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Retailer's optimal ordering policy under supplier credits

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Abstract. In the traditional inventory economic order quantity (or EOQ) model, it was assumed that the customer must pay for the items as soon as the items are received. However, in practices, the supplier frequently offers a cash discount and/or a permissible delay to the customer especially when the economy turns sour. As a result, in this paper, we establish an optimal ordering policy for a retailer when the supplier provides not only a cash discount to avoid the default risk but also a permissible delay to increase sales. We then characterize the optimal solution and provide an easy-to-use algorithm to find the optimal order quantity and replenishment time. Furthermore, we also compare the optimal order quantity under supplier credits to the classical economic order quantity. Finally, several numerical examples are given to illustrate the theoretical results and make the sensitivity of parameters on the optimal solution.

Key words: Inventory, Cash Discount, Finance, Delay payments, Deteriorating items

1. Introduction

In the classical inventory economic order quantity (or EOQ) model, it was tacitly assumed that the supplier is paid for the items immediately after the items are received. In reality, a supplier is always willing to provide the customer either a cash discount or a permissible delay of payments. A cash discount can encourage the customer pays cash on delivery and reduce the default risk. A permissible delay in payments is considered a type of price reduction and it can attract new customers and increase sales. As a result, the customer has two distinct alternatives (*i.e.*, either a cash discount or a permissible delay) to find the optimal order quantity and replenishment time. So far, this important and relevant problem has not drawn much attention in the operations literature.

In recent years, marketing researchers and practitioners have recognized the phenomenon that the supplier offers a permissible delay to the customer if the outstanding amount is paid within the permitted fixed settlement period. Goyal (1985) derived an EOQ model under the conditions of permissible delay in payments. Aggarwal and Jaggi (1995) then extended Goyal's model to allow for deteriorating items. Next, Jamal *et al.* (1997) further generalized the model to allow for shortages. Liao *et al.* (2000) developed an inventory model for stock-depend consumption rate when a delay in payment is permissible. Recently, Arcelus *et al.* (2001) analyzed the pros and cons of price discount vs. trade credit. There were several interesting and relevant papers related to trade credits such as Davis and Gaither (1985), Arcelus and Srinivasan (1993, 1995, and 2001), Shah (1993), Chang and Dye (2001), Teng (2002), Chang *et al.* (2003), Change (2003) and Teng *et al.* (2003).

During the past few years, many researchers have studied inventory models for deteriorating items such as volatile liquids, blood banks, medicines, electronic components and fashion goods. Ghare and Schrader (1963) were the first proponents for developing a model for an exponentially decaying inventory. Next, Covert and Philip (1973) extended Ghare and Schrader's constant deterioration rate to a two-parameter Weibull distribution. Shah and Jaiswal (1977) and Aggarwal (1978) presented and re-established an order level inventory model with a constant rate of deterioration, respectively. Dave and Patel (1981) considered an inventory model for deteriorating items with time-proportional demand when shortages were prohibited. Sachan (1984) then extended the model to allow for shortages. Later, Hariga (1996) generalized the demand pattern to any log-concave function. Teng *et al.* (1999) and Yang *et al.* (2001) further generalized the demand function to include any non-negative, continuous function that fluctuates with time. Recently, Goyal and Giri (2001) wrote an excellent survey on the recent trends in modeling the deteriorating inventory.

In this paper, we provide the optimal ordering policy for the customer to obtain its minimum cost when the supplier provides not only a cash discount but also a permissible delay to the customer. For example, the supplier offers a 2% discount off the price if the payment is made within 10 days; otherwise the full price of the merchandise is due within 30 days. This credit term is usually denoted as "2/10, net 30" (*e.g.*, see Brigham (1995, p. 741)). We establish an EOQ model for deteriorating items under supplier credits, and then study the necessary and sufficient conditions for finding the optimal solution to the problem, and provide an easily determined condition to find the optimal replenishment interval.

The rest of the paper is organized as follows. In Section 2, we describe the assumptions and notation used throughout this study. In Section 3, we develop the mathematical model to minimize the total relevant cost per year. In Section 4, the necessary and sufficient conditions are derived, an approximately closed-form solution to the optimal replenishment interval is developed, and an important theorem is established to determine the optimal replenishment interval. In addition, we also characterize the effect of the value of parameters on the optimal replenishment cycle. We then compare the optimal order quantity under a cash discount and/or a permissible delay in payment with the classical economic order quantity (in which the supplier must be paid for the items as soon as the customer receives them) in Section 5. Numerical examples are presented in Section 6 to illustrate the results. Finally, we draw the conclusions and the future research in Section 7.

