Math Meth Oper Res (2003) 58:331–349
DOI 10.1007/s001860300294

Mathematical Methods
of Operations Research

Robust facility location

Emilio Carrizosa[1], **Stefan Nickel**[2]

[1] Facultad de Matemáticas, Universidad de Sevilla, Spain (e-mail: ecarrizosa@us.es)
[2] Lehrstuhl für Operations Research und Logistik (Chair of Operations Research and Logistics)
Universität des Saarlandes, Postfach 15 11 50, D-66041 Saarbrücken, Germany
(e-mail: s.nickel@wiwi.uni-sb.de)

Manuscript received: March 2001/Final version received: March 2003

Abstract. Let A be a nonempty finite subset of the plane representing the geographical coordinates of a set of demand points (towns, ...), to be served by a facility, whose location within a given region S is sought. Assuming that the unit cost for $a \in A$ if the facility is located at $x \in S$ is proportional to $dist(x, a)$ – the distance from x to a – and that demand of point a is given by ω_a, minimizing the total transportation cost $TC(\omega, x)$ amounts to solving the *Weber problem*. In practice, it may be the case, however, that the demand vector ω is not known, and only an estimator $\hat{\omega}$ can be provided. Moreover the errors in such estimation process may be non-negligible. We propose a new model for this situation: select a threshold value $B > 0$ representing the highest admissible transportation cost. Define the *robustness* ρ of a location x as the minimum increase in demand needed to become inadmissible, i.e. $\rho(x) = \min\{\|\omega - \hat{\omega}\| : TC(\omega, x) > B, \omega \geq 0\}$ and find the x maximizing ρ to get the *most robust* location.

Key words: Facilities, Location, Continuous, Decision analysis, Risk, Programming, Fractional

1 Introduction

In location planning one is typically concerned with finding a good location for one or several new facilities with respect to a given set of existing facilities (clients). The most common model in planar location theory for increasing the quality of the location of one new facility is the so-called Weber problem, where the average (weighted) distance of the new to the existing facilities is taken into account (see [8] [23] [19]).

More precisely we are given a finite set A of existing facilities (represented by their geographical coordinates) and distances d_a assigned to each existing facility $a \in A$. Additionally, weights ω_a reflecting the *relative importance* of existing facility $a \in A$ are provided.

With these definitions the objective function for the Weber problem can be written as

$$TC(\omega, x) := \sum_{a \in A} \omega_a d_a(x),$$

which should be minimized over all x in the plane or over a nonempty closed subset $S \subseteq \mathbb{R}^2$ for given weight set $\omega = (\omega_a)_{a \in A}$.

When applying this model to real world problems, mainly two sets of parameters have to be determined:

1. What kind of distances d_a should be used in the model.
2. How can we determine the weights ω_a.

A lot of research for finding appropriate distance functions for applying the Weber problem to different geographical settings has been done in the last decades, starting with [24]. Other contributions to this topic can be found in [15], [19], [4] and references therein.

For the determination of the weights the situation is somehow different. The existing approaches can be divided roughly into three categories:

1. All weights are assumed to be known and reliable (situation of complete information).
2. All weights are again assumed to be known but a sensitivity analysis is performed in order to get information about the stability of the optimal solution with respect to small changes in the input data, e.g. [14].
3. All weights are assumed to be given with respect to a known distribution, e.g. [9], [7], [19] and references therein.

In practice, it may however be the case that the demand ω is not known and no probabilistic distribution can be provided. Examples are activities which concern new (generations of) products, the planning of unique and major events for which no knowledge of the demand exists, or the planning of installations which are supposed to serve potential clients over a long period of time for which the evaluation of demand is unknown.

A possible strategy for such situations can be found, e.g., in [1]: lower and upper bounds on the weights are assumed to be known, and a worst-case approach is suggested.

In this paper we propose a different approach: we assume the existence of an estimate $\hat{\omega}$ for ω, with all its components positive. However, when replacing the demand ω by its estimate $\hat{\omega}$ the errors made may be rather high and uncontrollable, (so that a sensitivity analysis would be of no help), with a considerable (perhaps unacceptable) increase in transportation costs.

To keep transportation costs under control, we select a threshold value $B > 0$, representing the highest admissible transportation cost or just the budget given. Now, define the robustness ρ of a location x as the minimum deviation in demand with respect to $\hat{\omega}$ for which the total cost for location x exceeds the budget. In other words: Given a norm $\|\cdot\|$ on the space of weights we have

$$\rho(x) = \inf\{\|\omega - \hat{\omega}\| : TC(\omega, x) > B, \omega \geq 0\}$$

By solving then the optimization problem

$$\max_{x \in S} \rho(x) \tag{1.1}$$

we get a most robust location x^* within S.

In practice, situations with an extreme amount of uncertainty on the demand may be rare, which limits at first glance the usefulness of (1.1) as an applicable decision-making tool.

However, robustness, as defined above, can be used as a secondary (mostly tie-breaking) criterion, yielding still a problem of type (1.1). Indeed, suppose for instance one seeks a robust solution x^*

- in a set S_0 defined by geographical or legal constraints
- not exceeding an upper bound on the transportation costs $TC(\hat{\omega}, x)$ when the estimate $\hat{\omega}$ is used as weight vector,

  $$TC(\hat{\omega}, x) \leq B_0$$

- not exceeding a threshold value R_0 for the distance separating the facility from each demand point a,

  $$d_a(x) \leq R_0 \ \forall a \in A.$$

By defining S as the set of points in S_0 satisfying the constraints above, finding the most robust location *within* S yields a problem of type (1.1).

The remaining of the paper is organized as follows: In the next section the model is discussed in detail, and a general solution technique is proposed. In Section 3 we discuss a particular case, namely, the case in which distances are measured by the Manhattan norm. The structure is then used to provide efficient algorithms for particular choices of norm $\|\cdot\|$. The paper ends with a detailed example, some conclusions and an outlook to further research.

2 A possible model

For any feasible location $x \in S \subset \mathbb{R}^2$, its *robustness* $\rho(x)$ is defined as the optimal value of the optimization problem

$$\begin{array}{ll} \inf & \|\omega - \hat{\omega}\| \\ s.t. & TC(\omega, x) > B \\ & \omega \geq 0 \end{array} \tag{2.1}$$

where $\|\cdot\|$ is a norm in the space of weights $\mathbb{R}^{|A|}$, such as

$$\|u\| = \max_{a \in A} \frac{|u_a|}{\hat{\omega}_a}, \tag{2.2}$$

thus measuring the highest relative deviation, or

$$\|u\| = \max_{a \in A} |u_a| \tag{2.3}$$

measuring the highest absolute deviation, or

$$\|u\| = \sum_{a \in A} |u_a|, \tag{2.4}$$

measuring the total absolute deviation, or

$$\|u\| = \left(\sum_{a\in A} u_a^2\right)^{\frac{1}{2}} \tag{2.5}$$

measuring the squared root of the sum of squares.

