Journal of Geodesy (2003) 76: 495–509
DOI 10.1007/s00190-002-0280-7

Journal of Geodesy

Estimation of the Earth's tensor of inertia from recent global gravity field solutions

A. N. Marchenko[1], P. Schwintzer[2]

[1] State University 'Lviv Polytechnik', Faculty of Geodesy, S. Bandera St. 12, UA-79013 Lviv, Ukraine
e-mail: march@polynet.lviv.ua; Tel.: +380-322-398-456
[2] GeoForschungsZentrum Potsdam (GFZ), Div. 1, Telegrafenberg A 17, 14473 Potsdam, Germany
e-mail: psch@gfz-potsdam.de; Tel.: +49-331-288-1131; Fax: +49-331-288-1169

Received: 7 January 2002 / Accepted: 3 June 2002

Abstract. The dynamic figure of the Earth, characterized by the principal axes and principal moments of inertia, is estimated from satellite-derived gravitational harmonic coefficients of second degree in recent global Earth gravity models and from the dynamic ellipticity resulting from the precession constant observed through very-long-baseline interferometry (VLBI). Closed, exact formulae for the determination of these parameters of the Earth's tensor of inertia are developed based on the exact solution of the eigenvalue–eigenvector problem, including a rigorous error propagation. These formulae are applied to determine (a) static components and accuracy of the Earth's tensor of inertia at epoch and (b) the variation with time of the Earth's tensor of inertia and its accuracy. The best-fitting principal moments of inertia and second-degree harmonic coefficients in the principal-axes system are found from an adjustment involving four global gravity field models and six different values for the dynamic ellipticity. The evolution with time of the dynamic figure of the Earth is determined from the mean pole path and the observed secular rate of change in the second-degree zonal coefficient. It is found that differences in the principal moments of inertia change significantly over the time interval from 1962 to 2000, whereas changes in the absolute values cannot be reliably resolved due to the uncertainty in the dynamic ellipticity.

Keywords: Earth's tensor of inertia – Dynamic ellipticity – Figure of the Earth – Dynamic reference frame – Global gravity field model

Correspondence to: P. Schwintzer

1 Introduction

The investigation of temporal variations in the Earth's gravitational potential has attracted increased attention with the launch of the CHAMP satellite on 15 July 2000. With CHAMP, it will be possible to derive a time series of the long-wavelength gravitational coefficients with a step size of one to two months. The consistency of the reference frame, and the proper modelling of the time evolution within this frame underlying the sequence of the gravity field solution, is of great importance for the separation and interpretation of geophysically relevant temporal gravity field variations. Therefore, this study aims to derive the dynamic figure of the Earth, the orientation of the principal axes and its evolution with time from geodetic parameters represented by the second-degree coefficients of the most recent global gravity field models, and polar motion data, and the astronomic dynamic ellipticity H_D.

Based upon satellite observations, the latest gravity field models contain precise values of the five harmonic coefficients of degree 2, $\bar{C}_{2m}$ and $\bar{S}_{2m}$, including temporal variations. In addition, very-long-baseline interferometry (VLBI) observations have led to essential improvements of precession–nutation theories and the determination of the dynamic ellipticity H_D. The components of the Earth's inertial tensor are derived from these six parameters, and the $\bar{C}_{21}(t)$, $\bar{S}_{21}(t)$ coefficients define the path of the position of the polar figure axis $\bar{C}$, supposed to coincide with the mean pole axis. Therefore, these coefficients represent a significant part of the time variations in a dynamic reference frame. Thus, the determination of the time-dependent components of the inertial tensor and other associated parameters from gravity field and astronomical parameters has to be consistent with the observed polar motion data.

This study focuses on (a) the estimation of the principal moments and products of inertia from an adjustment of various sets of coefficients $\bar{C}_{2m}$, $\bar{S}_{2m}$ and H_D at a given epoch of time, and (b) the modelling of the secular and long-periodic constituents in the $\bar{C}_{21}$, $\bar{S}_{21}$ coefficients to provide the time dependence in the components of the Earth's inertial tensor in view of a temporally evolving reference frame. The theoretical background is described in Sects. 2 to 6, and the formulae are worked out to be applied in Sects. 7 and 8.

2 Transformation of second-degree harmonic coefficients from initial to principal-axes coordinate system (eigenvalue problem)

First, some closed non-linear expressions for the transformation of the fully normalized gravitational harmonic coefficients $(\bar{C}_{2m}, \bar{S}_{2m})$ of second degree, defined in an adopted Earth-fixed geocentric Cartesian coordinate system (X, Y, Z), into the coordinate system of the Earth's principal axes of inertia $(\bar{A}, \bar{B}, \bar{C})$ are given. The potential V_2 of second degree (or potential of the central gravitational quadrupole) may be written in the form

$$V_2(P) = \frac{\sqrt{15}}{2}\frac{GMa^2}{r^5}\mathbf{r}^{\mathrm{T}}\mathbf{H}\mathbf{r} = \frac{\sqrt{15}}{2}\frac{GMa^2}{r^5}\tilde{\mathbf{r}}^{\mathrm{T}}\tilde{\mathbf{H}}\tilde{\mathbf{r}} \tag{1}$$

where

$$\mathbf{H} = \begin{pmatrix} \bar{C}_{22} - \frac{\bar{C}_{20}}{\sqrt{3}} & \bar{S}_{22} & \bar{C}_{21} \\ \bar{S}_{22} & -\bar{C}_{22} - \frac{\bar{C}_{20}}{\sqrt{3}} & \bar{S}_{21} \\ \bar{C}_{21} & \bar{S}_{21} & 2\frac{\bar{C}_{20}}{\sqrt{3}} \end{pmatrix} \tag{2}$$

$$\tilde{\mathbf{H}} = \begin{pmatrix} \bar{A}_{22} - \frac{\bar{A}_{20}}{\sqrt{3}} & 0 & 0 \\ 0 & -\bar{A}_{22} - \frac{\bar{A}_{20}}{\sqrt{3}} & 0 \\ 0 & 0 & 2\frac{\bar{A}_{20}}{\sqrt{3}} \end{pmatrix} \tag{3}$$

The matrices $\mathbf{H}$ and $\tilde{\mathbf{H}}$ are defined in the geocentric Cartesian coordinate system (X, Y, Z) and in the system of principal axes of inertia $(\bar{A}, \bar{B}, \bar{C})$, respectively; the vectors $\mathbf{r}^{\mathrm{T}}$ and $\tilde{\mathbf{r}}^{\mathrm{T}}$ contain the Cartesian coordinates of the current point P in these systems. GM is the product of the gravitational constant G and the Earth's mass M; a is the semi-major axis of the ellipsoid of revolution; r is the distance from the origin of a coordinate system to the current point P; $(\bar{C}_{2m}, \bar{S}_{2m})$ and $(\bar{A}_{20}, \bar{A}_{22})$ are fully normalized harmonic gravitational coefficients in the system (X, Y, Z) and in the Earth's principal axes of inertia system $(\bar{A}, \bar{B}, \bar{C})$, respectively.

The transformation of the harmonic coefficients requires a transformation of the matrix $\mathbf{H}$ [Eq. (2)] into the diagonal form $\tilde{\mathbf{H}}$ [Eq. (3)]. This corresponds to a solution of the eigenvalue problem and leads in the case of the quadratic form $\mathbf{r}^{\mathrm{T}}\mathbf{H}\mathbf{r}$ to the characteristic equation for the eigenvalues Λ_i

$$\Lambda^3 + I_2\Lambda - I_3 = 0 \qquad [I_1 = \mathrm{Trace}(\mathbf{H}) = 0] \tag{4}$$

The solution of Eq. (4) may be written in the following non-linear form:

$$\left.\begin{matrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \end{matrix}\right\} = 2\sqrt{\frac{k_2}{3}} \cdot \left\{\begin{matrix} \tilde{\lambda}_1 & \tilde{\lambda}_1 \\ \tilde{\lambda}_2\,, & \tilde{\lambda}_2 \\ \tilde{\lambda}_3 & \tilde{\lambda}_3 \end{matrix}\right\} = \left\{\begin{matrix} \sin\left(\frac{\tilde{\varphi}}{3} + \frac{\pi}{3}\right) \\ -\sin\frac{\tilde{\varphi}}{3} \\ \sin\left(\frac{\tilde{\varphi}}{3} - \frac{\pi}{3}\right) \end{matrix}\right. \tag{5}$$

where the auxiliary angle $\tilde{\varphi}$ is expressed by means of the invariants $I_2 = -k_2$ and I_3

$$\tilde{\varphi} = \sin^{-1}\left(\frac{3\sqrt{3}}{2} \cdot \frac{I_3}{\sqrt{k_2^3}}\right), \quad -\frac{\pi}{2} \le \tilde{\varphi} \le \frac{\pi}{2} \tag{6}$$

with

$$k_2 = -I_2 = \sum_{m=0}^{2}\left(\bar{C}_{2m}^2 + \bar{S}_{2m}^2\right) = \bar{A}_{20}^2 + \bar{A}_{22}^2 \tag{7}$$

$$\begin{aligned} I_3 = \det(\mathbf{H}) &= \frac{2\bar{C}_{20}^3}{3\sqrt{3}} + \frac{\bar{C}_{20}}{\sqrt{3}}\left(\bar{C}_{21}^2 + \bar{S}_{21}^2 - 2\bar{C}_{22}^2 - 2\bar{S}_{22}^2\right) \\ &\quad + \bar{C}_{22}\left(\bar{C}_{21}^2 - \bar{S}_{21}^2\right) + 2\bar{C}_{21}\bar{S}_{21}\bar{S}_{22} \\ &= \det(\tilde{\mathbf{H}}) \end{aligned} \tag{8}$$

Here, the second-degree variance k_2 and I_3 represent the invariant characteristics of the Earth's gravity field, which are independent of linear transformations of the coordinate system (X, Y, Z). The parameters $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ and $\tilde{\lambda}_3$ are eigenvalues of the so-called normalized quadratic form of Eq. (1), which admits the closed solution [Eq. (5)] of the eigenvalue problem (Marchenko and Abrikosov 2001).

Thus, if harmonic coefficients $(\bar{C}_{2m}, \bar{S}_{2m})$ are given, Eqs. (5)–(8) provide the computation of the harmonic coefficients $(\bar{A}_{20}, \bar{A}_{22})$ in the principal-axes coordinate system via the simple expressions

$$\bar{A}_{20} = \frac{\sqrt{3}\Lambda_3}{2}, \quad \bar{A}_{22} = \frac{\Lambda_1 - \Lambda_2}{2} \tag{9}$$

since the diagonal elements of Eq. (3) represent the eigenvalues Λ_1, Λ_2 and Λ_3.

Note now that the matrix $\tilde{\mathbf{H}}$ can be used in the following way:

$$\begin{aligned} &\begin{pmatrix} B + C - 2A & 0 & 0 \\ 0 & A + C - 2B & 0 \\ 0 & 0 & A + B - 2C \end{pmatrix} \\ &\quad = \sqrt{15}\tilde{\mathbf{H}} \quad = \sqrt{15}\begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{pmatrix} \end{aligned} \tag{10}$$

where A, B and C are respectively the first and second equatorial and polar Earth's principal moments of inertia normalized by the factor $1/Ma^2$. As a result, if the eigenvalues Λ_i are determined, the relationships for differences between these normalized moments of inertia follow after some easy algebra

$$B - A = \sqrt{15}\frac{\Lambda_1 - \Lambda_2}{3}, \quad C - A = \sqrt{15}\frac{\Lambda_1 - \Lambda_3}{3}, \quad C - B = \sqrt{15}\frac{\Lambda_2 - \Lambda_3}{3} \tag{11}$$

These differences may also be represented by means of the harmonic coefficients $(\bar{A}_{20}, \bar{A}_{22})$ in the system of principal axes [Eq. (9)]

$$B - A = \frac{2\sqrt{15}}{3}\bar{A}_{22}, \quad C - A = \frac{\sqrt{15}\bar{A}_{22}}{3} - \sqrt{5}\bar{A}_{20}, \quad C - B = -\frac{\sqrt{15}\bar{A}_{22}}{3} - \sqrt{5}\bar{A}_{20} \tag{12}$$

Similarly, these differences can be expressed through parameters of the Earth's gravitational quadrupole (Marchenko 1998)

$$C - A = \tilde{M}_2 = \frac{M_2}{Ma^2} = \frac{\sqrt{15}\bar{A}_{22}}{3} - \sqrt{5}\bar{A}_{20} \tag{13}$$

$$\frac{C - B}{C - A} = \frac{1 - \cos\tilde{\gamma}}{2} = \sin^2\frac{\tilde{\gamma}}{2}, \quad \cos\tilde{\gamma} = \frac{3\bar{A}_{22} + \sqrt{3}\bar{A}_{20}}{\bar{A}_{22} - \sqrt{3}\bar{A}_{20}} \tag{14}$$

where M_2 is the moment of the gravitational quadrupole and $\tilde{\gamma}$ is the angle between two quadrupole axes, located in the plane of the axes $\bar{A}$ and $\bar{C}$ of inertia.

