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A characterization of exponential functionals in finite Markov chains*

Rolando Cavazos–Cadena[1], **Daniel Hernández–Hernández**[2]

[1] Departamento de Estadística y Cálculo, Universidad Autónoma Agraria Antonio Narro, Buenavista, Saltillo Coah., 25315 México (e-mail: rcavazos@narro.uaaan.mx)
[2] Centro de Investigación en Matemáticas, Apartado postal 402, Guanajuato, Gto., 36000 México; Fax: +52-473-732-5749 (e-mail: dher@cimat.mx)

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Abstract. This work considers Markov chains with finite state space. It is supposed that the process has a single recurrent class, but the set of transient states is not necessarily empty. In this context, a Varadhan's function, giving the exponential grow rate of an aggregated cost function, is studied. The main result establishes that this functional is the optimal value of a minimization problem on the Euclidean space whose dimension equals the number of states.

Key words: Exponential grow rate, Single recurrent class, Transient states, Risk–sensitive average cost

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1. Introduction

This work concerns a discrete–time Markov chain $\{X_n\}$ evolving over a finite state space S according to a stationary transition mechanism. The basic assumption on the communication structure of the process is that the chain has a single (positive) recurrent class, but the set of transient states is possibly nonempty. Within this framework, given a real–valued function C defined on S and a positive number λ, the paper analyzes the function $V(\cdot)$ defined by

$$V(x) = \limsup_{n\to\infty} \frac{1}{\lambda(n+1)} \log\left(E_x\left[e^{\lambda \sum_{t=0}^{n} C(X_t)}\right]\right), \quad x \in S, \tag{1.1}$$

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which will be referred to as the Varadhan's function associated to λ and $C(\cdot)$; see [2], [9], [18], [20]. This function has been widely studied in the literature, and the classical Perron–Frobenious theory of positive matrices establishes that, if the one–step transition matrix $[p_{x\,y}]$ is aperiodic and the whole state space is a communicating class, then $V(\cdot)$ is constant. Moreover, its value v^* is such that $e^{\lambda v^*}$ is the largest eigenvalue of the matrix $\left[e^{\lambda C(x)} p_{x\,y}\right]$, and (1.1) holds with limit instead of limit superior [5], [6]. However, when the above conditions are not satisfied, for instance, if the class of transient states is non empty, Varadhan's function is not necessarily constant (see Example 2.1 below). This paper provides a characterization of $V(\cdot)$ in this case in terms of the Poisson inequality (2.4).

The main motivation to look for a characterization of function V when it is not constant comes from the area of ergodic risk sensitive control, which has been intensively studied in recent years from various points of view [3], [7], [15], [16], [17]. Among them, its connection with H^∞ optimization methods, via small noise asymptotic limits [10], as well as with large deviations theory. This latter link can be made not only when one is interested in the small noise limits, via the Friedlin-Wentzell theory, but also through the asymptotic behavior of the empirical measure of a Markov process; see [5], [6]. The present work is close in spirit to this latter one.

In the context of risk sensitive control, $C(\cdot)$ is interpreted as a running cost, so that a controller incurs a cost $C(y)$ every time that the chain visits y, whereas $\lambda > 0$ is her risk–aversion parameter; see, for instance, [4], [8], [11], [14] and the references therein. When the initial state is $X_0 = x$ and the decision maker drives the system in such way that it evolves as the Markov chain $\{X_n\}$, $V(x)$ represents the (worst) long–run average exponential cost per stage from the controller's perspective. The results obtained in the above papers are based in dynamic programming techniques, where the optimality equation has the form of a nonlinear eigenvalue problem and, applied to the present uncontrolled case, can be summarized as follows:

(*i*) Assume that the whole state space is a communicating class. In this case, for each $\lambda > 0$, there exist a real number v^* and a function $h(\cdot)$ defined on S satisfying the following Poisson equation:

$$e^{\lambda v^* + \lambda h(x)} = e^{\lambda C(x)} E_x\left[e^{\lambda h(X_1)}\right], \quad x \in S \tag{1.2}$$

and, moreover, $V(\cdot) \equiv v^*$. These results can also be obtained from [1], where multiplicative ergodic theorems were established.

(*ii*) Suppose that the Markov chain $\{X_n\}$ has a single recurrent class (the unichain case). Under this condition, there exists $\Lambda_0 > 0$ such that the Poisson equiation (1.2) has a solution if and only if $\lambda < \Lambda_0$ and, in this case, $V(\cdot) \equiv v^*$. As it was shown in [4], when the class of transient states is not empty, the positive number Λ_0 is generally finite, so that (1.2) does not admit a solution for every positive risk sensitivity parameter λ. Consequently, within the unichain framework, in general (1.2) characterizes $V(\cdot)$ only when λ is small enough. On the other hand, it should be mentioned that when the above Poisson equation holds the limit superior in (1.1) can always be replaced by limit.

In optimal control theory a central role is played by the optimality equation, analogous to (1.2), which provides a characterization of the optimal value function, as well as a way to obtain an optimal control strategy, result that is commonly called a verification theorem. In the present uncontrolled case, it will be shown in Example 1.2 below that, if the class of transient states is not empty and $\lambda > 0$ is large enough, then Varadhan's function is not constant and can not be characterized by a single equation similar to (1.2). Therefore, since $V(\cdot)$ has an important interpretation in risk sensitive decision making, it is interesting to provide a general characterization of Varadhan's function, which is precisely the problem this work is concerned with. *The main result* in this direction, stated below as Theorem 2.2, establishes that when the Markov chain has a single recurrent class, $V(\cdot)$ can be obtained through a finite–dimensional convex minimization problem, whose constraints are given in terms of solutions to the associated Poisson inequality; see Definition 2.1. This result opens the possibility to use convex analysis techniques to determine Varadhan's function, and represents a first step to understand the controlled case.

The organization of the paper is as follows: In Section 2 an explicit example showing that $V(\cdot)$ is not constant is given, and the main result is stated as Theorem 2.2. Next, Sections 3 and 4 contain the technical tools that will be used to establish the main result, which is proved in Section 5. Finally, the paper concludes in Section 6 with some brief comments.