The case in which A consists of exactly one point, $A = \{a\}$, is trivial: the total transportation cost $TC(\omega, a)$ from a equals 0, thus $\rho(a) = +\infty$, and then a is the most robust solution. Throughout the paper we will exclude this trivial case and assume hereafter

$$A \text{ has at least two points} \tag{A1}$$

2.1 Some reformulations

Under Assumption A1, $TC(\omega, x)$ is strictly positive for any ω with strictly positive components, which implies the following

Proposition 2.1. *For any $x \in \mathbb{R}^2$, the problem* (2.1) *is feasible. In particular,*

$$\rho(x) < +\infty \qquad \forall x \in \mathbb{R}^2$$

Moreover, $\rho(x)$ can also be expressed as

$$\begin{array}{rcll} \rho(x) & = & \min & \|\omega - \hat{\omega}\| \\ & & s.t. & TC(\omega, x) \geq B \\ & & & \omega \geq 0 \end{array} \tag{2.6}$$

By Proposition 2.1, measuring the robustness of a given x amounts to solving the nonlinear optimization problem (2.6). We will show below that, under very mild conditions, the optimal value of (2.6) can be obtained explicitly.

We first recall that a norm $\|\cdot\|$ in $\mathbb{R}^n$ is said to be *absolute* iff

$$\|(u_1, u_2, \ldots, u_n)\| = \|(|u_1|, |u_2|, \ldots, |u_n|)\| \qquad \forall u \in \mathbb{R}^n$$

In particular, weighted l_p norms, such as those given in (2.2)–(2.5) are absolute norms. For technical reasons we assume in the following that

$$\|\cdot\| \text{ is an absolute norm} \tag{A2}$$

Proposition 2.2. *For any $x \in \mathbb{R}^2$,*

$$\rho(x) = \max\left\{0, \frac{B - TC(\hat{\omega}, x)}{\|(d_a(x))_{a\in A}\|^\circ}\right\}, \tag{2.7}$$

where $\|\cdot\|^\circ$ denotes the dual norm of $\|\cdot\|$, defined as $\|u\|^\circ = \max_{\|x\|=1} u^\top x$.

For the proof, see the Appendix.

From Propositions 2.1 and 2.2 one immediately obtains

Proposition 2.3. *Define* $z^* = \min_{x \in S} TC(\hat{\omega}, x)$.

1. *If* $z^* \geq B$, *then*

$$\rho(x) = 0 \qquad \forall x \in S$$

In particular, any $x \in S$ *is a most robust location.*

2. *If* $z^* < B$, *then* $\max_{x \in S} \rho(x) > 0$. *Moreover, a feasible point* $x^* \in S$ *is a most robust location iff it solves the problem*

$$\max_{x \in S} \frac{B - TC(\hat{\omega}, x)}{\|(d_a(x))_{a \in A}\|^\circ}$$

Hence, for $z^* \geq B$, the problem is trivial, and will not be considered in the following, by assuming

$$z^* < B \tag{A3}$$

2.2 A general solution approach

Denote by $\tilde{\rho}$ the function

$$\tilde{\rho} : x \mapsto \frac{B - TC(\hat{\omega}, x)}{\|(d_a(x))_{a \in A}\|^\circ} \tag{2.8}$$

By Proposition 2.3, solving (2.1) may be reduced to maximizing on S the nonlinear function $\tilde{\rho}$ defined in (2.8). Function $\tilde{\rho}$ has, however, a rich structure which enables its maximization by existing methods. In particular, we can use the approach of Dinkelbach (see [6], [21]) to get the following iterative solution procedure for

$$\max_{x \in S} \tilde{\rho}(x) =: \frac{N(x)}{D(x)}.$$

1. Find an optimal solution x^* for problem $\max_{x \in S} N(x)$.
2. $q := \tilde{\rho}(x^*)$.
3. Compute an optimal solution x' for

$$\max_{x \in S} N(x) - qD(x) \tag{2.9}$$

4. If $N(x') - qD(x') = 0$ then STOP: x' is an optimal solution to the fractional program.
5. $q := \tilde{\rho}(x')$. Goto Step 3.

Hence, in order to use Dinkelbach's approach, at each iteration a problem of type (2.9) must be solved. In turns out that problems (2.9) are manageable at least for a wide class of distance measures. Indeed, one has

Lemma 2.4. *Suppose that, for each* $a \in A$, d_a *is induced by a norm in* $\mathbb{R}^2$. *Then, any problem of type (2.9) to be solved in Step 3 of Dinkelbach's algorithm has a concave (non-differentiable) objective.*

Proof. Since $\|\cdot\|$ is, by assumption, monotone, its dual $\|\cdot\|^\circ$ is also monotone [2]. Hence, the function $x \mapsto \|(d_a(x))_{a\in A}\|^\circ$ is convex, since it is the composition of the convex functions d_a with the monotonically increasing convex function $\|\cdot\|^\circ$. Moreover, by Assumption $A3$, $\tilde{\rho}(x^*) > 0$, and by construction of N and D, each q obtained in Step 5 is also positive, thus the function $x \mapsto q\|(d_a(x))_{a\in A}\|^\circ$ is convex, from which the result follows. □

Hence, as soon as the feasible region S is a convex set, the optimization problem in Step 3 is a maximization of a concave function over a convex set (or equivalently a minimization of a convex function over a convex set) for which numerous algorithms exist (see, for example, [13]).

Anyway, Dinkelbach's approach is not the only option to maximize $\tilde{\rho}$. We recall that a function f is said to be *explicitly quasiconcave* if both upper level sets and strict upper level sets are convex sets, see e.g. [16] for further details.

Lemma 2.5. *Suppose that, for each $a \in A$, d_a is induced by a norm in $\mathbb{R}^2$. Then, $\tilde{\rho}$ is explicitly quasiconcave. In particular, for S convex, any local maximum of $\tilde{\rho}$ is also a global maximum on S.*

Moreover, if $S = \mathbb{R}^2$, a most robust solution exists in the convex hull of the set A.

Proof. It has been shown in the proof of Lemma 2.4 that the function $x \mapsto \|(d_a(x))_{a\in A}\|^\circ$ is convex, and it is obviously positive. The result then follows from the algebra of convex functions, see [16]. It is known that, for any x not in $co(A)$, the convex hull of A there exists some $x' \in co(A)$ satisfying

$$d_a(x') \le d_a(x) \quad \forall a \in A,$$

see [12]. Since any absolute norm (such as $\|\cdot\|^\circ$) is monotone,

$$\|(d_a(x'))_{a\in A}\|^\circ \le \|(d_a(x))_{a\in A}\|^\circ,$$

and the result follows. □

Hence, any local-search procedure leads to global optimality. Moreover, when S is the whole plane, the search can be further reduced to the convex hull of A.

3 Solution procedures for the Manhattan metric

In this section we will develop particular solution procedures for the unconstrained case (i.e., $S = \mathbb{R}^2$), with the Manhattan metric, i.e.

$$d_a(x) = l_1(x, a) = |x_1 - a_1| + |x_2 - a_2| \ \forall a \in A$$

where the index 1 and 2 refers to the first and second coordinate, respectively.

Contrary to the iterative (and, in principle, infinite) general-purpose method, here we propose a finite algorithm that, for some particular important choices of $\|\cdot\|$, finds a most robust solution in subquadratic time.

