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Dynamic inventory strategies for profit maximization in a service facility with stochastic service, demand and lead time

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Abstract. This paper addresses an optimal inventory control in a supply chain in which customers arrive at a facility according to a Poisson process and the facility provides service which takes exponential amounts of time, using items supplied by an outside supplier with exponential lead time process. Formulating our model as a Markov decision problem, we identify a replenishment policy which maximizes the facility's profit subject to the costs of service delay, inventory holding, and replenishment setup and analytically examine how the changes in system parameters affect the optimal profit and the optimal replenishment policy. We show that these results can be extended to Erlang lead time process. We present numerical study for the optimal performance evaluation and comparison of the optimal replenishment policy with (Q, r) policy.

Key words: Inventory management, Optimal control, Markov decision processes, Erlang process

1. Introduction

This paper addresses an optimal inventory control in a supply chain in which customers arrive at a facility randomly and the facility provides service which takes random amounts of time using items supplied by an outside supplier. If the facility places a replenishment order, it arrives after random amounts of time. When inventory is depleted, a service is not provided until items are available. The facility earns a revenue whenever a service to each customer is completed while it faces the costs of service delay, inventory holding, and replenishment setup. The goal of this paper is to characterize an optimal replenishment policy which maximizes the facility's profit and to examine a monotonic impact of systems parameters on the optimal policy and optimal profit.

Several strategic issues emerge from our model. The first question is when the facility should replenish items to maximize its profits and how many items should be replenished. The second question is how the change in system parameters and the randomness in demand, service, and lead time processes affect the replenishment decision. To provide some insights into the nature of this problem, we first address these issues in the context of a $M/M/1$ queueing system with an exponential replenishment process. It is our hope that the insights gained here will remain useful in addressing other systems for which exponential distributions are inappropriate.

By extending our model to the one with an Erlang lead time process, this paper discusses how the variability in replenishment lead time processes affects the facility's profit and how much reduction in lead time variability leads to increase in the profit. New logistics relationships among business partners in the supply chain have been developed in an attempt to eliminate uncertainty in the logistics flow. One such relationship is vendor-managed inventory (VMI) under which the supplier controls the management of the facility's inventory, making such decisions as when and how much inventory to ship to the facility ([15]). VMI is known to achieve a greater reduction in actual lead time and its variability through information sharing which enables the supplier to observe the demands at the facility when they occur ([8]). The reduction in lead time and its variability brings the cost savings and both the supplier and the facility can be beneficiaries if they agree to share the savings ([6]).

Several recent papers ([1] [2] [3] [4] [10] [11] [12]) have considered the model of interest in cost minimization settings. Berman et al. [1] studied a model with deterministic demand rate, service and lead times, and found an optimal order quantity. Optimal replenishment policies for the Markovian model with instantaneous lead time process were analyzed in Berman and Kim [2] and He et al. [11]. He and Jewkes [10] and He et al. [12] focused on computing optimal replenishment policies. Berman and Sapna in [4] found an optimal stocking level for the finite queueing model with Poisson arrivals, general service times, and zero lead time.

Perhaps the model studied in Berman and Kim [3] is the closest to the work in this paper. Berman and Kim [3] assumed that the replenishment lead time process follows an Erlang distribution and characterized an optimal inventory policy as a monotonic threshold structure using a Markov decision process approach. The model proposed in the present paper can be viewed as an extension of the Berman and Kim model in the sense that we assume a revenue is generated upon the service, incorporate it into the inventory control, and find an optimal policy which maximize the profit subject to the same cost components as considered in the Berman and Kim model. In addition to that, we provide a sensitivity analysis of the optimal performance with respect to system parameters.

The rest of this paper is organized as follows. The next section provides the formulation of our Markov decision model. Sect. 3 completely characterizes the optimal replenishment policy and Sect. 4 considers how the changes in system parameters have an impact on the optimal policy and optimal profit. In Sect. 5, we numerically investigate monotonic properties of system parameters on the optimal performance and exhibit the beneficial effect of a replenishment decision based on both customer and inventory information over those based on inventory information only. Section 6 extends our model to the one with Erlang replenishment processes and numerically shows that a reduction in lead time variability results in a significant increase of profitability. In Sect. 7, we consider a model allowing multiple outstanding orders and demonstrate its importance under a long replenishment process and heavy traffic. The last section contains conclusions.

2. Model definition

A facility providing a single class of service faces customers arriving according to a Poisson process with rate $\lambda > 0$. Each customer requires exactly one item in inventory for service. Service times are independent and identically distributed (i.i.d.) exponential random variables with mean μ^{-1} and are independent of all else. Denote the capacity utilization by $\rho(< 1)$.

A revenue of R is generated whenever a service to each customer is completed. A linear cost c_1 is assessed for each unit of time per each customer queued. This cost can be viewed as providing an incentive to minimize the weighted flow times of customer orders. A holding cost c_2 is incurred for each unit of time per each unit of item in inventory. A replenishment order with size of Q units incurs a lump-sum cost of K and takes an exponential lead time with mean d^{-1}. We assume that a replenishment order in process is never interrupted until it is completed and for mathematical tractability there is at most one outstanding order.

At each decision epoch a policy specifies whether or not a replenishment order is placed. The set of decision epochs consists of customer arrival, service completion, and order completion epochs. The original problem is a continuous time Markov decision problem (MDP). Let the profits at time $w \in \mathbb{R}$ be discounted with a factor $e^{-\beta w}$. Following the uniformization process (Bertsekas 1987), it can be formulated with an equivalent discrete time MDP with a transition rate $\gamma \triangleq \lambda + \mu + d$ and a discount factor $\frac{\gamma}{\beta+\gamma}$. Without any loss of generality, we assume that $\beta + \gamma = 1$.

We denote the customer queue length and inventory level at time $n = 0, 1, 2, \ldots$ by $x_1(n)$ and $x_2(n)$, respectively. A state at a decision epoch n is described by the following vector: $(x_1(n), x_2(n), \delta)$. δ is such an indicator variable that no replenishment orders are in process if $\delta = 0$ whereas a replenishment order is in process if $\delta = 1$. The state space is denoted by $S = \mathbb{Z}^+ \times \mathbb{Z}^+ \times \{0, 1\}$. At a decision epoch n, there are two admissible actions only in each state $(x_1(n), x_2(n), 0)$: *Do not replenish* and *Replenish*.

The goal of this paper is to find a control policy π that maximizes the following expected discounted profits over an infinite horizon:

$$E\left[\sum_{n=0}^{\infty} \gamma^n \left(R\mathbb{1}\{n \in \Gamma^\pi\} - \sum_{i=1}^{2} c_i x_i(n) - K\mathbb{1}\{n \in \Delta^\pi\}\} \right) |(x_1(0), x_2(0), \delta)\right] \tag{2.1}$$

where Γ^π and Δ^π denote the set of random instances that a service is completed and a replenishment order is placed under policy π, respectively. For convenience, let $(x_1(0), x_2(0), \delta) = (x_1, x_2, \delta)$. Let $J(x_1, x_2, \delta)$ be the optimal expected discounted profit over an infinite horizon when the initial state is given by (x_1, x_2, δ). Then, $J(x_1, x_2, \delta)$ is the maximum of the expected discounted profits in (2.1).

We denote $p(x_1, x_2) = \mu R \mathbb{1}\{x_1 > 0, x_2 > 0\} - \sum_{i=1}^{2} c_i x_i$ and $D(x_1, x_2) = (x_1 - 1, x_2 - 1)$ if $x_1 > 0$ and $x_2 > 0$; (x_1, x_2) otherwise. Define the value iteration operator T_u, T_p, and T on any function f as

$$T_u f(x_1,x_2,\delta) = \begin{cases} p(x_1,x_2) + \mu f(D(x_1,x_2),\delta) + \lambda f(x_1+1,x_2,\delta) + d f(x_1,x_2,\delta) & \text{if } \delta = 0 \\ p(x_1,x_2) + \mu f(D(x_1,x_2),\delta) + \lambda f(x_1+1,x_2,\delta) + d f(x_1,x_2+Q,0) & \text{if } \delta = 1, \end{cases}$$

$$T_p f(x_1, x_2, 0) = -K + T_u f(x_1, x_2, 1),$$

$$Tf(x_1, x_2, \delta) = \begin{cases} \max\{T_u f(x_1, x_2, \delta), T_p f(x_1, x_2, \delta)\} & \text{if} \delta = 0 \\ T_u f(x_1, x_2, \delta) & \text{if } \delta = 1. \end{cases}$$

where operators T_u and T_p correspond to *Do not replenish* and *Replenish* action, respectively. Since T_u, T_p, and T are contraction operators, the optimal profit function J can be shown to satisfy the following optimality equation:

$$J(x_1, x_2, \delta) = \begin{cases} \max\{T_u J(x_1, x_2, \delta), T_p J(x_1, x_2, \delta)\} & \text{if } \delta = 0 \\ T_u J(x_1, x_2, \delta) & \text{if } \delta = 1. \end{cases}$$

3. Structure of the optimal replenishment policy

To establish the structural properties of the optimal replenishment policy, we show that certain properties of the functions defined on state space S are preserved under the operator T (Porteus [13]). Let F be the set of all functions defined on S such that if $f \in F$, then

$$\Delta_1 f(x_1, x_2, 0) \le \Delta_1 f(x_1, x_2, 1), \tag{3.1}$$

$$\Delta_2 f(x_1, x_2, 0) \ge \Delta_2 f(x_1, x_2, 1), \tag{3.2}$$

$$\Delta_1 f(x_1, x_2, 1) \le \Delta_1 f(x_1, x_2 + Q, 0), \tag{3.3}$$

$$\Delta_2 f(x_1, x_2, 1) \ge \Delta_2 f(x_1, x_2 + Q, 0), \tag{3.4}$$

$$\Delta_{11} f(x_1, x_2, \delta) \ge \Delta_{11} f(x_1 + 1, x_2, \delta), \tag{3.5}$$

$$\Delta_{11} f(x_1, x_2, \delta) \le \frac{1}{1 - (\lambda + d)} (\mu R - c_1 - c_2), \tag{3.6}$$

$$\mu \Delta_{11} f(x_1, x_2, \delta) \le \mu R, \tag{3.7}$$

$$\Delta_{11} f(x_1, x_2, 0) \ge \Delta_{11} f(x_1, x_2, 1), \tag{3.8}$$

$$\Delta_{11} f(x_1, x_2, 1) \ge \Delta_{11} f(x_1, x_2 + Q, 0) \tag{3.9}$$

where $\Delta_1 f(x_1, x_2, \delta) \triangleq f(x_1 + 1, x_2, \delta) - f(x_1, x_2, \delta)$, $\Delta_2 f(x_1, x_2, \delta) \triangleq f(x_1, x_2 + 1, \delta) - f(x_1, x_2, \delta)$, and $\Delta_{11} f(x_1, x_2, \delta) \triangleq f(x_1 + 1, x_2 + 1, \delta) - f(x_1, x_2, \delta)$.