Notation. Throughout the remainder $\mathbb{N}$ and $\mathbb{R}$ stand for the set of nonnegative integers and real numbers, respectively. The class of real–valued functions defined in the state space S is denoted by $\mathcal{B}(S)$, and for each $C \in \mathcal{B}(S)$, $\|C\| := \max_s |C(s)|$ denotes for the corresponding supremum norm.

2. Characterization of Varadhan's function

In this section the main result of this note is stated as Theorem 2.2. To begin with, it is convenient to consider the following simple example concerning a Markov chain with nonempty set of transient states.

Example 2.1. *Consider a Markov chain with state space $S = \{0, 1\}$ and transition matrix $[p_{xy}]$ determined by*

$$p_{00} = 1, \quad p_{11} = \beta = 1 - p_{10},$$

where $\beta \in (0, 1)$, and let the function C be given by $C(1) = 1$ and $C(0) = 0$. In this case $V(0) = 0$, since state 0 is absorbing and $C(0) = 0$, and it is not difficult to see that (cf. [4])

$$V(1) = \begin{cases} 0 & \text{if } e^{\lambda}\beta < 1, \\ \frac{1}{\lambda}\log(e^{\lambda}\beta) & \text{if } e^{\lambda}\beta \geq 1. \end{cases} \tag{2.1}$$

Notice that $V(\cdot)$ is constant if $e^{\lambda}\beta \leq 1$, and non constant when $e^{\lambda}\beta > 1$. Moreover, when $e^{\lambda}\beta \geq 1$, there is not any function $h: S \to \mathbb{R}$ such that for every $x \in S$,

$$e^{\lambda V(x)+\lambda h(x)} \geq e^{\lambda C(x)} E_x\left[e^{\lambda h(X_1)}\right]; \tag{2.2}$$

indeed, from the definition of the transition law and the function $C: S \to \mathbb{R}$, for $x = 1$ *this inequality is equivalent to*

$$e^{\lambda V(1)+\lambda h(1)} \geq e^{\lambda}\left[\beta e^{\lambda h(1)} + (1-\beta) e^{\lambda h(0)}\right].$$

However, by (2.1), $e^{\lambda V(1)+\lambda h(1)} = e^{\lambda}\beta e^{\lambda h(1)}$ *when* $e^{\lambda}\beta \geq 1$, *so that the above inequality can not be satisfied by any function* $h(\cdot)$. □

As this example shows, when the class of transient states is non empty Varadhan's function may not be constant, and $V(\cdot)$ does not necessarily satisfy an equation similar to (1.2). This fact provides the motivation to look for a characterization of $V(\cdot)$, which is precisely the problem this work is concerned with. The result in this direction is stated in Theorem 2.2 below and, under the assumption that the underlying Markov chain has a single recurrent class, shows that $V(x)$ is the optimal value of a minimization problem on a convex subset of $\mathcal{R}^d$, where d is the number of elements of S. The precise statement involves the following family of functions.

Definition 2.1. *The class* $\mathcal{G} \subset \mathcal{B}(S)$ *consists of all functions* $g \in \mathcal{B}(S)$ *satisfying conditions* (i) *and* (ii) *below.*

(i) *For each* $x \in S$,

$$g(x) \geq \max\{g(y) | p_{xy} > 0\}. \tag{2.3}$$

(ii) *There exists a function* $h \in \mathcal{B}(S)$ *such that*

$$e^{\lambda g(x)+\lambda h(x)} \geq e^{\lambda C(x)} \sum_y p_{x\,y} e^{\lambda h(y)}, \quad x \in S. \tag{2.4}$$

Remark 2.1. (i) *Family* $\mathcal{G}$ *is nonempty. In fact, if* $g(x) = \|C\|$ *for each* $x \in S$, *then* $g \in \mathcal{G}$, *since (2.3) is clearly satisfied by this function, whereas (2.2) holds with* $h(\cdot) = 0$.

(ii) *Given* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha > 0$, *notice that if* $g: S \to \mathbb{R}$ *satisfies (2.3), then this inequality also holds for* $\alpha g(\cdot) + \beta$.

(iii) *Class* $\mathcal{G}$ *is convex. Indeed, given* $g_0, g_1 \in \mathcal{G}$ *and* $\alpha \in (0,1)$, *it is not difficult to see that (2.3) is satisfied by* $g(\cdot) = \alpha g_0(\cdot) + (1-\alpha) g_1(\cdot)$. *On the other hand, there exist functions* $h_i: S \to \mathbb{R}$ *such that for every* $x \in S$, $e^{\lambda g_i(x)+\lambda h_i(x)} \geq e^{\lambda C(x)} \sum_y p_{xy} e^{\lambda h_i(y)}$, $i = 0, 1$, *and setting* $h(\cdot) = \alpha h_0(\cdot) + (1-\alpha) h_1(\cdot)$, *it follows that*

$$\begin{aligned} e^{\lambda g(x)+\lambda h(x)} &= \left(e^{\lambda g_0(x)+h_0(x)}\right)^{\alpha} \left(e^{\lambda g_1(x)+h_1(x)}\right)^{1-\alpha} \\ &\geq \left(e^{\lambda C(x)} \sum_y p_{xy} e^{\lambda h_0(y)}\right)^{\alpha} \left(e^{\lambda C(x)} \sum_y p_{xy} e^{\lambda h_1(y)}\right)^{1-\alpha} \\ &= e^{\lambda C(x)} \left(\sum_y p_{xy} e^{\lambda h_0(y)}\right)^{\alpha} \left(\sum_y p_{xy} e^{\lambda h_1(y)}\right)^{1-\alpha} \\ &\geq e^{\lambda C(x)} \sum_y p_{xy} e^{\lambda(\alpha h_0(y)} e^{\lambda(1-\alpha) h_1(y)} \end{aligned}$$

where Hölder's inequality was used in the last step. Therefore,

$$e^{\lambda g(x)+\lambda h(x))} \geq e^{\lambda C(x)} \sum_y p_{xy} e^{\lambda(\alpha h_0(y)+(1-\alpha)h_1(y))} = e^{\lambda C(x)} \sum_y p_{xy} e^{\lambda h(y)} \quad x \in S,$$

so that $g = \alpha g_0 + (1-\alpha) g_1$ *satisfies (2.4) with* $h = \alpha h_0 + (1-\alpha) h_1$.