2. Assumptions and notation

The following assumptions are similar to those in Goyal's (1985) EOQ model.

(1) The demand for the item is constant with time.
(2) Shortages are not allowed.
(3) Replenishment is instantaneous.
(4) During the time the account is not settled, generated sales revenue is deposited in an interest bearing account. At the end of this period (*i.e.*, M_1 or M_2), the customer pays the supplier the total amount in the interest bearing account, and then starts paying off the amount owed to the supplier whenever the customer has money obtained from sales.
(5) Time horizon is infinite.

In addition, the following notation is used throughout this paper.

D = the demand rate per year.
h = the unit holding cost per year excluding interest charges.
p = the selling price per unit.
c = the unit purchasing cost, with $c < p$.
I_c = the interest charged per \$ in stocks per year by the supplier or a bank.
I_d = the interest earned per \$ per year.
S = the ordering cost per order.
Q = the order quantity.
r = the cash discount rate, $0 < r < 1$.
θ = the constant deterioration rate, where $0 \leq \theta < 1$.
M_1 = the period of cash discount.
M_2 = the period of permissible delay in settling account, with $M_2 > M_1$.
T = the replenishment time interval.
$I(t)$ = the level of inventory at time t, $0 \leq t \leq T$.
$Z(T)$ = the total relevant cost per year,

where the total relevant cost consists of (a) cost of placing orders, (b) cost of purchasing units, (c) cost of carrying inventory (excluding interest charges), (d) cash discount earned if the payment is made at M_1, (e) interest earned from sales revenue during the permissible period $[0, M_1]$ or $[0, M_2]$, and (f) cost of interest charges for unsold items after the permissible delay M_1 or M_2.

3. Mathematical formulation

The level of inventory $I(t)$ gradually decreases mainly to meet demands and partly due to deterioration. Hence, the variation of inventory with respect to time can be described by the following differential equations:

$$\frac{dI(t)}{dt} + \theta I(t) = -D, \quad 0 \leq t \leq T, \tag{1}$$

with the boundary conditions: $I(0) = Q$, $I(T) = 0$. Consequently, the solution of (1) is given by

$$I(t) = \frac{D}{\theta}[e^{\theta(T-t)} - 1], \quad 0 \leq t \leq T, \tag{2}$$

and the order quantity is

$$Q = I(0) = \frac{D}{\theta}(e^{\theta T} - 1). \tag{3}$$

Total demand during one cycle is DT. Therefore, the number of deteriorating items during a replenishment cycle is

$$Q - DT. \tag{4}$$

The total relevant cost per year consists of the following elements.

