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Green's function solution to spherical gradiometric boundary-value problems

Z. Martinec

Department of Geophysics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 180 00 Prague 8, Czech Republic
e-mail: zdenek@hervam.troja.mff.cuni.cz; Tel.: +420-2-2191-2539; Fax: +420-2-2192-2555

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Abstract. Three independent gradiometric boundary-value problems (BVPs) with three types of gradiometric data, $\{\Gamma_{rr}\}$, $\{\Gamma_{r\vartheta}, \Gamma_{r\lambda}\}$ and $\{\Gamma_{\vartheta\vartheta} - \Gamma_{\lambda\lambda}, \Gamma_{\vartheta\lambda}\}$, prescribed on a sphere are solved to determine the gravitational potential on and outside the sphere. The existence and uniqueness conditions on the solutions are formulated showing that the zero- and the first-degree spherical harmonics are to be removed from $\{\Gamma_{r\vartheta}, \Gamma_{r\lambda}\}$ and $\{\Gamma_{\vartheta\vartheta} - \Gamma_{\lambda\lambda}, \Gamma_{\vartheta\lambda}\}$, respectively. The solutions to the gradiometric BVPs are presented in terms of Green's functions, which are expressed in both spectral and closed spatial forms. The logarithmic singularity of the Green's function at the point $\psi = 0$ is investigated for the component Γ_{rr}. The other two Green's functions are finite at this point. Comparisons to the paper by van Gelderen and Rummel [Journal of Geodesy (2001) 75: 1–11] show that the presented solution refines the former solution.

Keywords: Geodetic boundary-value problem – Gravitation tensor – Green's function – Addition theorem – Tensor spherical harmonics

1 Introduction

Satellite gradiometry is expected to improve our knowledge of the external global gravitational field of the Earth. This hope is based on the fact that the attenuation of the gravitational field with increasing distance from the Earth is partly compensated by the effect of the differentiation of the gravitational field. Prescribing various linear combinations of the second-order derivatives of the gravitational potential as boundary data on a sphere (e.g. a mean-orbit sphere), one of the main objectives is to convert them into gravitational potential, geoid height, gravity anomaly or any other desired gravity quantity. This aspect has recently been treated by van Gelderen and Rummel (2001, 2002) (vGR01 hereafter). Among solutions to various geodetic boundary-value problems (BVPs), they introduced Green's function solutions to the gradiometric BVPs.

From the theoretical point of view, the vGR01 gradiometric solutions can be refined. This motivates the work of this paper. We present a detailed and systematic derivation of the Green's function solutions to the gradiometric BVPs with boundary data prescribed on a sphere. The paper is organized as follows. We recall the definition and basic properties of the gravitation tensor. The tensor spherical-harmonics decomposition of the gravitation tensor is used to group gradiometric observables into three independent gradiometric data sets. For each data set, we formulate the gradiometric BVP and specify the conditions of the existence and uniqueness of solution. We then solve the three gradiometric BVPs in terms of Green's functions that are expressed in spectral and closed spatial forms. We use the closed-form solutions to discuss the singular behavior of the gradiometric Green's functions. Finally, we compare our solutions with the vGR01 gradiometric solutions.

2 Spherical harmonic representation of the gravitation tensor

The gravitation tensor $\mathbf{\Gamma}$ is defined as the double gradient of the gravitational potential V (see e.g. Rummel and van Gelderen, 1992),

$$\mathbf{\Gamma} := \text{grad grad } V \ . \tag{1}$$

By this definition, the tensor $\mathbf{\Gamma}$ is symmetric, $\mathbf{\Gamma}^{\mathrm{T}} = \mathbf{\Gamma}$, which reduces the nine components of $\mathbf{\Gamma}$ to six independent components. In addition, if the gravitational potential is harmonic, $\nabla^2 V = 0$, the trace of $\mathbf{\Gamma}$ vanishes, $\mathrm{tr}\mathbf{\Gamma} = 0$. As a result, only five components of the gravitation tensor are independent functions in the region of harmonicity of the potential V.

In this paper, we consider the gravitational potential V represented in terms of scalar spherical harmonics $Y_{jm}(\Omega)$ (see e.g. Varshalovich et al., 1989):

$$V(r,\Omega) = \sum_{j=0}^{\infty}\sum_{m=-j}^{j} V_{jm}(r)Y_{jm}(\Omega) \ , \tag{2}$$

where r and Ω are spherical coordinates and $\Omega := (\vartheta, \lambda)$. The double gradient of individual constituents may be expressed in terms of tensor spherical harmonics $\mathbf{Z}^{(\lambda)}_{jm}(\Omega)$ (Appendix A):

$$\begin{aligned} &\text{grad grad } [V_{jm}(r)Y_{jm}(\Omega)] \\ &= \frac{d^2V_{jm}}{dr^2}\mathbf{Z}^{(1)}_{jm}(\Omega) + 2\frac{d}{dr}\left(\frac{V_{jm}}{r}\right)\mathbf{Z}^{(2)}_{jm}(\Omega) + \frac{V_{jm}}{2r^2}\mathbf{Z}^{(3)}_{jm}(\Omega) \\ &\quad - \frac{1}{r}\left(\frac{1}{j(j+1)}\frac{dV_{jm}}{dr} - \frac{V_{jm}}{2r}\right)\mathbf{Z}^{(4)}_{jm}(\Omega). \end{aligned} \tag{3}$$

For a harmonic potential that vanishes at infinity, $V_{jm}(r)$ is proportional to r^{-j-1} and eq. (3) reduces to

$$\begin{aligned} \text{grad grad } [r^{-j-1}Y_{jm}(\Omega)] = r^{-j-3}\Big[&(j+1)(j+2)\mathbf{Z}^{(1)}_{jm}(\Omega) \\ &- 2(j+2)\mathbf{Z}^{(2)}_{jm}(\Omega) + \frac{1}{2}\mathbf{Z}^{(3)}_{jm}(\Omega) \\ &+ \frac{j+2}{2j}\mathbf{Z}^{(4)}_{jm}(\Omega)\Big] \ . \end{aligned} \tag{4}$$

Note that the trace of grad grad$[r^{-j-1}Y_{jm}(\Omega)]$ vanishes since the tensor spherical harmonics $\mathbf{Z}^{(2)}_{jm}(\Omega)$ and $\mathbf{Z}^{(3)}_{jm}(\Omega)$ are trace-free and

$$\text{tr } \mathbf{Z}^{(1)}_{jm}(\Omega) = Y_{jm}(\Omega), \quad \text{tr } \mathbf{Z}^{(4)}_{jm}(\Omega) = -2j(j+1)Y_{jm}(\Omega). \tag{5}$$