The following theorem shows that every member of $\mathcal{G}$ is an upper bound of Varadhan's function.

Theorem 2.1. (i) *Suppose that* $g: S \to \mathbb{R}$ *satisfies (2.3). In this case, the process* $\{g(X_t)\}$ *is almost surely decreasing. More precisely, for every* $x \in S$,

$$P_x[g(X_{t+1}) \leq g(X_t)] = 1, \quad x \in S, \quad t \in \mathbb{N}.$$

(ii) *For each* $g \in \mathcal{G}$, $g(\cdot) \geq V(\cdot)$.

Proof. (i) Let $t \in \mathbb{N}$ and $x \in S$ be fixed. Since g satisfies (2.3), $P[X_{t+1} = y | X_t = w] = p_{wy} > 0$ implies that $g(y) \leq g(w)$. Therefore, $P[g(X_{t+1}) \leq g(X_t) | X_t] = 1$ P_x–almost surely, and the conclusion follows taking expectation with respect to P_x.

(ii) Take $g \in \mathcal{G}$ and select $h: S \to \mathbb{R}$ in such a way that inequality (2.4) is satisfied. In this case

$$e^{\lambda h(x)} \geq E_x\left[e^{\lambda[C(X_0)-g(X_0)]} e^{\lambda h(X_1)}\right], \quad x \in S,$$

and an induction argument using the Markov property yields that, for each $x \in S$ and $n \in \mathbb{N}$,

$$e^{\lambda h(x)} \geq E_x\left[e^{\lambda \sum_{t=0}^{n}[C(X_t)-g(X_t)]} e^{\lambda h(X_{n+1})}\right].$$

Therefore,

$$e^{\lambda \|h\|} \geq e^{\lambda h(x)} \geq E_x\left[e^{\lambda \sum_{t=0}^{n}[C(X_t)-g(X_t)]} e^{\lambda h(X_{n+1})}\right] \geq E_x\left[e^{\lambda \sum_{t=0}^{n}[C(X_t)-g(X_t)]}\right] e^{-\lambda \|h\|},$$

and then

$$e^{2\lambda \|h\|} \geq E_x\left[e^{\lambda \sum_{t=0}^{n}[C(X_t)-g(X_t)]}\right], \quad x \in S, \quad n \in \mathbb{N}. \tag{2.5}$$

On the other hand, regardless of the initial state, part (i) yields that with probability one

$$g(X_n) \leq g(X_{n-1}) \leq \cdots \leq g(X_1) \leq g(X_0), \quad n \in \mathbb{N},$$

so that

$$\sum_{t=0}^{n}[C(X_t) - g(X_t)] \geq \sum_{t=0}^{n} C(X_t) - (n+1) g(X_0)$$

and then, since $P_x[X_0 = x] = 1$,

$$\sum_{t=0}^{n}[C(X_t) - g(X_t)] \geq \sum_{t=0}^{n} C(X_t) - (n+1) g(x) \quad P_x - \text{a. s.}$$

From this statement and (2.5) it follows that for each $x \in S$ and $n \in \mathbb{N}$

$$e^{2\lambda \|h\| + \lambda(n+1) g(x)} \geq E_x\left[e^{\lambda \sum_{t=0}^{n} C(X_t)}\right],$$

and using the definition of $V(\cdot)$ in (1.1), this implies that $g(x) \geq V(x)$ for every state x. □

According to this result, the functional $V(\cdot)$ is a lower bound for each member of $\mathcal{G}$. On the other hand, in general $V(\cdot)$ does not belong to $\mathcal{G}$ since, as it was shown in Example 1.1, it is possible that (2.2) does not hold for any $h: S \to \mathbb{R}$, and in this case $V(\cdot)$ does not satisfy the second condition in Definition 2.1. However, under the following unichain requirement, the main result of this work asserts that $V(\cdot)$ is the largest lower bound of $\mathcal{G}$.

Assumption 2.1. The Markov chain $\{X_n\}$ has a single recurrent class $\mathcal{R}$.

Theorem 2.2. *Under Assumption 2.1, the following equality occurs for every* $x \in S$:

$$V(x) = \inf_{g \in \mathcal{G}} g(x).$$

The somewhat technical proof of this theorem will be given in Section 5 after stating some auxiliary preliminaries in the following two sections. Essentially, although it can not be ensured that Varadhan's function is a member of $\mathcal{G}$, *the effort is dedicated to show that, for each* $\alpha \in (0,1)$, $\alpha V(\cdot) + (1-\alpha)\|C\|$ *lies in* $\mathcal{G}$. *By convenience, the following notation is used in the subsequent development:*

$$V_n(x) = \frac{1}{\lambda} \log\left(E_x\left[e^{\lambda \sum_{t=0}^{n} C(X_t)}\right]\right), \quad x \in S, \quad n \in \mathbb{N}, \tag{2.6}$$

so that (1.1) can be written as

$$V(x) = \limsup_{n\to\infty} \frac{1}{n+1} V_n(x), \quad x \in S. \tag{2.7}$$

3. Basic preliminaries

Throughout the remainder of the paper Assumption 2.1 is enforced. Now let state z be a fixed recurrent state, and define T as the first passage time to state z in a positive time, that is,

$$T = \min\{n > 0 | X_n = z\}. \tag{3.1}$$

The approach used to establish Theorem 2.2 relies heavily on the following result concerning the tails of the distribution of T.

Lemma 3.1. *There exists* $\beta \in (0,1)$ *and a positive constant* B *such that*

$$P_x[T \geq n] \leq B\beta^n, \quad x \in S, \quad n \in \mathbb{N}.$$

This result has been widely used in the literature; see, for instance, [12], [13], [19]. However, since Lemma 3.1 plays a central role in the subsequent argumentation, a short proof is given.

