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A solution to the downward continuation effect on the geoid determined by Stokes' formula

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Abstract. The analytical continuation of the surface gravity anomaly to sea level is a necessary correction in the application of Stokes' formula for geoid estimation. This process is frequently performed by the inversion of Poisson's integral formula for a sphere. Unfortunately, this integral equation corresponds to an improperly posed problem, and the solution is both numerically unstable, unless it is well smoothed, and tedious to compute. A solution that avoids the intermediate step of downward continuation of the gravity anomaly is presented. Instead the effect on the geoid as provided by Stokes' formula is studied directly. The practical solution is partly presented in terms of a truncated Taylor series and partly as a truncated series of spherical harmonics. Some simple numerical estimates show that the solution mostly meets the requests of a 1-cm geoid model, but the truncation error of the far zone must be studied more precisely for high altitudes of the computation point. In addition, it should be emphasized that the derived solution is more computer efficient than the detour by Poisson's integral.

Keywords: Analytical continuation – Downward continuation – Poisson's integral – Stokes' formula

1 Introduction

The determination of the geoidal height by Stokes' formula requires that the effect of the Earth's topography is reduced or removed and that the surface gravity anomaly is reduced to sea level. The second reduction yields the effect of downward continuation (DWC) on gravity anomaly and the geoid. Mathematically the DWC effect on gravity anomaly can be expressed by the Taylor series

$$\delta\Delta g_{\mathrm{DWC}} = \Delta g^* - \Delta g = -H\frac{\partial \Delta g}{\partial H} + \frac{H^2}{2}\frac{\partial^2 \Delta g}{\partial H^2} - \cdots \tag{1}$$

where Δg^* and Δg are the gravity anomalies analytically continued down to sea level and at the Earth's surface, respectively. H is the orthometric height of the topography.

As Eq. (1) is needed globally to determine the DWC effect on the geoid, it would be expedient to evaluate the vertical derivatives in terms of spherical harmonics. However, the main problem with such an approach is that these derivatives have significant signals at much higher degrees and orders than the available sets of potential coefficients. In addition, the convergence of the Taylor series may be questioned.

Instead of applying Eq. (1) it appears more fruitful to determine Δg^* by solving Poisson's integral equation (see e.g. Heiskanen and Moritz 1967, Chap. 8; Martinec and Vanicek 1994; Martinec 1998; Hunegnaw 2001). However, as this formula is a Fredholm integral equation of the first kind, it corresponds to an improperly posed problem. In practice, the precise solution needs some kind of smoothing in order to avoid numerical instabilities. For example, Martinec (1998, Chap. 8) solved Poisson's integral equation with satisfactory results in the Canadian Rocky Mountains using surface blocks of size down to $5' \times 5'$ (i.e. approximately 10×10 km^2), but for smaller blocks ($30'' \times 60''$) the solution became numerically unstable. Hunegnaw (2001) came to a similar conclusion in an application to Ethiopian gravity data.

However, the final goal of all these efforts is to determine the DWC effect on the geoid. Sjöberg (2001) emphasized that the Stokes integration implies a process of smoothing of the gravity anomalies. Hence, by avoiding the computation of the downward-continued gravity anomaly, and by targeting directly the DWC effect on the geoidal height, the task should more easily be solved. This will also be the spirit of the approach taken in this paper. Another goal is to formulate a more practical solution than that of Sjöberg (2001), where spectral smoothing stabilized the DWC effect on the geoidal height in an optimum way.

2 Formulation of the solution

The DWC effect on the geoid can be written

$$\delta N_{\mathrm{DWC}}(P) = k \iint_{\sigma} S(\psi)(\Delta g_Q^* - \Delta g_Q)\,\mathrm{d}\sigma_Q \tag{2}$$

where $k = R/(4\pi\gamma_0)$, R being the geocentric radius of mean sea level and γ_0 being normal gravity on the reference ellipsoid at the foot-print of the ellipsoidal normal passing through the computation point P. Furthermore, σ is the unit sphere and $S(\psi)$ is Stokes' function with the argument ψ being the geocentric angle between P and the running point Q on the sphere. Equation (2) can also be abbreviated to

$$\delta N_{\mathrm{DWC}}(P) = \tilde{\zeta}_P^* - (N_P) \tag{3a}$$

where

$$\tilde{\zeta}_P^* = k \iint_{\sigma} S(\psi)\Delta g_Q^*\,\mathrm{d}\sigma_Q \tag{3b}$$

is the downward-continued re-scaled height anomaly $\tilde{\zeta}_P$ (see below), and

$$(N_P) = k \iint_{\sigma} S(\psi)\Delta g_Q\,\mathrm{d}\sigma_Q \tag{3c}$$

is the approximate geoidal undulation (or height anomaly), directly determined from surface gravity anomalies without any topographic corrections.