By Proposition 2.1, the robust facility location problem can now be written as

$$\max_{x\in\mathbb{R}^2} \tilde{\rho}(x) = \frac{B - \sum_{a\in A} \omega_a l_1(x, a)}{\|l_1(x, a)_{a\in A}\|^\circ}.$$

Let $a'_{1_1}, \ldots, a'_{P_1}$ be the different values of the first coordinates of the existing facilities A sorted in increasing order, such that

$$a'_{1_1} < a'_{2_1} < \cdots < a'_{P_1}$$

holds. $a'_{1_2}, \ldots, a'_{Q_2}$ are defined analogously with respect to the second coordinates of $a \in A$. Additionally we define $a'_{0_1} = a'_{0_2} = -\infty$ and $a'_{P_1+1} = a'_{Q_2+1} = +\infty$ and we get a subdivision of the plane into $O(|A|^2)$ rectangular cells

$$\langle s,t\rangle := \left\{x = (x_1, x_2) : a'_{s_1} \le x_1 \le a'_{s_1+1},\ a'_{t_2} \le x_2 \le a'_{t_2+1}\right\},$$

for $s \in \{0,1,2,\ldots,P\}$ and $t \in \{0,1,2,\ldots,Q\}$.

By the structure of the l_1 norm, we can eliminate a part of the plane being candidate for containing a globally optimal solution. Indeed, one has

Lemma 3.1. *Let $R = [a'_{1_1}, a'_{P_1}] \times [a'_{1_2}, a'_{Q_2}]$ be the smallest rectangle containing all $a \in A$. Then all globally optimal solutions for the robust location problem are contained in R.*

Proof. Let $x' \notin R$ and x'' its orthogonal projection on R. Then we know from [11] that $l_1(x'', a) < l_1(x', a)$, for all $a \in A$. Using this fact we have for the numerator of $\tilde{\rho}$, that $B - \sum_{a\in A} \omega_a l_1(x'', a) > B - \sum_{a\in A} \omega_a l_1(x', a)$. For the denominator of $\tilde{\rho}$ we get using in addition that $\|\cdot\|^\circ$ is monotone $\|l_1(x', a)\|^\circ \ge \|l_1(x'', a)\|^\circ$. In total we get $\tilde{\rho}(x'') > \tilde{\rho}(x')$ and therefore only points in R can be globally optimal. □

As will be shown in Subsection 3.2, finding the most robust location with a cell $\langle s,t\rangle$, i.e., solving

$$\max_{x\in\langle s,t\rangle} \frac{B - \sum_{a\in A} \omega_a l_1(x,a)}{\left\| l_1(x,a)_{a\in A}\right\|^\circ} \qquad (P.\langle s,t\rangle)$$

can be efficiently done for particular choices of norm $\|\cdot\|$.

This fact and Lemma 3.1 suggest a procedure for finding the most robust location in the plane presumably more efficient than Dinkelbach's algorithm, namely, solve for each bounded cell $\langle s,t\rangle$ the corresponding problem $(P.\langle s,t\rangle)$. We will postpone to Subsection 3.2 a detailed discussion on how Problems $(P.\langle s,t\rangle)$ can be solved, and devote Subsection 3.1 to design more efficient search procedures which avoid complete enumeration of the $O(|A|^2)$ bounded cells.

3.1 A search procedure

In order to develop procedures with low computing times, it is of great importance to have good dominance rules, i.e., tests which enable us to eliminate cases without explicit evaluation.

Since $\tilde{\rho}$ is explicitly quasiconcave (see Lemma 2.5), we get the following

Lemma 3.2. *Let C be closed and convex, and let x^* be optimal to*

$$\max_{x\in C} \tilde{\rho}(x)$$

Denote by $T_C(x^*)$ *the set*

$$T_C(x^*) = \left\{x \in \mathbb{R}^2 :\ x = x^* + \lambda(x^\circ - x^*) \text{ for some } \lambda \geq 0,\ x^\circ \in C\right\}$$

Then, x^* *also solves*

$$\max_{x \in T_C(x^*)} \tilde{\rho}(x)$$

The interest of this result stems from the fact that, if C is a bounded cell $\langle s,t\rangle$, then the sets T_C are either the whole plane, a halfspace or a quadrant.

We introduce now the following notation: for any bounded cell $\langle s,t\rangle$, let us denote by $c^i_{\langle s,t\rangle}, i = 1,2,3,4$ its corner points, $c^1 = (a'_{s_1}, a'_{t_2})$, $c^2 = (a'_{s_1}, a'_{t_2+1})$, $c^3 = (a'_{s_1+1}, a'_{t_2+1})$ and $c^4 = (a'_{s_1+1}, a'_{t_2})$, see Figure 3.1, and let $x^*_{\langle s,t\rangle}$ denote an optimal solution to $P.\langle s,t\rangle$.

With this notation we obtain from Lemma 3.2 the following

Lemma 3.3. *Let* $\langle s,t\rangle$ *be a bounded cell, and let* $x^*_{\langle s,t\rangle} \in \arg\max_{x\in\langle s,t\rangle} \tilde{\rho}(x)$

- *If* $x^*_{\langle s,t\rangle} \in int(\langle s,t\rangle)$ *then* $x^*_{\langle s,t\rangle}$ *is also an optimal solution to Problem* (P.$\langle s,t\rangle$).
- *If* $x^*_{\langle s,t\rangle}$ *is contained in the relative interior of an edge of* $\langle s,t\rangle$ *then the complete halfspace defined by this edge and* $\langle s,t\rangle$ *can be excluded from the search*, (*see* Figure 3.2).
- *If* $x^*_{\langle s,t\rangle}$ *is a corner point of* $\langle s,t\rangle$, *then the cone generated by* $x^*_{\langle s,t\rangle}$ *and the two adjacent edges of* $\langle s,t\rangle$ *can be excluded* (*see* Figure 3.3).

If a part of the cells can be excluded from the search procedure we can delete them from the set of cells and perform a search procedure only for the remaining ones. We say row i can be deleted if all cells $\langle i,j\rangle$, for $j = 1,\ldots,Q$ can be excluded from the search procedure. We say column j can be deleted if all cells $\langle i,j\rangle$, for $i = 1,\ldots,P$ can be excluded from the search procedure.