3 Error propagation for the eigenvalue problem

In order to prepare for the error propagation from the starting values to the fundamental parameters of the Earth, shown in Sect. 4, the variance–covariance matrix of the eigenvalues Λ_i [Eq. (5)] is derived here. The vector $\mathbf{g}$ containing the second-degree harmonic coefficients

$$\mathbf{g} = [\bar{C}_{20}, \bar{C}_{21}, \bar{S}_{21}, \bar{C}_{22}, \bar{S}_{22}]^{\mathrm{T}} \tag{15}$$

(hereafter the symbol T denotes transposition) and the (5×5) variance–covariance matrix $\mathbf{C}_{gg}$ of the coefficients are given as initial information. Starting from Eqs. (5)–(8), the necessary matrices of partial derivatives and subsequently the variance–covariance matrix of the eigenvalues Λ_i are obtained by applying the error propagation rule. Thus, defining the vectors

$$\Lambda = [\Lambda_1(\mathbf{J}), \Lambda_2(\mathbf{J}), \Lambda_3(\mathbf{J})]^{\mathrm{T}}, \quad \mathbf{J} = [k_2(\mathbf{g}), I_3(\mathbf{g})]^{\mathrm{T}} \tag{16}$$

we require the (3×5) matrix

$$\frac{\partial\Lambda}{\partial\mathbf{g}} = \frac{\partial\Lambda}{\partial\mathbf{J}} \cdot \frac{\partial\mathbf{J}}{\partial\mathbf{g}} \tag{17}$$

of partial derivatives of Λ_1, Λ_2 and Λ_3 with respect to the harmonic coefficients $(\bar{C}_{2m}, \bar{S}_{2m})$. Differentiating Eq. (5), we obtain for the first component of the right-hand side in Eq. (17) the expression

$$\frac{\partial\Lambda}{\partial\mathbf{J}} = \frac{1}{2k_2}\Lambda_{\mathbf{0}} + \frac{1}{3\sqrt{3}}\mathbf{Q} \cdot \frac{\partial\tilde{\varphi}(\mathbf{J})}{\partial\mathbf{J}}, \quad \Lambda_{\mathbf{0}} = [\Lambda, \mathbf{0}], \quad \mathbf{Q} = \begin{bmatrix} \sqrt{4k_2 - 3\Lambda_1^2} \\ -\sqrt{4k_2 - 3\Lambda_2^2} \\ \sqrt{4k_2 - 3\Lambda_3^2} \end{bmatrix} \tag{18}$$

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \frac{\partial\tilde{\varphi}(\mathbf{J})}{\partial\mathbf{J}} = \tan\tilde{\varphi}\left[-\frac{3}{2k_2}, \frac{1}{I_3}\right] \tag{19}$$

The second component of the right-hand side of Eq. (17) may be expressed as

$$\frac{\partial\mathbf{J}}{\partial\mathbf{g}} = \begin{bmatrix} \frac{\partial k_2(\mathbf{g})}{\partial\mathbf{g}} \\ \frac{\partial I_3(\mathbf{g})}{\partial\mathbf{g}} \end{bmatrix} = \begin{bmatrix} 2\mathbf{g}^{\mathrm{T}} \\ \mathbf{h}_1^{\mathrm{T}}\mathbf{H}_2\mathbf{A}_3 + \mathbf{h}_2^{\mathrm{T}}\mathbf{H}_3\mathbf{A}_1 + \mathbf{h}_3^{\mathrm{T}}\mathbf{H}_1\mathbf{A}_2 \end{bmatrix} \tag{20}$$

where $\mathbf{h}_1$, $\mathbf{h}_2$ and $\mathbf{h}_3$ are three column vectors of the initial fundamental matrix $\mathbf{H}$

$$\mathbf{h}_1 = \begin{pmatrix} \bar{C}_{22} - \frac{\bar{C}_{20}}{\sqrt{3}} \\ \bar{S}_{22} \\ \bar{C}_{21} \end{pmatrix}, \quad \mathbf{h}_2 = \begin{pmatrix} \bar{S}_{22} \\ -\bar{C}_{22} - \frac{\bar{C}_{20}}{\sqrt{3}} \\ \bar{S}_{21} \end{pmatrix}, \quad \mathbf{h}_3 = \begin{pmatrix} \bar{C}_{21} \\ \bar{S}_{21} \\ 2\frac{\bar{C}_{20}}{\sqrt{3}} \end{pmatrix} \tag{21}$$

The matrices $\mathbf{H}_1$, $\mathbf{H}_2$ and $\mathbf{H}_3$ represent the skew-symmetric matrices

$$\mathbf{H}_1 = \begin{pmatrix} 0 & -\bar{C}_{21} & \bar{S}_{22} \\ \bar{C}_{21} & 0 & \frac{\bar{C}_{20}}{\sqrt{3}} - \bar{C}_{22} \\ -\bar{S}_{22} & \bar{C}_{22} - \frac{\bar{C}_{20}}{\sqrt{3}} & 0 \end{pmatrix} \tag{22}$$

$$\mathbf{H}_2 = \begin{pmatrix} 0 & -\bar{S}_{21} & -\bar{C}_{22} - \frac{\bar{C}_{20}}{\sqrt{3}} \\ \bar{S}_{21} & 0 & -\bar{S}_{22} \\ \bar{C}_{22} + \frac{\bar{C}_{20}}{\sqrt{3}} & \bar{S}_{22} & 0 \end{pmatrix} \tag{23}$$

$$\mathbf{H}_3 = \begin{pmatrix} 0 & -2\frac{\bar{C}_{20}}{\sqrt{3}} & \bar{S}_{21} \\ 2\frac{\bar{C}_{20}}{\sqrt{3}} & 0 & -\bar{C}_{21} \\ -\bar{S}_{21} & \bar{C}_{21} & 0 \end{pmatrix} \tag{24}$$

constructed for every vector $(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3)$, respectively. The (3×5) matrices $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$ are found from the following derivatives:

$$\mathbf{A}_1 = \frac{\partial\mathbf{h}_1(\mathbf{g})}{\partial\mathbf{g}} = \begin{bmatrix} -\frac{1}{\sqrt{3}} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{25}$$

$$\mathbf{A}_2 = \frac{\partial \mathbf{h}_2(\mathbf{g})}{\partial \mathbf{g}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ -\frac{1}{\sqrt{3}} & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{26}$$

$$\mathbf{A}_3 = \frac{\partial \mathbf{h}_3(\mathbf{g})}{\partial \mathbf{g}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{27}$$

Hence Eq. (20) allows us to apply the error propagation rule for the computation of the variance–covariance matrix $\mathbf{C}_{\mathrm{JJ}}$ of the invariants k_2 and I_3 from the variance–covariance matrix $\mathbf{C}_{\mathrm{gg}}$

$$\mathbf{C}_{\mathrm{JJ}} = \frac{\partial \mathbf{J}}{\partial \mathbf{g}} \mathbf{C}_{\mathrm{gg}} \left(\frac{\partial \mathbf{J}}{\partial \mathbf{g}} \right)^{\mathrm{T}} \tag{28}$$

Considering Eq. (18), the computation of the variance–covariance matrix $\mathbf{C}_{\Lambda\Lambda}$ of the eigenvalues Λ_1, Λ_2 and Λ_3 follows from

$$\mathbf{C}_{\Lambda\Lambda} = \frac{\partial \Lambda}{\partial \mathbf{g}} \mathbf{C}_{\mathrm{gg}} \left(\frac{\partial \Lambda}{\partial \mathbf{g}} \right)^{\mathrm{T}} = \frac{\partial \Lambda}{\partial \mathbf{J}} \mathbf{C}_{\mathrm{JJ}} \left(\frac{\partial \Lambda}{\partial \mathbf{J}} \right)^{\mathrm{T}} \tag{29}$$

4 Some fundamental parameters of the Earth as a planet and estimation of their uncertainty

Defining the vector $\mathbf{a}$ consisting of the harmonic coefficients $\bar{A}_{20}$ and $\bar{A}_{22}$ in the principal-axes system

$$\mathbf{a} = [\bar{A}_{20}, \bar{A}_{22}]^{\mathrm{T}} \tag{30}$$

and taking into account Eq. (9), we find the (2×3) matrix of partial derivatives with respect to the eigenvalues Λ_1, Λ_2 and Λ_3

$$\frac{\partial \mathbf{a}}{\partial \Lambda} = \begin{bmatrix} 0 & 0 & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \tag{31}$$

so that the variance–covariance matrix $\mathbf{C}_{\mathrm{aa}}$ becomes

$$\mathbf{C}_{\mathrm{aa}} = \frac{\partial \mathbf{a}}{\partial \Lambda} \mathbf{C}_{\Lambda\Lambda} \left(\frac{\partial \mathbf{a}}{\partial \Lambda} \right)^{\mathrm{T}} \tag{32}$$

with $\mathbf{C}_{\Lambda\Lambda}$ according to Eq. (29).

In order to create the covariance matrix $\mathbf{C}_{\mathrm{DD}}$ of the differences of normalized principal moments of inertia [Eq. (12)] and the covariance matrix $\mathbf{C}_{\mathrm{QQ}}$ of the quadrupole parameters $\bar{M}_2$ and $\cos\tilde{\gamma}$ [Eqs. (13), (14)], two new vectors are defined as

$$\mathbf{D} = [C - A, C - B, B - A]^{\mathrm{T}}, \quad \mathbf{Q} = [\bar{M}_2, \cos\tilde{\gamma}]^{\mathrm{T}} \tag{33}$$

Through the relation of these vectors to the harmonic coefficients in the vector $\mathbf{a}$ we obtain, after straightforward computations, the corresponding matrices of partial derivatives

$$\frac{\partial \mathbf{D}}{\partial \mathbf{a}} = \begin{bmatrix} -\sqrt{5} & \frac{\sqrt{15}}{3} \\ -\sqrt{5} & -\frac{\sqrt{15}}{3} \\ 0 & \frac{2\sqrt{15}}{3} \end{bmatrix},$$
$$\frac{\partial \mathbf{Q}}{\partial \mathbf{a}} = \begin{bmatrix} -\sqrt{5} & \frac{\sqrt{15}}{3} \\ \frac{4\sqrt{3}\bar{A}_{22}}{\left(\sqrt{3}\bar{A}_{20} - \bar{A}_{22}\right)^2} & \frac{4\sqrt{3}\bar{A}_{20}}{\left(\sqrt{3}\bar{A}_{20} - \bar{A}_{22}\right)^2} \end{bmatrix} \tag{34}$$

and the variance–covariance matrices $\mathbf{C}_{\mathrm{DD}}$ and $\mathbf{C}_{\mathrm{QQ}}$

$$\mathbf{C}_{\mathrm{DD}} = \frac{\partial \mathbf{D}}{\partial \mathbf{a}} \mathbf{C}_{\mathrm{aa}} \left(\frac{\partial \mathbf{D}}{\partial \mathbf{a}} \right)^{\mathrm{T}} \tag{35}$$

$$\mathbf{C}_{\mathrm{QQ}} = \frac{\partial \mathbf{Q}}{\partial \mathbf{a}} \mathbf{C}_{\mathrm{aa}} \left(\frac{\partial \mathbf{Q}}{\partial \mathbf{a}} \right)^{\mathrm{T}} \tag{36}$$

Thus, the determination of the harmonic coefficients $\bar{A}_{20}$ and $\bar{A}_{22}$ and their variance–covariance matrix $\mathbf{C}_{\mathrm{aa}}$ provides a simple method of estimating the values and uncertainties of the fundamental parameters of Eq. (33) and of the following characteristic planetary quantities. The estimation of the normalized principal moments of inertia can be obtained by involving the dynamic ellipticity H_D

$$C = -\frac{\sqrt{5}\bar{A}_{20}}{H_D} \quad \text{with } H_D = \frac{2C - A - B}{2C} \left(= -\frac{\sqrt{5}\bar{A}_{20}}{C} \right) \tag{37}$$

Substitution of Eq. (37) into Eq. (12) gives

$$A = \sqrt{5}\bar{A}_{20} - \frac{\sqrt{15}\bar{A}_{22}}{3} - \frac{\sqrt{5}\bar{A}_{20}}{H_D},$$
$$B = \sqrt{5}\bar{A}_{20} + \frac{\sqrt{15}\bar{A}_{22}}{3} - \frac{\sqrt{5}\bar{A}_{20}}{H_D} \tag{38}$$