As the correct and re-scaled height anomalies, ζ and $\tilde{\zeta}$ respectively, are both defined by Bruns' formula

$$\zeta_P = \frac{T_P}{\gamma} \quad \text{and} \quad \tilde{\zeta}_P = \frac{T_P}{\gamma_0} \tag{4}$$

where T_P is the disturbing potential and γ and γ_0 are normal gravity values at normal height and at the reference ellipsoid, respectively, they are related by the simple equation

$$\tilde{\zeta}_P = \frac{\gamma}{\gamma_0}\zeta_P \tag{5}$$

which we will apply below.

We now introduce the geocentric radius $r_P = R + H_P$ at the point P on the Earth's surface, and we subtract and add $R\tilde{\zeta}_P/r_P$ on the right-hand side of Eq. (3a). This yields

$$\delta N_{\mathrm{DWC}}(P) = \delta N_{\mathrm{DWC}}^{(1)}(P) + \delta N_{\mathrm{DWC}}^{(2)}(P) \tag{6a}$$

where

$$\delta N_{\mathrm{DWC}}^{(1)}(P) = \zeta_P^* - \frac{R}{r_P}\zeta_P + \Delta\zeta_P \tag{6b}$$

where, from Eq. (5)

$$\Delta\zeta_P = \tilde{\zeta}_P^* - \zeta_P^* - \frac{R}{r_P}(\tilde{\zeta}_P - \zeta_P) = \left(\frac{\gamma}{\gamma_0} - 1\right)\left(\zeta_P^* - \frac{R}{r_P}\zeta_P\right) \tag{6c}$$

and

$$\delta N_{\mathrm{DWC}}^{(2)}(P) = \frac{R}{r_P}\tilde{\zeta}_P - (N_P) \tag{6d}$$

As it can be seen from Eqs. (6b) and (6d), both components include all the wavelengths of the gravity field. However, as will be shown later, $\delta N_{\mathrm{DWC}}^{(1)}$ contributes mostly to the short-wavelength spectrum of the signal, while the significant parts of $\delta N_{\mathrm{DWC}}^{(2)}$ are more in the long wavelengths of the spectrum. To distinguish between the two components, we will therefore call them 'the short-wavelength effect' and the 'long-wavelength effect', respectively. Below we will develop them into practical formulae.

2.1 The short-wavelength effect on the geoid

By expanding the downward-continued height anomaly ζ^* into a Taylor series at the point P, we obtain

$$\zeta_P^* = \sum_{k=0}^{\infty} \frac{(-H_P)^k}{k!}\left(\frac{\partial^k \zeta}{\partial H^k}\right)_P \approx \zeta_P - H_P\left(\frac{\partial \zeta}{\partial H}\right)_P + \frac{H_P^2}{2}\left(\frac{\partial^2 \zeta}{\partial H^2}\right)_P \tag{7}$$

where we for the further analysis will omit terms of third and higher derivatives of the height anomaly.

Differentiating Bruns' formula, Eq. (4), with respect to H and applying 'the boundary condition' of physical geodesy [Heiskanen and Moritz 1967, Eq. (8–20)]

$$\left(\frac{\partial T}{\partial H}\right)_P = -\Delta g_P + \frac{\partial \gamma}{\partial H}\frac{T_P}{\gamma} \tag{8}$$

we obtain the derivatives

$$\left(\frac{\partial \zeta}{\partial H}\right)_P = -\frac{\Delta g_P}{\gamma} \tag{9}$$

and

$$\left(\frac{\partial^2 \zeta}{\partial H^2}\right)_P = -\frac{1}{\gamma}\left(\frac{\partial \Delta g}{\partial H}\right)_P - \frac{2\Delta g_P}{\gamma r_P} \tag{10}$$

where Eq. (10) was derived under the spherical approximation

$$(\partial\gamma/\partial r)_P \approx -2\gamma/r_P \tag{11}$$

Inserting Eqs. (7), (9) and (10) into Eq. (6b) we finally arrive at the following solution to the short-wavelength effect on the geoid:

$$\delta N^{(1)}_{\mathrm{DWC}}(P) \approx \frac{H_P \Delta g_P}{\gamma} + \frac{H_P}{r_P}\zeta_P - \frac{H_P^2}{2\gamma}\left(\frac{\partial \Delta g}{\partial H}\right)_P - \frac{H_P^2 \Delta g_P}{\gamma r_P} + \Delta\zeta_P \tag{12}$$

