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Gravity/magnetic potential of uneven shell topography

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Abstract. A fast spherical harmonic approach enables the computation of gravitational or magnetic potential created by a non-uniform shell of material bounded by uneven topographies. The resulting field can be evaluated outside or inside the sphere, assuming that density of the shell varies with latitude, longitude, and radial distance. To simplify, the density (or magnetization) source inside the sphere is assumed to be the product of a surface function and a power series expansion of the radial distance. This formalism is applied to compute the gravity signal of a steady, dry atmosphere. It provides geoid/gravity maps at sea level as well as satellite altitude. Results of this application agree closely with those of earlier studies, where the atmosphere contribution to the Earth's gravity field was determined using more time-consuming methods.

Keywords: Geopotential modeling – Spherical harmonics – Forward problem

1 Introduction

The deep structure of the Earth can be modeled as concentric shells of different densities and shapes, separated by irregular boundaries of complex morphology such as land and sea floor topographies. In order to compute the gravity signal from a given source geometry and exact theory, we can sum the contributions of each individual block of material, such as constant density topography columns (Calmant 1994; Ramillien and Wright 2000), but this process is time consuming, especially when the number of observations increases. In this paper, an alternative procedure for calculation of observed fields is presented. Mathematical expressions for gravitational and magnetic potentials due to the presence of an uneven shell interface are derived using 'external' or 'internal' spherical harmonics. This approach is similar to the one used by Parker (1972) in Cartesian geometry where Fourier transforms were used. Here, computation of gravity/magnetic fields is generalized to the entire terrestrial sphere using spherical harmonics and without assumptions on the geometry of sources such as topography, density, or magnetization.

Sections 2 and 3 deal with the gravitational and magnetic cases respectively. Section 3.2 presents an application of this formalism to evaluate gravity anomaly due to the atmosphere, taking land topography into account and assuming that the atmosphere is steady, free of lateral variations, and dry. For validation, these results are compared with previous calculations made by Anderson et al. (1975), who used a classical $5^\circ \times 5^\circ$ subzone model of atmosphere, and assumed that air density decreases linearly with altitude.

2 Method: the spherical harmonic approach

From a given distribution of sources (density or magnetization) inside a spheroid or a shell, the first task is to derive the expression of the corresponding outer and inner potential functions (i.e. forward problem). The sphere volume is assumed to be bounded by uneven surface topographies whose amplitudes h remain small compared to the mean radius of the whole sphere R_0 (i.e. $h/R_0 \ll 1$). The situation is presented in Fig. 1 for the cases of both a full and a hollow sphere in the geocentric reference system, where R is the radial distance, and θ and λ stand for the co-latitude and longitude respectively. The point of observation for the potential is located at the distance $R = a$, far enough from the reference level of the topography at $R = R_0$, so that the condition $|a - R_0| < h$ is always fulfilled. The gravitational potential is continuous in the whole space despite the fact that the Green function is singular (see e.g. Nagy et al. 2000).

Distributions of the sources inside the volume are generally continuous functions of the spatial variables R,

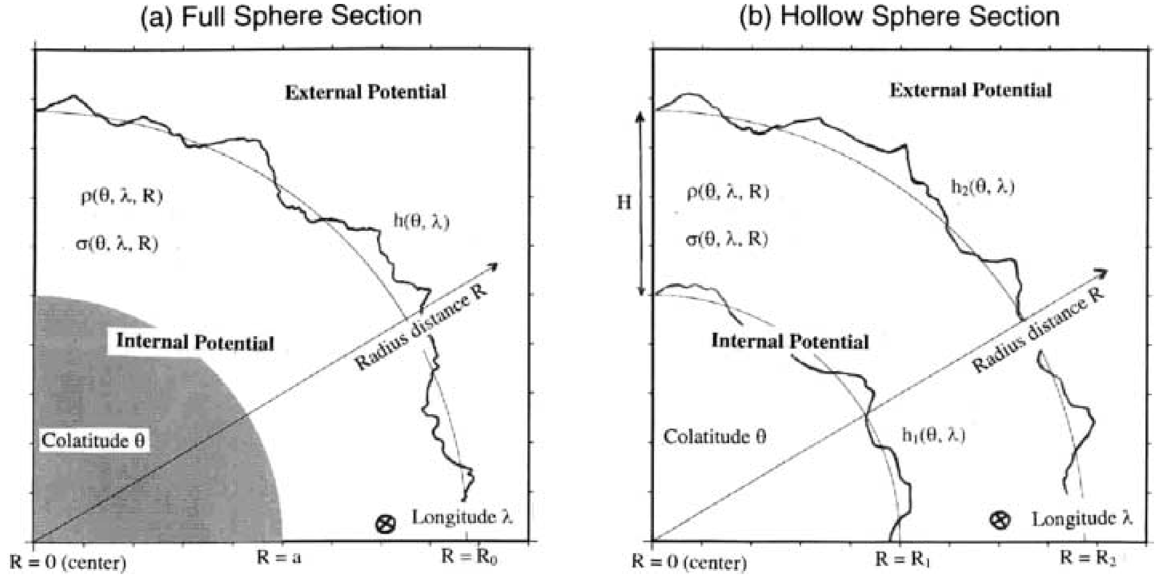

Fig. 1. Example of a planet bounded by uneven topographies, whose amplitudes are assumed to be smaller than the mean radius of the sphere R_0. Coordinates are expressed in the geocentric reference system: θ, λ, R, for co-latitude, longitude, and radial distance respectively. Density and magnetization distributions are $\rho(\theta, \lambda, R)$ and $\sigma(\theta, \lambda, R)$. **a** In the case of a 'full' volume, these sources of anomaly are defined from the center of the planet ($R = 0$). **b** For a 'hollow' sphere (i.e. a shell of material), ρ and σ depend not on R but on the altitude (i.e. the distance from the base of the shell located at $R = R_1$). Here the goal is to derive expressions for the potential at (θ, λ, R) when (i) $R > R_0$ or $R > R_2$ (i.e. external potential), and (ii) $R < R_0$ or $R < R_1$ (i.e. internal potential), assuming the volume distributions of source ρ and σ are known

θ, and λ, but their mathematical expressions may be so complicated that it is difficult to integrate them to obtain useful expressions in terms of spherical harmonics. In order to ease this derivation of harmonic coefficients, they are merely replaced by products of radial and surface functions which can be then integrated separately. Surface terms, like topographies, are continuous and integrated easily in spherical harmonics analysis. Radial terms are arbitrarily chosen as numerical approximations using power series expansions versus R to simplify: it is more convenient and faster to deal with constant coefficients of a development and power functions to integrate. Note that in practice we can also use more suitable developments for the radial term, such as 'negative' power series (i.e. $1/R, 1/R^2, \ldots$) or other simple basis functions.

Detailed derivations of the general expressions for the gravitational and magnetic potentials are presented in Appendixes A and C respectively. Harmonic coefficients are computed as the sums of those of the successive powers of the topographies, and weighted by the radial coefficients. In Appendix B, it is shown that the planar approximation of the final general expression [Eq. (A14a)] for outer gravitational potential coefficients is equivalent to the one proposed earlier by Parker (1972).