From Equations (3.1)–(3.4) the incremental profit of holding one more customer in queue and one more item in inventory is increasing and decreasing, respectively, in the progress of the replenishment process, which is straightforward because the larger inventory means less service delay but extra payment of holding costs. Equations (3.5)–(3.9) characterize the behavior of Δ_{11}. Equation (3.5) can be rewritten as $f(x_1+2, x_2+1, \delta) + f(x_1, x_2, \delta) \geq f(x_1+1, x_2+1, \delta) + f(x_1+1, x_2, \delta)$, which can be interpreted as a diagonal dominance (see [9] for terminology). Equations (3.6) and (3.7) provide an upper bound on Δ_{11}. Equations (3.8) and (3.9) have the same interpretations as of Equations (3.2) and (3.4). We first present the following result:

Lemma 1. *If $f \in F$,*

$$\Delta_1 T_u f(x_1, x_2, 0) \leq \Delta_1 T_p f(x_1, x_2, 0), \tag{3.10}$$

$$\Delta_2 T_u f(x_1, x_2, 0) \geq \Delta_2 T_p f(x_1, x_2, 0). \tag{3.11}$$

$$\Delta_{11} T_u f(x_1, x_2, 0) \geq \Delta_{11} T_p f(x_1, x_2, 0). \tag{3.12}$$

Proof. See the Appendix.

Equations (3.10) and (3.11) guarantee a threshold property and a monotonicity of the optimal replenishment policy, respectively. For the detailed explanation, refer to Berman and Kim [3]. Equation (3.12) is a stronger condition than Equation (3.11). To see this, suppose that $T_p f(x_1, x_2, 0) - T_u f(x_1, x_2, 0) \geq 0$. By Equation (3.12), $T_u f(x_1+1, x_2+1, 0) - T_p f(x_1+1, x_2+1, 0) \geq T_u f(x_1, x_2, 0) - T_p f(x_1, x_2, 0) \geq 0$, which implies a monotonicity with respect to the upward diagonal direction.

The following lemma says that Equations (3.1)–(3.9) are preserved under operator T.

Lemma 2. *If $f \in F$, then $Tf \in F$.*

Proof. See the Appendix.

We now state the optimal replenishment policy the structure of which is similar to the one in Berman and Kim[3]. Theorem 1 can be shown using Lemmas 1 and 2 and the proof is omitted.

Theorem 1. (i) *The optimal value function J satisfies Equation* (3.1)–(3.9), *that is, $J \in F$.*

(ii) *Let*

$$\Theta(x_1) := max\{x_2 \in \{0, 1, \ldots, \infty\} : T_p J(x_1, x_2, 0) > T_u J(x_1, x_2, 0)\} \tag{3.13}$$

If there does not exist such x_2 satisfying (3), *set $\Theta(x_1) := -\infty$. Then, it is optimal to replenish Q units of items if $x_2 \leq \Theta(x_1)$.*

(iii) *The threshold function $\Theta(x_1)$ is increasing in x_1*; i.e.,

$$\Theta(x_1) \leq \Theta(x_1+1),\ x_1 \geq 0. \tag{3.14}$$

We note that the optimal properties obtained under the discounted profit criterion can be extended to the average profit problem by applying the results of [7] and [14].

4. Monotonicity of optimal performance with respect to system parameters

In this section we analyze how the optimal discounted profit and optimal replenishment policy change as a function of system parameters. We first set the monotonicity of the total discounted profit as a function of some system parameters given the order quantity, which is an intuitive result.

Theorem 2. *The optimal profit function $J(x_1, x_2, \delta)$ is increasing in μ, d, and R and decreasing in c_1, c_2, and K.*

Proof. See the Appendix.

We next proceed to characterize the monotonicity of the replenishment policy with respect to the replenishment cost K and revenue R given the order quantity. Consider two instances of the scheduling problem described by (2). To differentiate each other, we use symbol A and B in the first and second case, respectively, for system parameters, optimal profit function, and optimal replenishment policy. We first show the monotonicity of the optimal replenishment policy $\Theta(x_1)$ with respect to the replenishment cost K.

Theorem 3. *Suppose that $\lambda^A = \lambda^B$, $\mu^A = \mu^B$, $d^A = d^B$, $R^A = R^B$, $c_1^A = c_1^B$, $c_2^A = c_2^B$, and $K^A < K^B$. Then, $\Theta^A(x_1) \geq \Theta^B(x_1)$ for all $x_1 \geq 0$.*

Proof. See the Appendix.

Figure 1 displays how the optimal replenishment policy shifts as a function of K for the example with $R = 50$, $c_1 = 3$, $c_2 = 1$, $\lambda = 0.6$, $\mu = 1$, $d = 0.1$, $Q = 20$.

The monotonicity of $\Theta(x_1)$ with respect to the revenue R is also preserved in the following theorem.

Theorem 4. *Suppose that $\lambda^A = \lambda^B$, $\mu^A = \mu^B$, $d^A = d^B$, $K^A = K^B$, $c_1^A = c_1^B$, $c_2^A = c_2^B$, and $R^A < R^B$. Then, $\Theta^A(x_1) \leq \Theta^B(x_1)$* for all $x_1 \geq 0$.

Proof. See the Appendix.

Figure 2 displays how the optimal replenishment policy shifts as a function of R for the example with $K = 100$, $c_1 = 3$, $c_2 = 1$, $\lambda = 0.6$, $\mu = 1$, $d = 0.1$, $Q = 20$. The same truncation levels as in Figure 1 are used here.

5. Numerical study

In this section, we evaluate the optimal performance with respect to system parameters and compare the optimal replenishment policy to (Q, r) policy. To this end, we compute the optimal average profits using value iteration [5]. The stopping rule given by Proposition 7, Ch 7 of Bertsekas [5] is used and the termination criterion ϵ is set to 10^{-3}.

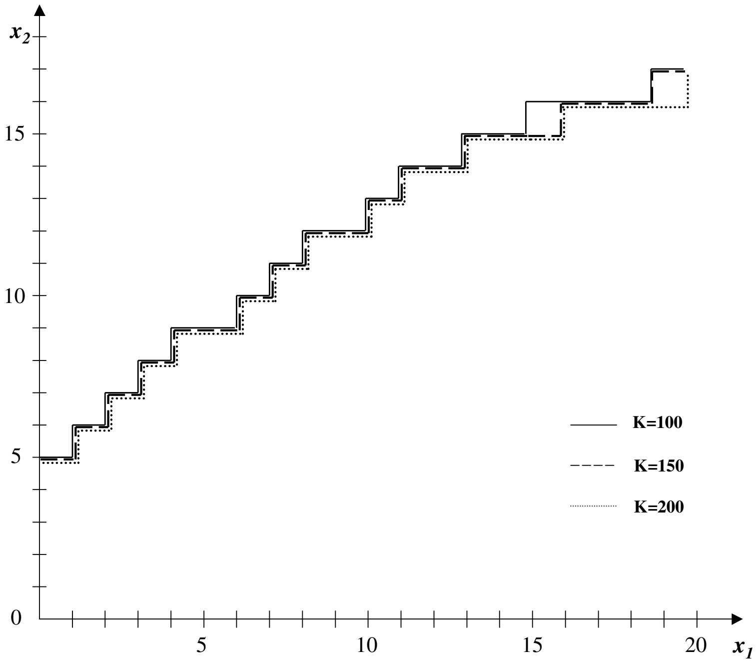

Fig. 1. Optimal policy as a function of the replenishment cost K

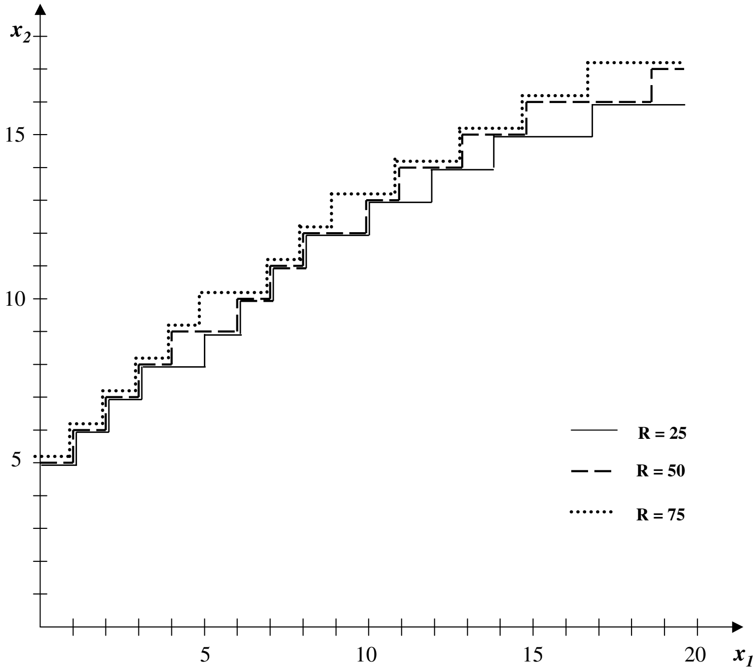

Fig. 2. Optimal policy as a function of the revenue R

5.1. Evaluation of optimal performance

Test examples and computational results are reported in Table 1–4. In these tables, Q^* and $\bar{J}^*$ represent the optimal order quantity and optimal average profit. To find Q^* and $\bar{J}^*$, we first compute the optimal average profit $\bar{J}(Q)$ given Q using value iteration. Then, varying Q, we find Q^* which maximizes $\bar{J}(Q)$. Numerical investigation indicates that $\bar{J}(Q)$ may be convex with respect to Q even though we could not prove it.

Example 1–24 in Table 1 are grouped into 2 sets by the revenue $R = 50, 100$. Each group is divided into 3 sub-groups by the lead time: $d = 0.1, 0.2, 1$. Each sub-group has 4 different arrival rates: $\lambda = 0.3$, 0.5, 0.7, 0.9. Table 2 examines the impact of the service rate on the optimal performance. In Table 3–4, we investigate the impact of c_1 and c_2, respectively, on the optimal performance. Example 37–48 in Table 3 (Example 49–50 in Table 4) are the same as Example 13–24 in Table 1 except for c_1 (c_2).