Proof. Using that the Markov chain has the single recurrent class $\mathcal{R}$ and $z \in \mathbb{R}$, for each $x \in S$ there exists a positive integer $n = n(x)$ such that $P_x[X_{n(x)} = z] > 0$, so that the definition of the hitting time T in (3.1) yields

$$P_x[T \geq n(x) + 1] < 1, \quad x \in S.$$

Therefore, setting $N = \max_{x\in S} n(x) + 1 > 0$, for each $x \in S$ the inequality $P_x[T \geq N] \leq P_x[T \geq n(x) + 1] < 1$ is valid. Defining

$$\tilde{\beta}_0 = \max_{x\in S} P_x[T \geq N], \quad \text{and} \quad \beta_0 = \max\left\{\tilde{\beta}_0, \frac{1}{2}\right\}$$

it follows that $\beta_0 \in (0, 1)$ and $P_x[T \geq N] \leq \beta_0$ for every initial state x. From this point, an induction argument using the Markov property leads to

$$P_x[T \geq kN] \leq \beta_0^k, \quad x \in S, \quad k \in \mathbb{N}. \tag{3.2}$$

To conclude, let $n \in \mathbb{N}$ be arbitrary and write $n = rN + d$, where r is the integral part of n/N and $0 \leq d < N$. With this notation, using (3.2) and the fact that the mapping $k \mapsto P_x[T \geq k]$ is always decreasing, it follows that for every state x and $n \in \mathbb{N}$,

$$P_x[T \geq n] \leq P_x[T \geq rN] \leq \beta_0^r = \beta_0^{-d/N}\left(\beta_0^{\frac{1}{N}}\right)^{rN+d} \leq \beta_0^{-1}\left(\beta_0^{\frac{1}{N}}\right)^n$$

and the conclusion follows setting $B = \beta_0^{-1}$ and $\beta = \beta_0^{\frac{1}{N}}$. □

The following lemma concerns a basic property of Varadhan's function. As already noted at the end of Section 2, in general $V(\cdot)$ does not satisfy the second requirement in the definition of the family $\mathcal{G}$; however, as it is shown below, at each state x the equality in condition (2.3) is satisfied by $V(\cdot)$.

Lemma 3.2. (*i*) *For every* $x \in S$

$$V(x) = \max\{V(y) | p_{xy} > 0\}.$$

Consequently,
(*ii*) *For every* $x, y \in S$ *and each positive integer* n,

$$P_x[X_n = y] > 0 \Longrightarrow V(x) \geq V(y).$$

(*iii*) $V(\cdot)$ *is constant in the class* $\mathcal{R}$ *of recurrent states*
(*iv*) $V(\cdot) \geq V(w)$ *whenever* $w \in \mathcal{R}$.

Proof. (*i*) Let $\varepsilon > 0$ be given, and notice that (2.6) and (2.7) together yield that there exists a positive integer N satisfying

$$\frac{1}{n} V_{n-1}(y) \leq V(y) + \varepsilon, \quad y \in S, \quad n \geq N. \tag{3.3}$$

On the other hand, from (2.6), an application of the Markov property yields that for each positive integer n and $x \in S$

$$\begin{aligned} e^{\lambda V_n(x)} &= E_x\left[e^{\lambda \sum_{t=0}^n C(X_t)}\right] \\ &= e^{\lambda C(x)} E_x\left[e^{\lambda \sum_{t=1}^n C(X_t)}\right] \\ &= e^{\lambda C(x)} E_x\left[E_x\left[e^{\lambda \sum_{t=1}^n C(X_t)} \middle| X_1\right]\right] \end{aligned}$$

$$\begin{aligned} &= e^{\lambda C(x)} E_x\left[E_{X_1}\left[e^{\lambda \sum_{t=0}^{n-1} C(X_t)}\right]\right] \\ &= e^{\lambda C(x)} E_x\left[e^{\lambda V_{n-1}(X_1)}\right] \end{aligned} \tag{3.4}$$

and, together with (3.3), this implies that for $n \geq N$ and $x \in S$,

$$e^{\lambda V_n(x)} \leq e^{\lambda C(x)} E_x\left[e^{n\lambda(V(X_1)+\varepsilon)}\right] \leq e^{\lambda C(x)} e^{n\lambda(M(x)+\varepsilon)}$$

where $M(x) := \max\{V(y) | p_{xy} > 0\}$. Therefore,

$$\frac{1}{n+1} V_n(x) \leq \frac{1}{n+1} C(x) + \frac{n}{n+1}(M(x)+\varepsilon), \quad x \in S, \quad n \geq N,$$

and taking limit superior as n goes to ∞, it follows that $V(\cdot) \leq M(\cdot) + \varepsilon$; consequently, since $\varepsilon > 0$ is arbitrary,

$$V(x) \leq M(x) = \max\{V(y) | p_{xy} > 0\}, \quad x \in S. \tag{3.5}$$

To obtain the reverse inequality, let $x, y \in S$ be such that $p_{x\,y} > 0$. In this case, (3.4) yields that for every $n \in \mathbb{N} \setminus \{0\}$,

$$e^{\lambda V_n(x)} = e^{\lambda C(x)} E_x\left[e^{\lambda V_{n-1}(X_1)}\right] \geq e^{\lambda C(x)} p_{x\,y} e^{\lambda V_{n-1}(y)},$$

and then

$$\frac{1}{n+1} V_n(x) \geq \frac{1}{\lambda(n+1)} \log\left(e^{\lambda C(x)} p_{x\,y}\right) + \frac{n}{n+1}\left(\frac{1}{n} V_{n-1}(y)\right)$$

and, after taking limit superior, this leads to $V(x) \geq V(y)$; see (2.7). Since the states x and y satisfying $p_{x\,y} > 0$ were arbitrary, it follows that

$$V(x) \geq \max\{V(y) | p_{xy} > 0\}, \quad x \in S,$$

and the conclusion follows combining this inequality with (3.5).