Since the general equations [Eqs. (A14a)–(A14c), (A17a) and (A17b), (C5a)–(C5c), (C12a)–(C12c), and (C13a)–(C13d)] are valid if the amplitude of the topography remains smaller than the radius of the sphere, the sum of the harmonic coefficients of ℓ-powers of h/R_0 converges naturally. In practice, convergence of these iterative expressions depends strongly on the chosen series expansion of the radial function, and more precisely on the absolute values of the coefficients α_k from Eqs. (A7) and (C1), and β_k from Eq. (A16) ($k = 0, 1, 2, \ldots$). This is ensured ideally when the coefficients decrease monotonically and fast enough with the power k, or for a truncated development (for example, when α_k or β_k are constant or become very small if $k > k_{\min}$ and $k < k_{\max}$). In that particular case, only a few iterations ($k = 0, 1, 2, \ldots$) are needed to reach a stable harmonic spectrum for the potential. This is illustrated by the following numerical example of the atmosphere density versus altitude (see the fast convergence of the computed spectrum in Fig. 2), where the coefficients associated with the exponential density function are so small when $k > 2$ that the computation can be stopped after the first three terms of the sum. Under other conditions on the series of radial coefficients, convergence does not occur, especially if the coefficients α_k or β_k increase drastically with k. Nevertheless, in the case of a complicated distribution of sources, the whole sphere volume can be divided into several domains where the conditions of convergence hold, and the expressions of potential can be extended using piecewise polynomial radial functions which are continuous inside each domain.

3 Numerical example: application to the terrestrial atmosphere

The formalism developed above is now applied to the real case of the atmosphere of our planet. Its mass is only about 10^{-6} that of the entire solid Earth, so its

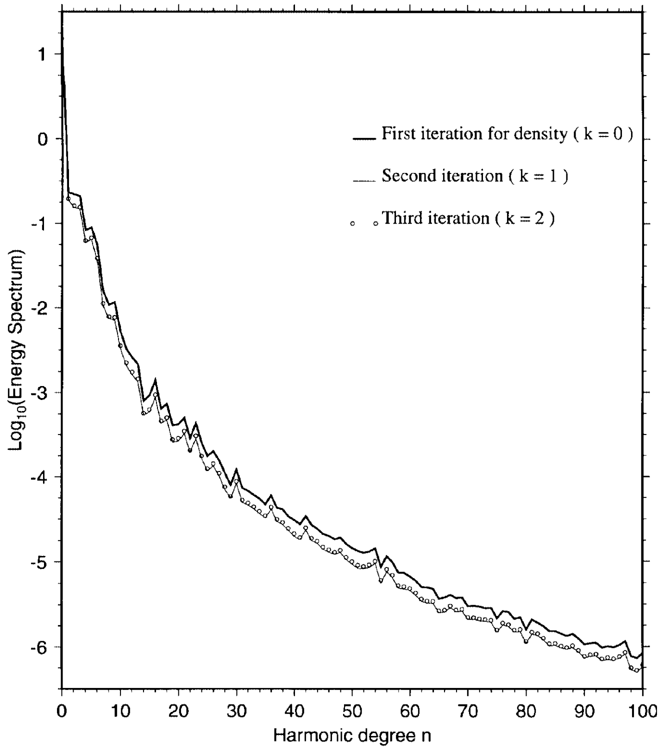

Fig. 2. Results of computing gravitational potential coefficients of atmosphere: fast convergence of the energy spectrum versus the degree n and for the three first terms in power series for density, down to a stable estimate

contributions to the static gravity field are much smaller than those from the rocky part of the planet. The reference level is the sea surface of altitude zero, or a perfect sphere surface of constant radius $R = 6371$ km.

The dry atmosphere shell is assumed to consist of a homogenous mixture of gas (nitrogen, oxygen, and rare gases). In order to simplify, we make these following crude assumptions: (1) the atmosphere consists only of dry air, so it contains no water vapor; (2) the atmosphere is in a steady state and there is no horizontal movement of air (no wind); (3) its density does not depend upon the geographical position on Earth (no dependency on longitude and latitude); (4) we consider only the first 40 km of the atmosphere because it contains 98% of its total mass (see Anderson et al. 1975); and (5) its density decays exponentially with altitude h from sea level, where the mean air density is equal to $\rho_0 = 1.225\,\mathrm{kg/m^3}$ on the reference sphere:

$$\rho(h) = \rho_0 \mathrm{e}^{-h/H} \tag{1}$$

where the parameter H is the scale height, and its value is about 8 km. In Eq. (1), the exponential term offers a very simple development into power series of the altitude h. Thus, referring to Appendix A, it is straightforward to determine the density coefficients α_k and β_k in the previous general equations for the gravitational potential. However, the above assumptions are not necessarily realistic for the complex moving wet atmosphere.

3.1 Gravity effects of atmosphere layer detected by satellite

Using Eq. (A17a) for the case of a dry and steady air layer, we can compute the external potential field observed by a satellite orbiting at an altitude of 400 km. The air layer is bounded by an uneven lower topography which is zero over ocean areas and corresponds to continent topography over lands, as well as an upper topography located at 40-km altitude and assumed to be smooth. Topography over continental areas is obtained from the 5-minute ETOPO-5 global grid (National Geophysical Data Center 1988). Vertical variations of dry air density are given by Eq. (1), once developed in powers of the altitude h, and from 0 (sea level) to 40-km height. As suggested by previous tests (Fig. 2), computation could be limited to the first three terms for density variations ($k = 0, 1, 2$) to reach a stable energy spectrum. More terms ($k > 2$) would not provide more accuracy since they are numerically too small to be significant. Harmonic coefficients of both topographies were computed by SPHEREPACK 3.0 software (Adams and Swarztrauber 1997) on a half-degree global grid. The result of this global simulation is presented in Fig. 3, and shows that the atmospheric effect detected from space is in the range of a few millimeters on the geoid. The contour interval is 0.05 mm. Negative values are still correlated to relief; the minimum is -1.1 mm over the Himalayas, and the maximum is $+0.5$ mm over the Central Pacific Ocean. Note that due to upward continuation operation, this map is smoother than the one presented in Fig. 4, which is computed at sea surface. Continental topography has a strong effect, especially over major relief such as the Himalayas where the air layer is thinner. In these areas, the geoid contribution is therefore negative.

3.2 Gravity effects of atmosphere layer observed at sea level

From the same assumptions and methodology, but using Eq. (A17b), the geoid contribution of atmosphere at sea level can be derived from the simple dry air model. The corresponding map is displayed in Fig. 4. Values at sea and away from the coasts are around 5.6 m, and less on continents due to the effect of the relief. These results are based on a spherical harmonic analysis degree and order 72. Geoid heights are expressed in meters, and the contour interval is 0.5 cm. Low potential values observed on high land topography are due to the lack of air mass. Away from the coast at sea, the atmospheric contribution to geoid is around 5.5–5.6 m. There is a good agreement between our result and the values proposed by Anderson et al. (1975) using another method of computation.