The gravitation tensor can alternatively be represented in terms of the symmetric spherical dyadics $\mathbf{e}_{ij}$ (Appendix A):

$$\begin{aligned} \boldsymbol{\Gamma} = &\Gamma_{rr}\mathbf{e}_{rr} + 2\Gamma_{r\vartheta}\mathbf{e}_{r\vartheta} + 2\Gamma_{r\lambda}\mathbf{e}_{r\lambda} + \frac{1}{2}(\Gamma_{\vartheta\vartheta} - \Gamma_{\lambda\lambda})(\mathbf{e}_{\vartheta\vartheta} - \mathbf{e}_{\lambda\lambda}) \\ &+ 2\Gamma_{\vartheta\lambda}\mathbf{e}_{\vartheta\lambda} + \frac{1}{2}(\Gamma_{\vartheta\vartheta} + \Gamma_{\lambda\lambda})(\mathbf{e}_{\vartheta\vartheta} + \mathbf{e}_{\lambda\lambda}) \ . \end{aligned} \tag{6}$$

Representing the tensor spherical harmonics in eq. (4) in terms of the symmetric spherical dyadics and comparing the result with Eq. (6), the five independent components of the gravitation tensor $\boldsymbol{\Gamma}$ can be grouped into three second-order gradiometric data tensors $\boldsymbol{\Gamma}^{(1)}$, $\boldsymbol{\Gamma}^{(2)}$ and $\boldsymbol{\Gamma}^{(3)}$ as follows:

$$\begin{aligned} \boldsymbol{\Gamma}^{(1)} &= \Gamma_{rr}\mathbf{e}_{rr}, \\ \boldsymbol{\Gamma}^{(2)} &= 2\Gamma_{r\vartheta}\mathbf{e}_{r\vartheta} + 2\Gamma_{r\lambda}\mathbf{e}_{r\lambda}, \\ \boldsymbol{\Gamma}^{(3)} &= \frac{1}{2}(\Gamma_{\vartheta\vartheta} - \Gamma_{\lambda\lambda})(\mathbf{e}_{\vartheta\vartheta} - \mathbf{e}_{\lambda\lambda}) + 2\Gamma_{\vartheta\lambda}\mathbf{e}_{\vartheta\lambda} \ . \end{aligned} \tag{7}$$

These combinations of gradiometric observables were proposed by Rummel and van Gelderen (1992).

Equation (6) shows that the gradiometric data combination $\Gamma_{\vartheta\vartheta} + \Gamma_{\lambda\lambda}$ standing at the dyadic $\mathbf{e}_{\vartheta\vartheta} + \mathbf{e}_{\lambda\lambda}$ can, in principle, also be considered. However, this data combination on a mean curvature of level surfaces does not contain independent information on the gravitational potential V because $\Gamma_{\vartheta\vartheta} + \Gamma_{\lambda\lambda}$ can be deduced from the observations of the component Γ_{rr} and the trace-free condition on $\boldsymbol{\Gamma}$, namely $\Gamma_{\vartheta\vartheta} + \Gamma_{\lambda\lambda} = -\Gamma_{rr}$.

3 Formulation of the gradiometric BVPs

The three combinations of gradiometric observables $\boldsymbol{\Gamma}^{(1)}$, $\boldsymbol{\Gamma}^{(2)}$ and $\boldsymbol{\Gamma}^{(3)}$ enable us to formulate three different gradiometric BVPs as follows. We aim at determining the gravitational potential $V(r,\Omega)$ on and outside the reference sphere of radius R that is governed by one of the following three gradiometric BVPs, which differ in the usage of three different kinds of gradiometric boundary data:

$$\nabla^2 V = 0 \quad \text{for } r > R \ , \tag{8}$$

$$\text{grad grad } V = \boldsymbol{\Gamma}^{(\lambda)} \quad \text{for } r = R \ , \tag{9}$$

$$V \sim O\left(\frac{1}{r}\right) \quad \text{for } r \to \infty \ , \tag{10}$$

where the gradiometric data $\boldsymbol{\Gamma}^{(\lambda)}$, $\lambda = 1, 2, 3$, introduced by Eq. (7) are assumed to be known tensor functions of the angular variable Ω. The asymptotic condition (10) implies that the harmonic function V approaches zero at infinity. The BVPs for gradiometric data $\boldsymbol{\Gamma}^{(1)}$, $\boldsymbol{\Gamma}^{(2)}$ and $\boldsymbol{\Gamma}^{(3)}$ will be called the *vertical-vertical gradiometric BVP*, the *vertical-horizontal gradiometric BVP* and the *horizontal-horizontal gradiometric BVP*, respectively, since $\boldsymbol{\Gamma}^{(1)}$ measures the vertical gradient of gravity, $\boldsymbol{\Gamma}^{(2)}$ measures the horizontal gradient of gravity and $\boldsymbol{\Gamma}^{(3)}$ measures the difference between the two principal radii of curvature and the direction of the maximum radius of curvature of the level surfaces of potential V.

The solution to the Laplace equation (8) can be written in terms of solid spherical harmonics $r^{-j-1}Y_{jm}(\Omega)$ as follows:

$$V(r,\Omega) = \sum_{j=0}^{\infty}\left(\frac{R}{r}\right)^{j+1}\sum_{m=-j}^{j} V_{jm}Y_{jm}(\Omega) \ , \tag{11}$$

where V_{jm} are expansion coefficients to be determined from the boundary condition (9). Making use of eq. (4), we compute the double gradient of V and substitute the result into the boundary condition (9):

$$\begin{aligned} &\frac{1}{R^2}\sum_{j=0}^{\infty}\sum_{m=-j}^{j}(j+1)(j+2)V_{jm}\mathbf{Z}^{(1)}_{jm}(\Omega) = \boldsymbol{\Gamma}^{(1)}(\Omega), \\ &\frac{1}{R^2}\sum_{j=0}^{\infty}\sum_{m=-j}^{j}(j+2)V_{jm}\mathbf{Z}^{(2)}_{jm}(\Omega) = -\frac{1}{2}\boldsymbol{\Gamma}^{(2)}(\Omega), \\ &\frac{1}{R^2}\sum_{j=0}^{\infty}\sum_{m=-j}^{j}V_{jm}\mathbf{Z}^{(3)}_{jm}(\Omega) = 2\boldsymbol{\Gamma}^{(3)}(\Omega) \ . \end{aligned} \tag{12}$$