Based on the computational results, we observe the following monotonic behavior of the optimal performance with respect to system parameters assuming all other parameters are held constant:

1. $Q^*$is non-decreasing as the arrival rate λ increases.
2. Q^* is non-decreasing and $\bar{J}^*$ is non-increasing as the mean lead time d^{-1} increases.
3. $\bar{J}^*$ is non-decreasing as the service rate μ increases.
4. Q^* is non-decreasing, and $\bar{J}^*$ is non-decreasing as the revenue R increases.

Table 1. Test examples and optimal average profits

Ex.	R	K	c_1	c_2	λ	μ	d	Q^*	$\bar{J}^*$
1	50	100	3	1	0.3	1	0.1	11	3.276
2					0.5			16	5.387
3					0.7			20	4.054
4					0.9			23	−6.175
5					0.3		0.2	10	5.265
6					0.5			13	9.352
7					0.7			16	10.230
8					0.9			18	1.371
9					0.3		1	9	6.274
10					0.5			11	11.795
11					0.7			13	15.101
12					0.9			14	8.990
13	100	100	3	1	0.3	1	0.1	11	18.263
14					0.5			16	30.252
15					0.7			20	38.553
16					0.9			24	36.657
17					0.3		0.2	10	20.264
18					0.5			13	34.343
19					0.7			16	45.139
20					0.9			18	45.235
21					0.3		1	9	21.274
22					0.5			11	36.795
23					0.7			13	50.089
24					0.9			15	53.304

Table 2. Optimal performance as a function of the service rate μ

Ex.	R	K	c_1	c_2	λ	μ	d	Q^*	$\bar{J}^*$
25	50	100	3	1	0.9	1	0.1	23	−6.175
26						1.2		23	6.972
27						1.4		24	12.558
28						1.6		24	15.480
29						1	0.2	18	1.371
30						1.2		19	14.500
31						1.4		19	19.651
32						1.6		18	22.225
33						1	1	14	8.990
34						1.2		15	21.094
35						1.4		15	25.290
36						1.6		14	27.232

Table 3. Optimal performance as a function of c_1

Ex.	R	K	c_1	c_2	λ	μ	d	Q^*	$\bar{J}^*$
37	100	100	4.5	1	0.3	1	0.1	12	16.490
38					0.5			17	26.990
39					0.7			21	32.750
40					0.9			24	24.410
41					0.3		0.2	10	18.992
42					0.5			13	31.888
43					0.7			16	40.364
44					0.9			19	34.104
45					0.3		1	8	20.414
46					0.5			11	35.089
47					0.7			13	46.256
48					0.9			15	43.255

Table 4. Optimal performance as a function of c_2

Ex.	R	K	c_1	c_2	λ	μ	d	Q^*	$\bar{J}^*$
49	100	100	3	1.5	0.3	1	0.1	10	15.995
50					0.5			15	26.079
51					0.7			18	31.879
52					0.9			21	27.423
53					0.3		0.2	8	18.543
54					0.5			11	31.451
55					0.7			14	40.462
56					0.9			16	38.333
57					0.3		1	7	19.834
58					0.5			9	34.605
59					0.7			11	47.089
60					0.9			12	49.197

5. Q^* is non-decreasing, and $\bar{J}^*$ is non-increasing as the customer holding cost c_1 increases.
6. Q^* is non-increasing, $\bar{J}^*$ is non-increasing as the inventory holding cost c_2 increases.

It is interesting to see from the first observation that the optimal profit does not necessarily increase in λ. The intuition behind this is that as the capacity utilization (ρ) becomes high, the revenue rate μR will be increasingly offset by service delay and inventory holding costs. In fact, computational results show that $\bar{J}^*$ is convex with respect to λ. Hence, it is conjectured that there exists λ^* which achieves the maximum of profit for the given system parameters.

5.2. Numerical comparison of optimal policy with (Q, r)

In this section, we numerically investigate the beneficial effect of utilizing information on both customers in queue and inventory over utilizing only inventory information when making replenishment decisions.

To this end, we numerically evaluate (Q, r) policy for the examples 1–24 in Table 1. In implementing (Q, r) policy, we consider three types of reorder point: $r = \lambda/d, 1.2\lambda/d, 1.4\lambda/d$. Note that λ/d is the average lead time demand. In Table 5, Q^h is the order quantity which maximizes the average profit under (Q, r) policy, and $\bar{J}_{(Q,r)}$ is the average profit corresponding to Q^h and r. The column of % is defined as the change in percentage of the profit under (Q, r) policy relative to the optimal profit.

The sub-optimality of (Q, r) policy varies with the system parameters. When $\rho = 0.3$ or 0.5, the performance of (Q, r) policy becomes improved as reorder point decreases. When $\rho = 0.7$ and 0.9, we have an opposite result.

Table 5. Performance of (Q, r) policy

Ex.	$r = \lambda/d$			$r = 1.2\lambda/d$			$r = 1.4\lambda/d$		
	Q^h	$\bar{J}_{(Q^h,r)}$	%	Q^h	$\bar{J}_{(Q^h,r)}$	%	Q^h	$\bar{J}_{(Q^h,r)}$	%
1	11	2.771	15.4	11	2.771	15.4	10	2.422	26.1
2	18	4.848	10.0	16	4.823	10.5	16	4.691	12.9
3	25	2.853	29.6	24	3.163	22.0	23	3.371	16.8
4	30	−7.892	27.8	29	−7.419	20.1	27	−6.738	9.1
5	9	4.805	8.7	9	4.805	8.7	8	4.262	19.1
6	13	8.954	4.3	12	8.800	5.9	12	8.800	5.9
7	20	8.950	12.5	18	9.399	8.1	18	9.399	8.1
8	26	−1.539	212.3	24	−0.569	141.5	23	0.153	88.8
9	8	5.611	10.6	8	5.611	10.6	8	5.611	10.6
10	11	11.434	3.1	11	11.434	3.1	11	11.434	3.1
11	14	14.400	4.6	14	14.400	4.6	14	14.400	4.6
12	20	5.761	35.9	17	7.944	11.6	17	7.944	11.6
13	12	17.963	1.6	12	17.963	1.6	10	17.413	4.7
14	18	29.688	1.9	17	29.675	1.9	16	29.560	2.3
15	28	36.606	5.1	25	37.445	2.9	24	37.703	2.2
16	33	33.844	7.7	32	34.484	5.9	29	35.468	3.2
17	9	19.805	2.3	9	19.805	2.3	8	19.262	4.9
18	13	33.939	1.2	12	33.789	1.6	12	33.789	1.6
19	20	43.745	3.1	18	44.223	2.0	18	44.223	2.0
20	28	41.466	8.3	26	42.616	5.8	24	43.509	3.8
21	8	20.611	3.1	8	20.611	3.1	8	20.611	3.1
22	11	36.434	1.0	11	36.434	1.0	11	36.434	1.0
23	14	49.377	1.4	14	49.377	1.4	14	49.377	1.4
24	20	49.651	6.9	17	52.080	2.3	17	52.080	2.3

This result means that the reorder point should be carefully determined based on the capacity utilization.

The (Q, r) policy performs much worse when $R = 50$ than when $R = 100$. We also note that it does not perform well even for $Q^h = Q^*$ when $R = 50$. The intuition behind this is that the optimal policy adjusts the reorder point depending on the system status in order to save costs and the effect of holding cost savings on the profit is more obvious when $R = 50$ than when $R = 100$.

Table 5 also indicates that even though (Q, r) policy is implemented, the order quantity Q and reorder point r should be chosen depending on the system parameters in order to achieve the best performance, which means (Q, r) policy requires almost the same effort as the optimal policy in terms of computational complexity.

6. Extension to Erlang lead time distributions

One can expect that as the lead time process becomes less variable, the facility's profit would increase and it would be willing to dedicate more of its effort to assist the supplier in lead time variability reduction. To discuss it, the replenishment lead time, D, is assumed to have a M-Erlang distribution where M is a positive integer, that is, $D = \sum_{i=1}^{M} D_i$ where D_i's are exponential random variables with rate Md. D_is are considered as order process phases that are performed in sequence. If an order is placed, then, it enters phase M. Spending a time exponentially distributed with rate Md, the order enters phase $M - 1$. A replenishment order completion occurs on leaving phase 1.

The state in the Erlang lead time model is described by (x_1, x_2, δ) where δ represents the remaining replenishment process phase(s). No orders are in process if $\delta = 0$ while δ phases are left for the order completion if $\delta > 0$. The operator T is the same as before but T_u and T_p can be defined as

$$T_u f(x_1, x_2, \delta)$$

$$= \begin{cases} p(x_1,x_2) + \lambda f(x_1+1, x_2, 0) + \mu f(D(x_1,x_2), 0) + Mdf(x_1,x_2,0) & \text{if} \delta = 0 \\ p(x_1,x_2) + \lambda f(x_1+1, x_2, \delta) + \mu f(D(x_1,x_2), \delta) + Mdf(x_1,x_2,\delta-1) & \text{if } 1 < \delta \le M \\ p(x_1,x_2) + \lambda f(x_1+1, x_2, 1) + \mu f(D(x_1,x_2), 1) + Mdf(x_1,x_2+Q,0) & \text{if } \delta = 1, \end{cases}$$

$$T_p f(x_1, x_2, 0) = -K + T_u f(x_1, x_2, M).$$

The followings are functional properties of f defined on state space S which establish the structure of the optimal replenishment policy for the Erlang model:

$$\Delta_1 Tf(x_1, x_2, \delta) \ge \Delta_1 Tf(x_1, x_2, \delta + 1) \ge \Delta_1 Tf(x_1, x_2, 0), \ 1 \le \delta \le M - 1, \tag{6.1}$$

$$\Delta_2 Tf(x_1, x_2, \delta) \le \Delta_2 Tf(x_1, x_2, \delta + 1) \le \Delta_2 Tf(x_1, x_2, 0), \ 1 \le \delta \le M - 1, \tag{6.2}$$

$$\Delta_1 f(x_1, x_2, 1) \le \Delta_1 f(x_1, x_2 + Q, 0), \tag{6.3}$$

$$\Delta_2 f(x_1, x_2, 1) \ge \Delta_2 f(x_1, x_2 + Q, 0), \tag{6.4}$$

$$\Delta_{11} f(x_1, x_2, \delta) \ge \Delta_{11} f(x_1 + 1, x_2, \delta),\ 0 \le \delta \le M, \tag{6.5}$$

$$\Delta_{11} f(x_1, x_2, \delta) \le \frac{1}{1 - (\lambda + d)} (\mu R - c_1 - c_2),\ 0 \le \delta \le M, \tag{6.6}$$

$$\mu \Delta_{11} f(x_1, x_2, \delta) \le \mu R, 0 \le \delta \le M, \tag{6.7}$$

$$\Delta_{11} f(x_1, x_2, \delta) \le \Delta_{11} f(x_1, x_2, \delta + 1) \le \Delta_{11} f(x_1, x_2, 0),\ 1 \le \delta \le M - 1, \tag{6.8}$$

$$\Delta_{11} f(x_1, x_2, 1) \ge \Delta_{11} f(x_1, x_2 + Q, 0). \tag{6.9}$$

Using the same argument as in the exponential lead time, it can be shown that the results in Lemma 1–2 and Theorem 1 continue to be valid with Erlang lead times. Figure 3 displays how the optimal average profit changes as a function of the arrival rate λ with a change in lead time variability for the example with $R = 50$, $K = 100$, $c_1 = 3$, $c_2 = 1$, $\mu = 1$, $d = 0.1$, $Q = 20$. In this figure, the profit is plotted as a function of λ when the replenishment process is exponential, Erlang-2, Erlang-3, and Erlang-4. As the replenishment process becomes less variable, the profit level at any λ increases and the level of λ which achieves the maximum of the profit given the replenishment process distribution also increases.