In practice, the height anomaly needs to be known only approximately, and the vertical gradient of gravity anomaly (needed only for high elevations) can be estimated from Heiskanen and Moritz [1967, Eq. (2–217)]

$$\left(\frac{\partial \Delta g}{\partial r}\right)_P = \frac{1}{16\pi r_P}\iint_\sigma \frac{\Delta g_Q - \Delta g_P}{\sin^3(\psi/2)}\,\mathrm{d}\sigma_Q - \frac{2\Delta g_P}{r_P} \tag{13}$$

where the last term is very small. Finally, considering Eq. (6c) and the approximations

$$\gamma \approx \gamma_0 - \frac{2\gamma_0 H_P}{r_P}$$

and, from Eqs. (7) and (9)

$$\zeta^*_P \approx \zeta_P + \frac{H_P \Delta g_P}{\gamma}$$

we arrive at the following expression for the small term $\Delta\zeta_P$:

$$\Delta\zeta_P \approx -2\frac{H_P}{r_P}\left(\frac{H_P}{r_P}\zeta_P + \frac{\Delta g_P}{\gamma_0}H_P\right) \tag{14}$$

2.2 The long-wavelength effect on the geoid

The long-wavelength DWC effect on the geoid is given by Eq. (6d) including $\tilde{\zeta}_P$.

Formally $\tilde{\zeta}_P$ is given by the extended Stokes integral [Bjerhammar 1963; Heiskanen and Moritz 1967, Eq. (8–89)]

$$\tilde{\zeta}_P = k\iint_\sigma S(r_P, \psi)\Delta g^*_Q\,\mathrm{d}\sigma_Q \tag{15a}$$

where $S(r_P, \psi)$ is the so-called extended Stokes function

$$S(r_P, \psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1}\left(\frac{R}{r_P}\right)^{n+1} P_n(\cos\psi) \tag{15b}$$

including the Legendre polynomial $P_n(\cos\psi)$, related to the fully normalized spherical harmonic Y_{nm} [see also Eqs. (18a) and (18b) below] through the addition theorem

$$P_n(\cos\psi) = \frac{1}{2n+1}\sum_{m=-n}^{n} Y_{nm}(P)Y_{nm}(Q) \tag{16}$$

For $r_P = R$, the kernel of Eq. (15b) becomes the standard Stokes function $S(\psi)$.

Let us first develop the downward-continued and the surface gravity anomalies into spherical harmonics

$$\Delta g^*_P = \sum_{n=2}^{\infty} \Delta g_n(P) \tag{17a}$$

and (Heiskanen and Moritz 1967, Sect. 2–23)

$$\Delta g_P = \sum_{n=2}^{\infty}\left(\frac{R}{r_P}\right)^{n+2} \Delta g_n(P) \tag{17b}$$

where

$$\Delta g_n(P) = \frac{n-1}{R}\sum_{m=-n}^{n} A_{nm} Y_{nm}(P) \tag{17c}$$

Here A_{nm} is the potential coefficient related to the fully normalized spherical harmonic (cf. Heiskanen and Moritz 1967, p. 31)

$$Y_{nm} = \bar{P}_{n|m|}(\cos\theta)\begin{cases}\cos m\lambda, & \text{if } m \geq 0\\ \sin|m|\lambda, & \text{if } m < 0\end{cases} \tag{18a}$$

where θ is the co-latitude, λ is the longitude and $\bar{P}_{nm}$ is the associated Legendre function, normalized in such a way that

$$\frac{1}{4\pi}\iint_\sigma Y_{nm}Y_{pq}\,\mathrm{d}\sigma = \begin{cases}1, & \text{if } n = p,\ m = q\\ 0, & \text{otherwise}\end{cases} \tag{18b}$$

Let us also introduce the expansion of the gravity anomaly at the arbitrary point Q on the sphere of radius r_P as

$$\Delta g(r_P, Q) = \sum_{n=2}^{\infty}\left(\frac{R}{r_P}\right)^{n+2}\Delta g_n(Q) \tag{19}$$