The last three equations allow us to investigate the existence and uniqueness of the gradiometric BVPs (8)–(10). Since

$$\mathbf{Z}^{(2)}_{00}(\Omega) = \mathbf{Z}^{(3)}_{00}(\Omega) = \mathbf{Z}^{(3)}_{1m}(\Omega) = 0 \ , \tag{13}$$

the existence of a solution to the problem (8)–(10) for gradiometric data $\mathbf{\Gamma}^{(2)}$ is guaranteed if $\mathbf{\Gamma}^{(2)}$ does not contain the zero-degree spherical harmonic. Likewise, a solution to the problem (8)–(10) for gradiometric data $\mathbf{\Gamma}^{(3)}$ exists if $\mathbf{\Gamma}^{(3)}$ does not contain the zero- and first-degree spherical harmonics. Mathematically, the conditions on the existence of a solution are

$$\int_{\Omega_0} \Gamma_{r\vartheta}(\Omega)\mathrm{d}\Omega = \int_{\Omega_0} \Gamma_{r\lambda}(\Omega)\mathrm{d}\Omega = \int_{\Omega_0} [\Gamma_{\vartheta\vartheta}(\Omega) - \Gamma_{\lambda\lambda}(\Omega)]\mathrm{d}\Omega = \int_{\Omega_0} \Gamma_{\vartheta\lambda}(\Omega)\mathrm{d}\Omega = 0,$$
$$\int_{\Omega_0} [\Gamma_{\vartheta\vartheta}(\Omega) - \Gamma_{\lambda\lambda}(\Omega)]Y^*_{1m}(\Omega)\mathrm{d}\Omega = \int_{\Omega_0} \Gamma_{\vartheta\lambda}(\Omega)Y^*_{1m}(\Omega)\mathrm{d}\Omega = 0, \quad m = -1, 0, 1 \ , \tag{14}$$

where the asterisk denotes complex conjugation, $\mathrm{d}\Omega := \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\lambda$ and Ω_0 is the full solid angle. Throughout the paper, we assume that the 10 existence conditions (14) are satisfied. If these conditions are violated by observational errors, the zero- and first-degree spherical harmonics must be removed from the gradiometric data $\mathbf{\Gamma}^{(2)}$ and $\mathbf{\Gamma}^{(3)}$, respectively. As far as the vertical-vertical gradiometric BVP for data type $\mathbf{\Gamma}^{(1)}$ is concerned, the existence of the solution to this problem is unconditionally guaranteed.

In addition, in order to ensure the uniqueness of a solution, the zero-degree spherical harmonic and the zero- and first-degree spherical harmonics must be removed from the potential V for the vertical-horizontal and horizontal-horizontal gradiometric BVPs, respectively. Mathematically, the asymptotic condition (10) must be replaced by a more precise condition of the form

$$V \sim O\left(\frac{1}{r^\lambda}\right) \text{ for } r \to \infty \ , \tag{15}$$

where $\lambda = 1$, 2 and 3 for the vertical-vertical, vertical-horizontal and horizontal-horizontal gradiometric BVP, respectively,

The existence, uniqueness and stability of a solution to the gradiometric BVPs was investigated by Schreiner (1994). Making use of the concept of the Sobolev space, the theory of pseudodifferential operators and the assumption that gradiometric observables satisfy the conditions (14), he proved (Schreiner 1994, Theorem 3.3.1.) that solutions to the gradiometric BVPs exist and are unique. In addition, he assumed that gradiometric observables are elements of the Sobolev space $h_{-1/2}(\Omega)$. Since the construction of the norm of this functional space may be difficult if gradiometric observables are not represented in terms of spherical harmonics, we impose a stronger constraint on the gradiometric observables and assume that they are square-integrable functions of Ω, $\mathbf{\Gamma}^{(\lambda)} \in L_2(\Omega)$. This is allowed, because the space of square-integrable functions $L_2(\Omega)$ is embedded in $h_{-1/2}(\Omega)$, i.e. $L_2(\Omega) \subset h_{-1/2}(\Omega)$.

4 Solution in the spectral domain

Under the assumption that the gradiometric data tensors are square-integrable functions of Ω, they can be expanded into a series of tensor spherical harmonics:

$$\mathbf{\Gamma}^{(\lambda)}(\Omega) = \sum_{j=\lambda-1}^{\infty} \frac{1}{[N_j^{(\lambda)}]^2} \times \sum_{m=-j}^{j} \int_{\Omega_0} \left(\mathbf{\Gamma}^{(\lambda)}(\Omega') : \left[\mathbf{Z}^{(\lambda)}_{jm}(\Omega')\right]^*\right) \mathrm{d}\Omega' \, \mathbf{Z}^{(\lambda)}_{jm}(\Omega) \ , \tag{16}$$

where the colon denotes the double-dot product of tensors and $[N_j^{(\lambda)}]^2$ is the square norm of $\mathbf{Z}^{(\lambda)}_{jm}(\Omega)$ (Appendix A). Considering expansion (16) in eq. (12) and comparing the coefficients standing at tensor spherical harmonics $\mathbf{Z}^{(\lambda)}_{jm}(\Omega)$ results in

$$V_{jm} = \begin{cases} \dfrac{R^2}{(j+1)(j+2)} \displaystyle\int_{\Omega_0} \left(\mathbf{\Gamma}^{(1)}(\Omega') : \left[\mathbf{Z}^{(1)}_{jm}(\Omega')\right]^*\right) \mathrm{d}\Omega', \\ -\dfrac{R^2}{j(j+1)(j+2)} \displaystyle\int_{\Omega_0} \left(\mathbf{\Gamma}^{(2)}(\Omega') : \left[\mathbf{Z}^{(2)}_{jm}(\Omega')\right]^*\right) \mathrm{d}\Omega', \\ \dfrac{R^2}{(j-1)j(j+1)(j+2)} \displaystyle\int_{\Omega_0} \left(\mathbf{\Gamma}^{(3)}(\Omega') : \left[\mathbf{Z}^{(3)}_{jm}(\Omega')\right]^*\right) \mathrm{d}\Omega' \ . \end{cases} \tag{17}$$

Applying the decomposition (7) of the gradiometric data tensors $\mathbf{\Gamma}^{(\lambda)}(\Omega)$ in Eq. (17) and evaluating the double-dot products of the symmetric spherical dyadics with the tensor spherical harmonics, the coefficients V_{jm} can be expressed in more explicit forms:

$$V_{jm} = \begin{cases} \dfrac{R^2}{(j+1)(j+2)} \displaystyle\int_{\Omega_0} \Gamma_{rr}(\Omega') Y^*_{jm}(\Omega') \mathrm{d}\Omega', \\ -\dfrac{R^2}{j(j+1)(j+2)} \displaystyle\int_{\Omega_0} \left[\Gamma_{r\vartheta}(\Omega') E^*_{jm}(\Omega') + \Gamma_{r\lambda}(\Omega') F^*_{jm}(\Omega')\right] \mathrm{d}\Omega', \\ \dfrac{R^2}{(j-1)j(j+1)(j+2)} \displaystyle\int_{\Omega_0} \left[[\Gamma_{\vartheta\vartheta}(\Omega') + \Gamma_{\lambda\lambda}(\Omega')] G^*_{jm}(\Omega') - 2\Gamma_{\vartheta\lambda}(\Omega') H^*_{jm}(\Omega')\right] \mathrm{d}\Omega' \ , \end{cases} \tag{18}$$

where the functions $E_{jm}(\Omega)$, $F_{jm}(\Omega)$, $G_{jm}(\Omega)$ and $H_{jm}(\Omega)$ are defined in Appendix A. Finally, substituting these expressions into eq. (11), interchanging the order of summation over J and m and integration over Ω' due to the uniform convergence of the series expansion (11), the solutions to the three gradiometric BVPs (8)–(10) read