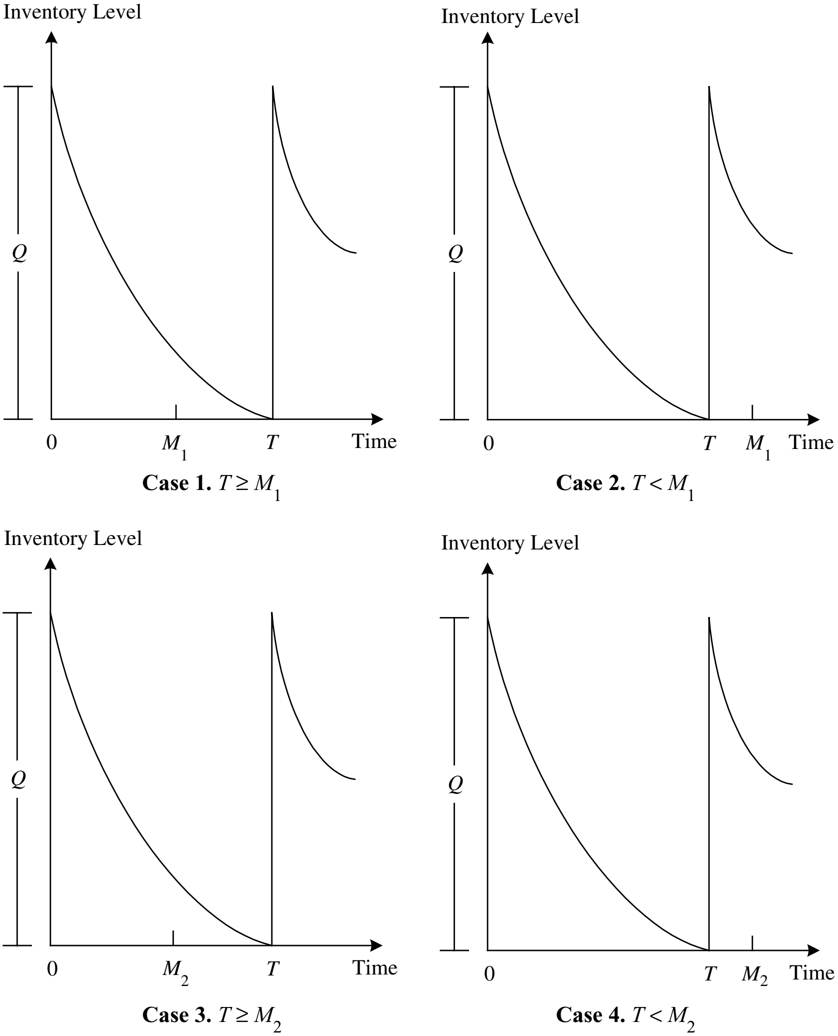

Fig. 1. Graphical representation of four inventory systems

(a) Cost of placing orders $= S/T$. (5)

(b) Cost of purchasing units $= cQ/T = \frac{cD}{\theta T}(e^{\theta T} - 1)$. (6)

(c) Cost of carrying inventory $= h\int_0^T I(t)dt/T = \frac{hD}{\theta^2 T}(e^{\theta T} - 1) - \frac{hD}{\theta}$. (7)

Regarding cash discount, interests charged and earned (*i.e.*, costs of (d) – (f)), we have four possible cases based on the customer's two choices (*i.e.*, pays at M_1 or M_2) and the length of T. In Case 1, the payment is paid at M_1 to get a cash discount and $T \geq M_1$. For Case 2, the customer pays in full at M_1 to get a cash discount but $T < M_1$. Similarly, if the payment is paid at time M_2 to get the permissible delay and $T \geq M_2$, then it is Case 3. As to Case 4, the customer pays in full at M_2 but $T < M_2$. Now, we can express the cash discount, the cost of interest charges and the interest earned for each of those four cases as shown in Figure 1.

Case 1. $T \geq M_1$

Since the payment is paid at time M_1, the customer saves rcQ per cycle due to price discount. From (3), we know that the discount savings per year is given by

$$\frac{rcQ}{T} = \frac{rcD}{\theta T}(e^{\theta T} - 1). \tag{8}$$

Next, during [0, M_1] period, the customer sells products and deposits the revenue into an account that earns I_d per dollar per year. Therefore, the interest earned per year is

$$pI_d \int_0^{M_1} Dt\, dt/T = \frac{pI_d D}{2T} M_1^2. \tag{9}$$

Finally, the customer buys $I(0)$ units at time 0, and owes $c(1\text{-}r)I(0)$ to the supplier. At time M_1, the customer sells (DM_1) units in total, and has pDM_1 plus interest earned $p\ I_d\ D\ M_1^2/\ 2$ to pay the supplier. From the difference between the total purchase cost $c(1-r)I(0)$ and the total amount of money in the account $pDM_1 + p\ I_d\ D\ M_1^2/\ 2$, we have the following two cases: $pDM_1 + p\, I_d\, D\, M_1^2/2 \geq c(1-r)I(0)$, and $pDM_1 + p\, I_d\, D\, M_1^2/2 < c(1-r)I(0)$. For simplicity, we will discuss only the case in which $pD\, M_1 + p\, I_d\, D\, M_1^2/2 < c(1-r)I(0)$. The reader can easily obtain the similar results for the other case.

