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Generalized vector equilibrium problems with set-valued mappings

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Abstract. In this paper, we introduce a more general form of vector equilibrium problems with a moving ordering cone and set-valued mappings, and obtain some existence theorems for generalized vector equilibrium problems, which extend and unify some existence results for similar problems.

Key Words: Vector equilibrium problem, moving cone, set-valued mapping, pseudo-monotonicity, topological vector space

1 Introduction

Let K be a nonempty subset, and $f : K \times K \to \mathbb{R}$ be a real valued function such that $f(x, x) \geq 0$, $\forall x \in K$. The equilibrium problem (in short, EP) is the problem of finding $x \in K$ such that

$$f(x, y) \geq 0, \quad \text{for all } y \in K.$$

The EP has many applications in physics, mathematical economics, and operations research, etc. Recently, the EP is extensively generalized to the vector valued functions (see [1–3, 6, 8, 11–13] and references therein).

In this paper, we consider a more general form of vector equilibrium problems (in short, VEP) with a moving ordering cone and set-valued mappings. Let X, Y and Z be real topological vector spaces, K be a nonempty convex subset of X and D be a nonempty subset of Y. Let $C : K \rightrightarrows Z$ be a set-valued mapping such that, $\forall x \in K$, $C(x)$ is a closed, convex and proper cone with apex at the origin and with nonempty interior, i.e. $\operatorname{int} C(x) \neq \emptyset$. Let $T : K \rightrightarrows D$ and $f : K \times K \times D \rightrightarrows Z$ be set-valued mappings such that, $\forall x \in K$, $T(x) \neq \emptyset$ and $\forall x \in K$, $t \in D$, $0 \in f(x, x, t) \subset C(x)$. Throughout this paper, unless otherwise specified, we fix these notations and assumptions.

We consider the following generalized vector equilibrium problems (in short, GVEP).

(GVEP 1) Find $y \in K$ such that $\forall x \in K$, $\exists v \in T(y)$, $f(x, y, v) \not\subset \operatorname{int} C(y)$.
(GVEP 2) Find $y \in K$ such that $\forall x \in K$, $\exists u \in T(x)$, $f(x, y, u) \not\subset \operatorname{int} C(y)$.
(GVEP 3) Find $y \in K$ and $v \in T(y)$ such that $f(x, y, v) \not\subset \operatorname{int} C(y)$, $\forall x \in K$.

The following problems are the special cases of (GVEP 1).

(1) If $X = Y$, $K = D$ and $\forall x \in K$, $T(x) = x$, $F : K \times K \rightrightarrows Z$, and let $f(x, y, t) := -F(x, y)$, then (GVEP 1) reduces to finding $y \in K$ such that

$$F(x, y) \not\subset -\operatorname{int} C(y), \quad \forall x \in K.$$

It was investigated in Konnov and Yao [12].

(2) If $Y = D = L(X, Z)$, the space of all continuous linear operators from X into Z, $T : K \rightrightarrows L(X, Z)$ and $f(x, y, t) := (t, y - x)$, then (GVEP 1) reduces to finding $y \in K$ such that $\forall x \in K$, $\exists t \in T(y)$,

$$(t, y - x) \notin \operatorname{int} C(y),$$

where (t, z) is the evaluation of $t \in L(X, Z)$ at $z \in Z$. This was studied in Konnov and Yao [11].

(3) If $\eta : K \times K \to X$, $\forall x \in K$, $\eta(x, x) = 0$, $Y = D = L(X, Z)$ and $T : K \rightrightarrows L(X, Z)$, let $f(x, y, t) := (t, \eta(x, y))$, then (GVEP 1) reduces to finding $y \in K$ such that $\forall x \in K$, $\exists t \in T(y)$,

$$(t, \eta(x, y)) \notin \operatorname{int} C(y).$$

It was considered in Ding and Tarafdar [8].