$$V(r,\Omega) = \begin{cases} \dfrac{R^2}{4\pi}\displaystyle\int_{\Omega_0} \Gamma_{rr}(\Omega')G_{rr}(t,\Omega,\Omega')\mathrm{d}\Omega', \\ \dfrac{R^2}{4\pi}\displaystyle\int_{\Omega_0} \Big[\Gamma_{r\vartheta}(\Omega')G_{r\vartheta}(t,\Omega,\Omega') + \Gamma_{r\lambda}(\Omega')G_{r\lambda}(t,\Omega,\Omega')\Big]\mathrm{d}\Omega', \\ \dfrac{R^2}{4\pi}\displaystyle\int_{\Omega_0} \Big[[\Gamma_{\vartheta\vartheta}(\Omega') - \Gamma_{\lambda\lambda}(\Omega')]G_{\vartheta\vartheta\lambda\lambda}(t,\Omega,\Omega') + 2\Gamma_{\vartheta\lambda}(\Omega')G_{\vartheta\lambda}(t,\Omega,\Omega')\Big]\mathrm{d}\Omega' \ , \end{cases} \tag{19}$$

where we have introduced five gradiometric Green's functions:

$$G_{rr}(t,\Omega,\Omega') := 4\pi\sum_{j=0}^{\infty}\frac{t^{j+1}}{(j+1)(j+2)}\sum_{m=-j}^{j}Y^*_{jm}(\Omega')Y_{jm}(\Omega),$$

$$G_{r\vartheta}(t,\Omega,\Omega') := -4\pi\sum_{j=1}^{\infty}\frac{t^{j+1}}{j(j+1)(j+2)}\sum_{m=-j}^{j}E^*_{jm}(\Omega')Y_{jm}(\Omega),$$

$$G_{r\lambda}(t,\Omega,\Omega') := -4\pi\sum_{j=1}^{\infty}\frac{t^{j+1}}{j(j+1)(j+2)}\sum_{m=-j}^{j}F^*_{jm}(\Omega')Y_{jm}(\Omega),$$

$$G_{\vartheta\vartheta\lambda\lambda}(t,\Omega,\Omega') := 4\pi\sum_{j=2}^{\infty}\frac{t^{j+1}}{(j-1)j(j+1)(j+2)} \times\sum_{m=-j}^{j}G^*_{jm}(\Omega')Y_{jm}(\Omega),$$

$$G_{\vartheta\lambda}(t,\Omega,\Omega') := 4\pi\sum_{j=2}^{\infty}\frac{t^{j+1}}{(j-1)j(j+1)(j+2)} \times\sum_{m=-j}^{j}H^*_{jm}(\Omega')Y_{jm}(\Omega) \ , \tag{20}$$

with the symbol $t := R/r$ for convenience. From a numerical point of view, the spectral forms (20) of gradiometric Green's functions may appear inconvenient, because some of them have a singularity at $(1,\Omega,\Omega)$. At this point, it is necessary to sum the spectral series up to high degrees and orders, which may be time consuming and numerically unstable. We thus aim at converting the spectral forms of the Green's functions to closed spatial forms. However, the actual satellite gradiometric data will be band limited and the summations over j will be finite and bounded.

5 Analytical forms of addition theorems for spherical harmonics

We now present the method of summing the series over the azimuthal order m occurring in eq. (20). The approach is based on the Laplace addition theorem for scalar spherical harmonics that we consider in the form (see e.g. Varshalovich et al., 1989, sect. 5.17.2)

$$\sum_{m=-j}^{j}Y^*_{jm}(\Omega')Y_{jm}(\Omega) = \frac{2j+1}{4\pi}P_j(\cos\psi) \ , \tag{21}$$

where $P_j(\cos\psi)$ is the Legendre polynomial of degree j and ψ is the angular distance between the computation point (ϑ,λ) and an integration point (ϑ',λ') referred to the point (ϑ,λ). Differentiating Eq. (21) with respect to ϑ' and λ', respectively, yields

$$\sum_{m=-j}^{j}\frac{\partial Y^*_{jm}(\Omega')}{\partial\vartheta'}Y_{jm}(\Omega) = \frac{2j+1}{4\pi}\frac{\partial P_j(\cos\psi)}{\partial\vartheta'},$$
$$\sum_{m=-j}^{j}\frac{1}{\sin\vartheta'}\frac{\partial Y^*_{jm}(\Omega')}{\partial\lambda'}Y_{jm}(\Omega) = \frac{2j+1}{4\pi}\frac{1}{\sin\vartheta'}\frac{\partial P_j(\cos\psi)}{\partial\lambda'} \ . \tag{22}$$

The partial derivatives of Legendre polynomials $P_j(\cos\psi)$ with respect to ϑ' and λ', respectively, are expressible in terms of the ordinary derivatives of the Legendre polynomials with respect to $\cos\psi$. By the chain rule of differentiation, Grafarend (2001) showed that

$$\frac{\partial}{\partial\vartheta'} = \cos\alpha'\sin\psi\frac{\partial}{\partial\cos\psi},$$
$$\frac{1}{\sin\vartheta'}\frac{\partial}{\partial\lambda'} = -\sin\alpha'\sin\psi\frac{\partial}{\partial\cos\psi} \ , \tag{23}$$

where α' is the azimuth of the computation point (ϑ,λ) with respect to an integration point (ϑ',λ'). Changing the differentiation on the right-hand side of eq. (22) according to the rule (23), we obtain

$$\sum_{m=-j}^{j}\frac{\partial Y^*_{jm}(\Omega')}{\partial\vartheta'}Y_{jm}(\Omega) = \frac{2j+1}{4\pi}\cos\alpha'\sin\psi\frac{\mathrm{d}P_j(\cos\psi)}{\mathrm{d}\cos\psi},$$
$$\sum_{m=-j}^{j}\frac{1}{\sin\vartheta'}\frac{\partial Y^*_{jm}(\Omega')}{\partial\lambda'}Y_{jm}(\Omega) = -\frac{2j+1}{4\pi}\sin\alpha'\sin\psi\frac{\mathrm{d}P_j(\cos\psi)}{\mathrm{d}\cos\psi} \ , \tag{24}$$

which may be regarded as the addition theorems for the first-order derivatives of scalar spherical harmonics.