A map of the vertical component of the gravity gradient was also derived using Eq. (A15d) and from harmonic coefficients of internal potential previously computed. This is presented in Fig. 5. Computation is based on a spherical harmonic analysis to degree and

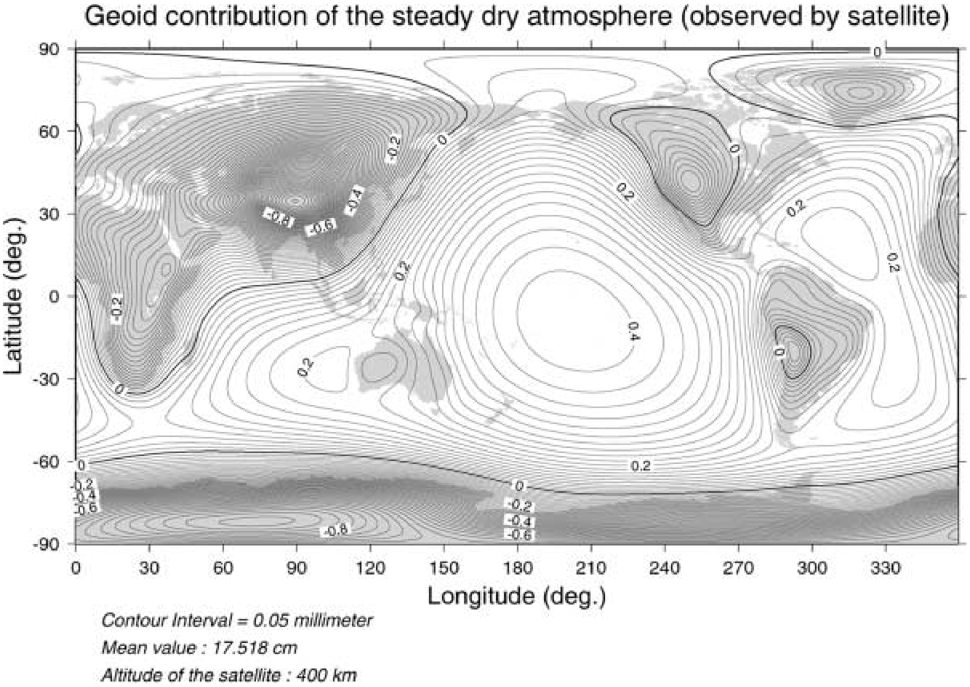

Fig. 3. Map of geoid contribution of a 40-km dry and steady atmosphere observed at satellite altitude of 400 km. Unit: mm

order 20; contour interval is 10 μGal. The anomaly is around zero at sea level and correlated negatively on continental areas. The minimum value is located in the Himalayas and reaches −200 μGal. Values are comparable to those from Anderson et al. (1975). Gravity anomaly is nearly zero over the ocean and negative on continents due to the ‘lack’ of atmospheric mass. Extreme values of −200 μGal are achieved over high mountains. Values from these two global maps evaluated are very consistent with the ones found by Anderson et al. (1975), who computed the atmosphere effect using a more time-consuming classical approach. Their method consisted of summing all the gravity effects of individual dry-air blocks, and assuming that air density was constant in each block of atmosphere.

4 Conclusion

We have derived useful and practical equations to compute potentials from uneven shell topographies associated with complicated distribution of density or magnetization, and using spherical harmonics. This ‘spectral’ approach presented in this paper was successfully implemented to evaluate globally the gravity contribution of a simplified dry and steady atmosphere

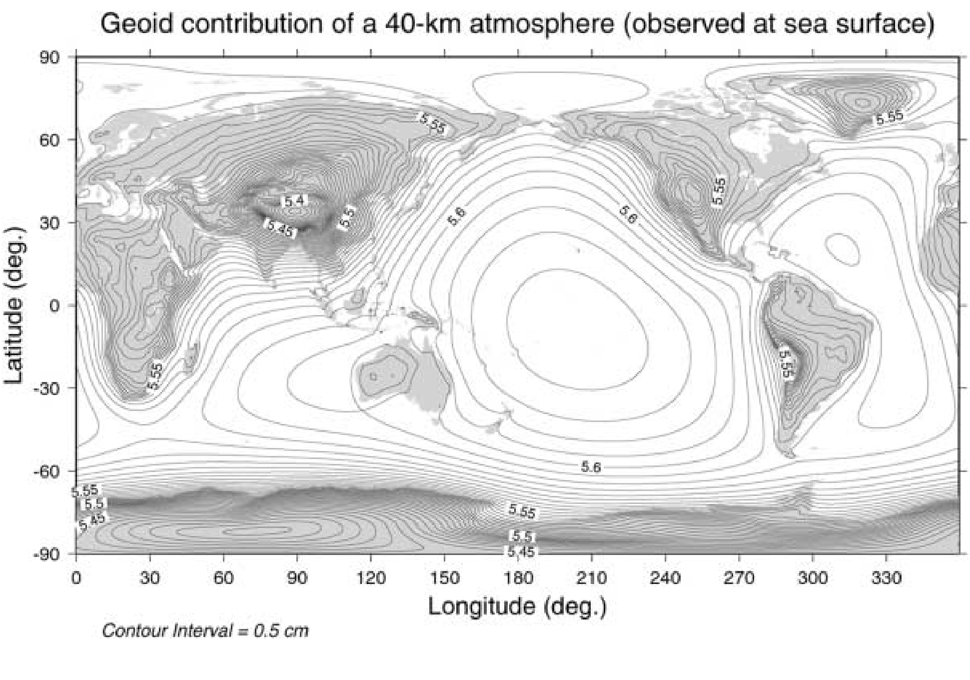

Fig. 4. Global map of geoid contribution of a 40-km thick dry and steady atmosphere observed at sea level. Unit: m

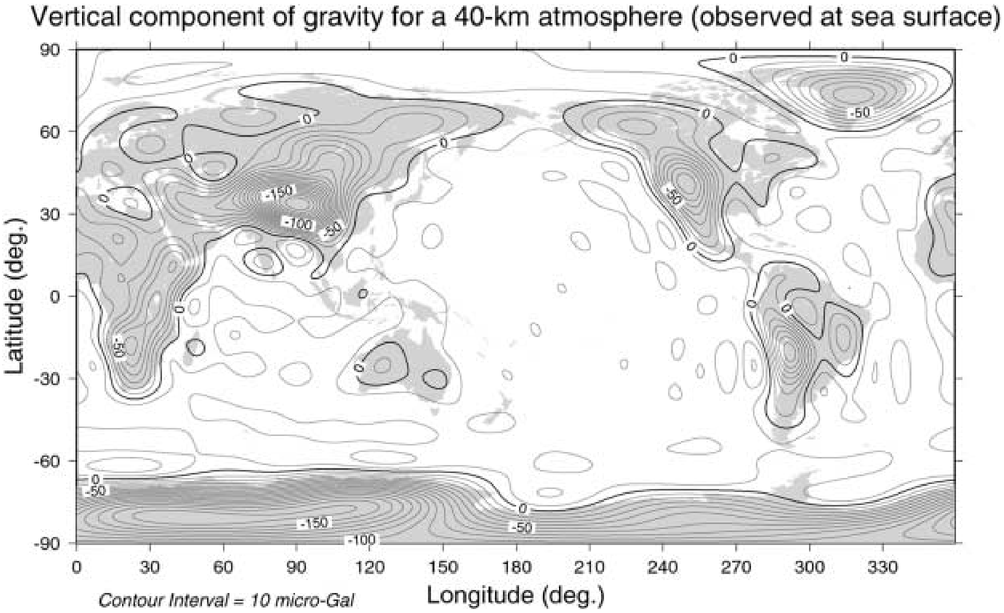

Fig. 5. The vertical component of the gravity gradient due to the dry atmosphere mass observed at sea level. Unit: μGal

as it would be measured at sea or by a satellite. The technique can also be applied to other geophysical problems, where we have to calculate the gravity/magnetic anomaly created by interface topographies deep inside the Earth or other planets. Possible applications could be in the mapping of the effect of the crust/mantle interface (assuming a Pratt isostasy model), and in prospecting methods to evaluate the global or regional effect of buried sources of anomaly. The anomaly generated by a complicated body may be computed as the sum of the contributions of its simple parts limited by different interfaces and when the source can be approximated by piecewise continuous functions.