(4) If $D \subset X^*$, the topological dual of X, $\eta : K \times K \to X$, $\eta(x, x) = 0$, $\forall x \in K$; $T : K \rightrightarrows D$ and $\theta : K \times D \to L(X, Z)$, and let $f(x, y, t) = (\theta(y, t), \eta(x, y))$, then (GVEP 1) reduces to finding $y \in K$ such that $\forall x \in K$, $\exists t \in T(y)$

$$(\theta(y, t), \eta(x, y)) \notin \operatorname{int} C(y).$$

It was investigated in Ansari, Siddiqi and Yao [2].

The purpose of this paper is to prove the existence theorems for (GVEP 1) under certain assumptions on f and T, which extend some results in [2, 11].

2 Preliminaries

In this section, we give some definitions and recall some well-known results we need.

Definition 1. Let $f : K \times K \times D \rightrightarrows Z$ be given.

(i) $f(x, y, t)$ is C_y-pseudomonotone with respect to T if, $\forall x, y \in K$, $\forall u \in T(x)$, $v \in T(y)$, $f(x, y, v) \not\subset \operatorname{int} C(y)$ implies $f(x, y, u) \not\subset \operatorname{int} C(y)$.

(ii) $f(x, y, t)$ is weakly C_y-pseudomonotone with respect to T if, $\forall x$, $y \in K$, $\forall v \in T(y)$, $f(x, y, v) \not\subset \operatorname{int} C(y)$ implies $f(x, y, u) \not\subset \operatorname{int} C(y)$ for some $u \in T(x)$.

(iii) $f(x, y, t)$ is u-hemicontinuous with respect to T if, $\forall x, y \in K$, $\alpha \in [0, 1]$, $x_\alpha = y + \alpha(x - y)$, then mapping $\alpha \to f(x, y, T(x_\alpha)) = \bigcup_{t \in T(x_\alpha)} f(x, y, t)$ is upper semicontinuous at $\alpha = 0$.

(iv) $f(x, y, t)$ is C_y-concave in x if, for any fixed $y \in K$, $t \in D$, $\forall x_1, x_2 \in K$, $\alpha \in [0, 1]$, $f(\alpha x_1 + (1 - \alpha)x_2, y, t) \subset \alpha f(x_1, y, t) + (1 - \alpha)f(x_2, y, t) + C(y)$.

(v) $f(x, y, t)$ is affine in x if, for any fixed $y \in K$, $t \in D$, $\forall x_1, x_2 \in K$, $\alpha \in [0, 1]$, $f(\alpha x_1 + (1 - \alpha)x_2, y, t) = \alpha f(x_1, y, t) + (1 - \alpha)f(x_2, y, t)$.

Remark 1. If f is a single valued mapping, and "$\subset$" is replaced with "$\in$" in some places, then the above definitions for the single valued mapping are obtained.

Definition 2. Let X and Y be topological spaces, $T : X \rightrightarrows Y$ a set-valued mapping. (i) T is said to be upper semicontinuous at $x \in X$ if, for any open set V containing $T(x)$, there is an open set U containing x such that for each $t \in U$, $T(t) \subset V$; T is called upper semicontinuous on X if it is upper semicontinuous at all $x \in X$. (ii) T is said to be closed if the graph of T, i.e., $G_r(T) := \{(x, y) : x \in X, y \in T(x)\}$, is a closed subset of $X \times Y$.