Next, we derive the analytical forms of the addition theorems for the second-order derivatives of spherical harmonics. The differentiation of Eq. (22) with respect to ϑ' and λ', respectively, yields

$$\sum_{m=-j}^{j} \frac{\partial^2 Y^*_{jm}(\Omega')}{\partial(\vartheta')^2} Y_{jm}(\Omega) = \frac{2j+1}{4\pi} \frac{\partial^2 P_j(\cos\psi)}{\partial(\vartheta')^2}$$
$$\sum_{m=-j}^{j} \frac{\partial}{\partial\vartheta'} \left(\frac{1}{\sin\vartheta'} \frac{\partial Y^*_{jm}(\Omega')}{\partial\lambda'} \right) Y_{jm}(\Omega) = \frac{2j+1}{4\pi} \frac{\partial}{\partial\vartheta'} \left(\frac{1}{\sin\vartheta'} \frac{\partial P_j(\cos\psi)}{\partial\lambda'} \right), \tag{25}$$
$$\sum_{m=-j}^{j} \frac{\partial^2 Y^*_{jm}(\Omega')}{\partial(\lambda')^2} Y_{jm}(\Omega) = \frac{2j+1}{4\pi} \frac{\partial^2 P_j(\cos\psi)}{\partial(\lambda')^2} \ .$$

The differential operators on the right-hand sides can again be expressed as ordinary differential operators in terms of $\cos\psi$. After some manipulations, we obtain the following second-order differential identities:

$$\frac{\partial^2}{\partial(\vartheta')^2} - \cot\vartheta' \frac{\partial}{\partial\vartheta'} - \frac{1}{\sin^2\vartheta'} \frac{\partial^2}{\partial(\lambda')^2} = \cos 2\alpha' \sin^2\psi \frac{\partial^2}{\partial(\cos\psi)^2},$$
$$\frac{\partial}{\partial\vartheta'} \left(\frac{1}{\sin\vartheta'} \frac{\partial}{\partial\lambda'} \right) = -\frac{1}{2} \sin 2\alpha' \sin^2\psi \frac{\partial^2}{\partial(\cos\psi)^2}. \tag{26}$$

This allows us to reduce eq. (25) to the form

$$\sum_{m=-j}^{j} \frac{\partial}{\partial\vartheta'} \left(\frac{1}{\sin\vartheta'} \frac{\partial Y^*_{jm}(\Omega')}{\partial\lambda'} \right) Y_{jm}(\Omega) = -\frac{2j+1}{8\pi} \sin 2\alpha' \sin^2\psi \frac{\mathrm{d}^2 P_j(\cos\psi)}{\mathrm{d}(\cos\psi)^2},$$
$$\sum_{m=-j}^{j} \left(\frac{\partial^2 Y^*_{jm}(\Omega')}{\partial(\vartheta')^2} - \cot\vartheta' \frac{\partial Y^*_{jm}(\Omega')}{\partial\vartheta'} - \frac{1}{\sin^2\vartheta'} \frac{\partial^2 Y^*_{jm}(\Omega')}{\partial(\lambda')^2} \right) \times Y_{jm}(\Omega) = \frac{2j+1}{4\pi} \cos 2\alpha' \sin^2\psi \frac{\mathrm{d}^2 P_j(\cos\psi)}{\mathrm{d}(\cos\psi)^2} \ . \tag{27}$$

Introducing functions $E_{jm}(\Omega)$, $F_{jm}(\Omega)$, $G_{jm}(\Omega)$ and $H_{jm}(\Omega)$ defined in Appendix A, the addition theorems (24) and (27) can be written in the compact forms

$$\sum_{m=-j}^{j} E^*_{jm}(\Omega') Y_{jm}(\Omega) = \frac{2j+1}{4\pi} \cos\alpha' \sin\psi \frac{\mathrm{d}P_j(\cos\psi)}{\mathrm{d}\cos\psi},$$
$$\sum_{m=-j}^{j} F^*_{jm}(\Omega') Y_{jm}(\Omega) = -\frac{2j+1}{4\pi} \sin\alpha' \sin\psi \frac{\mathrm{d}P_j(\cos\psi)}{\mathrm{d}\cos\psi},$$
$$\sum_{m=-j}^{j} G^*_{jm}(\Omega') Y_{jm}(\Omega) = \frac{2j+1}{4\pi} \cos 2\alpha' \sin^2\psi \frac{\mathrm{d}^2 P_j(\cos\psi)}{\mathrm{d}(\cos\psi)^2},$$
$$\sum_{m=-j}^{j} H^*_{jm}(\Omega') Y_{jm}(\Omega) = -\frac{2j+1}{4\pi} \sin 2\alpha' \sin^2\psi \frac{\mathrm{d}^2 P_j(\cos\psi)}{\mathrm{d}(\cos\psi)^2} \ . \tag{28}$$

6 Closed spatial forms of Green's functions

We are now ready to express the gradiometric Green's functions in closed spatial forms. Substituting the addition theorems (21) and (28) into eq. (20), the spectral forms of the gradiometric Green's functions are expressed as the products of the part depending on the azimuth α' and the part depending on the angular distance ψ (the exception is the Green's function for the vertical-vertical gradiometric BVP that depends on ψ only):

$$G_{rr}(t,\Omega,\Omega') = K_{rr}(t,\cos\psi),$$
$$G_{r\vartheta}(t,\Omega,\Omega') = -\cos\alpha' K_{r\Omega}(t,\cos\psi),$$
$$G_{r\lambda}(t,\Omega,\Omega') = \sin\alpha' K_{r\Omega}(t,\cos\psi) \tag{29}$$
$$G_{\vartheta\vartheta\lambda\lambda}(t,\Omega,\Omega') = \cos 2\alpha' K_{\Omega\Omega}(t,\cos\psi),$$
$$G_{\vartheta\lambda}(t,\Omega,\Omega') = -\sin 2\alpha' K_{\Omega\Omega}(t,\cos\psi) \ .$$

The three isotropic kernels $K_{rr}(t,\cos\psi)$, $K_{r\Omega}(t,\cos\psi)$ and $K_{\Omega\Omega}(t,\cos\psi)$ are given by infinite series of Legendre polynomials and their derivatives:

$$K_{rr}(t,x) := \sum_{j=0}^{\infty} \frac{2j+1}{(j+1)(j+2)} t^{j+1} P_j(x),$$
$$K_{r\Omega}(t,x) := \sqrt{1-x^2} \sum_{j=1}^{\infty} \frac{2j+1}{j(j+1)(j+2)} t^{j+1} \frac{\mathrm{d}P_j(x)}{\mathrm{d}x},$$
$$K_{\Omega\Omega}(t,x) := (1-x^2) \sum_{j=2}^{\infty} \frac{2j+1}{(j-1)j(j+1)(j+2)} t^{j+1} \frac{\mathrm{d}^2 P_j(x)}{\mathrm{d}x^2} \ , \tag{30}$$

where $x := \cos\psi$. We now replace the infinite series for the isotropic kernels by closed-form expressions. The fractions occurring in these series can be decomposed as

$$\frac{2j+1}{(j+1)(j+2)} = -\frac{1}{j+1} + \frac{3}{j+2},$$
$$\frac{2j+1}{j(j+1)(j+2)} = \frac{1}{2j} + \frac{1}{j+1} - \frac{3}{2(j+2)},$$
$$\frac{2j+1}{(j-1)j(j+1)(j+2)} = \frac{1}{2} \left(\frac{1}{j-1} - \frac{1}{j} - \frac{1}{j+1} + \frac{1}{j+2} \right) \ . \tag{31}$$

Substituting the partial-fraction decomposition (31) into Eq. (30) and summing up the particular constituents by making use of the formulae listed in Appendix B, we arrive, after some algebraic manipulations, at

$$K_{rr}(t,x) = \frac{3}{t}(g-1) + \left(\frac{3x}{t} - 1 \right) \ln\left(\frac{g+t-x}{1-x} \right),$$
$$K_{r\Omega}(t,x) = \sqrt{1-x^2} \left[\frac{3}{2g} + \frac{t^2(g+1)}{2g(g+1-tx)} + \left(1 - \frac{3x}{2t} \right) \times \left(\frac{1}{1-x} - \frac{g+t}{g(g+t-x)} \right) - \frac{3}{2t} \ln\left(\frac{g+t-x}{1-x} \right) \right],$$
$$K_{\Omega\Omega}(t,x) = -\frac{t}{2} + \frac{3}{2}xt^2 + gt + \frac{1}{t}(1-g) + \frac{x^2 t^3}{g+1-tx} + \frac{x(x-t)}{t(1-x)} - \frac{x^2}{t(g+t-x)} \ , \tag{32}$$

where $g \equiv g(t,x) := \sqrt{1+t^2-2tx}$, $-1 \le x \le 1$, $0 < t \le 1$, is the reciprocal generating function of Legendre polynomials.