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Appendix A: spherical harmonic coefficients of the gravity potential

General expressions for gravitational potential created by topography

Let us consider topographic heights $h(\theta', \lambda', R')$ at co-latitude θ', longitude λ', and mean distance R' from the center of the planet, such that the amplitude of $h(\theta', \lambda', R')$ is much smaller than the radius distance R'. The gravitational potential V due to the topographic mass distribution $\mathrm{d}m(\theta', \lambda', R')$ inside the sphere volume Ω, observed at the position $(\theta, \lambda, R = a)$ is given by Newton's formula of attraction

$$V(\theta, \lambda, a) = G \int\int\int_{\Omega} \frac{\mathrm{d}m(\theta', \lambda', R')}{\xi(\theta, \lambda, a, \theta', \lambda', R')} \tag{A1}$$

where G is the gravitational constant ($6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$) and ξ the distance operator between the locations (θ, λ, a) and (θ', λ', R') on the terrestrial sphere.

The mass $\mathrm{d}m(\theta', \lambda', R')$ is merely the product of its mean density $\rho(\theta', \lambda', R')$ and the small volume $\mathrm{d}v$ of this mass element, i.e.

$$\mathrm{d}m(\theta', \lambda', R') = \rho(\theta', \lambda', R')\mathrm{d}v = \rho(\theta', \lambda', R')R'^2 \sin\theta' \mathrm{d}\theta' \mathrm{d}\lambda' \mathrm{d}R' \tag{A2}$$

In spherical coordinates, the distance operator can be expressed as

$$\xi(\theta, \lambda, a, \theta', \lambda', R') = \sqrt{a^2 + R'^2 - 2aR' \cos\varphi} \tag{A3}$$

where φ is the distance in radians between the geographical locations (θ, λ) and (θ', λ'), given by the well-known relation in the spherical triangle

$$\cos\varphi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\lambda - \lambda') \tag{A4}$$

Hence, the general expression of the Newtonian potential from Eq. (A1) becomes

$$V(\theta, \lambda, a) = G \int_0^{2\pi} \int_0^{\pi} \sin\theta' \mathrm{d}\theta' \mathrm{d}\lambda' \int_R \frac{\rho(\theta', \lambda', R')R'^2 \mathrm{d}R'}{\sqrt{a^2 + R'^2 - 2aR'\cos\varphi}} \tag{A5}$$

The inverse of the distance ξ is a generating function for the Legendre polynomials P_n of degree n. Then ξ can be expanded as an n-power series of the ratio of the radii a and R', where P_n, are the coefficients of the development. Let us consider the following two cases: (i) when the point of observation is above the level R' of the topography, in other words $a > R'$; and (ii) when the point of observation is under the topography inside the Earth, i.e. $a < R'$ (Heiskanen and Moritz 1967, p 33)

$$a > R' \; \xi^{-1} = a^{-1}\sqrt{1 + \left(\frac{R'}{a}\right)^2 - 2\frac{R'}{a}\cos\varphi}^{\,-1} = \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{R'}{a}\right)^n P_n(\cos\varphi) \tag{A6a}$$

$$a < R' \; \xi^{-1} = R'^{-1}\sqrt{1 + \left(\frac{a}{R'}\right)^2 - 2\frac{a}{R'}\cos\varphi}^{\,-1} = \frac{1}{R'}\sum_{n=0}^{\infty}\left(\frac{a}{R'}\right)^n P_n(\cos\varphi) \tag{A6b}$$

In order to model the density variations inside Earth, let us assume that the density function ρ is the power series of the radial distance R' times a surface density function μ which only depends on geographical coordinates θ' and λ'

$$\rho(\theta', \lambda', R') = \rho_0\mu(\theta', \lambda')\sum_{k=0}^{\infty}\alpha_k R'^k \tag{A7}$$

where ρ_0 is a given density constant, and α_k are the real coefficients of the expansion.

By replacing Eqs. (A6a), (A6b), and (A7) in the last expression of V, we obtain new equations of the potential observed outside the Earth $(a > R')$ and for the whole sphere

$$V(\theta, \lambda, a) = \frac{G\rho_0}{a}\int_0^{2\pi}\int_0^{\pi}\sin\theta' \mathrm{d}\theta' \mathrm{d}\lambda' \sum_{n=0}^{\infty} a^{-n}P_n(\cos\varphi) \times \mu(\theta', \lambda')\sum_{k=0}^{\infty}\alpha_k\int_{R'} R'^{n+k+2}\mathrm{d}R' \tag{A8a}$$

as well as the one observed inside the Earth $(a < R')$, for the spherical shell which contains the topography

$$V(\theta, \lambda, a) = G\rho_0\int_0^{2\pi}\int_0^{\pi}\sin\theta' \mathrm{d}\theta' \mathrm{d}\lambda' \sum_{n=0}^{\infty} a^{n}P_n(\cos\varphi) \times \mu(\theta', \lambda')\sum_{k=0}^{\infty}\alpha_k\int_{R'} R'^{1+k-n}\mathrm{d}R' \tag{A8b}$$

The next step is to evaluate the last integrals versus the radial distance R. Finding a primitive function is easy for Eq. (A8a) since $n + k + 2$, the exponent of the radial distance R, is always a positive integer. In that case, the integration provides another power series of R. It is the same situation for Eq. (A8b) if n is different of $k + 2$. If the degree n equals $k + 2$, we have to consider the primitive of $\mathrm{d}R/R$, which simply corresponds to a natural logarithm.

For the external potential of Eq. (A8a), integration of the radial term is made from the center of the Earth $R = 0$ to the irregular surface $R = R_0 + h(\theta', \lambda')$, where h is the surface topography located at the mean radial distance R_0. In the case of computing the internal potential [Eq. (A8b)], the integration goes from $R = a$ (i.e. level of observation) to $R = R_0 + h(\theta', \lambda')$. Then, dividing these primitives by the mean reference distance R_0, they can be expressed as functions of h/R_0. Since this ratio is previously assumed to be small ($h \ll R_0$), we now expand the 'power' and 'log' functions using the binomial relation and an equivalent series for logarithms, such as

$$\left(1 + \frac{h}{R_0}\right)^{\eta} = 1 + \sum_{l=1}^{\infty}\frac{\eta}{l!R_0^l}\frac{\Gamma(\eta)}{\Gamma(\eta + 1 - l)}h^l \tag{A9}$$

where η is a real exponent and Γ the gamma function, and

$$\log\left(1 + \frac{h}{R_0}\right) = \sum_{l=1}^{\infty}\frac{(-1)^{-l}}{lR_0^l}h^l \tag{A10}$$

if the assumption $h/R_0 \ll 1$ is fulfilled. Taking these last relations into account, we can integrate Eqs. (A8a) and (A8b) to obtain new expressions for the potential from $R' = 0$ to $R' = R_0 + h$ outside, and from $R' = a$ to $R' = R_0 + h$ inside the sphere volume.