Lemma 1. *(i) T is closed if and only if for any net $\{x_\lambda\}$, $x_\lambda \to x$ and any net $\{y_\lambda\}$, $y_\lambda \in T(x_\lambda)$, $y_\lambda \to y$, one has $y \in T(x)$. (ii) If T is compact valued, then T is upper semicontinuous at x if and only if for any net $\{x_\lambda\}$, $x_\lambda \to x$ and any net $\{y_\lambda\}$, $y_\lambda \in T(x_\lambda)$, there exist $y \in T(x)$ and a subnet $\{y_{\lambda'}\}$ of $\{y_\lambda\}$, such that $y_{\lambda'} \to y$.*

Lemma 2. (i) *If y is a solution of (GVEP 3), then it is a solution of (GVEP 1).*

(ii) *If $f(x, y, t)$ is weakly C_y-pseudomonotone with respect to T and y is a solution of (GVEP 1), then it is a solution of (GVEP 2).*

(iii) *If $f(x, y, t)$ is C_y-concave in x and u-hemicontinuous with respect to T, and y is a solution of (GVEP 2), then it is a solution of (GVEP 1).*

Proof. (i) and (ii) are obvious. We need only to show (iii). Let $y \in K$ be a solution of (GVEP 2). Then, $\forall x \in K$, there is a $u \in T(x)$,

$$f(x, y, u) \not\subset \operatorname{int} C(y). \tag{1}$$

If y is not a solution of (GVEP 1), then there is an $\bar{x} \in K$ such that $\forall v \in T(y)$, $f(\bar{x}, y, v) \subset \operatorname{int} C(y)$, i.e., $f(\bar{x}, y, T(y)) \subset \operatorname{int} C(y)$. Since f is u-hemicontinuous with respect to T, there is a $\delta \in (0, 1)$ such that for all $\alpha \in (0, \delta)$, $x_\alpha = y + \alpha(\bar{x} - y) \in K$, $f(\bar{x}, y, T(x_\alpha)) \subset \operatorname{int} C(y)$, i.e., $\forall t \in T(x_\alpha)$,

$$f(\bar{x}, y, t) \subset \operatorname{int} C(y). \tag{2}$$

Since $f(x, y, t)$ is C_y-concave in x and $f(y, y, t) \subset C(y)$, by (2), we have $f(x_\alpha, y, t) \subset \alpha f(\bar{x}, y, t) + (1 - \alpha)f(y, y, t) + C(y) \subset \operatorname{int} C(y) + C(y) \subset \operatorname{int} C(y)$, a contradiction to (1). □

Let $C_+ := \operatorname{Co}\{C(x) : x \in K\}$ and $C_+^* := \{s \in Z^* : (s, x) \geq 0, \forall x \in C_+\}$, where $\operatorname{Co}(A)$ is the convex hull of a set A.

Lemma 3 ([11]). *Let $s \in C_+^* \backslash \{0\}$ and $H(s) = \{x \in Z : (s, x) \geq 0\}$. Then*

(i) *$H(s)$ is a closed convex cone in Z.*
(ii) *If $H(s) \neq Z$, then* $\operatorname{int} H(s) = s^{-1}((0, +\infty))$.

Proof. We need only to show (ii). If $x \in s^{-1}((0, +\infty))$, then $s(x) = (s, x) > 0$. Since s is continuous, there is a neighbourhood V of the origin in Z such that, $\forall z \in x + V$, $s(z) > 0$. Hence, $x \in \operatorname{int} H(s)$. On the other hand, if $x \in \operatorname{int} H(s)$, then there is a neighbourhood V of the origin in Z such that $x + V \subset \operatorname{int} H(s)$. We shall show $s(x) > 0$. If it is false, then $s(x) = 0$. Since V is absorbing, $\forall z \in Z$, there is an $r > 0$ such that $rz \in V$. We have $0 \leq s(x + rz) = rs(z)$. Hence $s(z) \geq 0$, i.e., $z \in H(s)$. Thus $z \subset H(s)$, a contradiction.

The following is a result of Chowdhury and Tan [6] which is a generalization of the well-known Fan-Browder fixed point theorem.