The vector $\mathbf{I}_{\mathrm{m}}$ of the Earth's normalized principal moments of inertia

$$\mathbf{I}_{\mathrm{m}} = [C, \quad B, \quad A]^{\mathrm{T}} \quad (C > B > A) \tag{39}$$

follows from the three parameters in vector $\mathbf{b} = \left[\bar{A}_{20}, \quad \bar{A}_{22}, \quad H_D\right]^{\mathrm{T}}$. In order to create the variance–covariance matrix of $\mathbf{I}_{\mathrm{m}}$, the partial derivatives are derived

$$\frac{\partial \mathbf{I}_{\mathrm{m}}}{\partial \mathbf{b}} = \begin{bmatrix} -\frac{\sqrt{5}}{H_D} & 0 & -\frac{C}{H_D} \\ \sqrt{5} - \frac{\sqrt{5}}{H_D} & \frac{\sqrt{15}}{3} & -\frac{C}{H_D} \\ \sqrt{5} - \frac{\sqrt{5}}{H_D} & -\frac{\sqrt{15}}{3} & -\frac{C}{H_D} \end{bmatrix} \tag{40}$$

The variance–covariance matrix $\mathbf{C}_{\mathrm{II}}$ of the normalized moments of inertia then follows from

$$\mathbf{C}_{\mathrm{II}} = \frac{\partial \mathbf{I}_{\mathrm{m}}}{\partial \mathbf{b}} \mathbf{C}_{\mathrm{bb}} \left(\frac{\partial \mathbf{I}_{\mathrm{m}}}{\partial \mathbf{b}} \right)^{\mathrm{T}} \tag{41}$$

Introducing the vector $\mathbf{F}$ containing the functions α, β and γ of the principal moments of inertia used in the integration of the Euler dynamical equations (see, for example, Bretagnon et al. 1998; Hartmann et al. 1999)

$$\mathbf{F} = \left[\alpha = \tfrac{C-B}{A}, \quad \beta = \tfrac{C-A}{B}, \quad \gamma = \tfrac{B-A}{C}\right]^{\mathrm{T}} \tag{42}$$

and deriving their (3 × 3) matrix of partial derivatives

$$\frac{\partial \mathbf{F}}{\partial \mathbf{I}_{\mathrm{m}}} = \begin{bmatrix} \frac{1}{A} & -\frac{1}{A} & -\frac{\alpha}{A} \\ \frac{1}{B} & -\frac{\beta}{B} & -\frac{1}{B} \\ -\frac{\gamma}{C} & \frac{1}{C} & -\frac{1}{C} \end{bmatrix} \tag{43}$$

we obtain the variance–covariance matrix $\mathbf{C}_{\mathrm{FF}}$

$$\mathbf{C}_{\mathrm{FF}} = \frac{\partial \mathbf{F}}{\partial \mathbf{I}_{\mathrm{m}}} \mathbf{C}_{\mathrm{II}} \left(\frac{\partial \mathbf{F}}{\partial \mathbf{I}_{\mathrm{m}}}\right)^{\mathrm{T}} = \frac{\partial \mathbf{F}}{\partial \mathbf{I}_{\mathrm{m}}} \frac{\partial \mathbf{I}_{\mathrm{m}}}{\partial \mathbf{b}} \mathbf{C}_{\mathrm{bb}} \left(\frac{\partial \mathbf{F}}{\partial \mathbf{I}_{\mathrm{m}}} \frac{\partial \mathbf{I}_{\mathrm{m}}}{\partial \mathbf{b}}\right)^{\mathrm{T}} \tag{44}$$

The variance–covariance matrix $\mathbf{C}_{\mathrm{ff}}$ of the vector $\mathbf{f}$ representing the inverse geometrical polar $(1/f)$ and equatorial $(1/f_e)$ flattening

$$\mathbf{f}^{\mathrm{T}} = [\,1/f, \quad 1/f_e\,] \tag{45}$$

can be derived in a similar manner. The relation between the $\bar{A}_{20}$ coefficient and the polar flattening $1/f$ reads, in an expansion up to the second order (Heiskanen and Moritz 1967, p. 78) with $q = \omega^2 a^3/GM$

$$-\sqrt{5}\bar{A}_{20} = \frac{2f}{3} - \frac{q}{3} - \frac{f^2}{3} + \frac{2fq}{21} \tag{46}$$

Here, instead of the coefficient $\bar{C}_{20}$, the coefficient $\bar{A}_{20}$ defined in the principal-axes system is used; ω is the angular velocity of the Earth's rotation. The root of the quadratic equation [Eq. (46)] is obtained as follows:

$$f = 1 + \frac{q-u}{7}, \quad u = \sqrt{147\sqrt{5}\bar{A}_{20} + q^2 - 35q + 49} \tag{47}$$

$$1/f = \frac{7}{q + 7 - u} \tag{48}$$

Like the polar flattening is independent of the coefficient $\bar{A}_{22}$, the inverse equatorial flattening $1/f_e$ is independent of the harmonic coefficient $\bar{A}_{20}$ (Marchenko 1979)

$$1/f_e = \frac{7 - 8q}{7\sqrt{15}\bar{A}_{22}} \tag{49}$$

Equations (48) and (49) are used to build up the (2 × 3) matrix of derivatives of the components of the vector $\mathbf{f}$ with respect to the components of the vector $\mathbf{c} = \left[\bar{A}_{20}, \quad \bar{A}_{22}, \quad q\right]^{\mathrm{T}}$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{c}} = \begin{bmatrix} \frac{1029\sqrt{5}}{2t(q-u+7)^2} & 0 & \frac{7(2q-2u-35)}{2t(q-u+7)^2} \\ 0 & \frac{\sqrt{15}(8q-7)}{105\bar{A}_{22}^2} & -\frac{8\sqrt{15}}{105\bar{A}_{22}} \end{bmatrix} \tag{50}$$

that allows the computation of the variance–covariance matrix $\mathbf{C}_{\mathrm{ff}}$

$$\mathbf{C}_{\mathrm{ff}} = \frac{\partial \mathbf{f}}{\partial \mathbf{c}} \mathbf{C}_{\mathrm{cc}} \left(\frac{\partial \mathbf{f}}{\partial \mathbf{c}}\right)^{\mathrm{T}} \tag{51}$$

5 Orientation of the Earth's principal axes in the initial coordinate system (eigenvector problem)

Any eigenvector $\mathbf{X}_j$ is found as the non-trivial solution of the following (in our case 3 × 3) homogeneous system of linear algebraic equations:

$$\begin{aligned}\left(\mathbf{H} - \Lambda_j \mathbf{I}\right) \cdot \mathbf{X} \\ = (\,\mathrm{h}_1 - \Lambda_1 \mathrm{e}_1, \quad \mathrm{h}_2 - \Lambda_2 \mathrm{e}_2, \quad \mathrm{h}_3 - \Lambda_3 \mathrm{e}_3\,) \cdot \mathrm{X} = 0\end{aligned} \tag{52}$$

where $\mathbf{h}_1$, $\mathbf{h}_2$ and $\mathbf{h}_3$ are the column vectors [Eq. (21)] of the initial matrix $\mathbf{H}$ in Eq. (2); $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ are the unit vectors of the identity matrix $\mathbf{I}$ for the adopted coordinate axes. The eigenvectors $\mathbf{X}_j$ may be efficiently computed from Marchenko and Abrikosov (2001) through the vectors $\mathbf{Z}_j$ $(j = 1, 2, 3)$, which coincide with the eigenvectors but are unnormalized

$$\mathbf{Z}_j = \mathbf{P} + \Lambda_j \mathbf{S} + \Lambda_j^2 \mathbf{E} \tag{53}$$

with [$\mathbf{H}_i$ from Eqs. (22)–(24)]

$$\begin{aligned}\mathbf{P} &= \mathbf{H}_2^{\mathrm{T}} \cdot \mathbf{h}_1 + \mathbf{H}_3^{\mathrm{T}} \cdot \mathbf{h}_2 + \mathbf{H}_1^{\mathrm{T}} \cdot \mathbf{h}_3 \\ &= \mathbf{h}_1 \times \mathbf{h}_2 + \mathbf{h}_2 \times \mathbf{h}_3 + \mathbf{h}_3 \times \mathbf{h}_1\end{aligned} \tag{54}$$

$$\mathbf{S} = \mathbf{h}_1 + \mathbf{h}_2 + \mathbf{h}_3 \tag{55}$$

and

$$\mathbf{E} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = [\,1 \quad 1 \quad 1\,]^{\mathrm{T}} \tag{56}$$

The three eigenvectors $\mathbf{X}_j$ resulting from the normalization of $\mathbf{Z}_j$ are nothing other than the unit vectors pointing in the same directions as the principal axes $\bar{A}$, $\bar{B}$ and $\bar{C}$ of inertia, expressed in the adopted coordinate system. For the following error propagation it is sufficient to use the vectors $\mathbf{Z}_1$, $\mathbf{Z}_2$ and $\mathbf{Z}_3$, which allow simpler expressions for the required partial derivatives than the normalized eigenvectors $\mathbf{X}_j$.

6 Error propagation for the eigenvector problem

The elements of the vectors $\mathbf{Z}_j$ in Eq. (53) are a nonlinear function of the five harmonic coefficients in the vector $\mathbf{g}$ of Eq. (15)

$$\mathbf{Z}_j(\mathbf{g}) = \mathbf{P}(\mathbf{g}) + \Lambda_j(\mathbf{g}) \cdot \mathbf{S}(\mathbf{g}) + \Lambda_j^2(\mathbf{g}) \cdot \mathbf{E} \tag{57}$$

Carrying out the differentiation of Eq. (57) with respect to the elements of the vector $\mathbf{g}$, we obtain the necessary expression for the (3 × 5) matrices

$$\frac{\partial \mathbf{Z}_j}{\partial \mathbf{g}} = \frac{\partial \mathbf{P}}{\partial \mathbf{g}} + \Lambda_j \frac{\partial \mathbf{S}}{\partial \mathbf{g}} + \left(\mathbf{S} + 2\Lambda_j \mathbf{E}\right) \frac{\partial \Lambda_j}{\partial \mathbf{g}}, \quad (j = 1, 2, 3) \tag{58}$$

of partial derivatives, where the (3 × 5) matrix $\partial \Lambda_j / \partial \mathbf{g}$ has already been determined by Eqs. (17)–(20). Omitting all auxiliary algebraic manipulations, we give only the final relationships for the (3 × 5) matrices

$$\frac{\partial \mathbf{P}}{\partial \mathbf{g}} = (\mathbf{H}_3 - \mathbf{H}_1)\mathbf{A}_1 + (\mathbf{H}_1 - \mathbf{H}_3)\mathbf{A}_2 + (\mathbf{H}_2 - \mathbf{H}_1)\mathbf{A}_3 \tag{59}$$

$$\frac{\partial \mathbf{S}}{\partial \mathbf{g}} = \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 \tag{60}$$

where $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ are defined by Eqs. (25)–(27). The variance–covariance matrices $\mathbf{C}_{\mathbf{Z}_j\mathbf{Z}_j}(j = 1, 2, 3)$ then follow from

$$\mathbf{C}_{\mathbf{Z}_j\mathbf{Z}_j} = \frac{\partial \mathbf{Z}_j}{\partial \mathbf{g}} \mathbf{C}_{\mathrm{gg}} \left(\frac{\partial \mathbf{Z}_j}{\partial \mathbf{g}}\right)^{\mathrm{T}} \tag{61}$$

The directions of the principal axes $\bar{A}, \bar{B}$ and $\bar{C}$ of inertia will now be expressed in spherical coordinates. Let φ_j and λ_j denote the geocentric latitude and longitude where the vector $\mathbf{Z}_j$ (i.e. the principal axis) intersects the unit sphere. The spherical coordinates are computed from the components of the vectors $\mathbf{Z}_j = \left[z_1^j, \quad z_2^j, \quad z_3^j\right]^{\mathrm{T}}$

$$\varphi_j = \sin^{-1}\left(\frac{z_3^j}{\rho_j}\right), \quad \lambda_j = \tan^{-1}\left(\frac{z_2^j}{z_1^j}\right), \quad \text{with}$$
$$\rho_j = \sqrt{\mathbf{Z}_j^{\mathrm{T}}\mathbf{Z}_j} \tag{62}$$

For the error propagation, the partial derivatives of the elements φ_j, λ_j of the vectors $\psi_j = [\varphi_i, \quad \lambda_j]^{\mathrm{T}}$ with respect to the components of the vector $\mathbf{Z}_j$ are derived

$$\frac{\partial \boldsymbol{\psi}_j}{\partial \mathbf{Z}_j} = \begin{bmatrix} -\frac{z_1^j z_3^j}{\tilde{r}_j \rho_j^2} & -\frac{z_2^j z_3^j}{\tilde{r}_j \rho_j^2} & \frac{\tilde{r}_j}{\rho_j^2} \\ -\frac{z_2^j}{\tilde{r}_j^2} & \frac{z_1^j}{\tilde{r}_j^2} & 0 \end{bmatrix}, \quad \tilde{r}_j = \sqrt{(z_1^j)^2 + (z_2^j)^2} \tag{63}$$

leading to the variance–covariance matrices $\mathbf{C}_{\psi_j\psi_j}$ of the spherical coordinates

$$\mathbf{C}_{\psi_j\psi_j} = \frac{\partial \boldsymbol{\psi}_j}{\partial \mathbf{Z}_j} \mathbf{C}_{\mathbf{Z}_j\mathbf{Z}_j} \left(\frac{\partial \boldsymbol{\psi}_j}{\partial \mathbf{Z}_j}\right)^{\mathrm{T}} = \frac{\partial \boldsymbol{\psi}_j}{\partial \mathbf{Z}_j}\frac{\partial \mathbf{Z}_j}{\partial \mathbf{g}} \mathbf{C}_{\mathrm{gg}} \left(\frac{\partial \boldsymbol{\psi}_j}{\partial \mathbf{Z}_j}\frac{\partial \mathbf{Z}_j}{\partial \mathbf{g}}\right)^{\mathrm{T}} \tag{64}$$

Given the second-degree spherical harmonic coefficients $(\bar{C}_{2m}, \bar{S}_{2m})$ of the Earth's gravity field, the geometric directions of all three principal axes $(\bar{A}, \bar{B}, \bar{C})$ of inertia can be derived by means of Eqs. (53) and (62), and the uncertainty of these directions as a function of the uncertainty of the coefficients follows from Eqs. (61) and (64).