Now, we need to define a general surface function $f_l(\theta', \lambda')$ as the product of the surface density function μ and the l-power of the topography, expanded using its cosine and sine harmonic coefficients Y_{nm} of degree n and order m

$$f_1(\theta', \lambda') = \mu(\theta', \lambda')h^l(\theta', \lambda') = \sum_{n=0}^{\infty}\sum_{m=0}^{n} Y_{nml}^c\cos m\lambda' + Y_{nml}^s\sin m\lambda' \tag{A11}$$

In any case, the surface potential function can also be developed in terms of spherical harmonic coefficients in the same way (Heiskanen and Moritz 1967, p 30)

$$V(\theta, \lambda) = \sum_{n=0}^{\infty} W_n = \sum_{n=0}^{\infty}\sum_{m=0}^{n} W_{nm}^c\cos m\lambda + W_{nm}^s\sin m\lambda \tag{A12}$$

where the coefficient W_n verifies

$$W_n = \frac{2n+1}{4\pi}\int_0^{2\pi}\int_0^{\pi} V(\theta', \lambda')P_n(\cos\varphi)\sin\theta' \mathrm{d}\theta' \mathrm{d}\lambda' \tag{A13}$$

Finally, the harmonic coefficients of the observed gravitational potential are deduced from those of the surface function f_1.

(1) outside the volume ($a > R_0$)

$$\begin{bmatrix} W^c_{nm} \\ W^s_{nm} \end{bmatrix} = \frac{4\pi G\rho_0 R_0^2}{2n+1}\left(\frac{R_0}{a}\right)^{n+1}\sum_{k=0}^{\infty}\frac{\alpha_k R_0^k}{n+k+3}\left\{\begin{bmatrix} Y^c_{nm0} \\ Y^s_{nm0} \end{bmatrix} + \sum_{l=1}^{\infty}\frac{n+k+3}{l!R_0^l}\frac{\Gamma(n+k+3)}{\Gamma(n+k+4-l)}\begin{bmatrix} Y^c_{nml} \\ Y^s_{nml} \end{bmatrix}\right\} \quad \text{(A14a)}$$

(2) inside the volume ($a < R_0$)

$$\begin{bmatrix} W^c_{nm} \\ W^s_{nm} \end{bmatrix} = \frac{4\pi G\rho_0 R_0^2}{2n+1}\left(\frac{a}{R_0}\right)^{n}\sum_{k=0}^{\infty}\frac{\alpha_k R_0^k}{2-n+k}\left\{\begin{bmatrix} Y^c_{nm0} \\ Y^s_{nm0} \end{bmatrix} + \sum_{l=1}^{\infty}\frac{2-n+k}{l!R_0^l}\frac{\Gamma(2-n+k)}{\Gamma(3-n+k-l)}\begin{bmatrix} Y^c_{nml} \\ Y^s_{nml} \end{bmatrix}\right\} - \frac{4\pi G\rho_0 a^2}{2n+1}\sum_{k=0}^{\infty}\frac{\alpha_k a^k}{2-n+k}\begin{bmatrix} Y^c_{nm0} \\ Y^s_{nm0} \end{bmatrix} \quad \text{(A14b)}$$

if $n \neq k+2$.

Otherwise

$$\begin{bmatrix} W^c_{nm} \\ W^s_{nm} \end{bmatrix} = \frac{4\pi G\rho_0}{2n+1}\left(\frac{a}{R_0}\right)^{n}\sum_{k=0}^{\infty}\alpha_k\left\{\begin{bmatrix} Y^c_{nm0} \\ Y^s_{nm0} \end{bmatrix}\log\left(\frac{R_0}{a}\right) + \sum_{l=1}^{\infty}\frac{(-1)^l}{lR_0^l}\begin{bmatrix} Y^c_{nml} \\ Y^s_{nml} \end{bmatrix}\right\} \quad \text{(A14c)}$$

These relations show that the harmonic coefficients W_{nm} of the gravitational potential can be computed as a series of harmonic coefficients of the powers of the topography. Since $h \ll R_0$, this series should converge rapidly, and only a few iterations should be necessary to reach a stable solution for W_{nm}.

In these equations, the terms $(R_0/a)^{n+1}$ and $(a/R_0)^n$ are for the upward and downward continuations respectively. They make the amplitude of the harmonic coefficients W_n decrease or increase with the distance between the point of observation and the reference level for the topography, and with the degree (or the wavelength of the topography).

Simple cases of the general equations (A14a), (A14b), and (A14c)

(1) External potential of a smooth sphere of constant density

$$W_0 = \frac{4}{3}\pi G\rho_0\frac{R_0^3}{a} = \frac{GM}{a} \quad \text{(A15a)}$$

where M is the mass of the sphere of constant density ρ_0.
(2) Internal potential of a smooth shell of constant density

$$W_0 = 2\pi G\rho_0(R_0^2 - a^2) \quad \text{(A15b)}$$

This relation is the same one found by McMillan (1930), Ramsey (1940), and Sigl (1985) in the case of internal potential due to a spherical shell of constant density.

(3) Geoid contribution of the topography. Given the mean gravity γ (around 9.78–9.81 m s^{-2} for the Earth), the corresponding harmonic coefficients N_{nm} of the geoid height can be easily deduced from those of the gravitational potential W_{nm} with the Bruns formula (Heiskanen and Moritz 1967, p 50)

$$\begin{bmatrix} N^c_{nm} \\ N^s_{nm} \end{bmatrix} = \frac{1}{\gamma}\begin{bmatrix} W^c_{nm} \\ W^s_{nm} \end{bmatrix} \quad \text{(A15c)}$$

(4) Corresponding gravity anomaly. This anomaly is computed by taking the radial derivative of the potential observed at the distance $R = a$ from the center of the terrestrial sphere (Heiskanen and Moritz 1967; Blakely 1995). The harmonic coefficients D_{nm} for vertical component of gravity anomaly are

$$\begin{bmatrix} D^c_{nm} \\ D^s_{nm} \end{bmatrix} = -\frac{n+1}{a}\begin{bmatrix} W^c_{nm} \\ W^s_{nm} \end{bmatrix} \quad \text{(A15d)}$$

Case of varying density only defined in a shell

A variable density in a full sphere was previously considered. It was defined from the center of the Earth ($R = 0$). Let us now evaluate the gravity potential from the density distribution within a shell, the boundary topographies still being non-uniform. For this purpose, the model density ρ is again a function of θ', λ', but of the altitude ε from the lower boundary limit of the shell, where β_k are the new coefficients of the expansion of the radial component of the density

$$\rho(\theta', \lambda', \varepsilon) = \rho_0\mu(\theta', \lambda')\sum_{k=0}^{\infty}\beta_k\varepsilon^k \quad \text{(A16)}$$

The lower and upper boundary limits are located at $R = R_1$ and $R = R_2$ respectively (see Fig. 1), from the center of the sphere ($R_2 > R_1$). Here, we can consider the mean shell thickness as $H = R_2 - R_1$. The surface topographies of the lower and upper boundary limits are $h_1(\theta', \lambda')$ and $h_2(\theta', \lambda')$, respectively. Considering the new distance $R' = R_1 + \varepsilon$ from the base of the shell and the approximation $\varepsilon \ll R_1$, we will use the binomial relation from Eq. (A9) again to integrate versus R' to obtain the harmonic coefficients of the potential.