Theorem 1. *Let $A, B : K \rightrightarrows K \cup \{\emptyset\}$ be two set-valued mappings such that*

(i) *$\forall z \in K$, $A(z) \subset B(z)$;*
(ii) *$\forall z \in K$, $B(z)$ is convex;*
(iii) *$\forall z \in K$, $A^{-1}(z)$ is compactly open (i.e., $A^{-1}(z) \cap L$ is open in L for each nonempty and compact subset L of K);*
(iv) *there exist a nonempty, closed and compact subset M of K and $\bar{z} \in M$, such that $K \backslash M \subset B^{-1}(\bar{z})$;*
(v) *$\forall z \in M$, $A(z) \neq \emptyset$.*

Then there an $x \in K$ such that $x \in B(x)$.

The following is the well-known Fan lemma in [10].

Theorem 2. *Let X be a Hausdorff topological vector space, and K be a nonempty convex subset of X. For each $x \in K$, let $F(x)$ be a closed subset of K such that the convex hull of every finite subset $\{x_1, \ldots, x_n\}$ of K is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$. If there is an $\bar{x} \in K$ such that $F(\bar{x})$ is compact, then $\bigcap_{x \in K} F(x) \neq \emptyset$.*

Definition 3 ([12]). A set-valued mapping $F : K \rightrightarrows K$ is called KKM-map if $\operatorname{Co}(x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$ for any finite subset $\{x_1, \ldots, x_n\}$ of K.

For properties of set-valued mappings and cones, we refer to Berge [4] and Jahn [9], respectively.

3 Solutions of (GVEP) with monotonicity

In this section, we use the technique of [2], [6] and [11] to get some existence results for (GVEP).

Theorem 3. *Let X, Y, Z, K, D, C and T be as in section 1. Let $f : K \times K \times$*

$D \rightrightarrows Z$ be such that, $\forall x, y \in K$, $t \in D$, $f(x, y, t)$ is a nonempty compact subset of Z. Assume that the following conditions hold:

(i) *$\forall x \in K$, $t \in D$, $0 \in f(x, x, t) \subset C(x)$;*
(ii) *$f(x, y, t)$ is C_y-pseudomonotone with respect to T and C_y-concave in x;*
(iii) *$f(x, y, t)$ is u-hemicontinuous with respect to T and upper semicontinuous in y;*
(iv) *the set-valued mapping $W : K \rightrightarrows Z$ defined by $W(x) := Z \backslash \operatorname{int} C(x)$, $\forall x \in K$, is closed;*
(v) *there are a nonempty, closed and compact subset M of K and a $\bar{z} \in M$ such that for each $z \in K \backslash M$, $f(\bar{z}, z, t) \subset \operatorname{int} C(z)$, $\forall t \in T(z)$;*

Then (GVEP 1) has a solution $y \in M$.

Proof. Define $A, B : K \rightrightarrows K \cup \{\emptyset\}$ by

$$A(z) := \{x \in K : \exists u \in T(x), f(x, z, u) \subset \operatorname{int} C(z)\}$$

and

$$B(z) := \{x \in K : \forall w \in T(z), f(x, z, w) \subset \operatorname{int} C(z)\}, \quad \forall z \in K.$$

The proof is divided into the following steps.

(i) $\forall z \in K$, $A(z) \subset B(z)$;

In fact, if $x \notin B(z)$, then there is a $w \in T(z)$ such that $f(x, z, w) \not\subset \operatorname{int} C(z)$. Since $f(x, z, t)$ is C_z-pseudomonotone with respect to T, we have $f(x, z, u) \not\subset \operatorname{int} C(z)$, $\forall u \in T(x)$. Thus $x \notin A(z)$.

(ii) $\forall z \in K$, $B(z)$ is a convex subset of K;

Let $x_1, x_2 \in B(z)$ and $\alpha \in (0, 1)$. Then, $\forall t \in T(z)$,

$$f(x_i, z, t) \subset \operatorname{int} C(z), \quad i = 1, 2. \tag{3}$$

By the condition (ii) and (3), we have $\forall t \in T(z)$,

$$\begin{aligned} f(\alpha x_1 + (1-\alpha)x_2, z, t) &\subset \alpha f(x_1, z, t) + (1-\alpha) f(x_2, z, t) + C(z) \\ &\subset \operatorname{int} C(z) + \operatorname{int} C(z) + C(z) \subset \operatorname{int} C(z). \end{aligned}$$

Therefore $\alpha x_1 + (1-\alpha)x_2 \in B(z)$.