If $pDM_1 + p\ I_d\ D\ M_1^2/\ 2 < c(1\text{-}r)I(0)$, then we need to finance $L = c(1-r)I(0) - (pDM_1 + p\ I_d\ D\ M_1^2/\ 2)$ (at interest rate I_c) at time M_1, and pay the supplier in full in order to get the cash discount. Thereafter, the customer gradually reduces the amount of financed loan due to constant sales and revenue received. By using (3), we obtain the interest payable per year is

$$I_c\ L[L/(pD)]/(2T) = \frac{I_c}{2pDT}\left[\frac{c(1-r)D}{\theta}(e^{\theta T}-1) - pDM_1(1+I_dM_1/2)\right]^2. \tag{10}$$

From (5)–(9) and (10), we have the total relevant cost per year $Z_1(T)$as follow:

$$\begin{aligned} Z_1(T) = {} & \frac{S}{T} + \frac{D[h+c\theta(1-r)]}{\theta^2 T}(e^{\theta T}-1) - \frac{hD}{\theta} - \frac{pI_dD}{2T}M_1^2 \\ & + \frac{I_c}{2pDT}\left[\frac{c(1-r)D}{\theta}(e^{\theta T}-1) - pDM_1(1+I_dM_1/2)\right]^2. \end{aligned} \tag{11}$$

Case 2. $T < M_1$

In this case, the customer sells DT units in total at time T, and has cDT to pay the supplier in full at time M_1. Consequently, there is no interest payable, while the cash discount is the same as that in Case 1. However, the interest earned per year is

$$pI_d\left[\int_0^T Dt\,dt + DT(M_1 - T)\right]/T = pI_dD(M_1 - T/2). \tag{12}$$

As a result, the total relevant cost per year $Z_2(T)$is

$$Z_2(T) = \frac{S}{T} + \frac{D[h+c\theta(1-r)]}{\theta^2 T}(e^{\theta T}-1) - \frac{hD}{\theta} - pI_d\ D(M_1 - \frac{T}{2}). \tag{13}$$

Case 3. $T \geq M_2$

Since the payment is paid at time M_2, there is no cash discount. The interest earned per year is

$$pI_d\int_0^{M_2} Dt\,dt/T = \frac{pI_dD}{2T}M_2^2. \tag{14}$$

For simplicity and generality, we will discuss only the case in which $pDM_2 + p\ I_d\ D\ M_2^2/\ 2 < cI(0)$. The reader can easily obtain the similar results for the other case in which $pD\ M_2 + p\ I_d\ D\ M_2^2/\ 2 \geq cI(0)$. By using an analogous as that in Case 1, if $pDM_2 + p\ I_d\ D\ M_2^2/\ 2 < cI(0)$, then the interest earned per year is

$$\frac{I_c}{2pDT}\left[\frac{cD}{\theta}(e^{\theta T}-1) - pDM_2(1+I_dM_2/2)\right]^2. \tag{15}$$

Therefore, the total relevant cost per year $Z_3(T)$is

$$\begin{aligned} Z_3(T) = {} & \frac{S}{T} + \frac{D(h+c\theta)}{\theta^2 T}(e^{\theta T}-1) - \frac{hD}{\theta} + \frac{I_c}{2pDT} \\ & \left[\frac{cD}{\theta}(e^{\theta T}-1) - pDM_2(1+I_dM_2/2)\right]^2 - \frac{pI_dD}{2T}M_2^2. \end{aligned} \tag{16}$$

Case 4. $T < M_2$

In this case, there is no interest charged. The interest earned per year is

$$pI_d\Big[\int_0^T \mathrm{D}t\,dt + DT(M_2 - T)\Big]/T = pI_dD\,(M_2 - T/2). \tag{17}$$

Hence, we get the total relevant cost per year$Z_4(T)$is

$$Z_4(T) = \frac{S}{T} + \frac{D(h + c\theta)}{\theta^2 T}(e^{\theta T} - 1) - \frac{hD}{\theta} - pI_dD(M_2 - \frac{T}{2}). \tag{18}$$