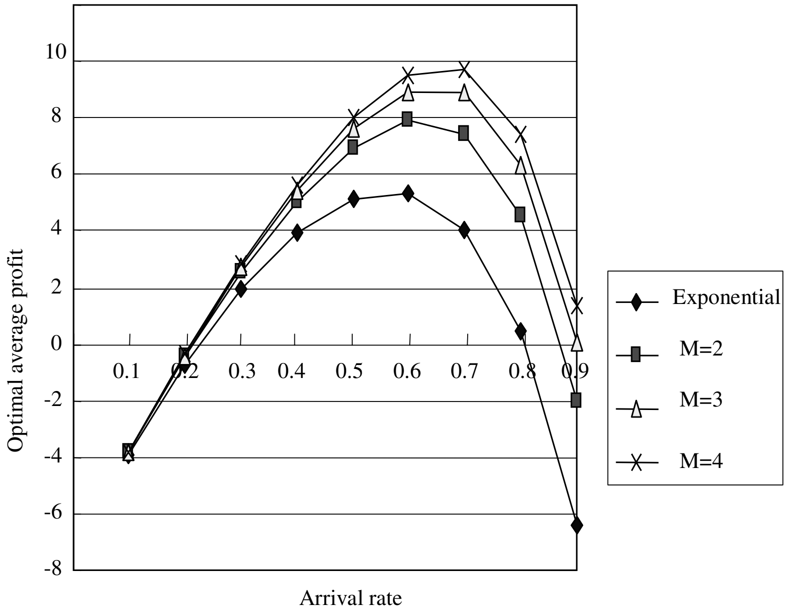

Fig. 3. Optimal profit as a function of λ with changes in lead time variability

Similarly, Figure 4 demonstrates the effect of reducing lead time variability. We plot the optimal profits as a function of the lead time rate d for cases of exponential, Erlang-2, Erlang-3, and Erlang-4 processes for the example with $R = 50$, $K = 100$, $c_1 = 3$, $c_2 = 1$, $\lambda = 0.6$, $\mu = 1$, $Q = 20$. Once again, we note that as the replenishment process becomes less variable, the profit increases. However, when $d \geq 0.5$, the effect of lowering lead time variability is minor and the profit is almost not affected by d.

These results clearly explain the beneficial effects of decreasing lead time variability. In fact, as lower variability leads to higher profits, the facility might be willing to offer higher replenishment prices to the supplier in exchange for guarantees in lead time variability.

7. Model with multiple outstanding replenishment orders

In the previous sections, our analysis is restricted to the case with at most one outstanding replenishment order. However, one might expect that multiple outstanding replenishment orders can contribute to dealing with demand fluctuations. To discuss it, we extend our model to the one that a replenishment order can be feasible in any states (x_1, x_2, δ) where δ $(0 \leq \delta \leq N)$ represents the number of outstanding replenishment orders. For example, δ replenishment orders are in process when $\delta > 0$ while no orders are in process when $\delta = 0$. The optimal cost function J is given by the following optimality equation:

$$J(x_1, x_2, \delta) = \max\{T_u J(x_1, x_2, \delta), T_p J(x_1, x_2, \delta)\}$$

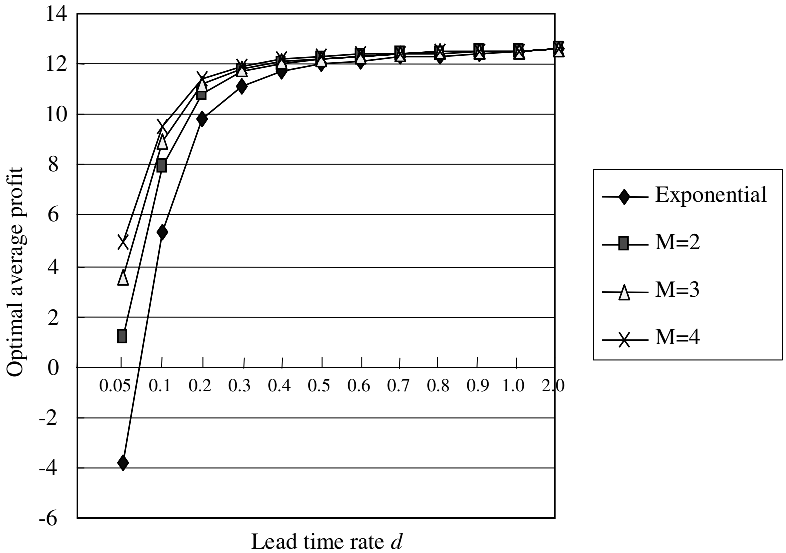

Fig. 4. Optimal profit as a function of d with changes in lead time variability

where

$$T_uJ(x_1,x_2,\delta) = \begin{cases} p(x_1,x_2)+\lambda J(x_1,x_2+1,\delta)+\mu J(D(x_1,x_2),\delta)+NdJ(x_1,x_2,\delta) & \text{if } \delta=0 \\ p(x_1,x_2)+\lambda J(x_1,x_2+1,\delta)+\mu J(D(x_1,x_2),\delta) & \\ \quad +\delta dJ(x_1,x_2+Q,\delta-1)+(N-\delta)dJ(x_1,x_2,\delta) & \text{if } \delta>0 \end{cases}$$

$$T_pJ(x_1,x_2,\delta) = -K + T_uJ(x_1,x_2,\delta+1).$$

Although we could not prove monotonicity of the optimal policy with respect to x_1 and δ, we provide the following conjecture based on numerical observations:

Conjecture.

(i) *Let*

$$\Theta(x_1,\delta) := \max\{x_2 \in \{0,1,\ldots,\infty\} : T_pJ(x_1,x_2,\delta) > T_uJ(x_1,x_2,\delta)\} \quad (7.1)$$

If there does not exist such x_2 satisfying (7.1), *set $\Theta(x_1,\delta) := -\infty$. Then, it is optimal to replenish Q units of items whenever $x_2 \le \Theta(x_1,\delta)$.*

(ii) *The reorder point $\Theta(x_1,\delta)$ is increasing in x_1 given δ, i.e.,*

$$\Theta(x_1,\delta) \le \Theta(x_1+1,\delta), \; x_1 \ge 0. \quad (7.2)$$

(iii) *The reorder point $\Theta(x_1,\delta)$ is decreasing in δ given x_1*, i.e.,

$$\Theta(x_1,\delta) \ge \Theta(x_1,\delta+1), \; \delta \ge 0. \quad (7.3)$$

Figure 5 presents the optimal replenishment policy for the example with $R=50$, $K=100$, $c_1=3$, $c_2=1$, $\lambda=0.6$, $\mu=1$, $d=0.1$, and $Q=20$. A replenishment order is allowed when $\delta=1$ as well as when $\delta=0$. Therefore, there can be two outstanding orders in the system. In this figure, symbols *o* and * represent that it is optimal to replenish in $(x_1,x_2,0)$ and $(x_1,x_2,1)$, respectively. As shown in Figure 5, the monotonic threshold structure of the optimal policy with respect to x_1 and x_2 is preserved when $\delta=0$ and $\delta=1$, respectively, and the monotonicity of the optimal policy with respect to δ is also preserved.

Table 6 displays the optimal performance of the model with two outstanding orders for examples 1–24 in Table 1. In this table, the second and third columns represent optimal order quantities and optimal average profits, respectively. The column of % in the fourth column shows how much the profit increases when two outstanding orders are allowed compared to when at most one outstanding order is allowed.

The results in Table 6 clearly demonstrate the beneficial effect of two outstanding orders when $d=0.1$ or $\rho \ge 0.8$. They also show that when $d=1$, allowing two outstanding orders does not contribute to the increase of profits. It is interesting to see that the beneficial effect is much larger when $R=50$ than when $R=100$. As one can expect, the model with two outstanding orders decreases the optimal replenishment quantities. In particular, when $d=0.1$ and $\rho \ge 0.8$, the optimal replenishment quantities decrease by 40%. This means that the optimal policy prefers more frequent replenishments with less quantities, which might contribute to the reduction in inventory holding