(iii) $\forall x \in K$, $A^{-1}(x)$ is compactly open;

Indeed, Let L be a nonempty compact subset of K, and $Q := A^{-1}(x) \cap L = \{z \in L : x \in A(z)\}$. We need to show that $L \backslash Q$ is closed in L. Let a net $\{z_\lambda\} \subset L \backslash Q$ be such that $z_\lambda \to z$. Then $x \notin A(z_\lambda)$. By the definition of A, $\forall u \in T(x)$, $f(x, z_\lambda, u) \not\subset \operatorname{int} C(z_\lambda)$. Hence, there is $t_\lambda \in f(x, z_\lambda, u)$ such that $t_\lambda \notin \operatorname{int} C(z_\lambda)$. Since $f(x, y, u)$ is upper semicontinuous in y, by Lemma 1, there exist a point $t \in f(x, z, u)$ and a subset $\{t_{\lambda'}\} \subset \{t_\lambda\}$ such that $t_{\lambda'} \to t$. Since the net $\{(z_{\lambda'}, t_{\lambda'})\} \subset G_r(W)$ and $(z_{\lambda'}, t_{\lambda'}) \to (z, t)$, and $G_r(W)$ is closed in $K \times Z$, we have $(z, t) \in G_r(W)$, i.e., $t \notin \operatorname{int} C(z)$. Hence, $\forall u \in T(x)$, $f(x, z, u) \not\subset \operatorname{int} C(z)$, i.e., $x \notin A(z)$. Thus $z \in L \backslash Q$.

(iv) By the condition (v), $K \backslash M \subset B^{-1}(\bar{z})$;

(v) We claim that there is a point $\bar{y} \in M$ such that $A(\bar{y}) = \emptyset$.

Suppose to the contrary that, $\forall y \in M$, $A(y) \neq \emptyset$. Then, by Theorem 1, B has a fixed point $x \in K$, i.e., $x \in B(x)$. Then, $\forall t \in T(x)$, $f(x,x,t) \subset \operatorname{int} C(x)$. But, by the condition (i), $0 \in f(x,x,t)$. Hence $0 \in \operatorname{int} C(x)$, a contradiction to $C(x) \neq Z$.

If $\bar{y} \in M$ and $A(\bar{y}) = \emptyset$, then for each $x \in K$, $\exists u \in T(x)$ such that $f(x,\bar{y},u) \not\subset \operatorname{int} C(\bar{y})$. This means $\bar{y}$ is a solution of (GVEP 2). By Lemma 2, $\bar{y}$ is a solution of (GVEP 1). □

Corollary 1. *Let X, Y, K, D and T be as in Theorem 3. Let $Z = \mathbb{R}$ and for each $x \in K$, $C(x) = \mathbb{R}_+ = [0,+\infty)$, and $f : K \times K \times D \rightrightarrows \mathbb{R}$ be such that $\forall x, y \in K$, $t \in D$, $f(x,y,t)$ is a nonempty, closed and bounded subset of $\mathbb{R}$. Assume that all conditions in Theorem 3 hold. Then there is a $\bar{y} \in M$ such that, $\forall x \in K$, $\exists v \in T(\bar{y})$, $f(x,\bar{y},v) \not\subset \operatorname{int} \mathbb{R}_+$.*

Corollary 2. *Let X, Y, Z, K, D, C and T be as in Theorem 3, and $f : K \times K \times D \to Z$ be a single valued mapping. Assume that Conditions (ii), (iv) and (v) in Theorem 3 hold. The conditions (i) and (iii) in Theorem 3 are replaced with the following*

(i)′ $\forall x \in K$, $t \in D$, $f(x,x,t) = 0$;
(iii)′ *$f(x,y,t)$ is u-hemicontinuous with respect to T and continuous in y.*

Then there is a $y \in M$ such that $\forall x \in K$, $\exists v \in T(y)$, $f(x,y,v) \notin \operatorname{int} C(y)$.