Inserting the last formula as well as the spectral form of Stokes' function, i.e. Eq. (15b) for r_P set to R, into the following integral, we obtain

$$k\iint_\sigma S(\psi)\Delta g(r_P, Q)\,\mathrm{d}\sigma_Q = \frac{R}{\gamma_0}\sum_{n=2}^{\infty}\left(\frac{R}{r_P}\right)^{n+2}\frac{\Delta g_n(P)}{n-1} = \frac{R}{r_P}\tilde{\zeta}_P \tag{20}$$

Here the last equality can be shown as follows: insert Eqs. (15b), (16) and (17a) into Eq. (15a) and note the orthogonality property of Eq. (18b). Then we readily arrive at the spectral form of the height anomaly

$$\tilde{\zeta}_P = \frac{R}{\gamma_0}\sum_{n=2}^{\infty}\left(\frac{R}{r_P}\right)^{n+1}\frac{\Delta g_n(P)}{n-1} \tag{21}$$

which proves the last equality of Eq. (20). Hence, by taking advantage of the relations of Eqs. (19) and (20) we can write the second term of the DWC effect on the geoid, Eq. (6d), in the following form:

$$\Delta N^{(2)}_{\mathrm{DWC}}(P) = k\iint_\sigma S(\psi)\{\Delta g(r_P, Q) - \Delta g_Q\}\,\mathrm{d}\sigma_Q \tag{22}$$

This integral has the nice property, in contrast to the original integral, Eq. (2), that its contribution from the integrand vanishes at the singularity point of Stokes' function (i.e. for $\psi = 0$). [See also Eq. (24) below.] This implies that Eq. (22) is less sensitive to short wavelengths of the gravity anomaly than the original Eq. (2). In addition, the equation does not include the downward-continued gravity anomaly at sea level, but the upward-or downward-continued anomaly to the level of the computation point P. Hence, except from a near-zone cap σ_0 around P, it should be possible to take advantage of a set of potential coefficients to estimate the integral. For that purpose we decompose the integral into

$$\delta N^{(2)}_{\mathrm{DWC}}(P) = \delta N^{(2,1)}_{\mathrm{DWC}}(P) + \delta N^{(2,2)}_{\mathrm{DWC}}(P) \tag{23a}$$

where

$$\delta N^{(2,1)}_{\mathrm{DWC}}(P) = k \iint\limits_{\sigma_0} S(\psi)\{\Delta g(r_P, Q) - \Delta g_Q\}\, \mathrm{d}\sigma_Q \tag{23b}$$

is the near-zone contribution and

$$\delta N^{(2,2)}_{\mathrm{DWC}}(P) = k \iint\limits_{\sigma-\sigma_0} S(\psi)\{\Delta g(r_P, Q) - \Delta g_Q\}\, \mathrm{d}\sigma_Q \tag{23c}$$

is the far-zone contribution. If the near zone is limited to a few kilometres, it can be approximated by a disk of radius $s_0 = R\psi_0$, Stokes' kernel be approximated by $2R/s$ and the surface element $R^2\,\mathrm{d}\sigma$ becomes $s\,\mathrm{d}s\,\mathrm{d}\alpha$, where s is the radius and α is the azimuth in polar coordinates. Thus we obtain

$$\delta N^{(2,1)}_{\mathrm{DWC}}(P) = \frac{1}{2\pi\gamma} \int_0^{2\pi} \int_0^{s_0} \{\Delta g(r_P, Q) - \Delta g_Q\}\, \mathrm{d}s\, \mathrm{d}\alpha \tag{24}$$

and by inserting the approximation

$$\Delta g(r_P, Q) - \Delta g_Q = \Delta H_{PQ} \left(\frac{\partial \Delta g}{\partial r}\right)_P \tag{25a}$$

where

$$\Delta H_{PQ} = H_P - H_Q \tag{25b}$$

into Eq. (24) we arrive at the practical formula

$$\delta N^{(2,1)}_{\mathrm{DWC}}(P) = \frac{\overline{\Delta Hs}}{\gamma} \left(\frac{\partial \Delta g}{\partial r}\right)_P \tag{26a}$$

where

$$\overline{\Delta Hs} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{s_0} \Delta H_{PQ}\, \mathrm{d}s\, \mathrm{d}\alpha = H_P s_0 - \frac{1}{2\pi} \int_0^{2\pi} \int_0^{s_0} H_Q\, \mathrm{d}s\, \mathrm{d}\alpha \tag{26b}$$

and $\left(\frac{\partial \Delta g}{\partial r}\right)_P$ is a representative value for the vertical gradient of the gravity anomaly in the cap. Although the latter component is typically in the high-frequency spectrum of gravity, it contributes relatively little to the DWC effect if s_0 is small. (See Chap. 4.)