The third axis $\bar{C}$ usually is close to the axis Z of the coordinate system (X, Y, Z). In order to avoid instabilities when computing the longitude for a point with a latitude close to 90°, a polar Cartesian coordinate system (x, y) is introduced for this axis, which is also used by the International Earth Rotation Service (IERS; McCarthy 1996) to describe the position of the pole.

With $\varphi_C = \varphi_3$, $\lambda_C = \lambda_3$ for axis $\bar{C}$ we obtain with the pole distance

$$d_{\mathrm{c}} = \frac{\pi}{2} - \varphi_{\mathrm{C}} \tag{65}$$

the coordinates

$$x_C = d_C \cdot \cos \lambda_C, \qquad y_C = -d_C \cdot \sin \lambda_C \tag{66}$$

For the error propagation we define the new vectors

$$\mathbf{w}_C = [x_C, \quad y_C]^{\mathrm{T}}, \quad \boldsymbol{\psi}_C = \boldsymbol{\psi}_3 = [\varphi_C, \quad \lambda_C]^{\mathrm{T}} \tag{67}$$

and obtain by straightforward differentiation

$$\frac{\partial \mathbf{w}_{\mathbf{C}}}{\partial \boldsymbol{\psi}_{\mathbf{C}}} = \begin{bmatrix} -\frac{x_C}{d_C} & y_C \\ -\frac{y_C}{d_C} & -x_C \end{bmatrix} \tag{68}$$

the variance–covariance [$\mathbf{C}_{\psi_3\psi_3}$ taken from Eq. (64)]

$$\mathbf{C}_{\mathbf{w}_C\mathbf{w}_C} = \frac{\partial \mathbf{w}_C}{\partial \boldsymbol{\psi}_C} \mathbf{C}_{\psi_3\psi_3} \left(\frac{\partial \mathbf{w}_C}{\partial \boldsymbol{\psi}_C}\right)^{\mathrm{T}} = \frac{\partial \mathbf{w}_C}{\partial \boldsymbol{\psi}_C}\frac{\partial \boldsymbol{\psi}_C}{\partial \mathbf{Z}_C}\frac{\partial \mathbf{Z}_C}{\partial \mathbf{g}} \mathbf{C}_{\mathrm{gg}} \left(\frac{\partial \mathbf{w}_C}{\partial \boldsymbol{\psi}_C}\frac{\partial \boldsymbol{\psi}_C}{\partial \mathbf{Z}_C}\frac{\partial \mathbf{Z}_C}{\partial \mathbf{g}}\right)^{\mathrm{T}} \tag{69}$$

with

$$\frac{\partial \boldsymbol{\psi}_C}{\partial \mathbf{Z}_C} = \frac{\partial \boldsymbol{\psi}_3}{\partial \mathbf{Z}_3}, \quad \frac{\partial \mathbf{Z}_C}{\partial \mathbf{g}} = \frac{\partial \mathbf{Z}_3}{\partial \mathbf{g}} \tag{70}$$

Setting

$$\frac{\partial \mathbf{w}_{\mathbf{C}}}{\partial \boldsymbol{\psi}_{\mathbf{C}}}\frac{\partial \boldsymbol{\psi}_{\mathbf{C}}}{\partial \mathbf{Z}_{\mathbf{C}}} = \frac{\partial \mathbf{w}_{\mathbf{C}}}{\partial \mathbf{Z}_{\mathbf{C}}} \tag{71}$$

and denoting $z_i = z_i^C = z_i^3$, $\tilde{r} = \tilde{r}_C = \tilde{r}_3$ and $\rho = \rho_C = \rho_3$, we obtain

$$\frac{\partial \mathbf{w}_C}{\partial \mathbf{Z}_C} = \begin{bmatrix} \frac{z_1^2 z_3 \tilde{r} + d_C \cdot z_2^2 \rho^2}{\tilde{r}^3 \rho^2} & \frac{z_1 z_2 (z_3 \tilde{r} - d_C \cdot \rho^2)}{\tilde{r}^3 \rho^2} & -\frac{z_1}{\rho^2} \\ \frac{z_1 z_2 (d_C \cdot \rho^2 - z_3 \tilde{r})}{\tilde{r}^3 \rho^2} & -\frac{z_2^2 z_3 \tilde{r} + d_C \cdot z_1^2 \rho^2}{\tilde{r}^3 \rho^2} & \frac{z_2}{\rho^2} \end{bmatrix} \tag{72}$$

for a more explicit formulation of Eq. (69).

In matrix notation, Eq. (69) changes to

$$\mathbf{C}_{\mathbf{w}_C\mathbf{w}_C} = \frac{\partial \mathbf{w}_C}{\partial \mathbf{Z}_C}\frac{\partial \mathbf{Z}_C}{\partial \mathbf{g}} \mathbf{C}_{\mathrm{gg}} \left(\frac{\partial \mathbf{w}_C}{\partial \mathbf{Z}_C}\frac{\partial \mathbf{Z}_C}{\partial \mathbf{g}}\right)^{\mathrm{T}} \tag{73}$$

which, applying Eqs. (58) and (72), allows the estimation of the uncertainty of the location of the axis $\bar{C}$ in the polar Cartesian coordinate system (x, y).

7 Estimation of the fundamental mechanical and geometrical parameters of the Earth

The harmonic coefficients of second degree and their temporal variations given in Table 1 are extracted from the following gravity field models.

Table 1. Initial second-degree normalized harmonic coefficients $\bar{C}_{2m}$ and $\bar{S}_{2m}$, and their secular variations $\dot{\bar{C}}_{2m}$ and $\dot{\bar{S}}_{2m}$

Harmonic coefficients	JGM-3[a] (epoch: 1986)	EGM96[b] (epoch: 1986)	GRIM5-S1[c] (epoch: 1997)	GRIM5-S1CH1[d] (epoch: 1997)
$\bar{C}_{20} \cdot 10^6$	−484.16954846	−484.16537174	−484.16511551	−484.16515921
	± 0.000047	± 0.000036	± 0.000020	± 0.0000042
$\dot{\bar{C}}_{20} \cdot 10^{11}$	1.16275534	1.16275534	1.36375910	1.16553188
[1/yr]	–	–	± 0.14	± 0.04
$\bar{C}_{21} \cdot 10^6$	0.00018699	−0.00018699	−0.00016286	−0.00016522
	(± 0.000041)	(± 0.000041)	± 0.000031	± 0.0000054
$\dot{\bar{C}}_{21} \cdot 10^{11}$	–	−0.32	–	–
[1/yr]		–		
$\bar{S}_{21} \cdot 10^6$	0.00119528	0.00119528	0.00132488	0.00136313
	(± 0.000041)	(± 0.000041)	± 0.000026	± 0.0000048
$\dot{\bar{S}}_{21} \cdot 10^{11}$	–	1.62	–	–
[1/yr]		–		
$\bar{C}_{22} \cdot 10^6$	2.43926075	2.43914352	2.43931947	2.43930284
	± 0.000037	± 0.000054	± 0.000014	± 0.0000031
$\bar{S}_{22} \cdot 10^6$	−1.40026640	−1.40016684	−1.40027653	−1.40029081
	± 0.000037	± 0.000054	± 0.000014	± 0.0000031

[a] Tapley et al. (1996)
[b] Lemoine et al. (1998)
[c] Biancale et al. (2000)
[d] Reigber et al. (2001)

1. JGM-3 (Tapley et al. 1996) and EGM96 (Lemoine et al. 1998), which both result from a combination of satellite tracking, altimetry and gravimetry data (combined solutions).
2. GRIM5-S1 (Biancale et al. 2000) and GRIM5-S1CH1 (Reigber et al. 2001), which are both based purely on the analysis of satellite orbit perturbations (satellite-only solutions). GRIM5-S1CH1 differs from GRIM5-S1 by the inclusion of 41 days (November/December 2000) of CHAMP Global Positioning System (GPS) satellite-to-satellite tracking data.

The four models are not fully independent, because there are overlaps in the underlying satellite data. These correlations between the models are neglected in the following.

The time-variable coefficients in these models are referred to different epochs with a spacing of 11 years in between. EGM96 contains secular variations for $\bar{C}_{20}, \bar{C}_{21}$ and $\bar{S}_{21}$.

The other solutions include only a drift in $\bar{C}_{20}$. To be consistent, the following transformations were applied to the initial values in Table 1: (a) prediction of $\bar{C}_{2m}(t)$ and $\bar{S}_{2m}(t)$ for a common epoch 1997; (b) reduction of $\bar{C}_{20}$ to a common permanent tide system; and (c) scaling of these coefficients to common values of GM and a.

For the transformation of $\bar{C}_{20}$ (JGM3) from the zero-frequency tide system $\bar{C}_{20}^{Z}$ to the tide-free system $\bar{C}_{20}^{f}$, and later on vice versa, the relation

$$\bar{C}_{20}^{f} = \bar{C}_{20}^{Z} + 3.1108 \cdot 10^{-8} \cdot 0.3/\sqrt{5} \tag{74}$$

is applied (Rapp 1989).

Among these four models, only the solutions of GRIM5-S1 and GRIM5-S1CH1 have a complete (5×5) variance–covariance matrix $\mathbf{C}_{gg}$ of $\bar{C}_{2m}$ and $\bar{S}_{2m}$. The coefficients $\bar{C}_{21}$ and $\bar{S}_{21}$ of JGM-3 and EGM96 are not adjusted, but based on the formulae (Lambeck 1971; Reigber 1981)

$$\bar{C}_{21} = (\sqrt{3}\bar{C}_{20} - \bar{C}_{22})\bar{x}_p + \bar{S}_{22}\bar{y}_p \tag{75a}$$

$$\bar{S}_{21} = -(\sqrt{3}\bar{C}_{20} + \bar{C}_{22})\bar{y}_p - \bar{S}_{22}\bar{x}_p \tag{75b}$$

or, in approximate form

$$\bar{C}_{21} \approx \sqrt{3}\bar{C}_{20}\bar{x}_p, \quad \bar{S}_{21} \approx -\sqrt{3}\bar{C}_{20}\bar{y}_p \tag{76}$$

with the mean pole coordinates $\bar{x}_p$, $\bar{y}_p$ taken for the epoch 1986, $\bar{x}_p = 0.046'', \bar{y}_p = 0.294''$ (McCarthy 1996). This is why for the models JGM-3 and EGM96 the mean pole coordinates and their assumed uncertainties of $0.01''$ (IERS 2001) in each component were used to compute $\mathrm{var}(\bar{C}_{21})$, $\mathrm{var}(\bar{S}_{21})$ and $\mathrm{cov}(\bar{C}_{21}, \bar{S}_{21})$ by error propagation [cf. Eq. (75)] to complete the (5×5) variance–covariance matrix $\mathbf{C}_{gg}$ for these two models.

Equations (37) and (38) were used to determine the Earth's normalized principal moments of inertia A, B, C, where H_D results from the observed precession of the Earth's pole. H_D is applied as a scale factor in precession and nutation theories. Table 2 gives six current estimations of H_D and the values of the underlying precession constant p_A, and the applied standards for these constants if given in the literature. The first five values of H_D are recommended in Dehant et al. (1999) as 'the best values to be used in nutation theory'. For only two H_D values are accuracy estimates found in the literature. The sixth value was recently determined by Mathews (2000).

