m|)}{35}\Delta g_{3m} \tag{56b}$$

in integral form

$$\iint_\sigma\Delta g(\varphi',\lambda')\cdot\left[1-\frac{4}{3}e^2\cdot P_2(\varphi')\right]\mathrm{d}\sigma=0 \tag{57a}$$

$$\iint_\sigma\Delta g(\varphi',\lambda')\cdot\left[P_1(\varphi')-\frac{4}{5}e^2\cdot P_3(\varphi')\right]\mathrm{d}\sigma=0 \tag{57b}$$

$$\iint_\sigma\Delta g(\varphi',\lambda')\cdot\left[P_{11}(\varphi')-\frac{4}{15}e^2\cdot P_{31}(\varphi')\right]\left\{\begin{matrix}\cos\lambda'\\ \sin\lambda'\end{matrix}\right\}\mathrm{d}\sigma=0 \tag{57c}$$

in general will not be satisfied by the real boundary data.

Finally, Eq. (26) can be summed, expressing the coefficients $\tilde{T}_{nm}$ by Eq. (53). Again a 'near-closed' solution on the ellipsoidal boundary surface is constructed in the sense of the decomposition Eq. (39)

$$T(r_E(\varphi),\varphi,\lambda)=T_S(\varphi,\lambda)+\delta T_E(\varphi,\lambda) \tag{58}$$

where the dominant 'spherical' part $T_S(\varphi,\lambda)$ is given by the Stokes integral [Eq. (40)]. Assuming that the conditions $T_{00}\equiv 0, T_{1m}\equiv 0$, or equivalently $T\sim(1/r^3)$ for $r\to\infty$, hold true, the 'ellipsoidal correction' takes the form

$$\begin{aligned}\delta T_E(r_E(\varphi),\varphi,\lambda) &= \frac{e^2}{5}\bigg[T_{20} + \frac{12}{7}T_{30}\cdot\sin\varphi \\ &+ \frac{24}{7}(T_{31}\cdot\cos\lambda + T_{3,-1}\cdot\sin\lambda)\cdot\cos\varphi\bigg] \\ &- \frac{e^2}{2}\cdot\sum_{n=2}^{N}\sum_{m=-n}^{+n}\bigg(\frac{n+1}{n-1}\alpha_{n-2,m}\cdot T_{n-2,m} \\ &+ \left(\beta_{nm}\frac{n+2}{n-1} - \frac{3}{n-1}\right)T_{nm} \\ &+ \frac{n+7}{n-1}\gamma_{n+2,m}\cdot T_{n+2,m}\bigg)\cdot Y_{nm}(\varphi,\lambda) + 0(e^4)\end{aligned} \tag{59}$$

Due to its fast numerical convergence, the series of Eq. (59) can be truncated at a finite maximum degree N and evaluated by means of a global geopotential model.

As discussed in Sect. 3, equivalent solutions to the linearized scalar-free GBVP including first-order 'ellipsoidal effects' can be constructed, corresponding to the basic types

$$\begin{aligned}T(r_E(\varphi),\varphi,\lambda) &= \frac{a}{4\pi}\iint_\sigma \Delta g(\varphi',\lambda')\cdot S(\psi)\,\mathrm{d}\sigma + \delta T_E(\varphi,\lambda) && \text{(60a)} \\ &= \frac{a}{4\pi}\iint_\sigma (\Delta g(\varphi',\lambda') + \delta g_E(\varphi',\lambda'))\cdot S(\psi)\,\mathrm{d}\sigma && \text{(60b)} \\ &= \frac{a}{4\pi}\iint_\sigma \Delta g(\varphi',\lambda')\cdot (S(\psi) + \delta S_E(\psi,\varphi,\varphi'))\,\mathrm{d}\sigma && \text{(60c)}\end{aligned}$$

or mixtures of these expressions (see e.g. Moritz 1980; Martinec and Grafarend 1997; Martinec 1998a, b; Fei 2000; Fei and Sideris 2000; Brovar et al. 2001). From the numerical point of view, the solution of Eq. (60a) – presented above in detail – might possess some advantages in comparison with the other alternatives, since the rapidly converging series of Eq. (59) can easily be evaluated by the aid of a geopotential model. Similar expressions can be obtained when the solution is scaled to a sphere with the mean radius R of the Earth, e.g. $R = (2a+b)/3 \approx a(1-e^2/6)$, instead of using the semi-major axis a.