The remaining far-zone contribution we develop into a series of spherical harmonics

$$\delta N^{(2,2)}_{\mathrm{DWC}}(P) = c \sum_{n=2}^{\infty} Q_n(\psi_0) \left\{ \left(\frac{R}{r_P}\right)^{n+2} \Delta g_n(P) - (\Delta g_s)_n(P) \right\} \tag{27}$$

where $c = R/(2\gamma_0)$, $Q_n(\psi_0)$ are the so-called Molodensky truncation coefficients (Heiskanen and Moritz 1967, p. 260)

$$Q_n(\psi_0) = \int_{\psi_0}^{\pi} S(\psi) P_n(\cos\psi) \sin\psi\, \mathrm{d}\psi \tag{28}$$

which can be developed into a recurrence relation (Paul 1973), $\Delta g_n(P)$ are the gravity anomaly harmonics of Eq. (17c), while $(\Delta g_s)_n(P)$ is the corresponding Laplace harmonic of the surface gravity anomaly, given by the formula

$$(\Delta g_s)_n(P) = \frac{n-1}{R} \sum_{m=-n}^{n} B_{nm} Y_{nm}(P) \tag{29a}$$

where B_{nm} is defined by

$$B_{nm} = \frac{R}{4\pi(n-1)} \iint\limits_{\sigma} \Delta g_Q Y_{nm}(Q)\, \mathrm{d}\sigma_Q \tag{29b}$$

In practice, the infinite sum of Eq. (27) must be approximated to some maximum upper limit, say $n_{\max} = 360$. The truncation error of such an approximation will be studied in the next section.

3 Practical considerations

Summarizing the derivations of Sect. 2, we have arrived at the following practical formulae for the effect of DWC of gravity anomaly on Stokes' formula:

$$\delta N_{\mathrm{DWC}}(P) = \delta N^{(1)}_{\mathrm{DWC}}(P) + \delta N^{(2)}_{\mathrm{DWC}}(P) \tag{30a}$$

where

$$\delta N^{(1)}_{\mathrm{DWC}}(P) = \frac{H_P \Delta g_P}{\gamma} + \frac{H_P}{r_P}\zeta_P - \frac{H_P^2}{2\gamma_0}\left(\frac{\partial \Delta g}{\partial r}\right)_P - \frac{H_P^2 \Delta g_P}{\gamma_0 r_P} + \Delta\zeta_P \tag{30b}$$

$$\Delta\zeta_P = -2\frac{H_P}{r_P}\left(\frac{H_P}{r_P}\zeta_P + H_P\frac{\Delta g_P}{\gamma}\right) \tag{30c}$$

$$\delta N^{(2)}_{\mathrm{DWC}}(P) = \frac{\overline{\Delta Hs}}{\gamma}\left(\frac{\overline{\partial \Delta g}}{\partial r}\right)_P + c\sum_{n=2}^{n_{\max}} Q_n(\psi_0) \times \left\{\left(\frac{R}{r_P}\right)^{n+2}\Delta g_n(P) - (\Delta g_s)_n(P)\right\} \tag{30d}$$

and $c = R/(2\gamma_0)$. The explicit expressions of the Laplace harmonics of the gravity anomalies in Eq. (30d) are given by Eqs. (17c), (29a) and (29b). This means that also the surface gravity anomaly must be expanded into a series of spherical harmonics complete through degree $n_{\max}$.

Let us now estimate the magnitude of several of the individual terms of Eqs. (30b)–(30d). Assuming that

$$|\zeta| \le 100\,\mathrm{m}, |\Delta g_P| \le 200\,\mathrm{mGal}, \left|\frac{\partial \Delta g}{\partial r}\right| \le 0.03\,\mathrm{mGal/m}$$

we obtain for H_P set to 5(8.8) km

$$\left|\frac{H\Delta g}{\gamma}\right| \le 1.02(1.79)\,\mathrm{m}, \quad \left|\frac{H\zeta}{r}\right| \le 0.08(0.14)\,\mathrm{m},$$

$$\left|\frac{H^2}{2\gamma}\left(\frac{\partial \Delta g}{\partial r}\right)\right| \le 0.38(1.18)\,\mathrm{m}$$

$$\left|\frac{H^2\Delta g}{\gamma r}\right| \le 1(3)\,\mathrm{mm}, |\Delta\zeta_P| \le 5\,\mathrm{mm}$$