With H_D known, the computation of the polar moment of inertia (normalized by the factor $1/Ma^2$), $C = -\sqrt{5}\bar{A}_{20}/H_D$, the sum

$$A + B = \sqrt{5}\bar{A}_{20}\left(2 - \frac{2}{H_D}\right) \tag{77}$$

and the trace $A + B + C$ of the Earth's tensor of inertia $\mathbf{I}$

$$\mathrm{Tr}(\mathrm{I}) = A + B + C = \sqrt{5}\bar{A}_{20}\left(2 - \frac{3}{H_D}\right) = 3I_m \tag{78}$$

according to Eqs. (37) and (38) is straightforward, if the fully normalized harmonic coefficient $\bar{A}_{20}$ is given.

Table 2. Various determinations of the dynamical ellipticity H_D

Reference	Precession constant p_A [″/yr], J2000	H_D	IERS standards
Williams (1994)	50.287700	0.0032737634	N/A[b]
Souchay and Kinoshita (1996)	50.287700	0.0032737548	IERS, 1992
Hartmann et al. (1997, 1999)	50.288200 (50.287700)[a]	0.003273792489 (0.0032737600)[a]	IERS, 1992
Bretagnon et al. (1998)	50.287700	0.003273766818 $\pm$0.000000000023	IERS, 1992; IERS, 1996
Roosbeek and Dehant (1998)	50.287700	0.0032737674	IERS, 1996
Mathews (2000)	50.288018 (50.287700)[a]	0.0032737875 $\pm$0.0000000005 (0.0032737668)[a]	N/A

[a] Transformed to a common value of the precession constant [Eq. (79)]
[b] N/A = Not applicable

From this we obtain a direct dependence of A, B and C and of the mean moment of inertia I_m on the adopted gravity field model and on the treatment of the permanent tide in the $\bar{C}_{20} \approx \bar{A}_{20}$ coefficient. The parameter $B - A = 2\sqrt{15}\bar{A}_{22}/3$ is also slightly dependent on the adopted permanent tide system because $\bar{C}_{20}$ enters into the computation of the coefficient $\bar{A}_{22}$ through Eq. (9). The indirect effect of the permanent tide (caused by the deformation potential) may be either included in the $\bar{C}_{20}$ coefficient (zero-frequency tide system) or excluded from it (tide-free system). The zero-frequency tide system approximates better the real figure of the Earth. It is assumed that the H_D values also are related to the zero-frequency tide system (Bursa 1995; Groten 2000), although this problem is not discussed in the precession–nutation literature as referenced in Table 2.

From the values of the dynamic ellipticity H_D given in Table 2 (assumed to refer to J2000), the Hartmann et al. (1999) and Mathews (2000) values differ in the precession constant adopted. In order to transform the associated quantities from $p_A = 50.2882''/\mathrm{yr}$ and $p_A = 50.288018''/\mathrm{yr}$, respectively, to the common value $p_A = 50.2877''/\mathrm{yr}$, the differential relationship of Souchay and Kinoshita (1996) is used; this is

$$\mathrm{d}H_D = \frac{\delta H_D}{\delta p_A}\mathrm{d}p_A = 6.4947 \cdot 10^{-7}\mathrm{d}p_A \tag{79}$$

where $\mathrm{d}p_A$ is expressed in arcseconds per Julian century (cy); with $\mathrm{d}p_A = -0.05''/\mathrm{cy}$ and $\mathrm{d}p_A = -0.0318''/\mathrm{cy}$, respectively, we obtain the values of H_D given in brackets in Table 2.

To reduce the values of H_D from epoch J2000 to 1997, an additional correction was applied. From Eqs. (37) and (38), taking into account that the non-tidal variation dC in the moment of inertia C is a function of $\dot{\bar{C}}_{20}$ only (Yoder et al. 1983), and the condition Tr(**I**) = constant (Rochester and Smylie 1974) 'as zonal forces do not change the revolution shape of the body' (Melchior 1978), we obtain for the secular variation of H_D

$$\dot{H}_D = \frac{\dot{\bar{A}}_{20}}{\bar{A}_{20}} H_D\left(1 - \frac{2}{3}H_D\right) \approx \frac{\dot{\bar{C}}_{20}}{\bar{C}_{20}} H_D\left(1 - \frac{2}{3}H_D\right) \tag{80}$$

Numerically this results in $\dot{H}_D \approx -7.864 \cdot 10^{-11}/\mathrm{yr}$ (with $\dot{\bar{C}}_{20}$ taken from the GRIM5-S1CH1 model) amounting to $\mathrm{d}h = 2.36 \cdot 10^{-10}$ for the reduction of H_D from the year 2000 to 1997. The H_D values used in the following are the reduced ones and refer to the common precession constant given above.

The Earth's principal moments of inertia A, B and C are determined from a least-squares (LS) adjustment of the astronomic and geodetic parameters, all referred to a common permanent tide system and one epoch, 1997. As 'observations' we take (a) the six values for H_D (Table 2) and (b) the four sets (Table 3) of harmonic coefficients $\bar{A}_{20}$, $\bar{A}_{22}$ in the principal-axes system, computed from the coefficients given in Table 1 by applying Eq. (9). Table 4 gives the geodetic parameters in the tide-free system for completness. Before, the harmonic coefficients $\bar{C}_{2m}, \bar{S}_{2m}$ were all transformed to the same GM and a values as given in Table 3. The variance–covariance matrices [Eq. (32)] are taken into account in the adjustment.

Table 3. Geodetic parameters in the zero-frequency system (GM = 398 600.4415 km^3/s^2; a = 6 378 136.49 m; epoch: 1997)

Parameter	JGM-3	EGM96	GRIM5-S1	GRIM5-S1CH1
$\bar{A}_{20} \cdot 10^6$	-484.169392 ± 0.000047	-484.169388 ± 0.000036	-484.169284 ± 0.000020	$-484.1693282 \pm 0.0000041$
$\bar{A}_{22} \cdot 10^6$	2.812603 ± 0.000037	2.812452 ± 0.000054	2.812660 ± 0.000014	2.8126522 ± 0.0000031
$(C-A) \cdot 10^6$	1086.26673 ± 0.00011	1086.26652 ± 0.00010	1086.26656 ± 0.00005	1086.266649 ± 0.00001
$(C-B) \cdot 10^6$	1079.00462 ± 0.00011	1079.00480 ± 0.00010	1079.00430 ± 0.00005	1079.004412 ± 0.00001
$(B-A) \cdot 10^6$	7.26211 ± 0.00010	7.26172 ± 0.00014	7.26226 ± 0.00004	7.262237 ± 0.00001
$1/f$	298.256479 ± 0.000016	298.256480 ± 0.000013	298.256511 ± 0.000009	298.256498 ± 0.000007
$1/f_e$	$91\,438.2 \pm 1.2$	$91\,443.1 \pm 1.7$	$91\,436.4 \pm 0.4$	$91\,436.6 \pm 0.1$

Table 4. Geodetic parameters in the tide-free system ($GM = 398\ 600.4415\ \mathrm{km}^3/\mathrm{s}^2$; $a = 6\ 378\ 136.46$ m; epoch: 1997)

Parameter	JGM-3	EGM96	GRIM5-S1	GRIM5-S1CH1
$\bar{A}_{20} \cdot 10^6$	-484.165223 ± 0.000047	-484.165220 ± 0.000036	-484.165116 ± 0.000020	$-484.1651592 \pm 0.0000041$
$\bar{A}_{22} \cdot 10^6$	2.812603 ± 0.000037	2.812452 ± 0.000054	2.812660 ± 0.000014	2.8126522 ± 0.0000031
$(C-A) \cdot 10^6$	1086.25741 ± 0.00011	1086.25720 ± 0.00010	1086.25724 ± 0.00005	1086.257327 ± 0.00001
$(C-B) \cdot 10^6$	1078.99529 ± 0.00011	1078.99548 ± 0.00010	1078.99498 ± 0.00005	1078.995090 ± 0.00001
$(B-A) \cdot 10^6$	7.26211 ± 0.00010	7.26172 ± 0.00014	7.26226 ± 0.00004	7.262237 ± 0.00001
$1/f$	298.257726 ± 0.000016	298.257727 ± 0.000013	298.257758 ± 0.000009	298.257745 ± 0.000007
$1/f_e$	$91\ 438.2 \pm 1.2$	$91\ 443.1 \pm 1.7$	$91\ 436.4 \pm 0.4$	$91\ 436.6 \pm 0.1$

Applying Eq. (12) and Eq. (37), we obtain the overdetermined system of non-linear observation equations

$$\begin{aligned} \frac{2C-A-B}{2C} &= H_D^{(i)} + \varepsilon_H^{(i)} \\ \frac{1}{2\sqrt{5}}(A+B-2C) &= \bar{A}_{20}^{(j)} + \varepsilon_{20}^{(j)} \\ \frac{3}{2\sqrt{15}}(B-A) &= \bar{A}_{22}^{(j)} + \varepsilon_{22}^{(j)} \end{aligned} \tag{81}$$

with respect to the three unknown parameters, i.e. the normalized principal moments A, B and C. $H_D^{(i)}$ ($i = 1, 2, \ldots, k$), $\bar{A}_{20}^{(j)}$, and $\bar{A}_{22}^{(j)}$ ($j = 1, 2, \ldots, l$) are treated as observations, with ε being an error component.

For k (here $k = 6$) values of $H_D^{(i)}$ and for l (here $l = 4$) sets of second-degree harmonic coefficients $\bar{A}_{20}^{(j)}, \bar{A}_{22}^{(j)}$ of l gravity field models, we obtain after linearization the system of $(k + 2l)$ observation equations

$$\begin{pmatrix} -\frac{1}{2C_0} & -\frac{1}{2C_0} & \frac{A_0+B_0}{2C_0^2} \\ \frac{1}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ -\frac{3}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}A \\ \mathrm{d}B \\ \mathrm{d}C \end{pmatrix} = \begin{pmatrix} H_D^{(i)} \\ \bar{A}_{20}^{(j)} \\ \bar{A}_{22}^{(j)} \end{pmatrix} + \begin{pmatrix} \varepsilon_H^{(i)} \\ \varepsilon_{20}^{(j)} \\ \varepsilon_{22}^{(j)} \end{pmatrix} \tag{82}$$

where A_0, B_0, C_0 are some approximate values of the principal moments A, B, C, and $\mathrm{d}A$, $\mathrm{d}B$, $\mathrm{d}C$ are the corrections provided by the solution of the normal equation system following from the linearized observation equation system of Eq. (82).

The initial values A_0, B_0, C_0 in Eq. (82) are solved straightforwardly from the system of the three non-linear equations of Eqs. (81), using only three 'observations', H_D, $\bar{A}_{20}$ and $\bar{A}_{22}$, which are obtained a priori as mean values from each set of $H_D^{(i)}$, $\bar{A}_{20}^{(j)}$ and $\bar{A}_{22}^{(j)}$, respectively, to determine the three unknowns $A_0 = A$, $B_0 = B$ and $C_0 = C$.

For each of the six values $H_D^{(i)}$, an identical standard deviation of $\pm 0.456 \cdot 10^{-8}$ derived from the scattering about the mean value was used for the weighting in the adjustment, because realistic accuracy estimates for these values are not given in the literature (also see Dehant and Capitaine 1997).

The final solution, computed for the epoch 1997, is derived from the six values of $H_D^{(i)}$ plus the second-degree harmonics of the gravity field models JGM-3, EGM96, GRIM5-S1 and GRIM5-S1CH1 (6 + 8 = 14 linear equations). The adjusted Earth's principal moments of inertia A, B, C are given in Table 5. For such an adjustment of astronomic and geodetic parameters, it is characteristic that the correlation coefficients between the solved parameters, i.e. the three moments of inertia, are close to +1. In addition to the solved parameters, the other fundamental parameters of the Earth derived from the three moments of inertia are also given in Table 5, together with their accuracy estimates from error propagation (cf. previous sections). Note finally that the polar moment of inertia C and the mean moment of inertia I_m from Table 5 agree well with those of Williams (1994): $C = 0.3307007$, $I_m = 0.3299789$.