(*ii*) From Theorem 2.1(*i*) and part (*i*), it follows that for each state x, the equality

$$P_x[V(X_n) \leq V(X_{n-1}) \leq \cdots \leq V(X_1) \leq V(X_0)] = 1$$

is valid for every positive integer n. Since $P_x[X_0 = x] = 1$, it follows that $P_x[V(X_n) \leq V(x)] = 1$, so that $V(y) \leq V(x)$ when $P_x[X_n = y] > 0$.

(*iii*) Since the Markov chain $\{X_n\}$ has the single recurrent class $\mathcal{R}$, by Assumption 2.1, given states $x, y \in \mathcal{R}$, there always exists an integer $m = m(x, y)$ such that

$$P_x[X_m = y] > 0,$$

so that $V(x) \geq V(y)$, by part (*ii*). Interchanging the roles of x and y the inequality $V(y) \geq V(x)$ is obtained, and then $V(x) = V(y)$ when x and y lie in $\mathcal{R}$.

(*iv*) Since the state space is finite and $\{X_n\}$ has the single recurrent class $\mathcal{R}$, this set must be visited regardless of the initial state, so that given $x \in S$, there exists a positive integer m and $y \in \mathcal{R}$ satisfying $P_x[X_m = y] > 0$, and in this case part (*iii*) yields $V(x) \geq V(y)$. Now, the conclusion follows using that $V(\cdot)$ is constant on $\mathcal{R}$, by part (*iii*). □

4. A key technical tool

The proof of Theorem 2.2 presented in the following section is based on the analysis of the discrepancy function $C(\cdot) - V(\cdot)$. The key fact to be established is that, when this function is divided by a number larger than one, then its aggregated value before the first visit to state z has a finite moment generating function at λ. This result is formally stated as follows.

Theorem 4.1. *Let $V(\cdot)$ and the stopping time T be as in (2.7) and (3.1), respectively. Under Assumption 2.1 the following holds: For every real number $p \in (1, \infty)$ and $x \in S$,*

$$E_x\left[e^{\lambda \sum_{t=0}^{T-1}[C(X_t) - V(X_t)]/p}\right] < \infty. \tag{4.1}$$

The somewhat technical proof of this theorem is based on the analysis of the different level sets of $V(\cdot)$ performed in Lemmas 4.1–4.3 below. First, it is convenient to introduce some notation.

Definition 4.1. *Let v_i, $i = 1, 2, \ldots, k$ be the different values of the functional $V(\cdot)$, where*

$$v_0 < v_1 < \cdots < v_k. \tag{4.2}$$

(*i*) *The set L_i is defined by*

$$L_i := \{y | V(y) = v_i\}, \quad i = 0, 2, \ldots, k. \tag{4.3}$$

(*ii*) *The exit time T_i is given by*

$$T_i := \min\{n \geq 1 | X_n \notin L_i\}. \tag{4.4}$$

(*iii*) *For each $i = 0, 1, \ldots, k$ set*

$$\mathcal{L}_i := \bigcup_{j=0}^{i} L_j \tag{4.5}$$

and define the arrival time $T_{\mathcal{L}_i}$ by

$$T_{\mathcal{L}_i} := \min\{n \geq 1 | X_n \in \mathcal{L}_i\}. \tag{4.6}$$

The following result concerns the jumps of $\{X_t\}$ among the different level sets L_i.

Lemma 4.1. (*i*) *The recurrent class $\mathcal{R}$ is contained in L_0.*
(*ii*) *The set $\mathcal{L}_i$ is absorbing for each $i = 0, \ldots, k$; see (4.5).*
(*iii*) *Suppose that the initial state $X_0 = x$ belongs to L_i, where $i = 1, 2, \ldots, k$. In this case, the first exit time from set L_i coincides with the first arrival time to set $\mathcal{L}_{i-1}$, i.e.,*

$$P_x[T_i = T_{\mathcal{L}_{i-1}}] = 1, \quad x \in L_i, \quad i = 1, 2, \ldots, k;$$

see (4.3)–(4.6). Therefore,

(iv) *For each* $i = 1, 2, \ldots, k$, $P_x[T_i \leq T] = 1$ *when* $x \in L_i$.
(v) *For each* $n \in \mathbb{N}$
$P_x[T_i \geq n] \leq B\beta^n, \quad x \in L_i, \quad i = 1, 2, \ldots, k,$

where the constants B and β are as in Lemma 3.1.

Proof. (i) By Lemma 3.2(iv), $V(\cdot)$ is constant on the recurrent class $\mathcal{R}$ and attains its minimum on $\mathcal{R}$. Since v_0 is the smallest value of $V(\cdot)$ (see (4.2)), it follows that $V(x) = v_0$ for every $x \in \mathbb{R}$, so that $\mathcal{R} \subset L_0$.
(ii) Suppose that $x \in \mathcal{L}_i$, so that $v_i \geq V(x)$, by Definition 4.1. Select a state y such that $p_{xy} > 0$, and notice that $V(x) \geq V(y)$, by Lemma 3.2(ii). Therefore, $v_i \geq V(y)$, and then (4.2) yields that $V(y) = v_r$ for some $r \leq i$, *i.e.*, $y \in L_r \subset \bigcup_{j=0}^{i} L_j = \mathcal{L}_i$. In short, if $x \in \mathcal{L}_i$ and $p_{xy} > 0$, then $y \in \mathcal{L}_i$, so that $\mathcal{L}_i$ is absorbing.
(iii) Define the event W by

$$W = \left[X_t \in \mathcal{L}_i = \bigcup_{j=0}^{i} L_j \quad \text{forevery } t \geq 0 \right],$$

(see (4.5)), and notice that, by part (ii), $P_x[W] = 1$ for every $x \in L_i$. Consider now a trajectory $\{X_t\}$ in W starting at $X_0 = x \in L_i$ and observe that, by Definition 4.1(iii), $T_i = m$ means that

$$X_j \in L_i, \quad j = 1, 2, \ldots, m-1, \quad X_m \notin L_i,$$

statement that, since the sample trajectory belongs to W, is equivalent to

$$X_j \in L_i, \quad j = 1, 2, \ldots, m-1, \quad X_m \in \bigcup_{j=0}^{i-1} L_j = \mathcal{L}_{i-1}.$$

Therefore, on the event W, $T_i = T_{\mathcal{L}_{i-1}}$ (see Definition 4.1), and the conclusion follows since, as already noted, $P_x[W] = 1$ for every state $x \in L_i$.