Moreover, for $s_0 = 1\,\mathrm{km}$ and $\left|\overline{\Delta Hs}\right| \le 5s_0\,\mathrm{km}^2$ for s_0 given in kilometres, we obtain that

$$\left|\frac{\overline{\Delta Hs}}{\gamma}\frac{\partial \Delta g}{\partial r}\right| \le 0.17\,\mathrm{m}$$

These calculations show that the DWC effect may be of the order of a few metres in the highest mountains. In addition we conclude that the last two terms of Eq. (30b) can be neglected for geoid estimates of 1-cm precision. Also, if the vertical gradient of the gravity anomaly is of the order of 0.03 mGal/m, it follows from above that the terms with the vertical gradient must be considered whenever the height of the computation point is about 1700 m or higher and/or the magnitude of the mean value $\overline{\Delta Hs}$ exceeds 700 m^2.

There are two types of truncations involved in Eqs. (30): the truncation of the Taylor series [Eq. (7)] resulting in Eq. (30b), and the truncation of the harmonic series [Eq. (30d)] at the maximum degree $n_{\max}$. The first two omitted terms of Eq. (30b) are

$$S_3 = -\frac{H_P^3}{6}\left(\frac{\partial^3\zeta}{\partial H^3}\right)_P \quad \text{and} \quad S_4 = \frac{H_P^4}{24}\left(\frac{\partial^4\zeta}{\partial H^4}\right)_P \tag{31}$$

Starting from Eq. (21), we can easily derive

$$S_3 = \frac{H_P^3}{6\gamma r_P^2}\sum_{n=2}^{\infty}\left(\frac{R}{r_P}\right)^{n+2}\frac{(n+1)(n+2)(n+3)}{n-1}\Delta g_n(P) \tag{32a}$$

and

$$S_4 = \frac{H_P^4}{24\gamma r_P^3}\sum_{n=2}^{\infty}\left(\frac{R}{r_P}\right)^{n+2} \times \frac{(n+1)(n+2)(n+3)(n+4)}{n-1}\Delta g_n(P) \tag{32b}$$

In the global RMS sense these sums become

$$\bar{S}_3 = \frac{H_P^3}{6\gamma r_P^2}\left\{\sum_{n=2}^{\infty}\left[\left(\frac{R}{r_P}\right)^{n+2} \times \frac{(n+1)(n+2)(n+3)}{n-1}\right]^2 c_n\right\}^{\frac{1}{2}} \tag{33a}$$

and

$$\bar{S}_4 = \frac{H_P^4}{24\gamma r_P^3}\left\{\sum_{n=2}^{\infty}\left[\left(\frac{R}{r_P}\right)^{n+2} \times \frac{(n+1)(n+2)(n+3)(n+4)}{n-1}\right]^2 c_n\right\}^{\frac{1}{2}} \tag{33b}$$

where c_n are the gravity anomaly degree variances. In Table 1 we estimate these terms for various heights with the degree variances taken from Tscherning and Rapp (1974). Table 1 shows that the first term is within 1.5 cm and the second one is less than a 1 cm even for the extreme elevation of Mt. Everest. This shows that the first type of truncation error can usually be omitted in practice.

The truncation error of the long-wavelength contribution [Eq. (30d)] is given by

$$\varepsilon_T = -c\sum_{n=n_{\max}+1}^{\infty} Q_n(\psi_0)\left\{\left(\frac{R}{r_P}\right)^{n+2}\Delta g_n(P) - (\Delta g_s)_n(P)\right\} \tag{34}$$

which can be estimated in the RMS sense by

$$m_T = \left\{\frac{1}{4\pi}\iint_\sigma \varepsilon_T^2\,\mathrm{d}\sigma\right\}^{\frac{1}{2}} \le c\left\{\sum_{n=n_{\max}+1}^{\infty} Q_n^2(\psi_0)c_n\left[\left(\frac{R}{r_P}\right)^{n+2} - 1\right]^2\right\}^{\frac{1}{2}} \tag{35}$$

where the right-hand side of the inequality corresponds to the worst (but not very likely) case that all the Earth of the far zone is without topography.

Table 1. Numerical values of the terms $\bar{S}_3$ and $\bar{S}_4$

H_P (km)	$\bar{S}_3$ (mm)	$\bar{S}_4$ (mm)
1	0	0
2	2	1
5	2	3
8.8	14	7

As illustrated in Table 2, for H_P set to the extreme elevation of Mt. Everest and the near-zone radius set to 1 km, the RMS error is within 11 cm for $n_{\max} = 360$ or higher. The error limit increases with elevation of computation point and decrease of near-zone disk radius. For s_0 and H_P set to 5 km and 4 km, respectively, the error is limited to 3.6 cm. Although the error limit provided by Eq. (35) might be too pessimistic, already these results are promising for most moderate elevations of topography.