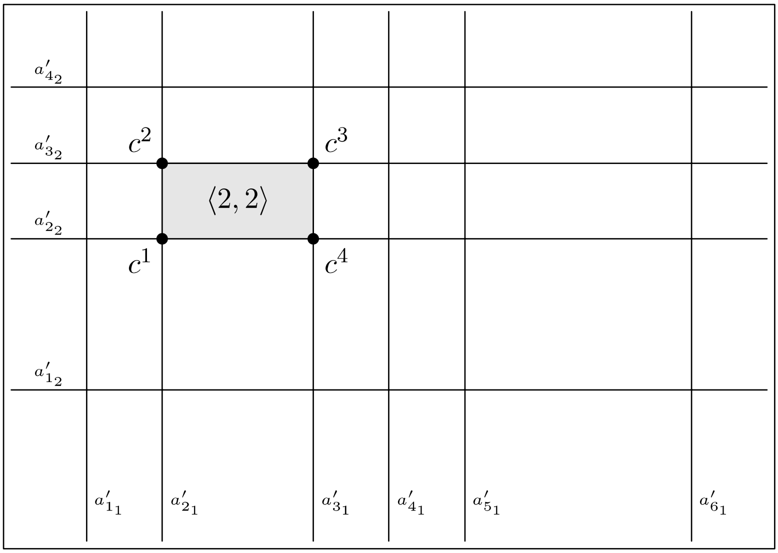

Fig. 3.1. A subdivision of the plane in cells $\langle s,t\rangle$, with $P = 6$ and $Q = 4$

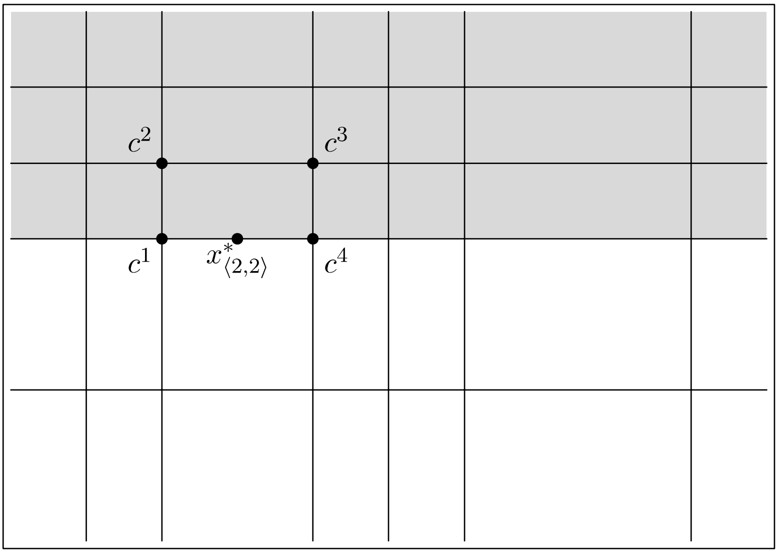

Fig. 3.2. If $x^*_{\langle s,t\rangle}$ is in the relative interior of the boundary then the whole halfspace containing $\langle s,t\rangle$ can be excluded

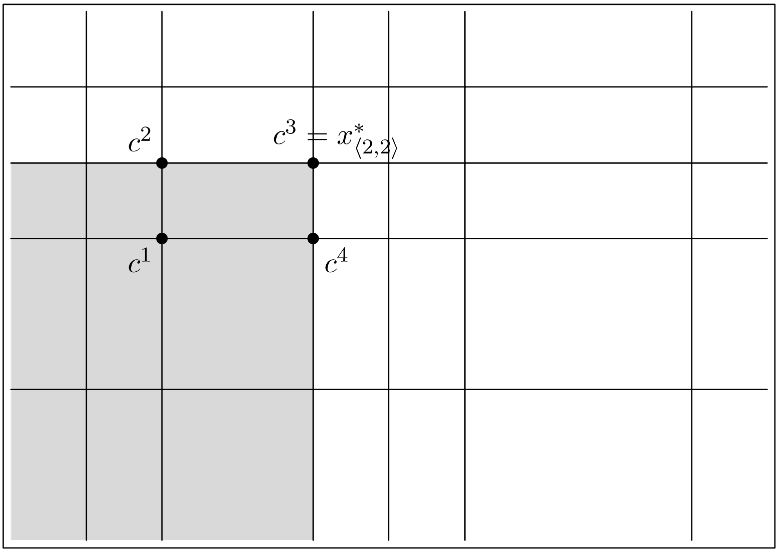

Fig. 3.3. If $x^*_{\langle s,t\rangle}$ is a corner point of the cell, only a cone can be excluded

Given two points u, v, let $(\overline{uv})$ denote the *open* segment with endpoints u, v. Using Lemma 3.1 and Lemma 3.3 we get the following corollary, which will serve as a start-point for a search procedure.

Corollary 3.4. *For cell $\langle 1,1\rangle$ with corner points c^1, c^2, c^3 and c^4 we have the following cases.*

- *If $x^*_{\langle 1,1\rangle} \in int(\langle 1,1\rangle)$ or $x^*_{\langle 1,1\rangle} \in (\overline{c^1c^2})$ or $x^*_{\langle 1,1\rangle} \in (\overline{c^4c^1})$ or $x^*_{\langle 1,1\rangle} = c^1$ then $x^*_{\langle 1,1\rangle}$ is also globally optimal.*
- *If $x^*_{\langle 1,1\rangle} \in (\overline{c^2c^3})$ or $x^*_{\langle 1,1\rangle} = c^2$ then row 1 can be deleted.*
- *If $x^*_{\langle 1,1\rangle} \in (\overline{c^3c^4})$ or $x^*_{\langle 1,1\rangle} = c^4$ then column 1 can be deleted.*
- *If $x^*_{\langle 1,1\rangle} = c^3$ then only cell $\langle 1,1\rangle$ can be excluded.*

From this result we get the following idea for an algorithm. We start with cell $\langle 1,1\rangle$ and apply Corollary 3.4. If a row or a column can be deleted we restart with a reduced cell system and a new cell $\langle 1,1\rangle$. Otherwise we perform diagonal steps to $\langle 2,2\rangle$, $\langle 3,3\rangle$, …, $\langle k,k\rangle$ until another dominance rule as the ones shown in the following lemmata is fulfilled.

Lemma 3.5. *If $x^*_{\langle l,l\rangle} = c^3_{\langle l,l\rangle}$ for all $l = 1, 2, \ldots, \min(P, Q)$ then, if $Q \le P$ (respect. $P \le Q$) we can eliminate the first Q columns (respect. the first P rows).*

Lemma 3.6 *Consider the robust location problem in cell $\langle k,k\rangle$ with $k > 1$ and $\min(P,Q) \ge k$ and corner points c^1, c^2, c^3 and c^4. Additionally we assume that $x^*_{\langle k,k\rangle} \neq c^3$ and in all cells $\langle l,l\rangle$, with $l < k$, $x^*_{\langle l,l\rangle} = c^3_{\langle l,l\rangle}$. Then the following cases can occur.*

- *If $x^*_{\langle k,k\rangle} \in int(\langle k,k\rangle)$ then $x^*_{\langle k,k\rangle}$ is also globally optimal.*
- *If $x^*_{\langle k,k\rangle} \in (\overline{c^2c^3})$ then the first k rows (row 1 up to row k) can be deleted.*
- *If $x^*_{\langle k,k\rangle} \in (\overline{c^3c^4})$ then the first k columns can be deleted.*
- *If $x^*_{\langle k,k\rangle} \in (\overline{c^1c^2})$ or $x^*_{\langle k,k\rangle} = c^2$ then the first $k-1$ rows can be deleted.*
- *If $x^*_{\langle k,k\rangle} \in (\overline{c^4c^1})$ or $x^*_{\langle k,k\rangle} = c^4$ then the first $k-1$ columns can be deleted.*