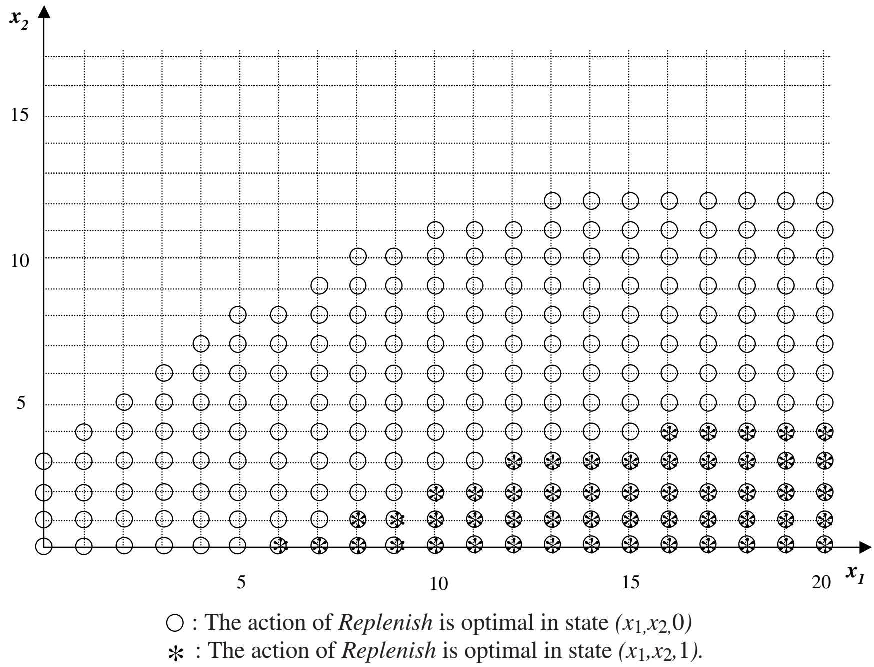

Fig. 5. Optimal policy when two outstanding orders are allowed

Table 6. Performance of the model allowing two outstanding orders

Ex.	Q^*	$\bar{J}^*$	%
1	9	4.098	25.1
2	11	7.800	44.8
3	12	8.378	106.7
4	15	−0.598	90.3
5	9	5.314	0.9
6	11	9.693	3.6
7	12	11.514	12.6
8	13	3.833	179.6
9	9	6.274	0.0
10	11	11.795	0.0
11	13	15.101	0.0
12	14	8.990	0.0
13	9	19.098	4.6
14	11	32.791	8.4
15	13	43.336	12.4
16	15	43.254	18.0
17	10	20.314	0.2
18	11	34.693	1.0
19	12	46.493	3.0
20	13	48.034	6.2
21	9	21.274	0.0
22	11	36.795	0.0
23	13	50.089	0.0
24	15	53.304	0.0

cost. Intuitively, it is likely that the effect of the savings in holding cost on the profit will be more apparent when $R = 50$ than when $R = 100$.

8. Conclusions

In this paper, we addressed an optimal inventory control in a service facility that earns a revenue upon each service completion but pays costs of service delay, inventory holding, and replenishment setup. Using Markov decision theory we were able to characterize the optimal replenishment policy that maximizes the facility's profit as a monotonic threshold policy. We were also able to establish the monotonic properties of optimal profit function and optimal reorder point with respect to system parameters.

The analysis of the Erlang lead time model confirmed that the lead time variability has a significant effect on the facility's profit, in particular, when the facility faces either high utilization or long replenishment lead time process. Numerical investigation for designed examples exhibited the outperformance of the policy using both customer and inventory information over the one using inventory information only.

Even though we could not prove it, the optimality of a monotonic threshold replenishment policy seems to be valid for the model with multiple outstanding orders. Numerical test showed that it performs better than the model with a single outstanding order.

Appendix

Proof of Lemma 1. The proof of Lemma 1 is very similar to that of (i) and (ii) of Theorem 2 in Berman and Kim [3] and we omit it.

Proof of Lemma 2. Denote by (u/p) the optimal action in state $(x_1, x_2, 0)$ where u and p represent *Do not replenish* and *replenish* actions, respectively. $\Delta_1 Tf(x_1, x_2, 0) \leq \Delta_1 Tf(x_1, x_2, 1)$ and $\Delta_2 Tf(x_1, x_2, 0) \geq \Delta_2 Tf(x_1, x_2, 1)$ can be shown in the same way as used in proving (iii) and (iv) of Theorem 2 in Berman and Kim [3] and we omit it.

(iii) $\Delta_1 Tf(x_1, x_2, 1) \leq \Delta_1 Tf(x_1, x_2 + Q, 0)$: We focus on the combinations of actions in $(x_1 + 1, x_2 + Q, 0)$ and $(x_1, x_2 + Q, 0)$. By (3.10) of Lemma 1, (u, p) is excluded. For (u, u),

$$\begin{aligned}\Delta_1 Tf(x_1,x_2,1) - \Delta_1 T_u f(x_1,x_2+Q,0) &= -\mu R \mathbb{1}\{x_1=0,x_2=0\} + \mu[\Delta_1 f(D(x_1,x_2),1) \\ &\quad - \Delta_1 f(D(x_1,x_2+Q),0)] + \lambda[\Delta_1 f(x_1+1,x_2,1) - \Delta_1 f(x_1+1,x_2+Q,0)] \leq 0.\end{aligned}$$

The inequality of μ term follows by (3.3) if $x_1 > 0$ and $x_2 > 0$. If $x_1 = 0$ and $x_2 > 0$, it becomes

$$\begin{aligned}&f(0,x_2-1,1) - f(0,x_2,1) - (f(0,x_2+Q-1,0) - f(0,x_2+Q,0)) \\ &= \Delta_2 f(0,x_2+Q-1,0) - \Delta_2 f(0,x_2-1,1) \leq 0 \quad (\text{by}(3.4)).\end{aligned}$$

If $x_1 > 0$ and $x_2 = 0$, it becomes $\Delta_1 f(x_1, 0, 1) - \Delta_1 f(x_1 - 1, Q - 1, 0) \leq \Delta_1 f(x_1, Q, 0) \quad -\Delta_1 f(x_1 - 1, Q - 1, 0)$ (by (3.3)) $= \Delta_{11} f(x_1, Q - 1, 0) - \Delta_{11} f(x_1 - 1, Q - 1, 0) \leq 0$ (by (3.5)).

Finally, if $x_1 = 0$ and $x_2 = 0$, it becomes $-\mu R + \mu[\Delta_1 f(0,0,1) - (f(0,Q-1,0) - f(0,Q,0))] \le \mu[\Delta_1 f(0,0,1) - (f(1,Q,0) - f(0,Q,0))]$ (by (3.7)) $= \mu[\Delta_1 f(0,0,1) - \Delta_1 f(0,Q,0)] \le 0$ (by (3.3)).

The inequality of λ term follows by (3.3). Case (p,p) can be shown using the result of (u,u) because

$$\Delta_1 Tf(x_1,x_2,1) - \Delta_1 T_p f(x_1,x_2+Q,0) \le \Delta_1 Tf(x_1,x_2,1) - \Delta_1 T_u f(x_1,x_2+Q,0) \text{ (by (3.10)} \le 0.$$

Case (p,u) can be shown using the result of (u,u).

(iv) $\Delta_2 Tf(x_1,x_2,1) \ge \Delta_2 Tf(x_1,x_2+Q,0)$: We focus on the combinations of actions in $(x_1,x_2+Q+1,0)$ and $(x_1,x_2+Q,0)$. By (3.11) (Monotonicity), (p,u) is excluded. For (u,u),

$$\Delta_2 Tf(x_1,x_2,1) - \Delta_2 T_u f(x_1,x_2+Q,0) = \mu R \mathbb{1}\{x_1>0, x_2=0\} + \mu[\Delta_2 f(D(x_1,x_2),1) - \Delta_2 f(D(x_1,x_2+Q),0)] + \lambda[\Delta_2 f(x_1+1,x_2,1) - \Delta_2 f(x_1+1,x_2+Q,0)] \ge 0.$$

If $x_1 > 0$ and $x_2 = 0$, the μ term becomes

$$\begin{aligned}
&\mu R + \mu[\Delta_2 f(D(x_1,0),1) - \Delta_2 f(D(x_1,Q),0)] \\
&= \mu R + \mu[f(x_1-1,0,1) - f(x_1,0,1) - (f(x_1-1,Q,0) - f(x_1-1,Q-1,0))] \\
&\quad \ge \mu[f(x_1-1,0,1) - f(x_1,0,1) - (f(x_1-1,Q,0) - f(x_1,Q,0))] \quad \text{(by (3.7))} \\
&= \mu[\Delta_1 f(x_1-1,Q,0) - \Delta_1 f(x_1-1,0,1)] \ge 0 \quad \text{(by (3.3))}.
\end{aligned}$$

Otherwise, it follows by (3.4). The λ term follows by (3.4). The d term is canceled out. Case (p,p) follows by the result of (u,u) because $\Delta_2 Tf(x_1,x_2,1) - \Delta_2 T_p f(x_1,x_2+Q,0) \ge \Delta_2 Tf(x_1,x_2,1) - \Delta_2 T_u f(x_1,x_2+Q,0)$ (by (3.11)) ≥ 0. Finally, case (u,p) follows by the result of (u,u).

(v) $\Delta_{11} Tf(x_1,x_2,\delta) \ge \Delta_{11} Tf(x_1+1,x_2,\delta)$: Suppose that $\delta = 1$.

$$\begin{aligned}
&\Delta_{11} Tf(x_1,x_2,1) - \Delta_{11} Tf(x_1+1,x_2,1) = \mu R \mathbb{1}\{x_1=0, x_2>0\} + \mu[\Delta_{11} f(D(x_1,x_2),1) \\
&\quad - \Delta_{11} f(D(x_1+1,x_2),1)]\mathbb{1}\{x_2>0\} + \lambda[\Delta_{11} f(x_1+1,x_2,1) - \Delta_{11} f(x_1+2,x_2,1)] \\
&\quad + d[\Delta_{11} f(x_1,x_2+Q,0) - \Delta_{11} f(x_1+1,x_2+Q+1,0)] \ge 0.
\end{aligned}$$

The term of μ follows by (3.5) if $x_1 > 0$ and $x_2 > 0$. If $x_1 = 0$ and $x_2 > 0$,

$$\mu R + \mu[\Delta_{11} f(D(0,x_2),1) - \Delta_{11} f(D(1,x_2),1)] = \mu R + \mu[f(0,x_2,1) - f(0,x_2,1) - \Delta_{11} f(0,x_2-1,1)] = \mu R - \mu\Delta_{11} f(0,x_2-1,1) \ge 0 \text{ (by (3.7))}.$$

The λ and d terms follow by (3.5). Assume that $\delta = 0$. Using (3.10)–(3.12) of Lemma 1, admissible actions in states $(x_1+1,x_2+1,0)$, $(x_1,x_2,0)$, $(x_1+2,x_2+1,0)$, and $(x_1+1,x_2,0)$ are the following 6 cases: (u,u,u,u), (u,p,u,p), (u,p,p,p), (p,p,p,p), (u,u,u,p), (u,u,p,p). For (u,u,u,u) and (p,p,p,p), an argument similar to one in case $\delta = 1$ is applied here. For (u,u,p,p), we have

$$\begin{aligned}
&\Delta_{11} T_u f(x_1,x_2,0) - \Delta_{11} T_p f(x_1+1,x_2,0) \\
&\ge \Delta_{11} T_p f(x_1,x_2,0) - \Delta_{11} T_p f(x_1+1,x_2,0) \quad \text{(by (3.12))} \ge 0 \text{ (by}(p,p,p,p)).
\end{aligned}$$

For (u, p, u, p), we have

$$T_u f(x_1+1, x_2+1, 0) - T_p f(x_1, x_2, 0) - [T_u f(x_1+2, x_2+1, 0) - T_p f(x_1+1, x_2, 0)]$$

$$\geq T_u f(x_1+1, x_2+1, 0) - T_u f(x_1, x_2, 0)$$

$$- [T_u f(x_1+2, x_2+1, 0) - T_u f(x_1+1, x_2, 0)] \quad \text{(by (3.10)}$$

$$\geq 0 \text{ (by case } (u, u, u, u)).$$

Case (u, u, p, u) follows by (u, u, u, u), and (p, u, p, p) by (p, p, p, p).