(iv) Since $z \in \mathcal{R} \subset \mathcal{L}_0 \subset \mathcal{L}_{i-1}$, (3.1) and (4.6) together imply that the inequality $T_{\mathcal{L}_{i-1}} \leq T$ always holds, so that $P_x[T_i \leq T] = 1$ for $x \in L_i$, by part (iii).
(v) By part (iv), the inequality $P_x[T_i \geq n] \leq P_x[T \geq n]$ occurs for every $n \in \mathbb{N}$ whenever $x \in L_i$, and the result follows from Lemma 3.1. □

In the following lemma it will be proved that the conclusion of Theorem 4.1 holds when the initial state x lies in L_0.

Lemma 4.2. *Given* $p \in (1, \infty)$, *inequality (4.1) occurs for every* $x \in L_0$.

Proof. Let $x \in L_0$ be arbitrary but fixed, and observe that L_0 is closed, by Lemma 4.1(ii), so that $P_x[V(X_t) = v_0] = 1$ for every $t \in \mathbb{N}$. Thus,

$$E_x\left[e^{\lambda\sum_{t=0}^{T-1}[C(X_t)-V(X_t)]/p}\right]=E_x\left[e^{\lambda\sum_{t=0}^{T-1}[C(X_t)-v_0]/p}\right],\quad x\in L_0. \tag{4.7}$$

Next, observe that

$$\begin{aligned}E_x\left[e^{\lambda\sum_{t=0}^{T-1}[C(X_t)-v_0]/p}\right]&=\sum_{m=0}^{\infty}E_x\left[e^{\lambda\sum_{t=0}^{m-1}[C(X_t)-v_0]/p}I[T=m]\right]\\&\le\sum_{m=0}^{\infty}\left(E_x\left[e^{\lambda\sum_{t=0}^{m-1}[C(X_t)-v_0]}\right]\right)^{1/p}(E_x[I[T=m]])^{1/q}\end{aligned}$$

where Hölder's inequality was used to set the inequality and $q:=p/(p-1)$. Since $E_x[I[T=m]]=P_x[T=m]$, the above displayed inequality and (4.7) together yield, using Lemma 3.1 and (2.6), that

$$E_x\left[e^{\lambda\sum_{t=0}^{T-1}[C(X_t)-V(X_t)]/p}\right]\le B^{1/q}\sum_{m=0}^{\infty}\left(e^{\lambda[V_{m-1}(x)-mv_0]}\right)^{1/p}\left(\beta^{1/q}\right)^{m}. \tag{4.8}$$

Recalling that $\beta\in(0,1)$, select $\varepsilon>0$ such that

$$e^{\lambda\varepsilon/p}\beta^{1/q}<1. \tag{4.9}$$

On the other hand, since $V(x)=v_0$, there exists a positive integer $N=N(\varepsilon)$ such that when $m\ge N$,

$$\frac{1}{m}V_{m-1}(x)\le v_0+\varepsilon,$$

and then

$$\left(e^{\lambda[V_{m-1}(x)-mv_0]}\right)^{1/p}\left(\beta^{1/q}\right)^{m}\le\left(e^{\lambda\varepsilon/p}\beta^{1/q}\right)^{m},\quad m\ge N.$$

Thus, (4.9) implies that the series in (4.8) is finite, and the conclusion follows. □

The following step is the last one before the proof of Theorem 4.1.

Lemma 4.3. *Let $x\in L_i$ be arbitrary where $i\ge 1$. In this case, for every $p\in(1,\infty)$*

$$E_x\left[e^{\lambda\sum_{t=0}^{T_i-1}[C(X_t)-V(X_t)]/p}\right]<\infty.$$

Proof. The argument is along the lines used to establish the previous lemma. To begin with, notice that the definition of T_i yields that $X_t\in L_i$ when $1\le t<T_i$ so that when $X_0=x\in L_i$, $V(X_t)=v_i$ if $0\le t\le T_i-1$. Therefore,

$$E_x\left[e^{\lambda\sum_{t=0}^{T_i-1}[C(X_t)-V(X_t)]/p}\right]=E_x\left[e^{\lambda\sum_{t=0}^{T_i-1}[C(X_t)-v_i]/p}\right],\quad x\in L_i. \tag{4.10}$$

Now, for $x\in L_i$, write

$$E_x\left[e^{\lambda\sum_{t=0}^{T_i-1}[C(X_t)-v_i]/p}\right]=\sum_{m=0}^{\infty}E_x\left[e^{\lambda\sum_{t=0}^{m-1}[C(X_t)-v_i]/p}I[T_i=m]\right]$$

and use Hölder's inequality and (4.10) to obtain

$$E_x\left[e^{\lambda\sum_{t=0}^{T_i-1}[C(X_t)-V(X_t)]/p}\right] \le \sum_{m=0}^{\infty}\left(E_x\left[e^{\lambda\sum_{t=0}^{m-1}[C(X_t)-v_i]}\right]\right)^{1/p}(E_x[I[T_i=m]])^{1/q}$$
$$= \sum_{m=0}^{\infty}\left(e^{\lambda[V_{m-1}(x)-mv_i]}\right)^{1/p}(P_x[T_i=m])^{1/q},$$

where (2.6) was used to set the equality. Applying Lemma 4.1(v) this yields

$$E_x\left[e^{\lambda\sum_{t=0}^{T_i-1}[C(X_t)-V(X_t)]/p}\right] \le B^{1/q}\sum_{m=0}^{\infty}\left(e^{\lambda[V_{m-1}(x)-mv_i]}\right)^{1/p}\left(\beta^{1/q}\right)^m, \quad x\in L_i. \tag{4.11}$$

Now, let $\varepsilon>0$ be as in (4.9). Given $x\in L_i$, so that $v_i=V(x)$, from (2.7) there exists a positive integer $N=N(\varepsilon)$ satisfying $V_{m-1}(x)/m\le v_i+\varepsilon$ if $m\ge N$, and in this case

$$\left(e^{\lambda[V_{m-1}(x)-mv_i]}\right)^{1/p}\left(\beta^{1/q}\right)^m \le \left(e^{\lambda\varepsilon/p}\beta^{1/q}\right)^m, \quad m\ge N,$$

and then (4.9) shows that the series in (4.11) is finite, completing the proof. □

After the preliminaries in the previous lemmas, Theorem 4.1 can be established as follows.