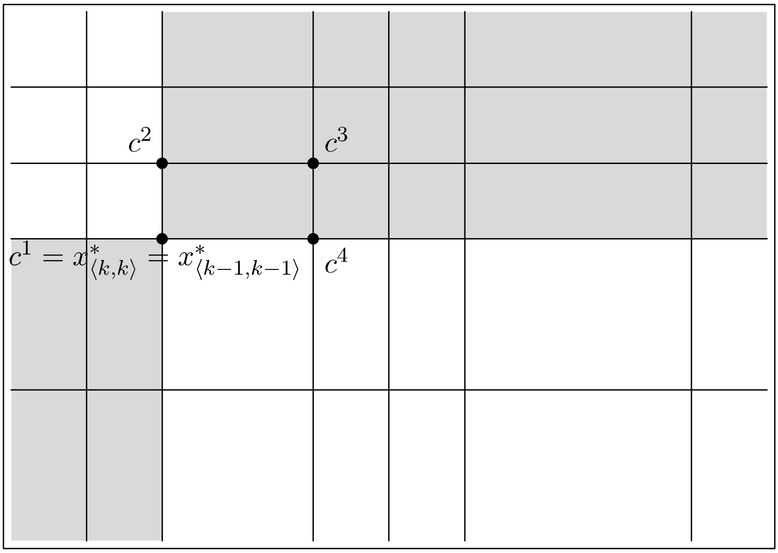

Fig. 3.4 The region which can be excluded if $x^*_{\langle k,k\rangle} = x^*_{\langle k-1,k-1\rangle}$

- *If* $x^*_{\langle k,k\rangle} = c^1$ *then the deletion rules depends on cell* $\langle k-1,k\rangle$ (*see* Figure 3.4).
  - *If* $x^*_{\langle k-1,k\rangle} \in int(\langle k-1,k\rangle)$ *then* $x^*_{\langle k-1,k\rangle}$ *is also globally optimal.*
  - *If* $x^*_{\langle k-1,k\rangle} \in (c^1_{\langle k-1,k\rangle} c^2_{\langle k-1,k\rangle})$ *or* $x^*_{\langle k-1,k\rangle} \in (c^2_{\langle k-1,k\rangle} c^3_{\langle k-1,k\rangle})$ *or* $x^*_{\langle k-1,k\rangle} = c^2_{\langle k-1,k\rangle}$ *then the first* $k-1$ *rows can be deleted.*
  - *If* $x^*_{\langle k-1,k\rangle} \in (c^3_{\langle k-1,k\rangle} c^4_{\langle k-1,k\rangle})$ *or* $x^*_{\langle k-1,k\rangle} \in (c^4_{\langle k-1,k\rangle} c^1_{\langle k-1,k\rangle})$ *or* $x^*_{\langle k-1,k\rangle} = c^4_{\langle k-1,k\rangle}$ *then the first* $k-1$ *columns can be deleted.*

Proof. The proof follows from Lemma 3.3 and using the fact that by assumption all cells $\langle i,j\rangle$ with $i,j \le k$ are already dominated. In the last case it should be noted that by the explicitly quasiconcavity of $\tilde{\rho}$ and the given solution in the adjacent cells isolated locally optimal points in $c^1_{\langle k-1,k\rangle}$ or $c^3_{\langle k-1,k\rangle}$ cannot occur. □

Now we have all technical details fixed to formulate a search algorithm to solve the problem.

Algorithm 3.1. *Algorithm to find the most robust location*

Input: Existing facilities A with corresponding weights $\hat{\omega}$.
Output: $x^* \in \arg\max_{x\in\mathbb{R}^2} \tilde{\rho}(x)$

1. Compute the data for the cells $\langle s,t\rangle$. Denote the set of all bounded cells by $\mathcal{C}$.
2. $k := 1$
3. While $P > 1$ and $Q > 1$ DO
   (a) Compute $x^*_{\langle k,k\rangle}$ and apply Corollary 3.4, Lemma 3.5 and Lemma 3.6.
   (b) If rows or columns can be deleted then reduce $\mathcal{C}$, P, Q accordingly and set $k := 1$. Goto Step 3.
   (c) $k := k+1$.
   (d) If $k > \min\{P,Q\}$ then delete the first k rows in the case $P = \min\{P,Q\}$ and the first k columns otherwise. Reduce $\mathcal{C}$, P, Q accordingly and set $k := 1$. Goto Step 3.
4. Now only one row or column is left. Do any search procedure to determine the cell containing an optimal solution x^*.
5. **Output:** x^*.

It is clear that the algorithm leads to an optimal solution. We discuss now its complexity. Since by the preceding results we are able to delete at least $k-1$ rows or columns after investigating $k+1$ cells, we have

Lemma 3.7. *Algorithm* 3.1 *solves* $O(|A|)$ *problems of type* $P.\langle s,t\rangle$.

Step 1 needs $O(|A|\log|A|)$ time for sorting. Moreover, by Lemma 3.7, the while loop needs $O(|A| \times \mathcal{K})$ time, where $\mathcal{K}$ is the complexity for finding an optimal solution with respect to a cell. Searching the last row or column needs also $O(|A| \times \mathcal{K})$ time. Summing up we have

Proposition 3.8. *If each problem* $P.\langle s,t\rangle$ *can be solved in* $O(\mathcal{K})$ *time*, *then a most robust location can be obtained in* $O(|A|\log(|A|) + |A|\mathcal{K})$ *time.*

In the following we will show how the problem in a cell can be solved and therefore determining the overall complexity of the algorithm.

3.2 Finding the most robust location in a cell

In the last section we have seen how we can search in linear time all cells $\langle s,t\rangle$. Now we will fix a cell $\langle s,t\rangle$ and solve $P.\langle s,t\rangle$. The following lemma shows that in a cell $\tilde{\rho}$ has an additional property.

Lemma 3.9 (see [8]) *$l_1(x,a)$, $a\in A$ is affine linear in $\langle s,t\rangle$ for all $s\in\{0,1,2,\dots,P\}$* and *$t\in\{0,1,2,\dots,Q\}$.*

We denote the numerator of $\tilde{\rho}$ in $\langle s,t\rangle$ by $N(x)$ and the denominator by $D(x)$. From Lemma 3.9 we know that $N(x)$ can be written as an affine linear function say $N(x)=\alpha^T_{\langle s,t\rangle}x+\beta_{\langle s,t\rangle}$. Therefore only the form of the denominator $D(x)$ has to be determined. In order to do that we have to look at possible choices for norm $\|\cdot\|$.

3.2.1 The maximum error

If we choose $\|\cdot\|$ as the maximum norm $\|\cdot\|_{l_\infty}$, we get $D(x)=\sum_{a\in A}l_1(x,a)$. Therefore we can also apply the cell subdivision for the denominator and get an affine linear representation of $D(x)$ in $\langle s,t\rangle$, i.e. $D(x)=\lambda^T_{\langle s,t\rangle}x+\mu_{\langle s,t\rangle}$. Summing up we can write $P.\langle s,t\rangle$ as

$$\max_{x\in\langle s,t\rangle}\frac{\alpha^T_{\langle s,t\rangle}x+\beta_{\langle s,t\rangle}}{\lambda^T_{\langle s,t\rangle}x+\mu_{\langle s,t\rangle}},\tag{3.1}$$

a linear fractional program. Using the fact that in this case $\tilde{\rho}(x)$ is pseudoconvex (see [3]) we get the following lemma.