(vi) $\Delta_{11} Tf(x_1, x_2, \delta) \leq \frac{1}{1-(\lambda+d)}(\mu R - c_1 - c_2)$: Suppose $\delta = 0$. We focus on the combinations of actions in $(x_1+1, x_2+1, 0)$ and $(x_1, x_2, 0)$. By (3.12), (p, u) is excluded. For (u, u), if $x_1 x_2 = 0$,

$$\Delta_{11} Tf(x_1, x_2, 0) = \mu R - c_1 - c_2 + \lambda \Delta_{11} f(x_1+1, x_2, 1) + d\Delta_{11} f(x_1, x_2, 0)$$

$$\leq \mu R - c_1 - c_2 + (\lambda + d)\frac{1}{1-(\lambda+d)}(\mu R - c_1 - c_2) \quad \text{(by (3))}$$

$$= \frac{1}{1-(\lambda+d)}(\mu R - c_1 - c_2).$$

Otherwise,

$$\Delta_{11} Tf(x_1, x_2, 0) = -c_1 - c_2 + \mu \Delta_{11} f(x_1-1, x_2-1, 1) + \lambda \Delta_{11} f(x_1+1, x_2, 1)$$

$$+ d\Delta_{11} f(x_1, x_2, 0)$$

$$\leq \mu R - c_1 - c_2 + \lambda \Delta_{11} f(x_1+1, x_2, 1) + d\Delta_{11} f(x_1, x_2, 0) \quad \text{(by (3.7))}$$

$$\leq \frac{1}{1-(\lambda+d)}(\mu R - c_1 - c_2).$$

For (p, p), $\Delta_{11} T_p f(x_1, x_2, 0) \leq \Delta_{11} T_u f(x_1, x_2, 0)$ (by (3.12)) $\leq \frac{1}{1-(\lambda+d)}(\mu R - c_1 - c_2)$. Case (u, p) can be shown using the result of (u, u). The proof of $\delta = 1$ is the same as that of (p, p).

(vii) $\mu \Delta_{11} Tf(x_1, x_2, \delta) \leq \mu R$:

$$\mu \Delta_{11} Tf(x_1, x_2, \delta) = \mu \frac{1}{1-(\lambda+d)}(\mu R - c_1 - c_2)$$

$$= \frac{\mu}{\beta+\mu}(\mu R - c_1 - c_2) \quad (\text{by } \beta + \mu + \lambda + d = 1) \leq \mu R - c_1 - c_2 \leq \mu R.$$

(viii) $\Delta_{11} Tf(x_1, x_2, 0) \geq \Delta_{11} Tf(x_1, x_2, 1)$: We focus on combinations of actions in $(x_1+1, x_2+1, 0)$ and $(x_1, x_2, 0)$. By (3.12), (p, u) is excluded. For (p, p), $\Delta_{11} T_p f(x_1, x_2, 0) - \Delta_{11} Tf(x_1, x_2, 1) = 0$ by the definition of value functions. For (u, u), $\Delta_{11} T_u f(x_1, x_2, 0) \geq \Delta_{11} T_p f(x_1, x_2, 0)$ (by (3.12)) $= \Delta_{11} Tf(x_1, x_2, 1)$. Case (u, p) can be shown using the result of (p, p).

(ix) $\Delta_{11} Tf(x_1, x_2, 1) \geq \Delta_{11} Tf(x_1, x_2 + Q, 0)$: We focus on combinations of actions in $(x_1+1, x_2+Q+1, 0)$ and $(x_1, x_2+Q, 0)$. By (3.12), (p, u) is excluded. For (u, u),

$$\begin{aligned}
&\Delta_{11}Tf(x_1,x_2,1)-\Delta_{11}T_uf(x_1,x_2+Q,0)\\
&\quad=\mu R\mathbb{1}\{x_1>0,x_2=0\}+\mu[\Delta_{11}f(D(x_1,x_2),1)-\Delta_{11}f(D(x_1,x_2+Q),0)]\\
&\qquad\times\mathbb{1}\{x_1>0\}+\lambda[\Delta_{11}f(x_1+1,x_2,1)-\Delta_{11}f(x_1+1,x_2+Q,0)]\geq 0.
\end{aligned}$$

The non-negativity of μ term follows by (3.9) if $x_1>0$ and $x_2>0$. If $x_1>0$ and $x_2=0$, it becomes

$$\begin{aligned}
&\mu R+\mu[\Delta_{11}f(D(x_1,0,1)-\Delta_{11}f(D(x_1,Q),0)]\\
&\quad=\mu R+\mu[f(x_1,0,1)-f(x_1,0,1)-\Delta_{11}f(x_1-1,Q-1,0)]\,.\\
&\quad=\mu R-\mu\Delta_{11}f(x_1-1,Q-1,0)\geq 0\quad(\text{by }(3.7))
\end{aligned}$$

The non-negativity of λ term follows by (3.9). For (p,p), $\Delta_{11}T_pf(x_1,x_2+Q,0)\leq\Delta_{11}T_uf(x_1,x_2+Q,0)$ (by (3.12)) $\leq\Delta_{11}Tf(x_1,x_2,1)$(by case$(u,u)$). Case (u,p) can be shown using the result of case (u,u). □

Proof of Theorem 2. The changes in c_1, c_2, R, and K guarantees higher profits if the policy which is the optimal before the change is also applied after the change is made. We now give the proof for μ. Consider two systems, labeled system A and system B, that are identical except for the service rates μ_A and μ_B where $\mu_A<\mu_B$. Let π_A^* be the optimal policy applied to system A. For each service in system B, we employ π_B allowing for idling for the period of $\mu_A^{-1}-\mu_B^{-1}$ appropriately along each sample path to make the evolution of system B identical to that of system A under π_A^*. It is clear that the policy π_B results in the equal performance to π_A^*. Because π_B may not necessarily be optimal, the optimal policy in system B will perform at least as well π_A^*.

Suppose $d_A<d_B$ and other parameters are identical for system A and system B. Let τ_k be the time epoch that the k^{th} order is placed under π_A^* in system A. By placing an order at $\tau_k+d_A^{-1}-d_B^{-1}$ in system B, both systems receive the k^{th} order at $\tau_k+d_A^{-1}$. Therefore, both systems have the same state realization along any sample path. Thus, a decrease in the lead time results in an equal or more expected profit under an optimal policy. □

Proof of Theorem 3. Consider the following functional properties established by J^A and J^B:

$$J^B(x_1,x_2,0)-J^B(x_1,x_2,1)\geq J^A(x_1,x_2,0)-J^A(x_1,x_2,1)-K^B+K^A,\quad(8.4)$$

$$J^B(x_1,x_2,1)-J^B(x_1,x_2+Q,0)\geq J^A(x_1,x_2,1)-J^A(x_1,x_2+Q,0),\quad(8.5)$$

$$\Delta_{11}J^B(x_1,x_2,\delta)\geq\Delta_{11}J^A(x_1,x_2,\delta),\quad(8.6)$$

$$\Delta_{11}J^B(x_1,x_2,0)\geq\Delta_{11}J^A(x_1,x_2,1),\quad(8.7)$$

$$\Delta_{11}J^B(x_1,x_2,0)\geq\Delta_{11}J^A(x_1,x_2+Q,0).\quad(8.8)$$

We first prove the following lemma:

Lemma 3. *If (8.4)–(8.8) hold,*

$$T_uJ^B(x_1,x_2,0)-T_pJ^B(x_1,x_2,0)\geq(T_uJ^A(x_1,x_2,0)-T_pJ^A(x_1,x_2,0).\quad(8.9)$$

Proof.

$$
\begin{aligned}
&T_uJ^B(x_1,x_2,0)-T_pJ^B(x_1,x_2,0)-(T_uJ^A(x_1,x_2,0)-T_pJ^A(x_1,x_2,0))\\
&=K^B-K^A+\mu[J^B(D(x_1,x_2),0)-J^B(D(x_1,x_2),1)\\
&\quad-(J^A(D(x_1,x_2),0)-J^A(D(x_1,x_2),1))]\\
&\quad+\lambda[J^B(x_1+1,x_2,0)-J^B(x_1+1,x_2,1)-(J^A(x_1+1,x_2,0)-J^A(x_1+1,x_2,1))]\\
&\quad+d[J^B(x_1,x_2,0)-J^B(x_1,x_2+Q,0)-(J^A(x_1,x_2,0)-J^A(x_1,x_2+Q,0))]\\
&\geq K^B-K^A-(\lambda+\mu+d)(K^B-K^A)\geq 0(\text{by}\lambda+\mu+d\leq 1).
\end{aligned}
$$

Equation (8.4) is applied to μ and λ terms. By (8.4) and (8.5), the d term becomes $d[J^B(x_1,x_2,0)-J^B(x_1,x_2+Q,0)-\ (J^A(x_1,x_2,0)-J^A(x_1,x_2+Q,0))]$ $\geq d(-K^B+K^A)$. □

We now prove $\Theta^A(x_1)\geq\Theta^B(x_1)$ using contradiction. Suppose $\Theta^A(x_1)<\Theta^B(x_1)$. Then, we have $T_uJ^A(\Theta^B(x_1),x_2,0)\geq T_pJ^A(\Theta^B(x_1),x_2,0)$ and $T_uJ^B(\Theta^B(x_1),x_2,0)<T_pJ^B(\Theta^B(x_1),x_2,0)$. It follows that $T_uJ^B(\Theta^B(x_1),x_2,0)-T_uJ^A(\Theta^B(x_1),x_2,0)<T_pJ^B(\Theta^B(x_1),x_2,0)\ -T_pJ^A(\Theta^B(x_1),x_2,0)$, which is a contradiction by (8.9) of Lemma 3.

To complete the proof of this theorem, we show that Equations (8.4)–(8.8) are preserved under T. Denote by (u^A/p^A) the optimal action for the first instance where u^A and p^A represent *Do not replenish* and *replenish* actions, respectively. Similarly, (u^B/p^B) correspond to the second instance.