Applying Bruns' formula, the disturbing potential on the surface of the ellipsoid can be transformed into the geoidal height N

$$N(\varphi,\lambda) = \frac{1}{\gamma(\varphi)}\cdot T(r_E(\varphi),\varphi,\lambda) \tag{61}$$

where $\gamma(\varphi)$ is the value of normal gravity on the ellipsoid at the geocentric latitude φ, consistent with the chosen Somigliana–Pizzetti normal field, e.g. GRS80. It should be pointed out that an additional 'ellipsoidal correction' term has to be considered if the latitude-dependent normal gravity value in Eq. (61) is replaced by a global constant value $\bar{\gamma}$ (see e.g. Ardalan and Grafarend 2001).

Using the $N = 360$ expansion of EGM96 (Lemoine et al. 1998) as prior information for the potential coefficients T_{nm}, the 'ellipsoidal correction' $\delta N(\varphi,\lambda) = \delta T_E(\varphi,\lambda)/\gamma(\varphi)$ based on Eq. (59) has been evaluated on a global grid (Fig. 3). This corroborates that the ellipsoidal correction to the (quasi-)geoidal height ranges between -0.5 and $+0.3$ m; compare also Huang et al. (2003). The power spectrum of δN (Fig. 4) features a strong decay for very small degrees n; in fact this decay is even stronger than the decay in the spectrum of the disturbing potential T itself. This property proves that $\delta T_E(\varphi,\lambda)$ and $\delta N(\varphi,\lambda)$ are rather smooth functions. This behaviour is also visible in Table 2, where the fully normalized harmonic coefficients of low degrees $n \leq 2$ have been printed, and from Figs. 5 and 6,

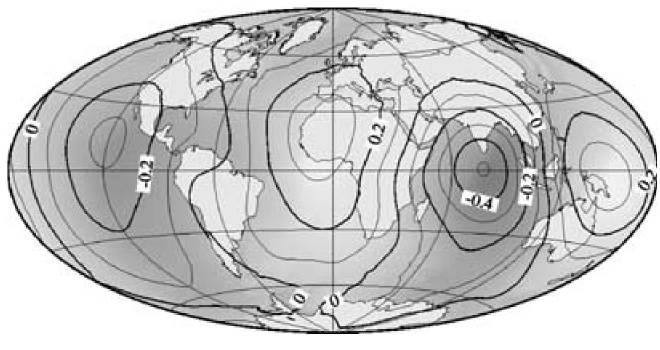

Fig. 3. Ellipsoidal correction $\delta N_E = \delta T_E(r_E(\varphi,\lambda),\varphi,\lambda)/\gamma(\varphi)$ in m, $0 \leq m \leq n \leq 360$ (Hammer equal-area projection)

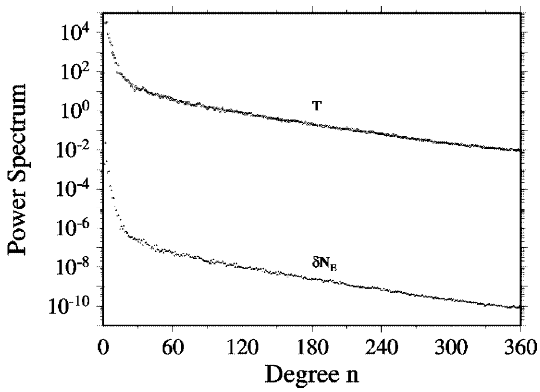

Fig. 4. Power spectrum of δN_E (in m^2) and T (in $m^4\ s^{-4}$)

Table 2. Fully normalized spherical harmonic coefficients of the ellipsoidal correction, Eq. (59), for small degrees and orders, $0 \leq m \leq n \leq 3$ (in mm)

	m	$\delta\bar{N}^c_{nm}$	$\delta\bar{N}^s_{nm}$
0	0	0.03	
1	0	21.41	
1	1	37.06	4.54
2	0	12.33	
2	1	24.03	21.25
2	2	115.35	−93.60
3	0	2.68	
3	1	16.04	4.19
3	2	−1.12	−1.87
3	3	25.28	39.98

showing the series terms up to $n = 1$ and $n = 2$, respectively. The sectorial terms of degree 2 and order 2 are dominant, possessing an 'energy content' of about 50% of the whole effect; in contrast, the sum of the terms between degrees 3 and 360 adds only another 50% (Fig. 7).

5 Conclusions

Several approximation levels concerning the representation of the boundary condition in the case of the scalar-free GBVP have been discriminated, starting from the general non-linear form and ending at the level of the spherical and constant-radius approximation, which provides a closed analytical solution in the form of Stokes' integral formula. From the discussion in Sect. 2 it follows that analytical solutions in closed form at higher, more realistic levels of approximation do not exist in a strict sense. First-order solutions with respect to the flattening f (or equivalently the square of the first numerical eccentricity e^2) related to an ellipsoidal boundary surface have been derived and discussed for two levels of approximation: (1) for the 'simple' Molodenskii problem, based on linear and spherical approximation, and (2) for the linearized scalar-free GBVP involving first-order ellipsoidal terms. The presented formulae make intensive use of the representation of the boundary condition in terms of spherical harmonics.