Lemma 3.10. *An optimal solution for* (3.1) *can always be found in one of the four corner points of $\langle s,t\rangle$.*

Since $\alpha_{\langle s,t\rangle},\beta_{\langle s,t\rangle},\lambda_{\langle s,t\rangle},\mu_{\langle s,t\rangle}$ can be found in $O(|A|)$ time, the total complexity for solving each $P.\langle s,t\rangle$ is linear, thus, by Proposition 3.8, a most robust location can be obtained in $O(|A|\log(|A|)+|A|^2)=O(|A|^2)$ time. Such complexity can be further improved by observing that, in Algorithm 3.1, one moves from a cell $\langle s,t\rangle$ to an adjacent one or eventually (case 3d) to a cell of the form $\langle s+i,t\rangle$ or $\langle s,t+j\rangle$.

It turns out that the linear fractional representation of $\tilde{\rho}$ in such new cell is easily obtained in terms of the coefficients for cell $\langle s,t\rangle$. Indeed, it is easily checked the following

Lemma 3.11. *Define*

$$\mathcal{I}_1\langle s,t\rangle=\{k_1: a'_{s_1}\le a_{k_1}\le a'_{t_1}\}$$
$$\mathcal{I}_2\langle s,t\rangle=\{k_2: a'_{s_2}\le a_{k_2}\le a'_{t_2}\}$$

One has:

$$\alpha_{\langle s+i,t+j\rangle} = \alpha_{\langle s,t\rangle} + \left(2 \sum_{k\in\mathcal{I}_1\langle s+1,s+i\rangle} \omega_k,\ 2 \sum_{l\in\mathcal{I}_2\langle t+1,t+j\rangle} \omega_l\right)$$
$$\beta_{\langle s+i,t+j\rangle} = \beta_{\langle s,t\rangle} - 2 \sum_{k\in\mathcal{I}_1\langle s+1,s+i\rangle} \omega_k a_{k_1} - 2 \sum_{l\in\mathcal{I}_2\langle t+1,t+j\rangle} \omega_l a_{l_2}$$
$$\lambda_{\langle s+i,t+j\rangle} = \lambda_{\langle s,t\rangle} + \left(2 \sum_{k\in\mathcal{I}_1\langle s+1,s+i\rangle} 1,\ 2 \sum_{l\in\mathcal{I}_2\langle t+1,t+j\rangle} 1\right)$$
$$\mu_{\langle s+i,t+j\rangle} = \mu_{\langle s,t\rangle} - 2 \sum_{k\in\mathcal{I}_1\langle s+1,s+i\rangle} a_{k_1} - 2 \sum_{l\in\mathcal{I}_2\langle t+1,t+j\rangle} a_{l_2}$$

Hence, after solving $P.\langle 1,1\rangle$ in $O(|A|)$ time, by Lemma 3.7, only $O(|A|)$ updates of parameters $\alpha, \beta, \lambda, \mu$ are required. By Lemma 3.11, it follows that such updates can be performed in total $O(|A|)$ time. Hence, Steps 2 to 5 of Algorithm 3.1 can be executed in $O(|A|)$ time. Since Step 1 requires $O(|A|\log(|A|))$ time, the overall complexity of the procedure is $O(|A|\log(|A|)+|A|) = O(|A|\log(|A|))$ time.

3.2.2 Sum of errors

If we measure the error as the absolute sum of errors, i.e., we choose $\|\cdot\|$ as the l_1 norm $\|\cdot\|_{l_1}$ we get $D(x) = \max\{l_1(x,a) : a \in A\}$.

The denominator can be simplified by using the following lemma (see [18]).

Lemma 3.12. *There exists a partition* $A^1.\langle s,t\rangle$, $A^2.\langle s,t\rangle$, $A^3.\langle s,t\rangle$, $A^4.\langle s,t\rangle$ of A, *such that for all* $x \in \langle s,t\rangle$

$$d_a(x) = x_1 + x_2 + c_a \ \forall a \in A^1.\langle s,t\rangle$$
$$d_a(x) = x_1 - x_2 + c_a \ \forall a \in A^2.\langle s,t\rangle$$
$$d_a(x) = -x_1 + x_2 + c_a \ \forall a \in A^3.\langle s,t\rangle$$
$$d_a(x) = -x_1 - x_2 + c_a \ \forall a \in A^4.\langle s,t\rangle$$

Furthermore, for any nonempty A^i *there exists* $a_i \in A^i.\langle s,t\rangle$, $i = 1,\ldots,4$, *such that for all* $x \in \langle s,t\rangle$

$$\max_{a\in A}\{d_a(x)\} = \max\{d_{a_i}(x) : A^i.\langle s,t\rangle \neq \emptyset\}.$$

With this result we can write the problem again as a linear fractional program of the following type

$$\max \frac{\alpha^T_{\langle s,t\rangle} x + \beta_{\langle s,t\rangle}}{z}$$

subject to

$$d_{a_i}(x) \leq z \ \forall i = 1,\ldots,4 \text{ with } A^i.\langle s,t\rangle \neq \emptyset$$
$$x_1 \geq a'_{s_1}$$
$$x_1 \leq a'_{s_1+1}$$
$$x_2 \geq a'_{t_2}$$
$$x_2 \leq a'_{t+2_2}$$

In addition, we know from [5] that a linear fractional program can be converted in a linear program by introducing one additional variable. Therefore the dimension is fixed and the problem

$$\max_{x\in\langle s,t\rangle} \tilde{\rho}(x)$$

can be solved in $O(1)$ time after building the sets $A^i.\langle s,t\rangle$, and then the coefficients $\alpha_{\langle s,t\rangle}, \beta_{\langle s,t\rangle}$ and the points a_i defined in Lemma 3.12 have been obtained. Since this information can be obtained in $O(|A|)$ time, it follows from Proposition 3.8 that a most robust location can be obtained in $O(|A|^2)$, although, as in Section 3.2, such complexity can be improved if, at each iteration, the problem $P.\langle s,t\rangle$ is not constructed from scratch but from the corresponding problem in the previous iteration. Such goal can be attained if, e.g., the elements of each $A^i.\langle s,t\rangle$ are stored in Fibonacci heaps, thus enabling the construction of the corresponding a_i in constant time, while insertions and deletions are done in logarithmic time. See [10] for details.

3.2.3 More general cases

The previous approach can directly be adapted to the case where $\|\cdot\|$ is a monotone polyhedral norm, because its dual is then also polyhedral and monotone, and each problem $P.\langle s,t\rangle$ can also be transformed in a fractional linear program using the fact that

$$\|x\|^\circ = \max e^T x \; \forall e \in Ext(B),$$

where $Ext(B)$ denotes the set of extreme points of the unit ball of $\|\cdot\|$. By substituting the constraints

$$d_{a_i}(x) \leq z$$

by

$$e^T(x - a_i) \leq z \; \forall e \in Ext(B)$$

we get a fractional linear program for the general polyhedral norm case, with three variables and $O(|A||Ext(B)|)$ constraints. By including one additional variable, this problem turns out to be equivalent to a linear problem with four variables and $O(|A||Ext(B)|)$ constraints, thus solvable in $O(|A||Ext(B)|)$ time by existing procedures, [17].