7 Singularity investigations of the isotropic kernels

The above formulae enable us to study the behaviour of the three isotropic kernels in the vicinity of the point $(t = 1, \psi = 0)$, i.e. the case when an integration point approaches the computation point, both lying on the reference sphere R. First, let us study the case $t = 1$. Then, the generating function $g(1, \cos\psi) = 2\sin\frac{\psi}{2}$, $1 - \cos\psi = 2\sin^2\frac{\psi}{2}$, and eq. (32) reduces to

$$K_{rr}(1,\cos\psi) = -3 + 6\sin\frac{\psi}{2} + (1-3\cos\psi)\ln\left(\frac{\sin\frac{\psi}{2}}{1+\sin\frac{\psi}{2}}\right),$$
$$K_{r\Omega}(1,\cos\psi) = 2\cos\frac{\psi}{2} + \frac{\sin\psi}{2\left(1+\sin\frac{\psi}{2}\right)} + \frac{3}{2}\sin\psi\ln\left(\frac{\sin\frac{\psi}{2}}{1+\sin\frac{\psi}{2}}\right),$$
$$K_{\Omega\Omega}(1,\cos\psi) = \frac{1}{2} + \frac{1}{2}\cos\psi\,. \tag{33}$$

Consequently, the function $K_{rr}(t, \cos\psi)$ has a logarithmic singularity at the point $(t = 1, \psi = 0)$, that is $K_{rr}(1, \cos\psi) \sim -2\ln(\psi/2)$ for $\psi \to 0$. On the other hand, functions $K_{r\Omega}(1, \cos\psi)$ and $K_{\Omega\Omega}(1, \cos\psi)$ are bounded at the point $\psi = 0$, $K_{r\Omega}(1, 1) = 2$ and $K_{\Omega\Omega}(1, 1) = 1$. Consequently, even after removing a low-frequency part of the gradiometric data, e.g. by a global gravity model, the contribution of the far-zone gradiometric data to the integral (19) remains relatively large and, in principle, cannot be neglected as, for instance, in the case of the Stokes's integral. The functions $K_{rr}(1, \cos\psi)$, $K_{r\Omega}(1, \cos\psi)$ and $K_{\Omega\Omega}(1, \cos\psi)$ within the interval $0^\circ \le \psi \le 180^\circ$ are shown in Fig. 1.

Second, let us study the limiting case $x = 1$ for $t < 1$. The generating function then reduces to $g(t, 1) = 1 - t$. Making use of the l'Hospital rule, we can find, after some manipulations, that

$$\lim_{x\to 1}\frac{g+t-x}{1-x} = \frac{1}{1-t},$$
$$\lim_{x\to 1}\left(\frac{x-t}{1-x} - \frac{x}{g+t-x}\right) = \frac{t(t-2)}{2(1-t)}\ , \tag{34}$$

which holds for $t < 1$. The isotropic kernels then reduce to the forms

$$K_{rr}(t, 1) = -3 + \left(1 - \frac{3}{t}\right)\ln(1-t)$$
$$K_{r\Omega}(t, 1) = K_{\Omega\Omega}(t, 1) = 0\ , \tag{35}$$

which are shown in Fig. 1.

8 Comparison with the van Gelderen and Rummel solution

Van Gelderen and Rummel (2001, 2002) presented solutions to geodetic BVPs in a spherical approximation for various types of observation. Since our solutions differ partly from those presented in vGR01 (Sects. 4.4 and 4.6), we briefly discuss the differences.

First, the summation of the spherical harmonic series for the Green's functions applied to the gradiometric data $\{\Gamma, z\}$ and $\{x, \Gamma\}$ runs from degree $\ell = 2$ up to infinity (vGR01, Table 2). The zero- and the first-degree spherical harmonics are removed in order to achieve the consistency with other BVPs solved there. However, as Eq. (15) shows, the zero- and the first-degree spherical harmonics can be considered, without any restrictions, in the solution of the vertical-vertical gradiometric BVP and the first-degree spherical harmonics in the solution of the vertical-horizontal gradiometric BVP. As far as the horizontal-horizontal gradiometric BVP is concerned, the summation of spherical harmonic series for the Green's function starts at degree two, as shown in Eq. (15). This is in agreement with the Green's function for data $\{\Gamma_{xx} - \Gamma_{yy}, 2\Gamma_{xy}\}$ considered in vGR01.

Second, the Green's functions in spatial domain are only tabulated for the computation point on the reference sphere R (vGR01, Table 2). Such forms do

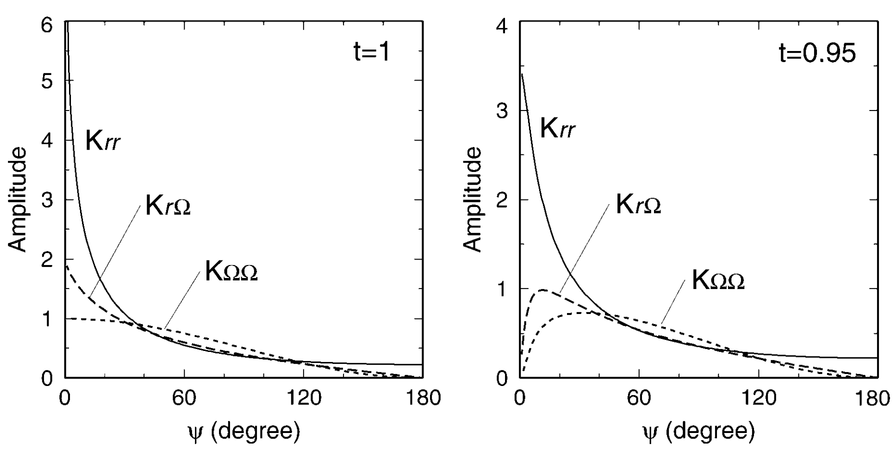

Fig. 1. The isotropic parts $K_{rr}(t, \cos\psi)$, $K_{r\Omega}(t, \cos\psi)$ and $K_{\Omega\Omega}(t, \cos\psi)$ of the gradiometric Green's functions for the cases $t = 1$ (left) and $t = 0.95$ (right)

not provide a solution to the gradiometric BVPs in the space outside the reference sphere, which is desired, for instance, in continuing gradiometric data from a non-spherical satellite orbit to a mean-orbit sphere. Equations (19), (29) and (32), derived in this paper, provide the solutions to the gradiometric BVPs on and outside the reference sphere. In particular, if the computation point is on the reference sphere, that is when $r = R$, the gradiometric Green's functions reduce to the form (33). Omitting the zero- and first-degree spherical harmonics in $K_{rr}(1, \cos\psi)$ and the first-degree spherical harmonics in $K_{r\Omega}(1, \cos\psi)$, the Green's functions (33) have the same forms as those tabulated in vGR01 (Table 2) for data $\{\Gamma, z\}$ and $\{x, \Gamma\}$, respectively. As far as the function $K_{\Omega\Omega}(1, \cos\psi)$ is concerned, it has the same form as that for data $\{\Gamma_{xx} - \Gamma_{yy}, 2\Gamma_{xy}\}$ considered in vGR01.