8 Dynamic reference system

8.1 Time-independent constituent

The vector $\mathbf{g}$ [Eq. (15)] of the second-degree harmonic coefficients $\bar{C}_{2m}$ and $\bar{S}_{2m}$, adopted in the Earth body-fixed frame XYZ, will be denoted by

$$\mathrm{g}_{Z'} = [\bar{A}_{20}, \bar{A}_{21}, \bar{B}_{21}, \bar{A}_{22}, \bar{B}_{22}]^{\mathrm{T}} \tag{83}$$

if given in the coordinate system $X'Y'Z'$, which is close to XYZ but with a difference in the orientation of the third axes with $Z - Z'$ being equal to the mean pole coordi-

Table 5. Results of the simultaneous adjustment of the astronomic H_D and geodetic $\bar{A}_{20}$, $\bar{A}_{22}$ parameters (zero-frequency tide system; $GM = 398\ 600.4415\ \mathrm{km}^3/\mathrm{s}^2$; $a = 6\ 378\ 136.49$ m, epoch: 1997)

Parameter	Six H_D (Table 2) + four gravity models (JGM-3, EGM96, GRIM5-S1, GRIM5-S1CH1)
Solved	
A	$0.32961433 \pm 0.00000032$
B	$0.32962159 \pm 0.00000032$
C	$0.33070060 \pm 0.00000032$
Derived	
I_m	$0.32997884 \pm 0.00000032$
$\alpha = (C-B)/A$	$(3273.5361 \pm 0.0032) \cdot 10^{-6}$
$\beta = (C-A)/B$	$(3295.4960 \pm 0.0032) \cdot 10^{-6}$
$\gamma = (B-A)/C$	$(21.9602 \pm 0.0001) \cdot 10^{-6}$
$H_D = H_D^{(i)} + \varepsilon_H^{(i)}$	$0.003273763447 \pm 0.0000000032$
$\bar{A}_{20} \cdot 10^6$	$-484.1693278 \pm 0.0000068$
$\bar{A}_{22} \cdot 10^6$	2.8126517 ± 0.0000088
$1/f$	298.256498 ± 0.000008
$1/f_e$	$91\ 436.6 \pm 0.3$

nates. Applying again the standard approach (Lambeck 1971; Reigber 1981), second-order terms will also be taken into account in the relationship

$$\mathbf{g} = \mathbf{P}_{xy} \cdot \mathbf{g}_{Z'} \tag{84}$$

where the matrix $\mathbf{P}_{xy}$ depends only on the coordinates $\bar{x}_p, \bar{y}_p$ of the mean pole and can be written in the following form:

$$\mathbf{P}_{xy} = \begin{pmatrix} 1-\frac{\bar{x}_p^2+\bar{y}_p^2}{2} & -\sqrt{3}\bar{x}_p & \sqrt{3}\bar{y}_p & \frac{\sqrt{3}(\bar{x}_p^2-\bar{y}_p^2)}{2} & -\sqrt{3}\bar{x}_p\bar{y}_p \\ \sqrt{3}\bar{x}_p & 1-\bar{x}_p^2 & \bar{x}_p\bar{y}_p & -\bar{x}_p & \bar{y}_p \\ -\sqrt{3}\bar{y}_p & \bar{x}_p\bar{y}_p & 1-\bar{y}_p^2 & -\bar{y}_p & -\bar{x}_p \\ \frac{\sqrt{3}(\bar{x}_p^2-\bar{y}_p^2)}{3} & \bar{x}_p & \bar{y}_p & 1 & 0 \\ -\frac{2\sqrt{3}\bar{x}_p\bar{y}_p}{3} & -\bar{y}_p & \bar{x}_p & 0 & 1 \end{pmatrix} \tag{85}$$

Since the determinant $\mathrm{Det}(\mathbf{P}_{xy}) \neq 0$, there exists also the inverse linear transformation

$$\mathbf{g}_{Z'} = \mathbf{P}_{xy}^{-1} \cdot \mathbf{g} \tag{86}$$

with the inverse matrix $\mathbf{P}_{xy}^{-1}$ written also in an analytical form

$$\mathbf{P}_{xy}^{-1} = \begin{pmatrix} \frac{2}{p_2p_1^2} & \frac{\sqrt{3}\bar{x}_p}{p_1^2} & -\frac{\sqrt{3}\bar{y}_p}{p_1^2} & \frac{\sqrt{3}p_3}{p_1p_2} & -\frac{2\sqrt{3}\bar{x}_p\bar{y}_p}{p_1p_2} \\ -\frac{2\sqrt{3}\bar{x}_pp_3}{3p_2p_1^2} & \frac{1-p_5}{p_1^2} & \frac{2\bar{x}_p\bar{y}_p}{p_1^2} & \frac{2\bar{x}_pp_4}{p_1p_2} & \frac{\bar{y}_p(p_5-2)}{p_1p_2} \\ \frac{2\sqrt{3}\bar{y}_pp_3}{3p_2p_1^2} & \frac{2\bar{x}_p\bar{y}_p}{p_1^2} & \frac{1+p_5}{p_1^2} & \frac{2\bar{y}_p(\bar{x}_p^2+1)}{p_1p_2} & \frac{\bar{x}_p(p_5+2)}{p_1p_2} \\ \frac{2\sqrt{3}p_5}{3p_1^2} & -\frac{\bar{x}_p(2\bar{y}_p^2+1)}{p_1^2} & -\frac{\bar{y}_p(2\bar{x}_p^2+1)}{p_1^2} & \frac{1}{p_1} & 0 \\ -\frac{4\sqrt{3}\bar{x}_p\bar{y}_p}{3p_6} & \frac{\bar{y}_p(1-p_5)}{p_6} & -\frac{\bar{x}_p(1+p_5)}{p_6} & 0 & \frac{1}{p_1} \end{pmatrix} \tag{87}$$

where the following notations are adopted:

$$\begin{aligned} p_1 &= \bar{x}_p^2 + \bar{y}_p^2 + 1 \\ p_2 &= \bar{x}_p^2 + \bar{y}_p^2 + 2 \\ p_3 &= \bar{x}_p^2 + \bar{y}_p^2 + 3 \\ p_4 &= \bar{y}_p^2 + 1 \\ p_5 &= \bar{x}_p^2 - \bar{y}_p^2 \\ p_6 &= (\bar{x}_p^2 + p_4)^2 \end{aligned} \tag{88}$$

If the coefficients $\bar{C}_{2m}$ and $\bar{S}_{2m}$ are given, Eq. (86) can be applied to compute the harmonic coefficients $\bar{A}_{2m}$ and $\bar{B}_{2m}$ related to the axis Z' (zero mean pole at epoch t_0). $\bar{A}_{21}$ and $\bar{B}_{21}$ then read

$$\bar{A}_{21} = -\frac{2\sqrt{3}\bar{x}_pp_3}{3p_2p_1^2}\bar{C}_{20} + \frac{1-p_5}{p_1^2}\bar{C}_{21} + \frac{2\bar{x}_p\bar{y}_p}{p_1^2}\bar{S}_{21} + \frac{2\bar{x}_pp_4}{p_1p_2}\bar{C}_{22} + \frac{\bar{y}_p(p_3-2)}{p_1p_2}\bar{S}_{22} \tag{89a}$$

$$\bar{B}_{21} = \frac{2\sqrt{3}\bar{y}_pp_3}{3p_2p_1^2}\bar{C}_{20} + \frac{2\bar{x}_p\bar{y}_p}{p_1^2}\bar{C}_{21} + \frac{1+p_5}{p_1^2}\bar{S}_{21} + \frac{2\bar{y}_p(\bar{x}_p^2+1)}{p_1p_2}\bar{C}_{22} + \frac{\bar{x}_p(p_5+2)}{p_1p_2}\bar{S}_{22} \tag{89b}$$

where the coefficients $\bar{A}_{21}$ and $\bar{B}_{21}$ must be zero by definition, if the axis Z' and the figure axis $\bar{C}$ coincide at t_0. From this, Eqs. (89a) and (89b) give a tool to test whether the four gravity field models used here are referred to a common axis $\bar{C}$.

Table 6 lists the obtained differences about zero and leads to the conclusion that the reference systems of the models do not exactly match (differences up to four times larger than the standard deviations). The other second-degree coefficients differ only insignificantly ($<10^{-15}$) between the two reference systems.

In order to avoid the differences in Table 6 when fixing a unique figure axis $\bar{C}$, one unique set of the coefficients $\bar{C}_{2m}$ and $\bar{S}_{2m}$ at epoch 1997 was determined from an LS adjustment of the given four sets, taking into account their variance–covariance matrices and the two natural conditions for the left-hand sides of Eq. (89): $\bar{A}_{21} = \bar{B}_{21} = 0$. For l adopted gravity models, the harmonic coefficients $\bar{A}_{2m}^{(j)}$, $\bar{B}_{2m}^{(j)}$ $(j = 1, 2, \ldots, l)$ are initially computed and treated further as observations.

Applying Eq. (86) we obtain the observation equations in the linear form

$$\mathbf{P}_{xy}^{-1} \cdot \begin{pmatrix} \bar{C}_{20} \\ \bar{C}_{21} \\ \bar{S}_{21} \\ \bar{C}_{22} \\ \bar{S}_{21} \end{pmatrix} = \begin{pmatrix} \bar{A}_{20}^{(j)} \\ \bar{A}_{21}^{(j)} \\ \bar{B}_{21}^{(j)} \\ \bar{A}_{22}^{(j)} \\ \bar{B}_{21}^{(j)} \end{pmatrix} + \begin{pmatrix} \varepsilon_1^{(j)} \\ \varepsilon_2^{(j)} \\ \varepsilon_3^{(j)} \\ \varepsilon_4^{(j)} \\ \varepsilon_5^{(j)} \end{pmatrix} \tag{90}$$

with the five unknown elements of the vector $\mathbf{g} = [\bar{C}_{20}, \bar{C}_{21}, \bar{S}_{21}, \bar{C}_{22}, \bar{S}_{22}]^{\mathrm{T}}$; $\varepsilon_i^{(j)}$ are error components. The matrix $\mathbf{P}_{xy}^{-1}$ of this system depends only on the mean pole coordinates selected for epoch 1997. The vector $\mathbf{g}$ results from the solution of the normal system following from Eq. (90) with the two additional conditions, i.e. zero left-hand sides in Eqs. (89a) and (89b).

Table 6. Harmonic coefficients $\bar{A}_{21}$ and $\bar{B}_{21}$ (expectation values are zero) at t_0 based on Eq. (89) for the adopted mean pole $\bar{x}_p = 0.040''$ and $\bar{y}_p = 0.340''$ (from IERS at epoch 1997)

Parameter	JGM-3	EGM96	GRIM5-S1	GRIM5 S1CH1
$\bar{A}_{21} \cdot 10^9$	-0.022 ± 0.041	-0.057 ± 0.041	0.002 ± 0.031	0.000 ± 0.005
$\bar{B}_{21} \cdot 10^9$	-0.183 ± 0.041	-0.005 ± 0.041	-0.054 ± 0.026	-0.015 ± 0.005

Taking the harmonic coefficients $\bar{A}_{2m}^{(j)}$ and $\bar{B}_{2m}^{(j)}$ in the $X'Y'Z'$ frame as observations from all $(l = 4)$ gravity models, we obtain the adjusted harmonic coefficients $\bar{C}_{2m}$ and $\bar{S}_{2m}$ at epoch 1997. The set of these $\bar{C}_{2m}$ and $\bar{S}_{2m}$ values (Table 7) restores exactly the adopted mean pole coordinate $\bar{x}_p = 0.040''$, $\bar{y}_p = 0.340''$ if inserted into Eq. (75)

$$\begin{aligned} \bar{x}_p &= \frac{(\sqrt{3}\bar{C}_{20} + \bar{C}_{22})\bar{C}_{21} + \bar{S}_{22}\bar{S}_{21}}{3\bar{C}_{20}^2 - \bar{C}_{22}^2 - \bar{S}_{22}^2}, \\ \bar{y}_p &= -\frac{(\sqrt{3}\bar{C}_{20} - \bar{C}_{22})\bar{S}_{21} + \bar{S}_{22}\bar{C}_{21}}{3\bar{C}_{20}^2 - \bar{C}_{22}^2 - \bar{S}_{22}^2} \end{aligned} \tag{91}$$

Thus, instead of the approximate formulae [Eq. (76)] recommended in the IERS Conventions (McCarthy 1996)

$$\bar{x}_p = \frac{\bar{C}_{21}}{\sqrt{3}\bar{C}_{20}}, \quad \bar{y}_p = -\frac{\bar{S}_{21}}{\sqrt{3}\bar{C}_{20}} \tag{92}$$

Eq. (91) is applied here since the differences between both formulae amount to $0.01''$, which is comparable to the accuracy of the mean pole coordinates.