If $\|\cdot\|$ is a general (non-polyhedral) monotone norm we can use the approach of Dinkelbach (see Section 2.2) for solving

$$\max_{x\in\langle s,t\rangle} \tilde{\rho}(x).$$

Example 3.1. We are given 10 existing facilities $a_1 = (0,0)$, $a_2 = (1,0)$, $a_3 = (1,-1)$, $a_4 = (0,1)$, $a_5 = (0,2)$, $a_6 = (5,6)$, $a_7 = (6,3)$, $a_8 = (8,4)$, $a_9 = (10,5)$ and $a_{10} = (6,10)$. The estimator for the weights is $\hat{\omega} = (4,4,4,4,4,1,1,1,1,1)$ and the budget B is 100.

The distance $d_a(x) = l_1(x,a)$ and the deviation in the space of weights is measured by the maximum error ($\|\cdot\| = l_\infty$). Therefore our objective function for finding the most robust location is now like in Section 3.2.1. We use

Algorithm 3.1 together with Lemma 3.10 to solve the problem. In our case we proceed as follows:

1. Start in cell $\langle 1,1\rangle$ and compute $\tilde{\rho}(x)$ for all corner points. We get $x^*_{\langle 1,1\rangle} = c^3$ (with objective value 0.22) and we continue with $\langle 2,2\rangle$. Now we get $x^*_{\langle 2,2\rangle} = c^2$ (with objective value 0.25) and by Lemma 3.6 we can delete the first row and restart.
2. Start in cell $\langle 1,2\rangle$. We get $x^*_{\langle 1,2\rangle} = c^3$ (with objective value 0.25) and we continue with $\langle 2,3\rangle$. Now we get $x^*_{\langle 2,3\rangle} = c^1$ (with objective value 0.25) and we have to look at cell $\langle 1,3\rangle$ according to Lemma 3.6. Here we have $x^*_{\langle 1,3\rangle} = c^4$ (with objective value 0.25) and we can delete the first column.
3. Restart with $\langle 2,2\rangle$, where we get $x^*_{\langle 2,2\rangle} = c^2$ and delete according to Corollary 3.4 the first row.
4. Restart in cell $\langle 2,3\rangle$ and get $x^*_{\langle 2,3\rangle} = c^1$ (with objective value 0.25) and conclude by Corollary 3.4 that $x^*_{\langle 2,3\rangle} = c^1 = (1,1)$ is globally optimal.

In Figure 3.5 the cell system with the deleted rows and columns is shown.

4 Conclusions and extension

In this paper we have addressed a planar single-facility location problem in which a high level of uncertainty is involved in the demand vector.

The concept of robustness of a feasible solution x as a measure of the acceptance of x is introduced, and the most robust location is then sought.

Finding the most robust location amounts to solving a nonlinear fractional problem, solvable by existing methods such as Dinkelbach's algorithm when distances are induced by norms, or by more efficient ad-hoc procedures when further assumptions (e.g. distances measured by the Manhattan norm) are made. In particular, an optimal solution can be found with an ad-hoc method in subquadratic time for some choices of $\|\cdot\|$. An empirical analysis of the performance of Dinkelbach's strategy for more general instances (e.g., when constraints exist) is an interesting area to be explored, which was outside the scope of the present paper.

The concept of robustness could also be used in another usual location setting, namely, location on networks, leading again to nonlinear fractional programs which, under further assumptions on the norm $\|\cdot\|$, can be solved by inspecting a finite set of candidate points.

Another interesting extension of this model is obtained if not only the robustness but also the actual transportation cost are taken into account via a biobjective problem, which again becomes piecewise linear and tractable under polyhedrality assumptions on $\|\cdot\|$.

These extensions are currently under research.

5 Appendix

Lemma 5.1. *Let $x \in \mathbb{R}^2$ such that $TC(\hat{\omega},x) < B$. Then,*

$$\min\{\|\omega - \hat{\omega}\| : TC(\omega,x) \geq B, \omega \geq 0\} = \min\{\|\omega - \hat{\omega}\| : TC(\omega,x) = B\} \tag{5.1}$$

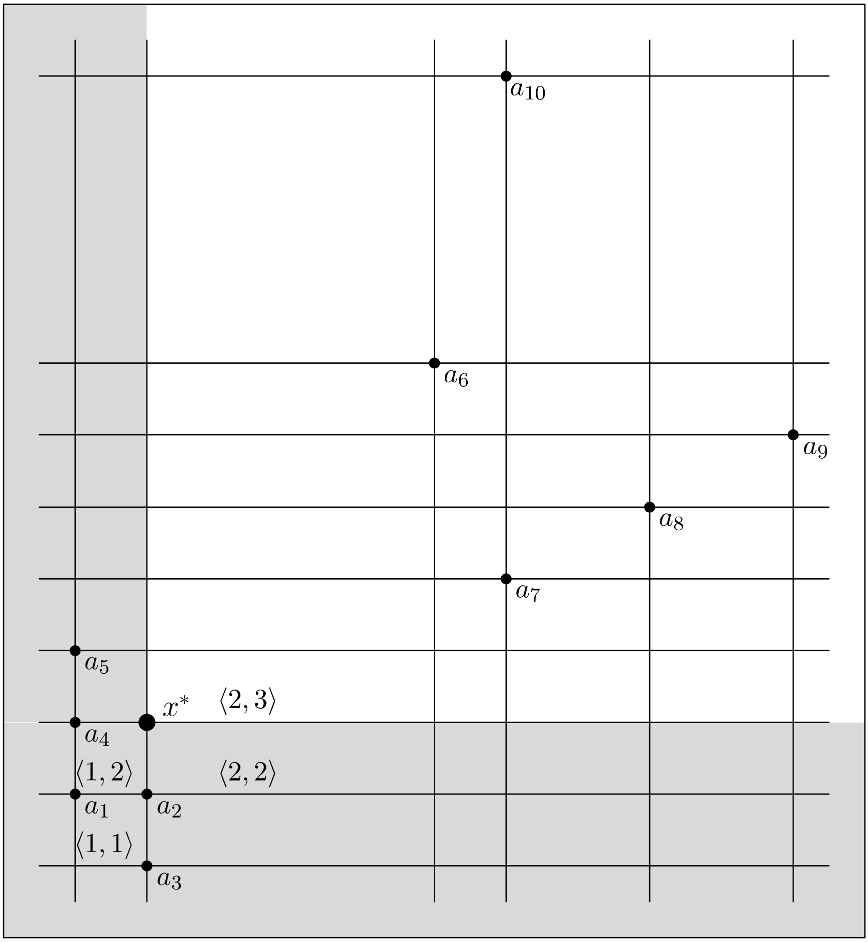

Fig. 3.5. Illustration for Example 3.1

Proof. For any $\omega^1 \in \mathbb{R}^{|A|}$ such that

$$TC(\omega^1, x) \geq B, \omega^1 \geq 0, \tag{5.2}$$

define λ as

$$\lambda = \frac{TC(\omega^1, x) - B}{TC(\omega^1 - \hat{\omega}, x)}$$

It follows from the assumptions and (5.2) that

$$TC(\omega^1, x) \geq B > TC(\hat{\omega}, x),$$

thus $\lambda \in [0, 1)$. Defining ω^2 as

$$\omega^2 = (1 - \lambda)\omega^1 + \lambda\hat{\omega},$$

it follows that

$$\begin{aligned} TC(\omega^2, x) &= TC(\omega^1, x) - \lambda TC(\omega^1 - \hat{\omega}, x) \\ &= B. \end{aligned}$$