4. Theoretical results

In reality, the value for the deterioration rate θ is sufficiently small. Utilizing the fact that

$$e^{\theta T} \approx 1 + \theta T + (\theta T)^2/2,\ \text{as } \theta T \text{ is small,}$$

we obtain

$$\begin{aligned} Z_1(T) \approx{}& \frac{S}{T} + \frac{D[h + c\theta(1-r)]}{\theta^2 T}\left(\theta T + \frac{\theta^2 T^2}{2}\right) - \frac{hD}{\theta} - \frac{pI_dD}{2T}M_1^2 \\ &+ \frac{I_cD}{2pT}\left[\frac{c(1-r)}{\theta}\left(\theta T + \frac{\theta^2 T^2}{2}\right) - pM_1(1 + I_dM_1/2)\right]^2, \end{aligned} \tag{19}$$

$$Z_2(T) \approx \frac{S}{T} + \frac{D[h + c\theta(1-r)]}{\theta^2 T}\left(\theta T + \frac{\theta^2 T^2}{2}\right) - \frac{hD}{\theta} - pI_dD\left(M_1 - \frac{T}{2}\right), \tag{20}$$

$$\begin{aligned} Z_3(T) \approx{}& \frac{S}{T} + \frac{D(h + c\theta)}{\theta^2 T}\left(\theta T + \frac{\theta^2 T^2}{2}\right) - \frac{hD}{\theta} \\ &+ \frac{I_cD}{2pT}\left[\frac{c}{\theta}\left(\theta T + \frac{\theta^2 T^2}{2}\right) - pM_2(1 + I_dM_2/2)\right]^2 - \frac{pI_dD}{2T}M_2^2, \end{aligned} \tag{21}$$

and

$$Z_4(T) \approx \frac{S}{T} + \frac{D(h + c\theta)}{\theta^2 T}\left(\theta T + \frac{\theta^2 T^2}{2}\right) - \frac{hD}{\theta} - pI_dD\left(M_2 - \frac{T}{2}\right). \tag{22}$$

The first-order condition for $Z_1(T)$in (19) to be minimized is $dZ_1(T)/dT = 0$, which leads to

$$\begin{aligned} &S + \frac{I_cD}{2p}\left[\frac{c(1-r)}{\theta}\left(\theta T + \frac{\theta^2 T^2}{2}\right) - pM_1(1 + I_dM_1/2)\right]^2 \\ &\quad= \frac{D[h + c\theta(1-r)]}{2}T^2 + \frac{pI_dD}{2}M_1^2 \\ &\quad+ \frac{cI_cD}{p}\left[\frac{c(1-r)}{\theta}\left(\theta T + \frac{\theta^2 T^2}{2}\right) - pM_1(1 + I_dM_1/2)\right](1 + \theta T)T. \end{aligned} \tag{23}$$

The optimal value of T for Case 1 (*i.e.,* T_1) can be determined by (23).

From $pDM_1 + p\, I_d\, D\, M_1^2/\, 2 < c(1-r)I(0)$, we obtain that

$$T_1 > (1/\theta)\{\ln[(pM_1\theta/c(1-r))(1+I_dM_1/2)+1]\}. \tag{24}$$

The second-order condition

$$\frac{d^2Z_1(T)}{dT^2} = \frac{1}{T^2}\left\{[h+c\theta(1-r)]TD + \frac{[c(1-r)]^2 I_c D}{p}(1+\theta T)^2 T + \frac{c(1-r)I_c D}{p}\left[\frac{c(1-r)}{\theta}\left(\theta T + \frac{\theta^2 T^2}{2}\right) - pM_1(1+I_dM_1/2)\right]\theta T\right\} > 0. \tag{25}$$

By using an analogous argument, we can easily obtain the first-order condition for finding the optimal value of T for Case 2 as

$$2S = D[h + c\theta(1-r) + pI_d]T^2, \tag{26}$$

and thus the optimal value of T for Case 2 is

$$T_2 \approx \sqrt{2S/\{D[h+c\theta(1-r)+pI_d]\}} \tag{27}$$

The second-order condition as

$$\frac{d^2Z_2(T)}{dT^2} = \frac{2S}{T^3} > 0. \tag{28}$$

Substituting (27) into inequality $T_2 < M_1$, we know that

$$\text{if and only if } 2S < D[h + c\theta(1-r) + pI_d]\, M_1^2, \quad \text{then } T_2 < M_1. \tag{29}$$

For Case 3, we obtain the first-order condition as

$$\begin{aligned} S &+ \frac{I_c D}{2p}\left[\frac{c}{\theta}\left(\theta T + \frac{\theta^2 T^2}{2}\right) - pM_2(1+I_dM_2/2)\right]^2 \\ &= \frac{D(h+c\theta)}{2}T^2 + \frac{pI_dD}{2}M_2^2 \\ &\quad + \frac{cI_cD}{p}\left[\frac{c}{\theta}(\theta T + \frac{\theta^2T^2}{2}) - pM_2(1+I_dM_2/2)\right](1+\theta T)T. \end{aligned} \tag{30}$$

The optimal value of Case 3 is T_3, which can be determined by (30).