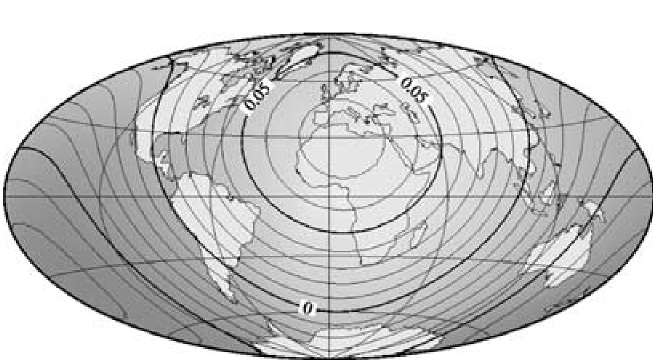

Fig. 5. Ellipsoidal correction (in m), terms of low degree, $0 \leq m \leq n \leq 1$ (Hammer equal-area projection)

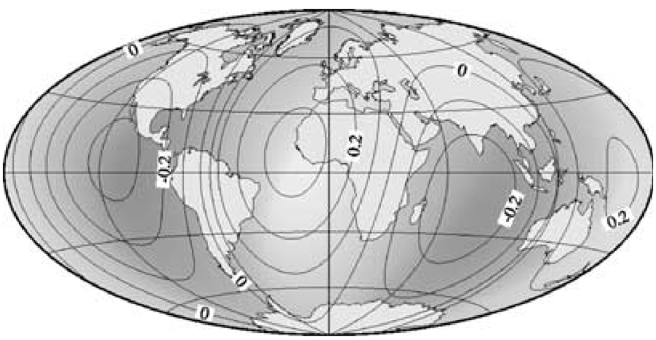

Fig. 6. Ellipsoidal correction (in m), terms of low degree, $0 \leq m \leq n \leq 2$ (Hammer equal-area projection)

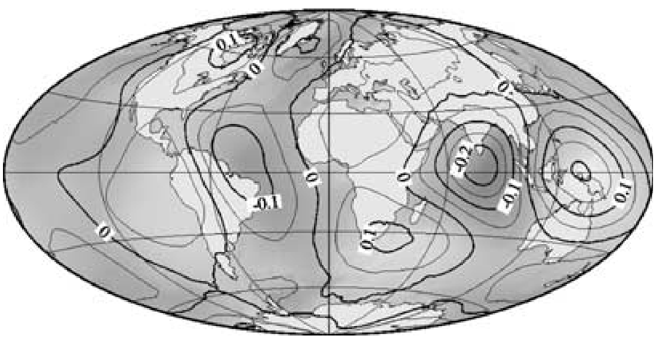

Fig. 7. Ellipsoidal correction (in m), terms of higher degree, $3 \leq m \leq n \leq 360$ (Hammer equal-area projection)

The discussion of the 'simple' Molodenskii problem shows the close relationship between the solution of the GBVP and the problem of upward/downward continuation between the ellipsoidal surface S_E and the (enclosing) sphere S_a. Although the ellipsoidal terms due to the non-sphericity of the boundary produce a 'roughening' effect on the spherical harmonic coefficients of the disturbing potential in space, this effect is no longer existent in the spectral relationship between the boundary data Δg and the potential considered at the ellipsoidal surface [Eqs. (30) and (52)].

Furthermore, the first-degree terms in the spherical harmonic representation of the disturbing potential T are no longer indeterminate for an ellipsoidal boundary surface, in contrast to the spherical case; on the other hand, the coefficients T_{10}, T_{11} and $T_{1,-1}$ can be determined with a poor precision only, due to small terms in the denominator. For this reason it is preferable to fix the first-degree coefficients to zero, tolerating a slight bias in the solution since the consistency conditions [Eqs. (38a) and (38b) or (57b) and (57c)] will not be satisfied by real data.

The solution of the ellipsoidal GBVP in first-order approximation concerning e^2 has been decomposed in two parts, namely a dominating 'spherical' term consisting of Stokes' integral and a small 'ellipsoidal correction' [Eqs. (42) and (58)]. The ellipsoidal correction is described in the form of a spherical harmonic series involving the coefficients of the disturbing potential; for this reason the ellipsoidal correction to be applied to Stokes' integral can easily be evaluated by utilizing a global geopotential model. The ellipsoidal correction to be added to the 'spherically' calculated (quasi-)geoidal height ranges between -0.5 and $+0.3$ m globally; it shows a dominant low-frequency behaviour, governed by the sectorial terms of degree 2. A first comparison with other, but principally equivalent, forms of ellipsoidal corrections indicates that the result derived in the present paper might possess some numerical advantages.

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