Hence, since $\lambda \in [0, 1)$,

$$\begin{aligned} \|\omega^1 - \hat{\omega}\| &\geq (1 - \lambda)\|\omega^1 - \hat{\omega}\| \\ &= \|(1 - \lambda)\omega^1 + \lambda\hat{\omega} - \hat{\omega}\| \\ &= \|\omega^2 - \hat{\omega}\| \\ &\geq \min\{\|\omega - \hat{\omega}\| : TC(\omega, x) = B\} \end{aligned}$$

Hence,

$$\rho(x) \geq \min\{\|\omega - \hat{\omega}\| : TC(\omega, x) = B\}$$

Conversely, given $\omega^3 \in \mathbb{R}^{|A|}$ such that $TC(\omega, x) = B$, define $\omega^4 \in \mathbb{R}^{|A|}$ as

$$\omega_a^4 = \max\{\omega_a^3, 0\} \qquad \forall a \in A$$

Then, $\omega^4 \geq 0$ and

$$\begin{aligned} TC(\omega^4, x) &= \sum_{a \in A} \omega_a^4 d_a(x) \\ &= \sum_{\{a \in A : \omega_a^3 \geq 0\}} \omega_a^4 d_a(x) + \sum_{\{a \in A : \omega_a^3 < 0\}} \omega_a^4 d_a(x) \\ &= \sum_{\{a \in A : \omega_a^3 \geq 0\}} \omega_a^3 d_a(x) \\ &\geq \sum_{a \in A} \omega_a^3 d_a(x) \\ &= TC(\omega^3, x) \\ &= B \end{aligned}$$

This implies that $\rho(x) \leq \|\omega^4 - \hat{\omega}\|$.

Moreover, since,

$$|\omega_a^4 - \hat{\omega}_a| \leq |\omega_a^3 - \hat{\omega}_a| \qquad \forall a \in A,$$

thus, since any absolute norm is a monotonic norm, [2],

$$\|\omega^4 - \hat{\omega}\| \leq \|\omega^3 - \hat{\omega}\|,$$

and hence

$$\|\omega^3 - \hat{\omega}\| \geq \min\{\|\omega - \hat{\omega}\| : TC(\omega, x) \geq B, \omega \geq 0\}$$

□

Proof of Proposition 2.2. If $TC(\hat{\omega}, x) \geq B$, then $\hat{\omega}$ is feasible for (2.6), thus $\rho(x) = 0$. If x satisfies $TC(\hat{\omega}, x) < B$, then, by Lemma 5.1, $\rho(x)$ is the distance (according to metric $\|\cdot\|$) from point $\hat{\omega} \in \mathbb{R}^{|A|}$ to the hyperplane $\{\omega \in \mathbb{R}^{|A|} : TC(\omega, x) = B\}$. Hence,

$$\rho(x) = \frac{B - TC(\hat{\omega}, x)}{\|(d_a(x))_{a \in A}\|^{\circ}},$$

e.g. [20], and then the result follows. □

Acknowledgment. The authors acknowledge the constructive remarks of the referees. The research of the first author has been supported in part by grant BFM2002-04525-C02-02 of MCYT, Spain. The research of the second author has been supported in part by a grant of the Deutsche Forschungsgemeinschaft.

References

[1] Averbakh I, Berman O (2000) Algorithms for the robust 1-center problem on a tree. European Journal of Operational Research 123:293–302
[2] Barker GP (1971) Monotone norms. Numer Math 18:321–326
[3] Bazaraa MS, Sherali HD, Shetty CM (1993) Nonlinear Programming, Wiley Second Edition
[4] Brimberg J, Love RF, (1995) Estimating distances. In: Drezner Z (ed.) Facility Location. A Survey of Applications and Methods, chapter 1, pp. 9–32, Springer
[5] Charnes A, Cooper WW (1962) Programming with linear fractionals. Naval Research Logistics Quaterly 9:181–186
[6] Dinkelbach W (1967) On nonlinear fractional programming. Management Science 13:492–497
[7] Drezner Z (1979) Bounds on the optimal location to the weber problem under conditions of uncertainty. Journal of the Operational Research Society 30:923–931
[8] Francis RL, McGinnis Jr LF, White JA (1992) Facility Layout and Location: an Analytical Approach. International Series in Industrial and Systems Engineering. Prentice Hall, Englewood Cliffs NJ 2nd edition
[9] Frank H (1966) Optimum locations on a graph with probabilistic demands. Operations Research 14:409–421
[10] Fredman ML, Tarjan RE (1987) Fibonacci heaps and their uses in improved network optimization algorithms. Journal of the ACM 34:596–615
[11] Hamacher HW, Nickel S (1994) Combinatorial algorithms for some 1-facility median problems in the plane. European Journal of Operational Research 79:340–351
[12] Perreur-J Hansen P, Thisse J-F (1980) Location theory dominance and convexity: Some further results. Operations Research 28:1241–1250
[13] Hiriart-Urruty J-B, Lemaréchal C (1993) Convex Analysis and Minimization Algorithms I,II Springer
[14] Labbé M, Thisse J-F, Wendell RE (1991) Sensitivity analysis in minisum facility location problems. Operations Research 39:961–969
[15] Love RF, Morris JG, Wesolowsky GO (1988) Facilities Location: Models and Methods North Holland, New York
[16] Martos B (1975) Nonlinear Programming Theory and Methods. North-Holland, Amsterdam
[17] Megiddo N (1982) Linear-time alogrithms for linear programming in R^3 and related problems. SIAM Journal on Computing 12:759–776
[18] Mehrez A, Sinuany-Stern Z, Stulman A (1986) An enhancement of the dreznerwesolowshy algorithm for single-facility location with maximum of rectilinear distance. Journal of the Operational Research Society 37:971–977
[19] Plastria F (1995) Continuous location problems. In: Drezner Z (ed.) Facility Location. A Survey of Applications and Methods, chapter 11, pp. 225–260, Springer
[20] Plastria F, Carrizosa E (2001) Gauge distances and median hyperplanes. Journal of Optimization Theory and Applications 110:173–182
[21] Schaible S (1976) Fractional programming II on Dinkelbach's algorithm. Management Science 22:868–873

[22] Wesolowsky GO (1997) Probabilisitic weights in the one-dimensional facility location problem. Management Science 24:224–229
[23] Wesolowsky GO (1993) The Weber Problem: History and Perspectives. Location Science 1:5–23
[24] Witzgall CJ (1964) Optimal Location of a Central Facility: Mathematical Models and Concepts. Technical Report 8388. National Bureau of Standards Report, US Department of Commerce, National Bureau of Standards, Washington DC