From $pDM_2 + p\, I_d\, D\, M_2^2/\, 2 < cI(0)$, we obtain that

$$T_3 > (1/\theta)\{\ln[(pM_2\theta/c)(1+I_dM_2/2)+1]\}. \tag{31}$$

The second-order condition as

$$\frac{d^2Z_3(T)}{dT^2} = \frac{1}{T^2}\left\{(h+c\theta)TD + \frac{c^2 I_c D}{p}(1+\theta T)^2 T + \frac{cI_cD}{p}\left[\frac{c}{\theta}\left(\theta T + \frac{\theta^2T^2}{2}\right) - pM_2(1+I_dM_2/2)\right]\theta T\right\} > 0. \tag{32}$$

For Case 4, we obtain the first-order condition as

$$2S = D(h + c\theta + pI_d)T^2, \tag{33}$$

the second-order condition as

$$\frac{d^2Z_4(T)}{dT^2} = \frac{2S}{T^3} > 0, \tag{34}$$

and the optimal value of T for Case 4 as

$$T_4 \approx \sqrt{\frac{2S}{D(h + c\theta + pI_d)}}. \tag{35}$$

Substituting (35) into inequality $T_4 < M_2$, we obtain that

$$\text{if and only if } 2S < (h + c\theta + pI_d)DM_2^2, \text{ then } T_4 < M_2. \tag{36}$$

Combining the above four cases, we obtain the following theorem.

Theorem 1.

(1) If $2S < [h + c\theta(1 - r) + pI_d]D\,M_1^2$, then $T^* = T_2$.
(2) If $2S = [\mathrm{h} + \mathrm{c}\theta(1 - r) + pI_d]D\,M_1^2$, then $T^* = M_1$.
(3) If $[\mathrm{h} + \mathrm{c}\ \theta(1 - r) + pI_d]D\,M_1^2 < 2S < (h + c\theta + pI_d)D\,M_2^2$, then we know :
  (a) If T_1 satisfies Equation (24) and $Z_4(T_4) \geq Z_1(T_1)$, then $T^* = T_1$.
  (b) Otherwise, $T^* = T_4$.
(4) If $2S = (h + c\theta + pI_d)D\,M_2^2$, then $T^* = M_2$.
(5) If $2S > (h + c\theta + pI_d)D\,M_2^2$ and T_3 satisfies Equation (31), then $T^* = T_3$.

Proof. It immediately follows from (24), (29), (31) and (36).

In fact, the supplier usually demands its customers to pay cash on delivery (*i.e.*, $M_1 = 0$) when it offers a cash discount. Consequently, we need to discuss the case in which $M_1 = 0$. Again, the value for the deterioration rate θ is, in general, sufficiently small. Applying Theorem 1 to the case in which $M_1 = 0$ and $\theta << 1$, we have the following corollary.

Corollary 1. If $M_1 = 0$ and θ is sufficiently small, then

(1) If $2\,S > (h + c\theta + pI_d)D\,M_2^2$ and T_3 satisfies equation (31), then $T^* = T_3$.
(2) If $2\,S = (h + c\theta + pI_d)D\,M_2^2$, then $T^* = M_2$.
(3) If $2\,S < (h + c\theta + pI_d)D\,M_2^2$, T_1 satisfies Equation (24) and $cr - pI_dM_2 \geq \left(\frac{[c(1-r)]^2Ic}{2p} - \frac{pI_d}{2}\right)T_4$, then $T^* = T_1$.
(4) If $2S < (h + c\theta + pI_d)D\,M_2^2$, T_1 satisfies Equation (24) and $cr - pI_dM_2 \leq \left(\frac{[c(1-r)]^2Ic}{2p} - \frac{pI_d}{2}\right)T_1$, then $T^* = T_4$.