(1) For an external point of observation

$$\begin{bmatrix} W^c_{nm} \\ W^s_{nm} \end{bmatrix} = \frac{4\pi G\rho_0 R_1}{2n+1}\left(\frac{R_1}{a}\right)^{n+1}\sum_{k=0}^{\infty}\beta_k \times\left\{\frac{1}{k+1}\begin{bmatrix} Y^c_{nm}(f_{2,k+1}) - Y^c_{nm}(f_{1,k+1}) \\ Y^s_{nm}(f_{2,k+1}) - Y^s_{nm}(f_{1,k+1}) \end{bmatrix} + \sum_{l=1}^{\infty}\frac{n+2}{(k+l+1)l!R_1^l}\frac{\Gamma(n+2)}{\Gamma(n+3-l)} \times\begin{bmatrix} Y^c_{nm}(f_{2,k+l+1}) - Y^c_{nm}(f_{1,k+l+1}) \\ Y^s_{nm}(f_{2,k+l+1}) - Y^s_{nm}(f_{1,k+l+1}) \end{bmatrix}\right\} \quad \text{(A17a)}$$

(2) For an internal point of observation

$$\begin{bmatrix} W_{nm}^c \\ W_{nm}^s \end{bmatrix} = \frac{4\pi G\rho_0 R_1}{2n+1}\left(\frac{a}{R_1}\right)^n \sum_{k=0}^{\infty}\beta_k \times \left\{ \frac{1}{k+1}\begin{bmatrix} Y_{nm}^c(f_{2,k+1}) - Y_{nm}^c(f_{1,k+1}) \\ Y_{nm}^s(f_{2,k+1}) - Y_{nm}^s(f_{1,k+1}) \end{bmatrix} + \sum_{l=1}^{\infty}\frac{l-n}{(k+l+1)l!R_1^l}\frac{\Gamma(l-n)}{\Gamma(2-n-1)} \times \begin{bmatrix} Y_{nm}^c(f_{2,k+l+1}) - Y_{nm}^c(f_{1,k+l+1}) \\ Y_{nm}^s(f_{2,k+l+1}) - Y_{nm}^s(f_{1,k+l+1}) \end{bmatrix} \right\} \tag{A17b}$$

where the corresponding surface functions to deal with are

$$f_{1,k}(\theta',\lambda') = \mu(\theta',\lambda')h_1^k(\theta',\lambda') \tag{A18a}$$

and

$$f_{2,k}(\theta',\lambda') = \mu(\theta',\lambda')[H + h_2(\theta',\lambda')]^k \tag{A18b}$$

Appendix B: Parker's formula – the flat-earth approximation of Eq. (A14a)

Suppose now that the thickness d of the shell we consider is small compared to the radial distances R_0 and a. In that case, the ratio R_0/a can be expanded using the classical binomial relation ($d \ll a$)

$$\left(\frac{R_0}{a}\right)^n = \left(\frac{a-d}{a}\right)^n = \left(1-\frac{d}{a}\right)^n \approx 1 - n\frac{d}{a} + \cdots \tag{B1}$$

Note that the first terms of this development are the same as the exponential ones

$$\mathrm{e}^{-n\frac{d}{a}} \approx 1 - n\frac{d}{a} + \cdots \tag{B2}$$

So, the following 'local' approximation on the downward continuation term can be made

$$\left(\frac{\mathrm{R}_0}{\mathrm{a}}\right)^{\mathrm{n}} \approx \mathrm{e}^{-n\frac{d}{a}} \tag{B3}$$

Moreover, the planar wavelength Λ relates to the degree of harmonics

$$\Lambda = \frac{2\pi a}{\sqrt{n(n+1)}} \tag{B4}$$

For high-degree harmonic coefficients, the amplitude κ of the corresponding wave number is then

$$\kappa = \frac{2\pi}{\Lambda} = \frac{1}{a}\sqrt{n(n+1)} \approx \frac{n}{a} \tag{B5}$$

If the density ρ_0 of the topography is assumed to be constant, as well as the associated surface density ($\mu = 1$), then Eq. (A14a) becomes

$$\begin{bmatrix} W_{nm}^c \\ W_{nm}^s \end{bmatrix} = 2\pi G\rho_0 \mathrm{e}^{-kd}\sum_{l=1}^{\infty}\frac{k^{l-2}}{l!}\begin{bmatrix} Y_{nml}^c \\ Y_{nml}^s \end{bmatrix} \tag{B6}$$

which is the formula proposed by Parker (1972), if W_{nm} and Y_{nml} are now the Fourier coefficients of the gravitational potential and the lth-order component of the topography, respectively. In other words, Eq. (A14a) is equivalent to Parker's formula for high-degree harmonic coefficients which correspond to short wavelengths, and low topographic amplitude compared to the radius of the Earth. Equations (A14a), (A14b) and (A14c) are for the spherical case and for a non-constant density. The first term of Parker's formula (Parker 1972) [i.e. Eq. (B6)] is largely used in marine geophysics to define the theoretical 'admittance' (or linear filter) between the observed field at the sea surface and the sea floor topography, in the case of regional studies.

Appendix C: spherical harmonic coefficients of the magnetic potential

General expressions for magnetic potential created by a magnetized shell

Let us consider a spatial distribution of magnetization σ, where each spherical component of this vector (i.e. σ_θ, σ_λ, σ_R) is defined, as before, to be a power series expansion of the radial distance R'

$$\sigma_{\theta,\lambda,R}(\theta',\lambda',R') = \sigma^0_{\theta,\lambda,R}\nu_{\theta,\lambda,R}(\theta',\lambda')\sum_{k=0}^{\infty}\alpha_k^{\theta,\lambda,R}R'^k \tag{C1}$$

where $\sigma^0_{\theta,\lambda,R}$ and $\alpha_k^{\theta,\lambda,R}$ are given coefficients of a function development. Here, $\nu_{\theta,\lambda,R}(\theta',\lambda')$ is the surface magnetization. The corresponding magnetic potential observed at location (θ,λ,a) is (see Blakely 1995, p 97)

$$U(\theta,\lambda,a) = \frac{\chi_0}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin\theta'\mathrm{d}\theta'\mathrm{d}\lambda'\int_R R'^2\sigma_{\theta,\lambda,R}\nabla\frac{1}{\xi}\mathrm{d}R' \tag{C2}$$

where χ_0 is the magnetic permeability of free space, $\chi_0/4\pi \approx 10^{-7}\,\mathrm{H\,m^{-1}}$, and ∇ is the gradient operator in the spherical coordinates. The scalar product of the two three-component vectors σ and ∇ corresponds to $\sigma_\theta(\nabla\xi^{-1})_\theta + \sigma_\lambda(\nabla\xi^{-1})_\lambda + \sigma_R(\nabla\xi^{-1})_R$. For this reason, Eq. (C2) can be expressed as a sum of three separate terms for co-latitude, longitude, and radial distance

$$U(\theta,\lambda,a) = U_\theta(\theta,\lambda,a) + U_\lambda(\theta,\lambda,a) + U_R(\theta,\lambda,a) \tag{C3}$$

This represents a major advantage because each of these terms can be integrated versus R separately.

Expression of the radial term U_R

We have to differentiate the inverse of the spherical distance ξ versus the radius of the sphere.