Applying Eqs. (62) and (64), the orientation of the principal axes $\bar{A}, \bar{B}$ and $\bar{C}$ is computed for each of the four individual gravity field models and for the adjusted set of second-degree coefficients. The results are given in spherical coordinates in Table 8 and for the axis C, following Eqs. (66) and (73), also in polar Cartesian coordinates (Table 9). Note, that the uncertainty of x_c and y_c in Table 9 for the two models JGM-3 and EGM96 recovers the $0.01''$ which were initially introduced for the mean pole coordinates in Eq. (75). It has to be pointed out, however, that the directions of the principal axes are practically independent of the treat-

Table 7. Results of a simultaneous adjustment of the $\bar{C}_{2m}$, $\bar{S}_{2m}$ parameters (zero-frequency tide system; GM = 398 600.4415 km^3/s^2; a = 6 378 136.49 m; $\bar{x}_p = 0.040''$, $\bar{y}_p = 0.340''$; H_D = 0.003273763447 ± 0.0000000032; epoch: 1997)

Parameter	JGM-3, EGM96, GRIM5-S1, GRIM5-S1CH1
Solved	
$\bar{C}_{20} \cdot 10^6$	−484.169328 ± 0.000007
$\bar{C}_{21} \cdot 10^6$	−0.000165 ± 0.000009
$\bar{S}_{21} \cdot 10^6$	0.001379 ± 0.000008
$\bar{C}_{22} \cdot 10^6$	2.439303 ± 0.000006
$\bar{S}_{22} \cdot 10^6$	−1.400290 ± 0.000006
Derived	
$(C - A) \cdot 10^6$	1086.266648 ± 0.000018
$(C - B) \cdot 10^6$	1079.004412 ± 0.000018
$(B - A) \cdot 10^6$	7.262236 ± 0.000014
A	0.32961433 ± 0.00000033
B	0.32962159 ± 0.00000033
C	0.33070060 ± 0.00000033
I_m	0.32997884 ± 0.00000033
$\alpha = (C - B)/A$	$(3273.5361 \pm 0.0032) \cdot 10^{-6}$
$\beta = (C - A)/B$	$(3295.4960 \pm 0.0033) \cdot 10^{-6}$
$\gamma = (B - A)/C$	$(21.9602 \pm 0.00005) \cdot 10^{-6}$
$1/f$	298.256498 ± 0.000008
$1/f_e$	91 436.6 ± 0.2

Table 8. Spherical coordinates of the principal axes and their uncertainties (epoch: 1997)

Gravity field model	$\varphi(\bar{A})$ [degree]	$\lambda(\bar{A})$ [degree]	$\varphi(\bar{B})$ [degree]	$\lambda(\bar{B})$ [degree]	$\varphi(\bar{C})$ [degree]	$\lambda(\bar{C})$ [degree]
JGM-3	−0.000033 ± 0.000003	345.0709 ± 0.0004	0.000076 ± 0.000003	75.0709 ± 0.0004	89.999917 ± 0.000003	278.75 ± 1.92
EGM96	−0.000039 ± 0.000003	345.0712 ± 0.0006	0.000087 ± 0.000003	75.0712 ± 0.0006	89.999905 ± 0.000003	279.05 ± 1.67
GRIM5-S1	−0.000034 ± 0.000002	345.0711 ± 0.0001	0.000085 ± 0.000002	75.0711 ± 0.0001	89.999908 ± 0.000002	276.88 ± 1.33
GRIM5-S1CH1	−0.0000348 ± 0.0000004	345.07093 ± 0.00004	0.0000874 ± 0.0000003	75.07093 ± 0.00003	89.9999060 ± 0.0000003	276.78 ± 0.22
Adjusted $\bar{C}_{2m}$, $\bar{S}_{2m}$ (Table 7)	−0.0000351 ± 0.0000006	345.07094 ± 0.00006	0.0000884 ± 0.0000006	75.07094 ± 0.00006	89.9999049 ± 0.0000006	276.71 ± 0.39

Table 9. Polar Cartesian coordinates of the principal axis $\bar{C}$ and their uncertainties (epoch: 1997)

Gravity field model	x_C (0.001″)	y_C (0.001″)
JGM-3	45.4 ± 10.0	294.8 ± 10.0
EGM96	53.9 ± 10.0	338.7 ± 10.0
GRIM5-S1	39.4 ± 7.6	326.8 ± 6.4
GRIM5-S1CH1	40.0 ± 1.3	336.2 ± 1.2
Adjusted $\bar{C}_{2m}$, $\bar{S}_{2m}$ (Table 7)	40.0 ± 2.3	340.0 ± 2.1

ment of the permanent tide effect in the harmonic coefficient $\bar{C}_{20}$ (Melchior 1978).

Because all parameters from Table 7 agree with the values in Table 5, the adjusted coefficients $\bar{C}_{2m}$ and $\bar{S}_{2m}$ are used in the following to study the time-dependent components.

8.2 Secular and long-periodic variations

Here we start from the simple linear model representing the mean pole's drift

$$\left.\begin{matrix}\bar{x}_p \\ \bar{y}_p\end{matrix}\right\} = \left\{\begin{matrix}\bar{x}_p^0 \\ \bar{y}_p^0\end{matrix}\right\} + \left\{\begin{matrix}\dot{\bar{x}}_p \\ \dot{\bar{y}}_p\end{matrix}\right\}(t - t_0) \tag{93}$$

where $\bar{x}_p^0, \bar{y}_p^0$ are the mean pole coordinates at some reference epoch t_0; $\dot{\bar{x}}_p, \dot{\bar{y}}_p$ are the secular variations in $\bar{x}_p^0, \bar{y}_p^0$ valid in the vicinity of t_0. According to McCarthy and Luzum (1996) we have for $t_0 = 1950$: $\bar{x}_p^0 = -0.0007''$, $\bar{y}_p^0 = 0.1794''$, $\dot{\bar{x}}_p = 0.000862''/\mathrm{yr}$, $\dot{\bar{y}}_p = 0.003217''/\mathrm{yr}$.

Inserting Eq. (93) into Eq. (75) or Eq. (85) with $\bar{A}_{21} = \bar{B}_{21} = 0$, we can split up the resulting expressions into two parts. The first part represents the epoch values of the harmonic coefficients in Eq. (75) with $\bar{x}_p^0, \bar{y}_p^0$ used instead of $\bar{x}_p, \bar{y}_p$. The second part represents the temporal variations $\dot{\bar{C}}_{21}$ and $\dot{\bar{S}}_{21}$ in the harmonic coefficients caused by a linear drift of the mean pole

$$\dot{\bar{C}}_{21} = (\sqrt{3}\bar{C}_{20} - \bar{C}_{22})\dot{\bar{x}}_p + \bar{S}_{22}\dot{\bar{y}}_p \tag{94a}$$

$$\dot{\bar{S}}_{21} = -(\sqrt{3}\bar{C}_{20} + \bar{C}_{22})\dot{\bar{y}}_p - \bar{S}_{22}\dot{\bar{x}}_p \tag{94b}$$

$$\dot{\bar{C}}_{21} \approx \sqrt{3}\bar{C}_{20}\dot{\bar{x}}_p, \quad \dot{\bar{S}}_{21} \approx -\sqrt{3}\bar{C}_{20}\dot{\bar{y}}_p \tag{95}$$

For the other second-degree coefficients we obtain from Eq. (85) with $\bar{A}_{21} = \bar{B}_{21} = 0$

$$\dot{\bar{C}}_{20} = \dot{\bar{C}}_{22} = \dot{\bar{S}}_{22} \approx 0 \tag{96}$$

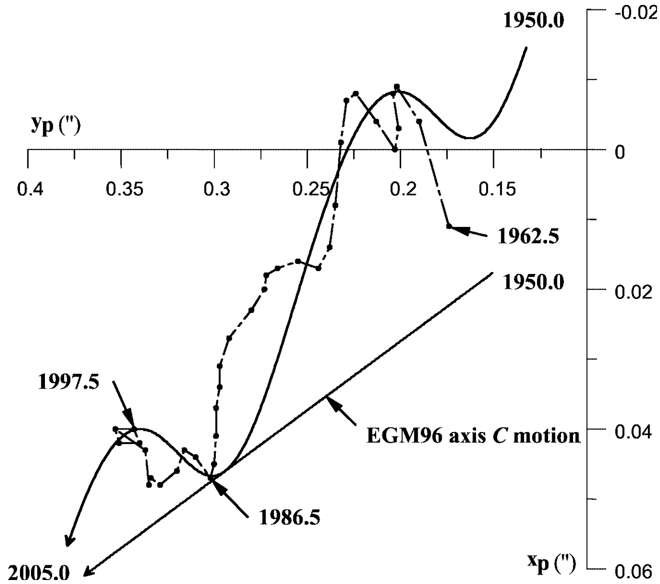

Fig. 1. Secular (*straight line*) and secular plus long-periodic (*solid line*) motion of the mean pole $\bar{C}$ vs IERS annual mean pole positions (*dash–dotted* line)

The computed values for the secular variations in the second-degree coefficients due to the McCarthy and Luzum (1996) drift in the mean pole, shown in Table 10, also confirm Eq. (96).

Now another representation for the mean pole motion (and by this for the motion of the figure axis $\bar{C}$) is introduced by adopting the following model:

$$\left.\begin{matrix}\bar{x}_p \\ \bar{y}_p\end{matrix}\right\} = \left\{\begin{matrix}\bar{x}_p^0 \\ \bar{y}_p^0\end{matrix}\right\} + \left\{\begin{matrix}\dot{\bar{x}}_p \\ \dot{\bar{y}}_p\end{matrix}\right\}(t - t_0) + \left\{\begin{matrix}A_x \\ A_y\end{matrix}\right\}\cos\left(\frac{2\pi}{P}(t - t_0) - \left\{\begin{matrix}\phi_x \\ \phi_y\end{matrix}\right\}\right) \tag{97}$$

where the components of an oscillation with a $P = 30.6$-yr period, detected by McCarthy and Luzum (1996), are added to the linear model of Eq. (93). To determine the parameters in the function, the x_p, y_p pole coordinates of the series EOP(IERS)93C01 at 0.05-yr intervals from 1962 to 2000 are then fitted to Eq. (97) by an LS adjustment with the two additional conditions that the mean pole coordinates at the epochs 1986 ($\bar{x}_p = 0.046''$, $\bar{y}_p = 0.294''$) and 1997 ($\bar{x}_p = 0.040''$, $\bar{y}_p = 0.340''$) coincide with the IERS values. The conditions allow us to connect the resulting model to the two different reference epochs used in the four gravity field models under consideration. The numerical values for the

Table 10. Secular variations per annum in the harmonic coefficients caused by the mean pole drift

Adopted linear model of the mean pole drift	$\dot{\bar{C}}_{20} \cdot 10^{16}$	$\dot{\bar{C}}_{21} \cdot 10^{11}$	$\dot{\bar{S}}_{21} \cdot 10^{11}$	$\dot{\bar{C}}_{22} \cdot 10^{16}$	$\dot{\bar{S}}_{22} \cdot 10^{16}$
McCarthy and Luzum (1996); epoch: 1950	0.449	−0.354	1.305	0.284	0.103
This study; epoch: 1997	0.728	−0.646	1.831	0.482	0.243

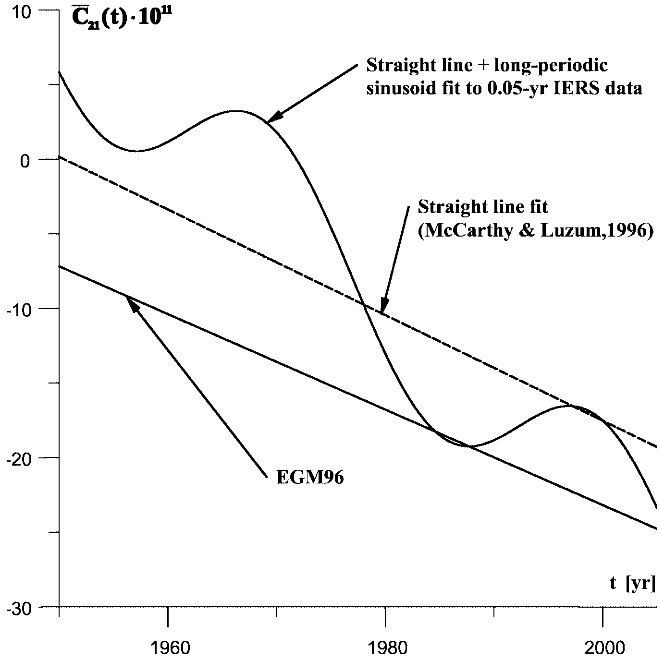

Fig. 2. Temporal changes in the harmonic coefficient $\bar{C}_{21}(t)$

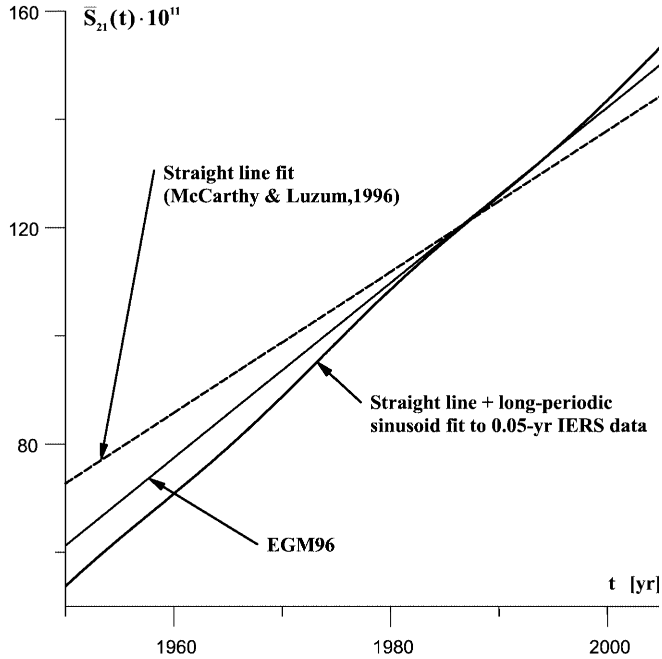

Fig. 3. Temporal changes in the harmonic coefficient $\bar{S}_{21}(t)$

Table 11. Consistent system of fundamental geodetic and astronomic parameters and their temporal differences over 38 years versus uncertainty (zero-frequency tide system; $GM = 398\ 600.4415\ \mathrm{km}^3/\mathrm{s}^2$; $a = 6\ 378\ 136.49$ m; $\bar{C}_{22}(t) = \text{const}$, $\bar{S}_{22}(t) = \text{const}$)