Proof. When θ is sufficiently small, we can reduce (11) and (18) as follows:

$$Z_1(T) \approx \frac{S}{T} + \frac{D[h + c\theta(1 - r)]}{\theta} - \frac{hD}{\theta} + \frac{[c(1 - r)]^2 I_c DT}{2p}, \tag{37}$$

and

$$Z_4(T) \approx \frac{S}{T} + \frac{D[h + c\theta]}{\theta} - \frac{hD}{\theta} - pI_d D\left(M_2 - \frac{T}{2}\right), \tag{38}$$

respectively. If $cr - pI_d M_2 \geq \left\{[c^2(1-r)^2 I_c/2p] - (pI_d/2)\right\} T_4$, then $Z_1(T_1) \leq Z_1(T_4) \leq Z_4(T_4)$which implies $T^* = T_1$. Similarly, if $cr - pI_d M_2 \leq \left\{[c^2(1-r)^2 I_c/2p] - (pI_d/2)\right\} T_1$, then $Z_1(T_1) \geq Z_4(T_1) \geq Z_4(T_4)$ and $T^* = T_4$. This completes the proof.

5. Comparisons

We use (3), (27) and (35), and obtain the economic order quantity for the following two cases is as follow:

The optimal economic order quantity for Case 2 is

$$\begin{aligned} Q^*(T_2) &= \frac{D}{\theta}(e^{\theta T_2} - 1) \approx D(T_2 + \theta T_2^2/2) \\ &= \sqrt{2SD/[h + c\theta(1-r) + pI_d]} + \theta S/[h + c\theta(1-r) + p\, I_d], \end{aligned} \tag{39}$$

and the optimal economic order quantity for Case 4 is

$$Q^*(T_4) \approx D(T_4 + \theta T_4^2/2) = \sqrt{2SD/(h + c\theta + pI_d)} + \theta S/(h + c\theta + pI_d), \tag{40}$$

respectively.

In the classical economic order quantity model, the supplier must be paid for the items as soon as the customer receives them. Therefore, the optimal replenishment cycle $T^* \approx \sqrt{2S/[D(h + c\theta + cI_c)]}$. As a result, the classical optimal economic order quantity is

$$Q^* = \frac{D}{\theta}(e^{\theta T^*} - 1) \approx \sqrt{2SD/(h + c\theta + cI_c)} + [\theta S/(h + c\theta + cI_c)]. \tag{41}$$

By comparing (39) – (41), we have the following theorem.

Theorem 2.

(a) If $c/p < I_d/I_c$, then $Q^*(T_4) < Q^*$.
(b) If $c/p > I_d/I_c$, then $Q^*(T_1), Q^*(T_2), Q^*(T_3)$ and $Q^*(T_4) > Q^*$.
(c) If $c/p = I_d/I_c$, then $Q^*(T_1), Q^*(T_2)$ and $Q^*(T_3) > Q^* = Q^*(T_4)$.

Proof. Since $T_1 \geq M_1$, $T_2 < M_1$, $T_3 \geq M_2$ and $T_4 < M_2$, we get $Q^*(T_1) > Q^*(T_2)$ and $Q^*(T_3) > Q^*(T_4)$. Using this result and equations (39) – (41), we obtain Theorem 2.

From Theorem 2, we know these results that (1) if $p\, I_{\mathrm{d}} > c\, I_{\mathrm{c}}$, then $Q^*(T_4) < Q^*$. This result reveals that by comparing with the classical optimal economic order quantity Q^*, the customer will order less quantity than Q^* in order to take the benefits of the permissible delay more frequently. (2) if $p\, I_{\mathrm{d}} \leq c\, I_{\mathrm{c}}$, then $Q^*(T_1)$ and $Q^*(T_2) > Q^*$. This implies that the price discount will encourage the customer to buy more quantity than Q^*.

6. Numerical examples

Example 1. Given $D = 1000$ units/year, $h = \$4$/unit/year, $I_c = 0.09$/year, $I_d = 0.06$ /year, $c = \$30$ per unit, $p = \$45$ per unit, $r = 0.02$, $\theta = 0.03$, $M_1 = 20$ days $= 20/365$ years, and $M_2 = 30$ days $= 30/365$ years, we obtain $[h + c\theta(1-r) + pI_d]\ DM_1^2 = 22.7645$ and $(h + c\theta + pI_d)\ DM_2^2 = 51.3417$. Consequently, we know from Theorem 1 that (1) if $S = 10$, then $2S < [h + c\theta(1-r) + pI_d]DM_1^2$, and $T^* = T_2$; (2) if $S = 25$, then $(h + c\theta + pI_d)\ DM_2^2 > 2S > [h + c\theta(1-r) + pI_d]DM_1^2$, and $T^* = T_1$ or T_4; (3) if $S = 50$, then $2S > (h + c\theta + pI_d)\ DM_2^2$, and $T^* = T_3$. The computational results in the sensitivity analysis on S are shown in Table 1. It indicates that a higher value of ordering cost S implies higher values of order quantity $Q(T^*)$, replenishment cycle T^* and total relevant cost $Z(T^*)$. In addition, the optimal order quantity $Q(T^*)$ is larger than classical economic Q^* and $c\ /\ p = I_d\ /\ I_c$. This result demonstrates Theorem 2(c).