Proof of Theorem 4.1. For each $m=0,1,2,\ldots,k$, consider the following claim:

$$\mathbf{C}_m:\quad E_x\left[e^{\lambda\sum_{t=0}^{T-1}[C(X_t)-V(X_t)]/p}\right]<\infty \quad \text{for every } x\in\mathcal{L}_m=\bigcup_{j=0}^{m}L_j.$$

It will be shown, by induction, that each $\mathbf{C}_m$ is true. First, notice that $\mathbf{C}_0$ is valid, by Lemma 4.2. Assume that C_{i-1} holds for some positive integer i, let $x\in L_i$ be arbitrary and observe that, by Lemma 4.1(iv), $T_i\le T$ P_x–a. s., so that

$$\begin{aligned}E_x\left[e^{\lambda\sum_{t=0}^{T-1}[C(X_t)-V(X_t)]/p}\right] &= E_x\left[e^{\lambda\sum_{t=0}^{T-1}[C(X_t)-V(X_t)]/p}I[T_i=T]\right]\\ &\quad+E_x\left[e^{\lambda\sum_{t=0}^{T-1}[C(X_t)-V(X_t)]/p}I[T_i<T]\right]\end{aligned} \tag{4.12}$$

It will be shown that both terms in the right hand side of this equation are finite. First, notice that

$$E_x\left[e^{\lambda\sum_{t=0}^{T-1}[C(X_t)-V(X_t)]/p}I[T_i=T]\right]\le E_x\left[e^{\lambda\sum_{t=0}^{T_i-1}[C(X_t)-V(X_t)]/p}\right]<\infty, \tag{4.13}$$

where the second inequality is due to Lemma 4.3. Next, observe that for each positive integer r and $y\in\mathcal{L}_{i-1}\setminus\{z\}$, the Markov property yields

$$E_x\left[e^{\lambda\sum_{t=0}^{T-1}[C(X_t)-V(X_t)]/p}I[T_i<T]\middle|X_0,X_1,\ldots,X_r=y,T_i=r\right]$$

$$
\begin{aligned}
&= I[r<T] e^{\lambda \sum_{t=0}^{r-1}[C(X_t)-V(X_t)]/p} E_x\left[e^{\lambda \sum_{t=r}^{T-1}[C(X_t)-V(X_t)]/p} \middle| X_0, X_1, \ldots, X_r = y, T_i = r \right] \\
&= I[r<T] e^{\lambda \sum_{t=0}^{r-1}[C(X_t)-V(X_t)]/p} E_y\left[e^{\lambda \sum_{t=0}^{T-1}[C(X_t)-V(X_t)]/p} \right] \\
&\leq \mathcal{M}_{i-1} e^{\lambda \sum_{t=0}^{r-1}[C(X_t)-V(X_t)]/p}
\end{aligned}
\tag{4.14}
$$

where

$$
\mathcal{M}_{i-1} := \max\left\{ E_y\left[e^{\lambda \sum_{t=r}^{T-1}[C(X_t)-V(X_t)]/p} \right] \middle| y \in \mathcal{L}_{i-1} \right\} < \infty
$$

and the inequality is due to the induction hypothesis. Observe now that, by Lemma 4.1(*iii*), $X_{T_i} \in \mathcal{L}_{i-1}$ P_x–almost surely; since (4.14) holds for every positive integer r and $y \in \mathcal{L}_i \setminus \{z\}$, it follows that

$$
E_x\left[e^{\lambda \sum_{t=0}^{T-1}[C(X_t)-V(X_t)]/p} I[T_i < T] \middle| T_i \right] \leq \mathcal{M}_{i-1} e^{\lambda \sum_{t=0}^{T_i-1}[C(X_t)-V(X_t)]/p}
$$

and then

$$
E_x\left[e^{\lambda \sum_{t=0}^{T-1}[C(X_t)-V(X_t)]/p} I[T_i < T] \right] \leq \mathcal{M}_{i-1} E_x\left[e^{\lambda \sum_{t=0}^{T_i-1}[C(X_t)-V(X_t)]/p} \right] < \infty
$$

where, again, Lemma 4.3 was used to set the second inequality. Combining this expression with (4.12) and (4.13), it follows that

$$
E_x\left[e^{\lambda \sum_{t=0}^{T-1}[C(X_t)-V(X_t)]/p} \right] < \infty, \quad x \in L_i. \tag{4.15}
$$

In short, assuming that $\mathbf{C}_{i-1}$ holds, it has been shown that (4.15) occurs. Since $\mathcal{L}_i = L_i \cup \mathcal{L}_{i-1}$ this proves that $\mathbf{C}_i$ is valid, completing the induction argument. Therefore, $\mathbf{C}_k$ is true, establishing Theorem 4.1. □

5. Proof of the main result

In this section a proof of Theorem 2.2 will be given. The argument relies on Theorem 4.1, as well as on the following lemma.