(i) $TJ^B(x_1,x_2,0)-TJ^B(x_1,x_2,1)\geq TJ^A(x_1,x_2,0)-TJ^A(x_1,x_2,1)-\ K^B+K^A$: We focus on admissible actions in $(x_1,x_2,0)^B$ and $(x_1,x_2,0)^A$. Case (p^B,u^A) is excluded by Lemma 3. For (p^B,p^A), $T_pJ^B(x_1,x_2,0)-\ TJ^A(x_1,x_2,1)-(T_pJ^A(x_1,x_2,0)-TJ^A(x_1,x_2,1))=-K^B+K^A$. For (u^B,u^A),

$$
\begin{aligned}
&T_uJ^B(x_1,x_2,0)-TJ^B(x_1,x_2,1)-(T_uJ^A(x_1,x_2,0)-TJ^A(x_1,x_2,1))\geq T_pJ^B(x_1,x_2,0)\\
&\quad-TJ^B(x_1,x_2,1)-(T_pJ^A(x_1,x_2,0)-TJ^A(x_1,x_2,1))(\text{by Lemma3})=-K^B+K^A.
\end{aligned}
$$

For (u^B,p^A),

$$
\begin{aligned}
&T_uJ^B(x_1,x_2,0)-TJ^B(x_1,x_2,1)-(T_pJ^A(x_1,x_2,0)-TJ^A(x_1,x_2,1))\\
&\geq T_pJ^B(x_1,x_2,0)-TJ^B(x_1,x_2,1)-(T_pJ^A(x_1,x_2,0)\\
&\quad-TJ^A(x_1,x_2,1))=-K^B+K^A.
\end{aligned}
$$

(ii) $TJ^B(x_1,x_2,1)-TJ^B(x_1,x_2+Q,0)\geq TJ^A(x_1,x_2,1)-TJ^A(x_1,x_2+Q,0)$: We focus on admissible actions in $(x_1,x_2+Q,0)^B$ and $(x_1,x_2+Q,0)^A$. Case (p^B,u^A) is excluded by Lemma 3. For (u^B,u^A),

$$
\begin{aligned}
&TJ^B(x_1,x_2,1)-T_uJ^B(x_1,x_2+Q,0)-(TJ^A(x_1,x_2,1)-T_uJ^A(x_1,x_2+Q,0))\\
&=\mu[J^B(D(x_1,x_2),1)-J^B(D(x_1,x_2+Q),0)-(J^A(D(x_1,x_2),1)\\
&\quad-J^A(D(x_1,x_2+Q),0))]\\
&\quad+\lambda[J^B(x_1+1,x_2,1)-J^B(x_1+1,x_2+Q,0)-(J^A(x_1+1,x_2,1)\\
&\quad-J^A(x_1+1,x_2+Q,0))]\geq 0.
\end{aligned}
$$

When $x_1 = 0$ or $x_2 > 0$, the μ term is non-negative by (8.5). When $x_1 > 0$ and $x_2 = 0$, it becomes

$$\begin{aligned}
&J^B(D(x_1,0),1) - J^B(D(x_1,Q),0) - (J^A(D(x_1,0),1) - J^A(D(x_1,Q),0))\\
&= J^B(x_1,0,1) - J^B(x_1-1,Q-1,0) - (J^A(x_1,0,1) - J^A(x_1-1,Q-1,0))\\
&\geq J^B(x_1,0,1) - J^B(x_1,Q,0)\\
&\quad - (J^A(x_1,0,1) - J^A(x_1,Q,0)) \quad (\text{by } (8.6)) \geq 0 \quad (\text{by } (8.5))
\end{aligned}$$

The λ term is non-negative by (8.5) and the d term is canceled out. For case (p^B, p^A),

$$\begin{aligned}
&TJ^B(x_1,x_2,1) - T_pJ^B(x_1,x_2+Q,0) - (TJ^A(x_1,x_2,1) - T_pJ^A(x_1,x_2+Q,0))\\
&\geq TJ^B(x_1,x_2,1) - T_uJ^B(x_1,x_2+Q,0) - (TJ^A(x_1,x_2,1)\\
&\quad - T_uJ^A(x_1,x_2+Q,0))(\text{by Lemma 3})\\
&\geq 0 \ (\text{by case } (u^B,u^A))
\end{aligned}$$

Case (u^B, p^A) can be shown using the result of case (u^B, u^A) because

$$\begin{aligned}
&TJ^B(x_1,x_2,1) - T_uJ^B(x_1,x_2+Q,0) - (TJ^A(x_1,x_2,1) - T_pJ^A(x_1,x_2+Q,0))\\
&\geq TJ^B(x_1,x_2,1) - T_uJ^B(x_1,x_2+Q,0) - (TJ^A(x_1,x_2,1)\\
&\quad - T_uJ^A(x_1,x_2+Q,0)) \geq 0.
\end{aligned}$$

(iii) $\Delta_{11}TJ^B(x_1,x_2,\delta) \geq \Delta_{11}TJ^A(x_1,x_2,\delta)$: Suppose $\delta = 1$. The non-negativity of $\Delta_{11}TJ^B(x_1,x_2,\delta) - \Delta_{11}TJ^A(x_1,x_2,\delta)$ follows by applying (8.6) to λ, μ, and d terms. Suppose $\delta = 0$. We focus on admissible actions in states $(x_1+1,x_2+1,0)^B$, $(x_1,x_2,0)^B$, $(x_1+1,x_2+1,0)^A$, and $(x_1,x_2,0)^A$. Using (3.12) of Lemmas 1 and 3, the following 6 cases are feasible: (u^B,u^B,u^A,u^A), (u^B,u^B,u^A,p^A), (u^B,u^B,p^A,p^A), (u^B,p^B,u^A,p^A), (u^B,p^B,p^A,p^A), and (p^B,p^B,p^A,p^A). Cases (u^B,u^B,u^A,u^A) and (p^B,p^B,p^A,p^A) be shown in a similar way used in $\delta = 1$. For (u^B,u^B,p^A,p^A),

$$\begin{aligned}
&\Delta_{11}T_uJ^B(x_1,x_2,0) - \Delta_{11}T_pJ^A(x_1,x_2,0)\\
&= \mu[\Delta_{11}J^B(D(x_1,x_2),0) - \Delta_{11}J^A(D(x_1,x_2),1)]\mathbb{1}\{x_1>0,x_2>0\}\\
&\quad + \lambda[\Delta_{11}J^B(x_1+1,x_2,0)\\
&\quad - \Delta_{11}J^A(x_1+1,x_2,1)] + d[\Delta_{11}J^B(x_1,x_2,0) - \Delta_{11}J^A(x_1,x_2+Q,0)] \geq 0.
\end{aligned}$$

The non-negativity corresponding to μ and λ terms follows by (8.7). The non-negativity of d term follows by (8.8). For (u^B,p^B,u^A,p^A),

$$\begin{aligned}
&T_uJ^B(x_1+1,x_2+1,0) - T_pJ^B(x_1,x_2,0) - (T_uJ^A(x_1+1,x_2+1,0) - T_pJ^A(x_1,x_2,0))\\
&\geq T_uJ^B(x_1+1,x_2+1,0) - T_uJ^B(x_1,x_2,0) - (T_uJ^A(x_1+1,x_2+1,0) - T_uJ^A(x_1,x_2,0))\\
&(\text{by Lemma 3}) \geq 0 \ (\text{by case } (u^B,u^B,u^A,u^A)).
\end{aligned}$$

(u^B,u^B,u^A,p^A) and (u^B,p^B,p^A,p^A) follows by (u^B,u^B,p^A,p^A) and (p^B,p^B,p^A,p^A), respectively.

(iv) $\Delta_{11}TJ^B(x_1,x_2,0) \geq \Delta_{11}TJ^A(x_1,x_2,1)$: $\Delta_{11}TJ^B(x_1,x_2,0) \geq \Delta_{11}TJ^B(x_1,x_2,1)$ (by (3.8)) $\geq \Delta_{11}TJ^A(x_1,x_2,1)$(by (iii)).

(v) $\Delta_{11}TJ^B(x_1,x_2,0) \geq \Delta_{11}TJ^A(x_1,x_2+Q,0)$: $\Delta_{11}TJ^A(x_1,x_2+Q,0) \leq \Delta_{11} TJ^A(x_1,x_2,1)$ (by (3.9)) $\leq \Delta_{11}TJ^B(x_1,x_2,0)$ (by (iv)). □

Proof of Theorem 4. Consider the following functional properties established by J^A and J^B:

$$J^B(x_1,x_2,1) - J^B(x_1,x_2,0) \geq J^A(x_1,x_2,1) - J^A(x_1,x_2,0), \tag{8.10}$$

$$J^B(x_1,x_2+Q,0) - J^B(x_1,x_2,1) \geq J^A(x_1,x_2+Q,0) - J^A(x_1,x_2,1), \tag{8.11}$$

$$\Delta_{11}J^B(x_1,x_2,\delta) \leq R^B - R^A + \Delta_{11}J^A(x_1,x_2,\delta), \tag{8.12}$$

$$\Delta_{11}J^B(x_1,x_2,1) \leq R^B - R^A + \Delta_{11}J^A(x_1,x_2,0), \tag{8.13}$$

$$\Delta_{11}J^B(x_1,x_2+Q,0) \leq R^B - R^A + \Delta_{11}J^A(x_1,x_2,0). \tag{8.14}$$

We first prove the following lemma:

Lemma 4. *If (8.10)–(8.14) hold,*

$$T_pJ^B(x_1,x_2,0) - T_uJ^B(x_1,x_2,0) \geq (T_pJ^A(x_1,x_2,0) - T_uJ^A(x_1,x_2,0)). \tag{8.15}$$

Proof.

$$\begin{aligned}
&T_pJ^B(x_1,x_2,0) - T_uJ^B(x_1,x_2,0) - (T_pJ^A(x_1,x_2,0) - T_uJ^A(x_1,x_2,0))\\
&= \mu[J^B(D(x_1,x_2),1) - J^B(D(x_1,x_2),0) - (J^A(D(x_1,x_2),1) - J^A(D(x_1,x_2),0))]\\
&\quad + \lambda[J^B(x_1+1,x_2,1) - J^B(x_1+1,x_2,0) - (J^A(x_1+1,x_2,1) - J^A(x_1+1,x_2,0))]\\
&\quad + d[J^B(x_1,x_2+Q,0) - J^B(x_1,x_2,0) - (J^A(x_1,x_2+Q,0) - J^A(x_1,x_2,0))] \geq 0.
\end{aligned}$$

(8.10) is applied to μ and λ terms. The non-negativity of d term follows by (8.10) and (8.11) □

We now prove $\Theta^A(x_1) \leq \Theta^B(x_1)$ using contradiction. Suppose $\Theta^A(x_1) > \Theta^B(x_1)$. Then, we have $T_uJ^A(\Theta^A(x_1),x_2,0) < T_pJ^A(\Theta^A(x_1),x_2,0)$ and $T_uJ^B(\Theta^A(x_1),x_2,0) > T_pJ^B(\Theta^A(x_1),x_2,0)$. It follows that $T_pJ^B(\Theta^A(x_1),x_2,0) - T_pJ^A(\Theta^A(x_1),x_2,0) < T_uJ^B(\Theta^A(x_1),x_2,0) - T_uJ^A(\Theta^A(x_1),x_2,0)$, which is a contradiction by (8.10) of Lemma 4.