(1) Outside the volume: $a > R'$

$$\frac{\partial}{\partial R'}\xi^{-1} = \frac{1}{aR'}\sum_{n=0}^{\infty} n\left(\frac{R'}{a}\right)^n P_n(\cos\varphi) \tag{C4a}$$

(2) Inside the volume: $a < R'$

$$\frac{\partial}{\partial R'}\xi^{-1} = -\frac{1}{R'^2}\sum_{n=0}^{\infty} n\left(\frac{a}{R'}\right)^n P_n(\cos\varphi) \tag{C4b}$$

For the external potential, we then integrate from the center of the sphere, $R = 0$ to $R = R_0 + h(\theta, \lambda)$

$$\begin{bmatrix} Z^c_{nm} \\ Z^s_{nm} \end{bmatrix}_R = \frac{\chi_0 n}{2n+1}\sigma^0_R R_0 \left(\frac{R_0}{a}\right)^{n+1} \sum_{k=0}^{\infty} \frac{\alpha^R_k R^k_0}{n+k+2} \left\{ \begin{bmatrix} Y^c_{nm0} \\ Y^s_{nm0} \end{bmatrix} + \sum_{l=1}^{\infty} \frac{n+k+2}{l! R^l_0} \frac{\Gamma(n+k+2)}{\Gamma(n+k-l+3)} \begin{bmatrix} Y^c_{nml} \\ Y^s_{nml} \end{bmatrix} \right\} \tag{C5a}$$

where Y_{mnl} are the harmonic coefficients of the surface function $f_l(\theta', \lambda') = v_R(\theta', \lambda') h^l(\theta', \lambda')$.

For the internal potential, the integration from $R = a$ to $R = R_0 + h(\theta', \lambda')$ leads to

$$\begin{bmatrix} Z^c_{nm} \\ Z^s_{nm} \end{bmatrix}_R = -\frac{n\chi_0}{2n+1}\sigma^0_R R_0 \left(\frac{a}{R_0}\right)^n \sum_{k=0}^{\infty} \frac{\alpha^R_k R^k_0}{1-n+k} \times \left\{ \begin{bmatrix} Y^c_{nm0} \\ Y^s_{nm0} \end{bmatrix} + \sum_{l=1}^{\infty} \frac{1-n+k}{l! R^l_0} \times \frac{\Gamma(1-n+k)}{\Gamma(2-n+k-l)} \begin{bmatrix} Y^c_{nml} \\ Y^s_{nml} \end{bmatrix} \right\} \tag{C5b}$$

if $n \neq k+1$

$$\begin{bmatrix} Z^c_{nm} \\ Z^s_{nm} \end{bmatrix}_R = -\frac{n\chi_0}{2n+1}\left(\frac{a}{R_0}\right)^n \sum_{k=0}^{\infty} \alpha_k \left\{ \begin{bmatrix} Y^c_{nm0} \\ Y^s_{nm0} \end{bmatrix} \log\frac{R_0}{a} + \sum_{l=1}^{\infty} \frac{(-l)^l}{l R_0} \begin{bmatrix} Y^c_{nml} \\ Y^s_{nml} \end{bmatrix} \right\} \tag{C5c}$$

otherwise.

Expression of the co-latitude and longitude terms U_θ and U_λ

Let us now differentiate ξ^{-1} with respect to θ' and λ' to express the other two spherical surface components, U_θ and U_λ.

(1) Outside the volume: $a > R'$

$$\frac{1}{R'}\frac{\partial}{\partial\theta'}\xi^{-1} = \frac{1}{a^2}\xi^{-3}[-\sin\theta'\cos\theta + \cos\theta'\sin\theta\cos(\lambda-\lambda')] \tag{C6a}$$

$$\frac{1}{R'}\frac{1}{\sin\theta'}\frac{\partial}{\partial\lambda'}\xi^{-1} = \frac{1}{a^2}\xi^{-3}\sin\theta\sin(\lambda'-\lambda) \tag{C6b}$$

(2) Inside the volume: $a < R'$

$$\frac{1}{R'}\frac{\partial}{\partial\theta'}\xi^{-1} = \frac{a}{R'^3}\xi^{-3}[-\sin\theta'\cos\theta + \cos\theta'\sin\theta\cos(\lambda-\lambda')] \tag{C7a}$$

$$\frac{1}{R'}\frac{1}{\sin\theta'}\frac{\partial}{\partial\lambda'}\xi^{-1} = \frac{a}{R'^3}\xi^{-3}\sin\theta\sin(\lambda'-\lambda) \tag{C7b}$$

These latter equations for the gradient in spherical coordinates cannot be integrated, as previously, to express the corresponding spherical harmonic coefficients as desired, since they also depend on the location of observation point (θ, λ). One solution is to consider a simple case, such as observation points at poles (i.e. $\theta = 0$ or $\theta = \pi$), where the longitude term given by Eqs. (C6b) and (C7b) vanishes. Nevertheless, we may still want to evaluate the co-latitude and longitude contributions to the magnetic potential just at a given single location (θ_0, λ_0), where the gradient only depend upon (θ', λ'), as a sum of spherical harmonics. In that case, the new surface co-latitude functions to integrate are as follows.

(1) For the co-latitude component

$$f_{\theta,1}(\theta', \lambda') = v_\theta(\theta', \lambda')_h{}^1(\theta', \lambda')[-\sin\theta'\cos\theta_0 + \cos\theta'\sin\theta_0\cos(\lambda_0-\lambda')] \tag{C8a}$$

(2) For the longitude component

$$f_{\lambda,1}(\theta', \lambda') = v_\lambda(\theta', \lambda')_h{}^1(\theta', \lambda')\sin\theta_0\sin(\lambda'-\lambda_0) \tag{C8b}$$

In order to find a development for ξ^{-3}, we say that it is the generating function for the first derivative P'_n of Legendre polynomials.

(1) Outside the volume

$$\xi^{-3} = \frac{1}{a}\sum_{n=1}^{\infty}\left(\frac{R'}{a}\right)^{n-1} P'_n(\cos\varphi). \tag{C9a}$$

(2) Inside the volume

$$\xi^{-3} = \frac{1}{R'}\sum_{n=1}^{\infty}\left(\frac{a}{R'}\right)^{n-1} P'_n(\cos\varphi) \tag{C9b}$$

From the general recurrent expression for degree n greater than 1 (McMillan 1930)

$$P'_{n+1} - P'_{n-1} = (2n+1)P_n \tag{C10}$$

Simple relations for P'_n can be derived as a sum of Legendre polynomials

$$P'_n = \begin{cases} \sum_{j=1}^{n/2}(4j-1)P_{2j-1} & \text{if } n \text{ is even} \\ \sum_{j=1}^{(n-1)/2}(4j-1)P_{2j} & \text{if } n \text{ is odd} \end{cases} \tag{C11}$$

After integration versus the radial distance R, we obtain the harmonic coefficients Z_{nm} for co-latitude and longitude components of the magnetic potential as follows.