Lemma 5.1. *Under Assumption 2.1, the following inequality holds:*

$$
E_z\left[e^{\lambda \sum_{t=0}^{T-1}[C(X_t)-V(X_t)]} \right] \leq 1. \tag{5.1}
$$

Proof. By Assumption 2.1, if $x, y \in \mathcal{R}$ then $P_x[X_n = y] > 0$ for some integer n, so that the restriction of the Markov chain $\{X_n\}$ to the recurrent class R is completely communicating. Therefore, there exists a function $h: S \to \mathbb{R}$ such that

$$
e^{\lambda v_0 + \lambda h(x)} = e^{\lambda C(x)} \sum_y p_{xy} e^{\lambda h(y)}, \quad x \in \mathcal{R}; \tag{5.2}
$$

see, for instance, [5], [14]. This equation is equivalent to

$$e^{\lambda h(x)} = e^{\lambda[C(x)-v_0]}P_x[T=1]e^{\lambda h(z)} + E_x\left[e^{\lambda[C(X_0)-v_0]}e^{\lambda h(X_1)}I[T>1]\right],$$

and from this point, an induction argument combining (5.2) and the Markov property yields that, for every positive integer n and $x \in \mathcal{R}$,

$$e^{\lambda h(x)} = \sum_{m=1}^{n} E_x\left[e^{\lambda \sum_{t=0}^{m-1}[C(X_t)-v_0]}I[T=m]\right]e^{\lambda h(z)} + E_x\left[e^{\lambda \sum_{t=0}^{n-1}[C(X_t)-v_0]}e^{\lambda h(X_n)}I[T>n]\right].$$

Setting $x = z$ in this equation, it follows that $e^{\lambda h(z)} \geq \sum_{m=1}^{n} E_z\left[e^{\lambda \sum_{t=0}^{m-1}[C(X_t)-v_0]}I[T=m]\right]e^{\lambda h(z)}$ and after letting n go to ∞, this implies

$$e^{\lambda h(z)} \geq \sum_{m=1}^{\infty} E_z\left[e^{\lambda \sum_{t=0}^{m-1}[C(X_t)-v_0]}I[T=m]\right]e^{\lambda h(z)} = E_z\left[e^{\lambda \sum_{t=0}^{T-1}[C(X_t)-v_0]}\right]e^{\lambda h(z)}$$

and then

$$E_z\left[e^{\lambda \sum_{t=0}^{T-1}[C(X_t)-v_0]}\right] \leq 1.$$

Since $\mathcal{R}$ is closed and $V(\cdot)$ assumes the value v_0 in $\mathcal{R}$, by Lemma 4.1(i), this inequality is equivalent to (5.1). □

Proof of Theorem 2.2. It will be shown that

$$g_p(\cdot) := \frac{1}{p}V(\cdot) + \left(1 - \frac{1}{p}\right)\|C\| \in \mathcal{G} \quad \text{for every } p \in (1,\infty), \tag{5.3}$$

assertion that combined with Theorem 2.1(ii) yields that $V(x) = \inf_{g\in\mathcal{G}} g(x)$ for every $x \in S$, which is the desired conclusion. To verify (5.3), let $p \in (1,\infty)$ be given and notice that Lemma 3.2(i) and Remark 2.1(ii) together yield that inequality (2.3) is satisfied by g_p. To verify condition (ii) in Definition 2.1, define $h_p: S \to \mathcal{R}$ by

$$h_p(x) := \frac{1}{\lambda}\log\left(E_x\left[e^{\lambda \sum_{t=0}^{T-1}[C(X_t)-V(X_t)]/p}\right]\right), \quad x \in S; \tag{5.4}$$

notice that $h_p(\cdot) < \infty$, by Theorem 4.1, whereas $h_p(\cdot) > -\infty$, since $P_x[T<\infty] = 1$, by Lemma 3.1. Observe now that, by Hölder's inequality,

$$e^{\lambda h_p(z)} = E_z\left[e^{\lambda \sum_{t=0}^{T-1}[C(X_t)-V(X_t)]/p}\right] \leq \left(E_z\left[e^{\lambda \sum_{t=0}^{T-1}[C(X_t)-V(X_t)]}\right]\right)^{1/p}$$

and using Lemma 5.1 this implies

$$e^{\lambda h_p(z)} \leq 1. \tag{5.5}$$

To conclude observe that, for every $x \in S$, (5.4) and the Markov property together yield

$$
\begin{aligned}
e^{\lambda h_p(x)} &= E_x\left[e^{\lambda \sum_{t=0}^{T-1}[C(X_t)-V(X_t)]/p}\right] \\
&= e^{\lambda[C(x)-V(x)]/p} p_{x\,z} + e^{\lambda[C(x)-V(x)]/p} \sum_{y \neq z} p_{x\,y} E_y\left[e^{\lambda \sum_{t=0}^{T-1}[C(X_t)-V(X_t)]/p}\right] \\
&= e^{\lambda[C(x)-V(x)]/p} p_{x\,z} + e^{\lambda[C(x)-V(x)]/p} \sum_{y \neq z} p_{x\,y} e^{\lambda h_p(y)},
\end{aligned}
$$

and then (5.5) allows to write, for each $x \in S$,

$$
e^{\lambda h_p(x)} \geq e^{\lambda[C(x)-V(x)]/p} \sum_y p_{x\,y} e^{\lambda h_p(y)}.
$$

Multiplying both sides of this inequality by $e^{\lambda g_p(x)}$ (see (5.3)), it follows that

$$
e^{\lambda g_p(x)+\lambda h_p(x)} \geq e^{\lambda[C(x)/p+(1-1/p)\|C\|]} \sum_y p_{x\,y} e^{\lambda h_p(y)}.
$$

and observing that $[C(\cdot)/p + (1-1/p)\|C\|] \geq C(\cdot)$, the above inequality yields

$$
e^{\lambda g_p(x)+\lambda h_p(x)} \geq e^{\lambda C(x)} \sum_y p_{x\,y} e^{\lambda h_p(y)}, \quad x \in S.
$$

Therefore, function $g_p(\cdot)$ in (5.3) also satisfies the second part of Definition 2.1, establishing (5.3). As already mentioned, this completes the proof of Theorem 2.2. □

6. Concluding remarks

This work considered Markov chains with finite state space. Under the assumption that the process has a single recurrent class, the Varadhan's function in (1.1) associated to a given function $C: S \to \mathbb{R}$ and $\lambda > 0$ was studied, and it was shown in Theorem 2.2 that $V(\cdot)$ is the optimal value of a convex minimization in $\mathbb{R}^d$, where d is the number of states. As already noted, when the initial state x belongs to the recurrent class $\mathbb{R}$ the limit superior in the definition of Varadhan's function can be replaced by limit, and it is interesting to see if the same can be done at a transient state. Research on this direction is currently in progress.

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