Third, since the gradiometric Green's functions are harmonic outside the reference sphere, their spatial dependence is of the form $t^{-j-1}P_j(\cos\psi)$, as eq. (20) demonstrates. The form $t^{-j-3}P_j(\cos\psi)$ presented in vGR01 (section 4.4) is incorrect in the radial dependence. In the notation used in vGR01, the formula for $\Gamma^i_{\ell m}$ is to be divided by R^2.

Fourth, vGR01 does not discuss the conditions on the existence of a solution. The existence conditions associated with the vGR01 gradiometric solutions are stronger than those considered in the present paper or by Schreiner (1994). The 10 general existence conditions (14) are to be supplemented by the conditions requiring that the zero- and first-degree spherical harmonics are removed from the observations Γ_{zz} and the first-degree spherical harmonics from Γ_{xz} and Γ_{yz}. The vGR01 gradiometric solutions then exist.

9 Conclusion

This work was motivated by the recent approval of the GOCE gradiometry mission and the effort to create an adequate mathematical tool for inverting gradiometric observables to information on the external gravitational potential of the Earth. We have managed to solve gradiometric BVPs in terms of Green's functions that have been expressed in spectral form as series of tensor spherical harmonics. This form of the solution can be applied to develop the gravitational field in terms of spherical harmonics from the GOCE data. Alternatively, by means of the addition theorems for spherical harmonics and the infinite-sum formulae for Legendre polynomials, the spectral forms have been converted to closed spatial forms. These forms can used to construct the upward- and downward-continuation operators that can be applied to transform the measured gradiometric data from a non-spherical satellite orbit to a mean-orbit sphere. The results presented in this paper have been compared with those recently published by van Gelderen and Rummel (2001). We found that the vGR01 gradiometric solution can be refined.

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Appendix A

Tensor spherical harmonics

The spherical harmonic representation of a tensor field was considered by Regge and Wheeler (1957), Backus (1967), Zerilli (1970), Phinney and Burridge (1973), James (1976), Jones (1985) and others. The Zerilli and James tensors are defined by the irreducible tensor product of the scalar spherical harmonics $Y_{jm}(\Omega)$ and the second-rank cyclic-covariant base dyadics. The Regge–Wheeler and Backus tensors are the result of applying the operators $\mathbf{e}_r$, ∇_Ω and $\mathbf{L}_\Omega$ to the scalar spherical harmonics $Y_{jm}(\Omega)$. Throughout this paper, we use the Regge–Wheeler definition and restrict ourselves to the second-order, symmetric, spherical harmonic tensors with trace. There are six such tensor spherical harmonics (Zerilli, 1970):

$$\begin{aligned}
\mathbf{Z}^{(1)}_{jm}(\Omega) &:= [\mathbf{e}_r\mathbf{e}_r Y_{jm}(\Omega)]_s, \\
\mathbf{Z}^{(2)}_{jm}(\Omega) &:= [\mathbf{e}_r\nabla_\Omega Y_{jm}(\Omega)]_s, \\
\mathbf{Z}^{(3)}_{jm}(\Omega) &:= [\nabla_\Omega\nabla_\Omega Y_{jm}(\Omega) + 2\mathbf{e}_r\nabla_\Omega Y_{jm}(\Omega) - \mathbf{L}_\Omega\mathbf{L}_\Omega Y_{jm}(\Omega)]_s, \\
\mathbf{Z}^{(4)}_{jm}(\Omega) &:= [\nabla_\Omega\nabla_\Omega Y_{jm}(\Omega) + \mathbf{L}_\Omega\mathbf{L}_\Omega Y_{jm}(\Omega)]_s, \\
\mathbf{Z}^{(5)}_{jm}(\Omega) &:= [\mathbf{e}_r\mathbf{L}_\Omega Y_{jm}(\Omega)]_s, \\
\mathbf{Z}^{(6)}_{jm}(\Omega) &:= [\mathbf{L}_\Omega\nabla_\Omega Y_{jm}(\Omega) + \mathbf{e}_r\mathbf{L}_\Omega Y_{jm}(\Omega)]_s,
\end{aligned} \tag{A1}$$

where ∇_Ω is the angular part of the gradient operator

$$\nabla_\Omega := \mathbf{e}_\vartheta \frac{\partial}{\partial\vartheta} + \mathbf{e}_\lambda \frac{1}{\sin\vartheta}\frac{\partial}{\partial\lambda}, \tag{A2}$$

and $\mathbf{L}_\Omega$ stands for the angular part of the angular momentum operator

$$\mathbf{L}_\Omega := \mathbf{e}_r \times \nabla_\Omega. \tag{A3}$$

The subscript s denotes the symmetric part of second-order tensor $\mathbf{A}$ with trace, i.e. $[\mathbf{A}]_s := \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$. Creating the dyadic products of spherical unit base vectors $\mathbf{e}_r$, $\mathbf{e}_\vartheta$ and $\mathbf{e}_\lambda$ and taking the symmetric part of the result, we define the symmetric spherical dyadics

$$\mathbf{e}_{ij} := [\mathbf{e}_i \otimes \mathbf{e}_j]_s \quad i, j \in \{r, \vartheta, \lambda\}. \tag{A4}$$