In order to reduce the truncation error of the long-wavelength contribution $\Delta N^{(2)}_{\mathrm{DWC}}$ we may modify the Stokes-type integral of Eq. (22) into the form

$$\delta N^{(2)}_{\mathrm{DWC}}(P) = k \iint\limits_{\sigma} S^M(\psi)\{\}\, \mathrm{d}\sigma_Q + c\sum_{n=2}^{M} t_n \left[\left(\frac{R}{r_P}\right)^{n+2} \Delta g_n(P) - (\Delta g_s)_P\right] \quad (36a)$$

where the modified Stokes function reads (see e.g. Sjöberg 1991)

$$S^M(\psi) = S(\psi) - \sum_{n=0}^{M} \frac{2n+1}{2} t_n P_n(\cos\psi) \quad (36b)$$

and M is the same as $n_{\max}$. Also, the bracket { } is the same as in Eq. (22), and the parameters t_n, which are formally arbitrary, should in the present application be selected, e.g. according to Molodensky (Molodensky et al. 1962), to reduce the truncation error of the Stokes integral. If t_0 is set to zero, this modification of Eq. (22) will not change the estimate of the planar approximation of the near-zone integral $N^{(2,1)}_{\mathrm{DWC}}$, as all the Legendre polynomials of the modified Stokes kernel vanish. Then the long-wavelength contribution can be estimated by the formula

$$\delta N^{(2)}_{\mathrm{DWC}}(P) = \delta N^{(2,1)}_{\mathrm{DWC}}(P) + \delta N^{(2,2)}_{\mathrm{DWC}}(P) \quad (37a)$$

where $\delta N^{(2,1)}_{\mathrm{DWC}}$ was given by Eqs. (26a) and (26b) and

$$\delta N^{(2,2)}_{\mathrm{DWC}}(P) = c\sum_{n=1}^{M} \left[Q^M_n(\psi_0) + t_n\right] \times \left\{\left(\frac{R}{r_P}\right)^{n+2} \Delta g_n(P) - (\Delta g_s)_n(P)\right\} \quad (37b)$$

Table 2. The RMS truncation error m_T (cm) for various elevations H_P (km), near-zone radius s_0 (km) and degrees of truncation $n_{\max}$

$n_{\max}$	s_0:	1	5	5
	H_P:	8.85	4	2
180		15.9	5.9	3.2
360		11.1	3.6	2.0
720		6.3	1.5	0.9
1080		4.0	0.9	0.6
1440		2.6	0.9	0.6

Here Q^M_n are the Molodensky-type truncation parameters related to the modified Stokes kernel S^M (see e.g. Sjöberg 1991)

$$Q^M_n(\psi_0) = Q_n(\psi_0) - \sum_{k=0}^{M} \frac{2k+1}{2} t_k e_{nk}(\psi_0) \quad (38a)$$

and

$$e_{nk}(\psi_0) = \int_{-1}^{\cos\psi_0} P_n(t) P_k(t)\, \mathrm{d}t \quad (38b)$$

Moreover, if the modification parameters t_n are selected according to Molodensky's method of modification (see e.g. Sjöberg 1991), all Q^M_n of Eq. (37b) vanish, which simplifies the formula. Then we can expect that the truncation error of the modified Stokes formula of Eq. (36)

$$\varepsilon_T = -c\sum_{n=M+1}^{\infty} Q^M_n(\psi_0)\left\{\left(\frac{R}{r_P}\right)^{n+2} \Delta g_n(P) - (\Delta g_s)_n(P)\right\} \quad (39)$$

is generally smaller than the original truncation error of Eq. (34).