(1) Outside the volume

$$\begin{bmatrix} Z^c_{nm} \\ Z^s_{nm}\end{bmatrix}_{\theta,\lambda}(\theta_0,\lambda_0) = \frac{\chi_0\sigma^0_{\theta,\lambda}}{2n+1}\left(\frac{R_0}{a}\right)^{n+2}\sum_{j=J}^{J_{\max}} Q_j \sum_{k=0}^{\infty}\frac{\alpha_k^{\theta,\lambda}R_0^k}{j+k+2}\left\{\begin{bmatrix}Y^c_{jm0}\\Y^s_{jm0}\end{bmatrix}_{\theta,\lambda} + \sum_{l=1}^{\infty}\frac{j+k+2}{l!R_0^l}\frac{\Gamma(j+k+2)}{\Gamma(j+k-l+3)}\begin{bmatrix}Y^c_{jml}\\Y^s_{jml}\end{bmatrix}_{\theta,\lambda}\right\} \qquad \text{(C12a)}$$

(2) Inside the volume

$$\begin{bmatrix} Z^c_{nm} \\ Z^s_{nm}\end{bmatrix}_{\theta,\lambda}(\theta_0,\lambda_0) = \frac{\chi_0\sigma^0_{\theta,\lambda}}{2n+1}\left(\frac{a}{R_0}\right)^{n}\sum_{j=J}^{J_{\max}} Q_j \sum_{k=0}^{\infty}\frac{\alpha_k^{\theta,\lambda}R_0^k}{k-j}\left\{\begin{bmatrix}Y^c_{jm0}\\Y^s_{jm0}\end{bmatrix}_{\theta,\lambda} + \sum_{l=1}^{\infty}\frac{k-j}{l!R_0^l}\frac{\Gamma(k-j)}{\Gamma(k-j+l-1)}\begin{bmatrix}Y^c_{jml}\\Y^s_{jml}\end{bmatrix}_{\theta,\lambda}\right\} - \frac{\chi_0\sigma^0_{\theta,\lambda}}{2n+1}\sum_{j=J}^{J_{\max}} Q_j \sum_{k=0}^{\infty}\frac{\alpha_k^{\theta,\lambda_a^k}}{k-j}\begin{bmatrix}Y^c_{jm0}\\Y^s_{jm0}\end{bmatrix}_{\theta,\lambda} \qquad \text{(C12b)}$$

In the case that $j = k$, we have

$$\begin{bmatrix} Z^c_{nm} \\ Z^s_{nm}\end{bmatrix}_{\theta,\lambda}(\theta_0,\lambda_0) = \frac{\chi_0\sigma^0_{\theta,\lambda}}{2n+1}\left(\frac{a}{R_0}\right)^{n}\sum_{j=J}^{J_{\max}} Q_j \sum_{k=0}^{\infty}\alpha_k^{\theta,\lambda} \times \left\{\begin{bmatrix}Y^c_{jm0}\\Y^s_{jm0}\end{bmatrix}_{\theta,\lambda}\log\left(\frac{R_0}{a}\right) + \sum_{l=1}^{\infty}\frac{(-l)^l}{lR_0^l}\begin{bmatrix}Y^c_{jml}\\Y^s_{jml}\end{bmatrix}_{\theta,\lambda}\right\} \qquad \text{(C12c)}$$

where Y_{nml} are the harmonic coefficients of the surface functions $f_{\theta,1}$ and $f_{\lambda,1}$ from Eqs. (C8a) and (C8b) respectively, and for a given reference location θ_0 and λ_0. When the degree n is even $Q_j = 4j-1; J = 1$ and $J_{\max} = n/2$. Otherwise, $Q_j = 4j+1;\ J = 0$ and $J_{\max} = (n-1)/2$. As for gravitational potential, it could be interesting to see what the latter general equations would become when magnetization is only defined inside the shell. In that particular case, we have for each component R, θ, and λ the following.

(1) Outside the volume

$$\begin{bmatrix} Z^c_{nm} \\ Z^s_{nm}\end{bmatrix}_{R} = \frac{n\chi_0}{2n+1}\sigma^0_R\left(\frac{R_1}{a}\right)^{n+1} \times \sum_{k=0}^{\infty}\beta_k\left\{\frac{1}{k+1}\begin{bmatrix}Y^c_{nm}(f_{2,k+1}) - Y^c_{nm}(f_{1,k+1})\\ Y^s_{nm}(f_{2,k+1}) - Y^s_{nm}(f_{1,k+1})\end{bmatrix} + \sum_{l=1}^{\infty}\frac{n+2}{(k+l+1)l!R_1^l}\frac{\Gamma(n+2)}{\Gamma(n+3-l)} \times \begin{bmatrix}Y^c_{nm}(f_{2,k+l+1}) - Y^c_{nm}(f_{1,k+l+1})\\ Y^s_{nm}(f_{2,k+l+1}) - Y^s_{nm}(f_{1,k+l+1})\end{bmatrix}\right\} \qquad \text{(C13a)}$$

$$\begin{bmatrix} Z^c_{nm} \\ Z^s_{nm}\end{bmatrix}_{\theta,\lambda} = \frac{\chi_0}{2n+1}\sigma^0_{\theta,\lambda}\left(\frac{R_1}{a}\right)^{n+1}\sum_{j=J}^{J_{\max}} Q_j\sum_{k=0}^{\infty}\beta_k \times \left\{\frac{1}{k+1}\begin{bmatrix}Y^c_{jm}(f_{2,k+1}) - Y^c_{jm}(f_{1,k+1})\\ Y^s_{jm}(f_{2,k+1}) - Y^s_{jm}(f_{1,k+1})\end{bmatrix} + \sum_{l=1}^{\infty}\frac{n+2}{(k+l+1)l!R_1^l}\frac{\Gamma(n+2)}{\Gamma(n+3-l)} \times \begin{bmatrix}Y^c_{jm}(f_{2,k+l+1}) - Y^c_{jm}(f_{1,k+l+1})\\ Y^s_{jm}(f_{2,k+l+1}) - Y^s_{jm}(f_{1,k+l+1})\end{bmatrix}\right\} \qquad \text{(C13b)}$$

(2) Inside the volume

$$\begin{bmatrix} Z^c_{nm} \\ Z^s_{nm}\end{bmatrix}_{R} = \frac{n\chi_0}{2n+1}\sigma^0_R\left(\frac{a}{R_1}\right)^{n} \times \sum_{k=0}^{\infty}\beta_k\left\{\frac{1}{k+1}\begin{bmatrix}Y^c_{nm}(f_{2,k+1}) - Y^c_{nm}(f_{1,k+1})\\ Y^s_{nm}(f_{2,k+1}) - Y^s_{nm}(f_{1,k+1})\end{bmatrix} + \sum_{l=1}^{\infty}\frac{1-n}{(k+l+1)l!R_1^l}\frac{\Gamma(1-n)}{\Gamma(2-n-l)} \times \begin{bmatrix}Y^c_{nm}(f_{2,k+l+1}) - Y^c_{nm}(f_{1,k+l+1})\\ Y^s_{nm}(f_{2,k+l+1}) - Y^s_{nm}(f_{1,k+l+1})\end{bmatrix}\right\} \qquad \text{(C13c)}$$

$$\begin{bmatrix} Z^c_{nm} \\ Z^s_{nm}\end{bmatrix}_{\theta,\lambda} = \frac{\chi_0}{2n+1}\sigma^0_{\theta,\lambda}\left(\frac{\mathrm{a}}{R_1}\right)^{n} \times \sum_{j=J}^{J_{\max}} Q_j\sum_{k=0}^{\infty}\beta_k + \left\{\frac{1}{k+1}\begin{bmatrix}Y^c_{jm}(f_{2,k+1}) - Y^c_{jm}(f_{1,k+1})\\ Y^s_{jm}(f_{2,k+1}) - Y^s_{jm}(f_{1,k+1})\end{bmatrix} + \sum_{l=1}^{\infty}\frac{1-n}{(k+l+1)l!R_1^l}\frac{\Gamma(1-n)}{\Gamma(2-n-l)} \times \begin{bmatrix}Y^c_{jm}(f_{2,k+l+1}) - Y^c_{jm}(f_{1,k+l+1})\\ Y^s_{jm}(f_{2,k+l+1}) - Y^s_{jm}(f_{1,k+l+1})\end{bmatrix}\right\} \qquad \text{(C13d)}$$

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