Parameter	$P(t_1)$ Epoch: $t_1 = 1962$	$P(t_2)$ Epoch: $t_2 = 2000$	Difference: $P(t_2) - P(t_1)$	Standard deviation σ for P at epoch
Satellite geodesy only				
$\bar{C}_{20} \cdot 10^6$	−484.169736	−484.169293	$+4.43 \cdot 10^{-10}$	$0.7 \cdot 10^{-11}$
$\bar{C}_{21} \cdot 10^6$	0.000020	−0.000175	$-1.95 \cdot 10^{-10}$	$0.9 \cdot 10^{-11}$
$\bar{S}_{21} \cdot 10^6$	0.000741	0.001436	$+6.95 \cdot 10^{-10}$	$0.8 \cdot 10^{-11}$
$(C\text{–A}) \cdot 10^6$	1086.267560	1086.266570	$-0.990 \cdot 10^{-9}$	$1.8 \cdot 10^{-11}$
$(C\text{–B}) \cdot 10^6$	1079.005324	1079.004334	$-0.990 \cdot 10^{-9}$	$1.8 \cdot 10^{-11}$
$(B\text{–A}) \cdot 10^6$	7.262236	7.262236	$-1.6 \cdot 10^{-15}$	$1.4 \cdot 10^{-11}$
$1/f$	298.256377	298.256508	$+1.31 \cdot 10^{-4}$	$0.8 \cdot 10^{-5}$
$1/f_e$	91 436.6	91 436.6	$+1.8 \cdot 10^{-5}$	0.2
Astronomy + geodesy				
H_D	0.0032737662	0.0032737632	$-3.0 \cdot 10^{-9}$	$3.2 \cdot 10^{-9}$
A	0.32961433107	0.32961433140	$+3.3 \cdot 10^{-10}$	$3.3 \cdot 10^{-7}$
B	0.32962159330	0.32962159364	$+3.3 \cdot 10^{-10}$	$3.3 \cdot 10^{-7}$
C	0.33070059863	0.33070059797	$-6.6 \cdot 10^{-10}$	$3.3 \cdot 10^{-7}$
I_m	0.32997884100	0.32997884100	$+0.1 \cdot 10^{-12}$	$3.3 \cdot 10^{-7}$
$(C\text{–B})/A$	$3273.5389 \cdot 10^{-6}$	$3273.5359 \cdot 10^{-6}$	$-3.0 \cdot 10^{-9}$	$3.2 \cdot 10^{-9}$
$(C\text{–A})/B$	$3295.4988 \cdot 10^{-6}$	$3295.4958 \cdot 10^{-6}$	$-3.0 \cdot 10^{-9}$	$3.3 \cdot 10^{-9}$
$(B\text{–A})/C$	$21.96015 \cdot 10^{-6}$	$21.96015 \cdot 10^{-6}$	$+4.0 \cdot 10^{-14}$	$0.5 \cdot 10^{-10}$

parameters in Eq. (97) are obtained as follows (with $t_0 = 1997$):

$$\bar{x}_p^0 = 0.050755'', \quad \bar{y}_p^0 = 0.342324'',$$
$$\dot{\bar{x}}_p = 0.001577''/\mathrm{yr}, \quad \dot{\bar{y}}_p = 0.004515''/\mathrm{yr}$$
$$A_x = 0.013091'', \quad A_y = 0.002331'',$$
$$\phi_x = 214.7568^\circ, \quad \phi_y = 175.5864^\circ$$

The secular variations of the second-degree coefficients following from these adjusted values $\dot{\bar{x}}_p$, $\dot{\bar{y}}_p$ are given in the second row of Table 10.

The resulting path of the mean pole axis defined by the model of Eq. (97) with the values obtained from the adjustment is compared in Fig. 1 with the annual values of the IERS mean pole coordinates (within the time interval from 1962.5 to 1997.5). The estimated root mean squares of differences of the smooth sinusoidal curve with respect to the annual IERS values are $\pm 0.006''$ and $\pm 0.012''$, respectively, which correspond to the IERS estimated level of accuracy of $\pm 0.01''$ for the annual values (IERS 2001). Figure 1 also shows the EGM96 adopted model in the mean pole evolution (straight line).

Thus, Eq. (75) together with Eq. (97) can be considered as an analytical representation of the time-dependent harmonic coefficients $\bar{C}_{21}(t)$ and $\bar{S}_{21}(t)$ with appropriate consideration of the secular and low-frequency changes in the mean pole motion and an exact agreement with the IERS mean pole coordinates for the two chosen epochs, 1986 and 1997.

Figures 2 and 3 show the secular [in the case of the McCarthy and Luzum (1996) and the EGM96 model] and the secular plus long-periodic (this study) changes in $\bar{C}_{21}(t)$ and $\bar{S}_{21}(t)$ deduced from the adopted model of mean pole motion. It can be seen that Eq. (97) gives a better long-term representation of the mean pole motion (Fig. 1) and therefore also an improved approximation of the $\bar{C}_{21}, \bar{S}_{21}$ temporal variations compared to linear drift models. This especially holds for the $\bar{C}_{21}$ component (Fig. 2).

9 Conclusions

In conclusion, Table 11 summarizes the geodetic and astronomic fundamental Earth parameters given at two different epochs in order to demonstrate their temporal changes caused by $\bar{x}_p(t)$, $\bar{y}(t)$, and the secular variation $\dot{\bar{C}}_{20}$, taken from the GRIM5-S1CH1 model. The condition $dA = dB = -dC/2$ to conserve the trace of the inertial tensor when changing $\bar{C}_{20}$ was applied.

All these values represent one consistent system of parameters, which were found by a simultaneous adjustment of the most recent geodetic and astronomic observations. Nevertheless, the comparison of their time evolution with the standard deviations σ gives significant results only for the first set of parameters coming purely from satellite geodesy. Due to the uncertainty in the astronomic dynamic ellipticity H_D, the second parameter subset reveals only insignificant temporal changes over the given time interval of 38 years.

It has to be noted that the geodetic observations, represented by the four global gravity field models, are not completely independent due to overlaps in the underlying satellite tracking data.

References

Biancale R, Balmino G, Lemoine J-M, Marty J-C, Moynot B, Barlier F, Exertier P, Laurain O, Gegout P, Schwintzer P, Reigber Ch, Bode A, König R, Massmann F-H, Raimondo JC, Schmidt R, Zhu SY (2000) A new global Earth's gravity field model from satellite orbit perturbations: GRIM5-S1. Geophys Res Lett 27: 3611–3614

Bretagnon P, Francou G, Rocher P, Simon JL (1998) SMART97: a new solution for the rotation of the rigid Earth. Astron Astrophys 329: 329–338

Bursa M (1995) Primary and derived parameters of common relevance of astronomy, geodesy, and geodynamics. Earth, Moon Plan 69: 51–63

Dehant V, Capitaine N (1997) On the precession constant: values and constraints on the dynamical ellipticity; link with Oppolzer terms and tilt-over-mode. Celest Mech Dynam Astron 65: 439–458

Dehant V, Arias F, Bizouard Ch, Bretagnon P, Brzezinski A, Buffett B, Capitaine N, Defraigne P, DeViron O, Feissel M, Fliegel H, Forte A, Gambis D, Getino J, Gross R, Herring T, Kinoshita H, Klioner S, Mathews PM, McCarthy D, Moisson X, Petrov S, Ponte RM, Roosbeek F, Salstein D, Schuh H, Seidelmann K, Soffel M, Souchay J, Vondrak J, Wahr J, Wallace P, Weber R, Williams J, Yatskiv Y, Zharov V, Zhu SY (1999) Considerations concerning the non-rigid Earth nutation theory. Celest Mech Dynam Astron 72: 245–309

Groten E (2000) Parameters of common relevance of astronomy, geodesy, and geodynamics. J Geod 74: 134–140

Hartmann T, Soffel M (1997) A new nutation series for a rigid Earth model. In: Wytrzyszczak IM, Lieske JH, Feldman RA (eds.) Dynamics and Astrometry of Natural and Artificial Celestial Bodies, Kluwer Academic Publishers, The Netherlands 287–293

Hartmann T, Soffel M, Ron C (1999) The geophysical approach towards the nutation of a rigid Earth. Astron Astrophys, Suppl Ser 134: 271–286

Heiskanen WA, Moritz H (1967) Physical geodesy. Freeman, San Francisco

International Earth Rotation Service (2001) Polar motion. International Earth Rotation Service, Web site www.iers.org/iers/earth

Kinoshita H, Souchay J (1990) The theory of the nutation for the rigid earth model at the second order. Celest Mech Dynam Astron 48: 187–265

Lambeck K (1971) Determination of the Earth's pole of rotation from laser range observations to satellites. Bull Géod 101: 263–281

Lemoine FG, Kenyon SC, Factor JK, Trimmer RG, Pavlis NK, Chinn DS, Cox CM, Klosko SM, Luthcke SB, Torrence MH, Wang YM, Williamson RG, Pavlis EC, Rapp RH, Olson TR (1998) The development of the joint NASA GSFC and the National IMagery and Mapping Agency (NIMA) geopotential model EGM96. NASA tech paper NASA/TP-1998-206861, Goddard Space Flight Center, Greenbelt, MD

Marchenko AN (1979) The gravitational quadrupole of a planet. Sov Astron Lett 5: 106–107

Marchenko AN (1998) Parameterization of the Earth's gravity field. Point and line singularities. Lviv Astron Geod Soc, Lviv

Marchenko AN, Abrikosov OA (2001) Evolution of the Earth's principal axes and moments of inertia: the canonical form of solution. J Geod 74: 655–669

Mathews PM (2000) Improved models for precession and nutation. In: Johnston KJ, McCarthy DD, Luzum BJ, Kaplan GH (eds.) Towards Models and constants for Sub-Microarcsecond Astronomy. Proc IAU Colloquium 180, U.S. Naval Observatory Special Publications, USNO Washington 212–222

McCarthy DD (1996) IERS Conventions. IERS Technical Note 21, Observatoire de Paris, Paris

McCarthy DD, Luzum BJ (1996) Path of the mean rotational pole from 1899 to 1994. Geophys J Int 125: 623–629

Melchior P (1978) The tides of the planet Earth. Pergamon, Oxford

Rapp RH (1989) The treatment of permanent tidal effects in the analysis of satellite altimeter data for sea surface topography. Manuscr Geod 14: 368–372

Reigber Ch (1981) Representation of orbital element variations and force function with respect to various reference systems. Bull Géod 55: 111–131

Reigber Ch, Balmino G, Schwintzer P, Biancale R, Barthelmes F, Bode A, König R, Lemoine J-M, Loyer S, Neumayer K-H, Perosanz F (2001) Global gravity field recovery with CHAMP. Paper presented at the IAG 2001 Scientific Assembly, Budapest, 2–7 September 2001

Rochester MG, Smylie DE (1974) On changes in the trace of the Earth's inertia tensor. J Geophys Res 79(32): 4948–4951

Roosbeek F, Dehant V (1998) RDAN97: an analytical development of rigid Earth nutation series using the torque approach. Celest Mech Dynam Astron 70: 215–253

Souchay J, Kinoshita H (1996) Corrections and new developments in rigid Earth nutation theory: I. Lunisolar influence including indirect planetary effects. Astron Astrophys 312: 1017–1030

Tapley BD, Watkins MM, Ries JC, Davis GW, Eanes RJ, Poole SR, Rim HJ, Schutz BE, Shum CK, Nerem RS, Lerch FJ, Marshall JA, Klosko SM, Pavlis NK, Williamson RG (1996) The joint gravity model 3. J Geophys Res 101(B12): 28 029–28 049

Williams JG (1994) Contributions to the Earth's obliquity rate, precession and nutation. Astron J 108: 711–724

Yoder CF, Williams JG, Dickey JO, Schutz BE, Eanes RJ, Tapley BD (1983) Secular variation of Earth's gravitational harmonic J2 coefficient from Lageos and nontidal acceleration of Earth rotation. Nature 303: 757–762