Making use of these dyadics and of the spherical components of the operators ∇_Ω and $\mathbf{L}_\Omega$, the dyadic components of the tensor spherical harmonics are

$$
\begin{aligned}
\mathbf{Z}^{(1)}_{jm}(\Omega) &= Y_{jm}(\Omega)\mathbf{e}_{rr},\\
\mathbf{Z}^{(2)}_{jm}(\Omega) &= E_{jm}(\Omega)\mathbf{e}_{r\vartheta} + F_{jm}(\Omega)\mathbf{e}_{r\lambda},\\
\mathbf{Z}^{(3)}_{jm}(\Omega) &= G_{jm}(\Omega)(\mathbf{e}_{\vartheta\vartheta} - \mathbf{e}_{\lambda\lambda}) + 2H_{jm}(\Omega)\mathbf{e}_{\vartheta\lambda},\\
\mathbf{Z}^{(4)}_{jm}(\Omega) &= -j(j+1)Y_{jm}(\Omega)(\mathbf{e}_{\vartheta\vartheta} + \mathbf{e}_{\lambda\lambda}),\\
\mathbf{Z}^{(5)}_{jm}(\Omega) &= -F_{jm}(\Omega)\mathbf{e}_{r\vartheta} + E_{jm}(\Omega)\mathbf{e}_{r\lambda},\\
\mathbf{Z}^{(6)}_{jm}(\Omega) &= G_{jm}(\Omega)\mathbf{e}_{\vartheta\lambda} - H_{jm}(\Omega)(\mathbf{e}_{\vartheta\vartheta} - \mathbf{e}_{\lambda\lambda}),
\end{aligned}
\tag{A5}
$$

where the abbreviations have the following meanings:

$$
\begin{aligned}
E_{jm}(\Omega) &:= \frac{\partial Y_{jm}(\Omega)}{\partial\vartheta},\\
F_{jm}(\Omega) &:= \frac{1}{\sin\vartheta}\frac{\partial Y_{jm}(\Omega)}{\partial\lambda},\\
G_{jm}(\Omega) &:= \left(\frac{\partial^2}{\partial\vartheta^2} - \cot\vartheta\frac{\partial}{\partial\vartheta} - \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\lambda^2}\right)Y_{jm}(\Omega),\\
H_{jm}(\Omega) &:= 2\frac{\partial}{\partial\vartheta}\left(\frac{1}{\sin\vartheta}\frac{\partial Y_{jm}(\Omega)}{\partial\lambda}\right).
\end{aligned}
\tag{A6}
$$

The orthogonality property of the spherical base vectors and the scalar harmonics combine to give the orthogonality property of the tensor spherical harmonics

$$
\int_{\Omega_0} \mathbf{Z}^{(\lambda)}_{jm}(\Omega) : \left[\mathbf{Z}^{(\lambda')}_{j'm'}(\Omega)\right]^* \mathrm{d}\Omega = \left[N^{(\lambda)}_j\right]^2 \delta_{jj'}\delta_{mm'}\delta_{\lambda\lambda'} \tag{A7}
$$

where the colon denotes the double-dot product of tensors and $[N^{(\lambda)}_j]^2$ is the square norm of $\mathbf{Z}^{(\lambda)}_{jm}(\Omega)$

$$
\left[N^{(\lambda)}_j\right]^2 = \begin{cases}
1 & \text{for } \lambda = 1,\\
\frac{1}{2}j(j+1) & \text{for } \lambda = 2,\\
2(j-1)j(j+1)(j+2) & \text{for } \lambda = 3,\\
2j^2(j+1)^2 & \text{for } \lambda = 4,\\
\frac{1}{2}j(j+1) & \text{for } \lambda = 5,\\
\frac{1}{2}(j-1)j(j+1)(j+2) & \text{for } \lambda = 6.
\end{cases} \tag{A8}
$$

A collection of six tensor harmonics of all possible values of j and m forms a complete set of tensor functions in the domain $0 \le \vartheta \le \pi$, $0 \le \lambda < 2\pi$. Any second-order symmetric tensor $\boldsymbol{\tau}(\vartheta, \lambda)$ whose components are square-integrable functions of the angular variable Ω can be expanded in a series of tensor spherical harmonics $\mathbf{Z}^{(\lambda)}_{jm}(\Omega)$

$$
\boldsymbol{\tau}(\Omega) = \sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{\lambda=1}^{6} \tau^{(\lambda)}_{jm}\mathbf{Z}^{(\lambda)}_{jm}(\Omega), \tag{A9}
$$

where the expansion coefficients $\tau^{(\lambda)}_{jm}$ are obtained by a systematic application of the orthogonality relation of (A7).

Appendix B

Summation of infinite series of Legendre polynomials and their derivatives

Let us recall some summation formulae for infinite series of the Legendre polynomials and their derivatives [others can be found in e.g. Pick et al. (1973, Sect. D-18)].

$$
S_1(t,x) := \sum_{j=2}^{\infty}\frac{t^{j-1}}{j-1}P_j(x) = -x + \frac{1}{t} - \frac{g}{t} - x\ln\left(\frac{g+1-tx}{2}\right), \tag{B1}
$$

$$
S_2(t,x) := \sum_{j=1}^{\infty}\frac{t^j}{j}P_j(x) = -\ln\left(\frac{g+1-tx}{2}\right), \tag{B2}
$$

$$
S_3(t,x) := \sum_{j=0}^{\infty}\frac{t^{j+1}}{j+1}P_j(x) = \ln\left(\frac{g+t-x}{1-x}\right), \tag{B3}
$$

$$
S_4(t,x) := \sum_{j=0}^{\infty}\frac{t^{j+2}}{j+2}P_j(x) = -1 + g + x\ln\left(\frac{g+t-x}{1-x}\right), \tag{B4}
$$

where $g \equiv g(t,x) := \sqrt{1+t^2-2tx}$, $-1 \le x \le 1$ and $0 < t \le 1$. The partial derivatives of the sums S_i, $i = 1, \ldots, 4$, with respect to x read

$$
\frac{\partial S_1(t,x)}{\partial x} = -1 + \frac{1}{g} + \frac{tx(g+1)}{g(g+1-tx)} - \ln\left(\frac{g+1-tx}{2}\right) \tag{B5}
$$

$$
\frac{\partial S_2(t,x)}{\partial x} = \frac{t(g+1)}{g(g+1-tx)} \tag{B6}
$$

$$
\frac{\partial S_3(t,x)}{\partial x} = \frac{1}{1-x} - \frac{g+t}{g(g+t-x)} \tag{B7}
$$

$$
\frac{\partial S_4(t,x)}{\partial x} = -\frac{t}{g} + \frac{x}{1-x} - \frac{x(g+t)}{g(g+t-x)} + \ln\left(\frac{g+t-x}{1-x}\right) \tag{B8}
$$

The partial second-order derivatives of the sums S_i with respect to x multiplied by $1-x^2$ are

$$
(1-x^2)\frac{\partial^2 S_1(t,x)}{\partial x^2} = 3x + \frac{2(t-x)}{g} + \frac{t-x}{g^3} + \frac{2tx^2(g+1)}{g(g+1-tx)}, \tag{B9}
$$

$$
(1-x^2)\frac{\partial^2 S_2(t,x)}{\partial x^2} = 1 - \frac{1}{g} + \frac{t(t-x)}{g^3} + \frac{2tx(g+1)}{g(g+1-tx)}, \tag{B10}
$$

$$
(1-x^2)\frac{\partial^2 S_3(t,x)}{\partial x^2} = \frac{2x}{1-x} + \frac{t^2(t-x)}{g^3} - \frac{2x(g+t)}{g(g+t-x)}, \tag{B11}
$$

$$
(1-x^2)\frac{\partial^2 S_4(t,x)}{\partial x^2} = 2 - g - \frac{1}{g} + \frac{2x^2}{1-x} + \frac{t^3(t-x)}{g^3} - \frac{2x^2(g+t)}{g(g+t-x)}. \tag{B12}
$$

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