To complete the proof of this theorem, we show that Equations (8.4)–(8.8) are preserved under T.

(i) $TJ^B(x_1,x_2,1) - TJ^B(x_1,x_2,0) \geq TJ^A(x_1,x_2,1) - TJ^A(x_1,x_2,0)$: We focus on admissible actions in $(x_1,x_2,0)^B$ and $(x_1,x_2,0)^A$. Case (u^B,p^A) is excluded by Lemma 4. For (p^B,p^A), $TJ^B(x_1,x_2,1) - T_pJ^B(x_1,x_2,1) - (TJ^A(x_1,x_2,1) - T_pJ^A(x_1,x_2,0)) = 0$. For (u^B,u^A),

$$\begin{aligned}
&TJ^B(x_1,x_2,1) - T_uJ^B(x_1,x_2,1) - (TJ^A(x_1,x_2,1) - T_uJ^A(x_1,x_2,0))\\
&\geq TJ^B(x_1,x_2,1) - T_pJ^B(x_1,x_2,1) - (TJ^A(x_1,x_2,1)\\
&- T_pJ^A(x_1,x_2,0))(\text{by Lemma } 4) = 0.
\end{aligned}$$

For (p^B, u^A),

$$TJ^B(x_1,x_2,1) - T_pJ^B(x_1,x_2,1) - (TJ^A(x_1,x_2,1) - T_uJ^A(x_1,x_2,0))$$

$$\geq TJ^B(x_1,x_2,1) - T_pJ^B(x_1,x_2,1) - (TJ^A(x_1,x_2,1) - T_pJ^A(x_1,x_2,0)) = 0.$$

(ii) $TJ^B(x_1,x_2+Q,0) - TJ^B(x_1,x_2,1) \geq TJ^A(x_1,x_2+Q,0) - TJ^A(x_1,x_2,1)$: We focus on admissible actions in $(x_1,x_2+Q,0)^B$ and $(x_1,x_2+Q,0)^A$. Case (u^B,p^A) is excluded by Lemma 4. For (u^B,u^A),

$$T_uJ^B(x_1,x_2+Q,0) - TJ^B(x_1,x_2,1) - (T_uJ^A(x_1,x_2+Q,0) - TJ^A(x_1,x_2,1))$$

$$= \mu(R^B - R^A)\mathbb{1}\{x_1 > 0, x_2 = 0\} + \mu[J^B(D(x_1,x_2+Q),0) - J^B(D(x_1,x_2),1)$$

$$- (J^A(D(x_1,x_2+Q),0)) - J^A(D(x_1,x_2),1)] + \lambda[J^B(x_1+1,x_2+Q,0)$$

$$- J^B(x_1+1,x_2,1) - (J^A(x_1+1,x_2+Q,0)) - J^A(x_1+1,x_2,1)]$$

$$+ d[J^B(x_1,x_2+Q,0) - J^B(x_1,x_2+Q,0) - (J^A(x_1,x_2+Q,0)$$

$$- J^A(x_1,x_2+Q,0))] \geq 0.$$

When $x_1 > 0$ and $x_2 > 0$ or when $x_1 = 0$ and $x_2 \geq 0$, μ term is non-negative by (8.11). When $x_1 > 0$ and $x_2 = 0$, it becomes

$$\mu(R^B - R^A) + \mu[J^B(D(x_1,Q),0) - J^B(D(x_1,0),1) - (J^A(D(x_1,Q),0)$$

$$- J^A(D(x_1,0),1))]$$

$$= \mu(R^B - R^A) + \mu[J^B(x_1-1,Q-1,0) - J^B(x_1,0,1) - (J^A(x_1-1,Q-1,0)$$

$$- J^A(x_1,0,1))$$

$$\geq \mu[J^B(x_1,Q,0) - J^B(x_1,0,1) - (J^A(x_1,Q,0)) - J^A(x_1,0,1)] (\text{by}(8.12))$$

$$\geq 0 \text{ (by (8.11))}.$$

The λ term is non-negative by (8.11) and the d term is canceled out. For case (p^B,p^B),

$$T_pJ^B(x_1,x_2+Q,0) - TJ^B(x_1,x_2,1) - (T_pJ^A(x_1,x_2+Q,0) - TJ^A(x_1,x_2,1))$$

$$\geq T_uJ^B(x_1,x_2+Q,0) - TJ^B(x_1,x_2,1) - (T_uJ^A(x_1,x_2+Q,0)$$

$$- TJ^A(x_1,x_2,1)) \text{(by Lemma 3)}$$

$$\geq 0 \text{ (by case } (u^B,u^A))$$

Case (p^B,u^A) can be shown using the result of case (u^B,u^A) because

$$T_pJ^B(x_1,x_2+Q,0) - TJ^B(x_1,x_2,1) - (T_uJ^A(x_1,x_2+Q,0) - TJ^A(x_1,x_2,1))$$

$$\geq T_uJ^B(x_1,x_2+Q,0) - TJ^B(x_1,x_2,1) - (T_uJ^A(x_1,x_2+Q,0) - TJ^A(x_1,x_2,1)) \geq 0.$$

(iii) $\Delta_{11}TJ^B(x_1,x_2,\delta) \leq R^B - R^A + \Delta_{11}TJ^A(x_1,x_2,\delta)$: Suppose $\delta = 1$. Then,

$$\begin{aligned}&\Delta_{11}TJ^B(x_1,x_2,1)-\Delta_{11}TJ^A(x_1,x_2,1)\\&=\mu(R^B-R^A)\mathbb{1}\{x_1x_2=0\}+\mu[\Delta_{11}J^B(D(x_1,x_2),1)\\&\quad-\Delta_{11}J^A(D(x_1,x_2),1)]\mathbb{1}\{x_1>0,x_2>0\}\\&\quad+\lambda[\Delta_{11}J^B(x_1+1,x_2,1)-\Delta_{11}J^A(x_1+1,x_2,1)]+d[\Delta_{11}J^B(x_1,x_2+Q,0)\\&\quad-\Delta_{11}J^A(x_1,x_2+Q,0))]\le(\lambda+\mu+d)\ (R^B-R^A)\ (\text{by }(8.12))\le R^B-R^A.\end{aligned}$$

Suppose $\delta=0$. We focus on admissible actions in states $(x_1+1,x_2+1,0)^B$, $(x_1,x_2,0)^B$, $(x_1+1,x_2+1,0)^A$, and $(x_1,x_2,0)^A$. Using (3.12) of Lemma 1 and Lemma 5, the following 6 cases are feasible: (u^B,u^B,u^A,u^A), (p^B,p^B,p^A,p^A), $((p^B,p^B,u^A,u^A))$, (p^B,p^B,u^A,p^A), (u^B,p^B,u^A,p^A), and (u^B,p^B,u^A,u^A). Cases (u^B,u^B,u^A,u^A) and (p^B,p^B,p^A,p^A) be shown in a similar way used in proving case $\delta=1$. For (p^B,p^B,u^A,u^A),

$$\begin{aligned}&\Delta_{11}T_pJ^B(x_1,x_2,0)-\Delta_{11}T_uJ^A(x_1,x_2,0)\\&=\mu(R^B-R^A)\mathbb{1}\{x_1x_2=0\}+\mu[\Delta_{11}J^B(D(x_1,x_2),1)-\Delta_{11}J^A(D(x_1,x_2),0)]\\&\quad\times\mathbb{1}\{x_1>0,x_2>0\}\\&\quad+\lambda[\Delta_{11}J^B(x_1+1,x_2,1)-\Delta_{11}J^A(x_1+1,x_2,0)]+d[\Delta_{11}J^B(x_1,x_2+Q,0)\\&\quad-\Delta_{11}J^A(x_1,x_2,0))]\\&\le(\lambda+\mu+d)(R^B-R^A)(\text{by}(8.13)\text{ to }\mu\text{ and }\lambda\text{ terms and }(8.14)\text{ to }d\text{ term})\le R^B-R^A.\end{aligned}$$

For (u^B,p^B,u^A,p^A),

$$\begin{aligned}&T_uJ^B(x_1+1,x_2+1,0)-T_pJ^B(x_1,x_2,0)-(T_uJ^A(x_1+1,x_2+1,0)-T_pJ^A(x_1,x_2,0))\\&\le T_uJ^B(x_1+1,x_2+1,0)-T_uJ^B(x_1,x_2,0)-(T_uJ^A(x_1+1,x_2+1,0)-T_uJ^A(x_1,x_2,0))\\&(\text{by Lemma 4})\le R^B-R^A\ (\text{by case }(u^B,u^B,u^A,u^A)).\end{aligned}$$

(u^B,p^B,u^A,u^A) and (p^B,p^B,u^A,p^A) follows by (u^B,u^B,p^A,p^A) and (p^B,p^B,p^A,p^A), respectively.

(iv) $\Delta_{11}TJ^B(x_1,x_2,1)\le R^B-R+\Delta_{11}TJ^A(x_1,x_2,0)$:

$$\begin{aligned}&\Delta_{11}TJ^B(x_1,x_2,1)\le\Delta_{11}TJ^B(x_1,x_2,0)\ (\text{by }(3.8))\le R^B-R^A\\&\quad+\Delta_{11}TJ^A(x_1,x_2,0)(\text{by (iii)}).\end{aligned}$$

(v) $\Delta_{11}TJ^B(x_1,x_2+Q,0)\le R^B-R+\Delta_{11}TJ^A(x_1,x_2,0)$:

$$\begin{aligned}&\Delta_{11}TJ^B(x_1,x_2+Q,0)\le\Delta_{11}TJ^B(x_1,x_2,1)\ (\text{by }(3.9))\le R^B-R^A\\&\quad+\Delta_{11}TJ^A(x_1,x_2,0)(\text{by (iv)}).\end{aligned}$$

□

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