Example 2. Given $D = 1000$ units/year, $h = \$4$ /unit/year, $I_c = 0.09$/year, $S = 10$, $c = \$20$ per unit, $p = \$30$ per unit, $r = 0.02$, $\theta = 0.03$, $M_1 = 20$ days $= 20/365$ years, and $M_2 = 30$ days $= 30/365$ years, we know from (41) that the classical economic order quantity $Q^* = 51.3384$. The computational results in the sensitivity analysis on interest rates I_d are shown in Table 2. It implies that if the ratio of purchasing cost to selling price $(c\ /\ p)$ is equal to (less than) the ratio of the interest earned to interest charge $(I_d\ /\ I_c)$, then the optimal order quantity under supplier credits $Q^*(T_2)$ is more than the classical economic order quantity Q^*. These results demonstrate the results of Theorem 2(b) and (c). In addition, using this table, we obtain that a higher value of interest rates I_d causes lower values of order quantity $Q(T^*)$, replenishment cycle T^* and total relevant cost $Z(T^*)$.

Table 1. Optimal solutions for different ordering costs

Ordering Cost S	Replenishment Cycle T^*	EOQ $Q(T^*)$	Classical EOQ Q^*	Total Relevant Cost $Z(T^*)$
10	$T_2 = 0.051360$	$Q^*(T_2) = 51.3994$	51.3384	$Z_2(T_2) = 29641.543$
25	$T_1 = 0.090389$	$Q^*(T_1) = 90.5116$	81.2094	$Z_1(T_1) = 29853.004$
50	$T_3 = 0.127630$	$Q^*(T_3) = 127.8745$	114.9052	$Z_3(T_3) = 30633.503$

Table 2. Optimal solutions for different interest rates I_d

Interest Rates I_d	Replenishment Cycle T^*	Economic Order Quantity $Q(T^*)$	Total Relevant Cost $Z(T^*)$
0.04	$T_2 = 0.054709$	$Q^*(T_2) = 54.7543$	$Z_2(T_2) = 19667.019$
0.05	$T_2 = 0.052955$	$Q^*(T_2) = 52.9974$	$Z_2(T_2) = 19654.436$
0.06	$T_2 = 0.051360$	$Q^*(T_2) = 51.3994$	$Z_2(T_2) = 19641.534$
0.07	$T_2 = 0.049900$	$Q^*(T_2) = 49.9377$	$Z_2(T_2) = 19628.252$
0.08	$T_2 = 0.048559$	$Q^*(T_2) = 48.5940$	$Z_2(T_2) = 19614.716$

7. Conclusions

We develop an EOQ model for a retailer to determine the optimal ordering policy when the supplier provides a cash discount and/or a permissible delay in payments. In order to obtain the explicit solution of the optimal replenishment cycle, we use Taylor's series approximation. Moreover, we also provide a simple way to obtain the optimal replenishment interval by examining the explicit conditions in Theorem 1. Furthermore, we establish Theorem 2, which compares the optimal economic order quantities with a cash discount and/or a permissible delay in payments with the classical economic order quantity under the different conditions. Finally, some numerical examples are studied to illustrate the theoretical results. There are both managerial phenomena: (1) a higher value of ordering cost causes higher values of order quantity, replenishment cycle and total relevant cost; (2) a higher value of interest rate implies lower values of total relevant cost, order quantity and replenishment cycle.

The proposed model can be extended in several ways. For instance, we may extend the constant deterioration rate to a two-parameter Weibull distribution. In addition, we could consider the demand as a function of time, selling price, product quality, and others. Finally, we could generalize the model to allow for shortages, quantity discounts, and others.

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