4 The DWC effect on the height anomaly

The DWC effect on the height anomaly is given by the integral

$$\delta\zeta_{\mathrm{DWC}}(P) = k \iint\limits_{\sigma} S(r_P,\psi)\{\Delta g^*_Q - \Delta g_Q\}\, \mathrm{d}\sigma_Q = \tilde{\zeta}_P - (N_P) = \delta N^{(2)}_{\mathrm{DWC}}(P) + \frac{H_P}{r_P}\tilde{\zeta}_P \quad (40)$$

where the last member stems from Eq. (6d). The difference between the effects on the geoid and the height anomaly can thus be estimated to

$$\delta N_{\mathrm{DWC}}(P) - \delta\zeta_{\mathrm{DWC}}(P) = \delta N^{(1)}_{\mathrm{DWC}}(P) - \frac{H_P}{r_P}\tilde{\zeta}_P \quad (41)$$

5 The DWC effect on the modified Stokes formula

Today Stokes' formula is usually modified (e.g. according to Sjöberg 1991) to take advantage of the fact that the long-wavelength part of the geoid is better determined by an Earth gravity model than by the original Stokes formula. The DWC effect on the geoid by the modified Stokes formula becomes

$$\delta N^L_{\mathrm{DWC}}(P) = k \iint\limits_{\sigma_0} S^L(\psi)\left\{\Delta g^*_Q - \Delta g_Q\right\} \mathrm{d}\sigma_Q \quad (42a)$$

where

$$S^L(\psi) = S(\psi) - \sum_{n=0}^{L} \frac{2n+1}{2} s_n P_n(\cos\psi) \quad (42b)$$

Here s_n are arbitrary parameters of modification with the upper degree L. By inserting Eq. (42b) into Eq. (42a) we obtain

$$\delta N^L_{\mathrm{DWC}}(P) = \delta N_{\mathrm{DWC}}(P) - k \iint_{\sigma-\sigma_0} S^L(\psi)\left\{\Delta g^*_Q - \Delta g_Q\right\} \mathrm{d}\sigma_Q - c\sum_{n=2}^{L} s_n\left\{\Delta g_n(P) - (\Delta g_s)_n(P)\right\} \tag{43}$$

where the second term on the right-hand side is caused by the limitation of the area of integration to a spherical cap around the computation point. It can also be written as a spectral series

$$k \iint_{\sigma-\sigma_0} S^L(\psi)\left\{\Delta g^*_Q - \Delta g_Q\right\}\mathrm{d}\sigma_Q = c\sum_{n=2}^{\infty} Q^L_n(\psi_0)\left\{\Delta g_n(P) - (\Delta g_s)_n(P)\right\} \tag{44}$$

which in practice must be truncated at the upper limit $n_{\max}$ of the sets of harmonics of anomalies. Thus we obtain also the practical formula

$$\delta N^L_{\mathrm{DWC}}(P) = \delta N_{\mathrm{DWC}}(P) - c\sum_{n=2}^{n_{\max}}\left\{Q^L_n + s^*_n\right\} \times \left\{\Delta g_n(P) - (\Delta g_s)_n(P)\right\} \tag{45a}$$

where

$$s^*_n = \begin{cases} s_n, & \text{if } 0 \le n \le \min(L, n_{\max}) \\ 0, & \text{otherwise} \end{cases} \tag{45b}$$

This concludes our derivations of a practical solution to the effect on the geoid of the downward continuation of gravity anomaly under Stokes' integral.

6 Concluding remarks

In the current struggle to achieve a 1-cm geoid model, we have derived what we believe is a practical tool to evaluate the effect on Stokes' formula of the analytical continuation of the gravity anomaly. In most cases the effect can be estimated from the orthometric height of the computation point, its surface gravity anomaly and the approximate height anomaly, as well as the set of spherical harmonics representing the long-wavelength effect. The truncation error of the harmonic series should be analysed numerically in more detail. For elevations roughly of the order of 1 km and higher, the vertical gradient of the surface gravity anomaly at the computation point must also be included as a significant contributor.

Now, the reader might wonder what happened to the original numerical instability of the analytical continuation of the surface gravity anomaly. We can explain this apparent controversy as follows. First, the intermediate step of computing the difference between the surface and the continued gravity anomaly is avoided by the direct estimation of the integrated effect on the geoid. This led us to a smoother difference between the true and downward-continued height anomaly. Second, the latter difference, $\delta N^{(1)}_{\mathrm{DWC}}$, is evaluated by the Taylor series of its vertical derivatives truncated after a few terms with a negligible truncation error. Third, a remaining contribution, $\delta N^{(2,2)}_{\mathrm{DWC}}$, is expanded as a long-wavelength series of spherical harmonics of the difference between the downward-continued and the surface gravity anomaly. The truncated higher-degree harmonics, representing the remaining instability of the process, should be numerically insignificant. If this is not the case, in particular for high elevations, the problem might be solved by some kind of smoothing of the gravity field. However, this is beyond the scope of the present article.

Last, but not least, the derived technique should be very computer efficient compared to current techniques of first solving Poisson's integral equation for Δg^* before applying Stokes' formula.

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