Corollary 3. *Let X, Y, Z, K, D, C and T be as in Theorem 3. Assume that the following conditions hold:*

(i) *the single valued mapping $\eta : K \times K \to X$ is affine in the first argument and continuou in the second argument; $\forall x \in K$, $\eta(x,x) = 0$;*
(ii) *the single valued mapping $\theta : K \times D \to L(X,Z)$ is continuous in the first argument and $(\theta(y,t), \eta(x,y))$ is C_y-pseudomonotone with respect to T;*
(iii) *the bilinear form $(\cdot,\cdot)$ between $L(X,Z)$ and X is continuous;*
(iv) *$T : K \rightrightarrows D$ is u-hemicontinuous with respect to θ, i.e., $\forall x, y \in K$, $\alpha \in [0,1]$, $x_\alpha = \alpha x + (1-\alpha)y$, the mapping $\alpha \to (\theta(y, T(x_\alpha)), \eta(x,y))$ is upper semicontinuous at $\alpha = 0$;*
(v) *the set-valued mapping $W : K \rightrightarrows Z$ defined by $W(x) := Z \backslash \operatorname{int} C(x)$ is closed;*
(vi) *there exist a nonempty, closed and compact subset M of K and $\bar{z} \in M$ such that for each $z \in K \backslash M$, $(\theta(z,t), \eta(\bar{z},z)) \in \operatorname{int} C(z)$, $\forall t \in T(z)$;*

Then there is a $y \in M$ such that $\forall x \in K$, $\exists v \in T(y)$,

$$(\theta(y,v), \eta(x,y)) \notin \operatorname{int} C(y).$$

Proof. Let $f(x,y,t) = (\theta(y,t), \eta(x,y))$, $\forall x, y \in K$ and $t \in D$. Then it is easy to check that all the conditions in Corollary 2 hold. Corollary 2 yields the conclusion. □

Remark 2. Theorem 3.1 in [2] is similar to the above corollary.

Corollary 4. *Let X and Z be real Banach spaces, and K be a nonempty convex subset of X. Let $C : K \rightrightarrows Z$ be as in section 1. Let $T : K \rightrightarrows L(X,Z)$ be a set-*

valued mapping with $T(x) \neq \emptyset$, $\forall x \in K$. *Assume that the following conditions hold:*

(i) T *is* C_x*-pseudomonotone, i.e.,* $\forall x, y \in K$, $\forall t' \in T(x)$, $t'' \in T(y)$, $(t', x - y) \notin \operatorname{int} C(x)$ *implies* $(t'', x - y) \notin \operatorname{int} C(x)$;
(ii) T *is u-hemicontinuous, i.e.,* $\forall x, y \in K$, $\alpha \in [0, 1]$, *the mapping* $\alpha \rightarrow (T(\alpha x + (1 - \alpha) y), y - x)$ *is upper semicontinuous at* $\alpha = 0$.
(iii) *the set-valued mapping* $W : K \rightrightarrows Z$, $W(x) := Z \backslash \operatorname{int} C(x)$, $\forall x \in K$, *has a weakly closed graph* $G_r(W)$ *in* $X \times Z$;
(iv) *there exist a nonempty, and weakly compact subset* M *of* K *and a* $\bar{z} \in M$ *such that,* $\forall z \in K \backslash M$, $(t, z - \bar{z}) \subset \operatorname{int} C(z)$, $\forall t \in T(z)$.

Then there is $y \in M$ *such that,* $\forall x \in K$, $\exists v \in T(y)$, $(v, y - x) \notin \operatorname{int} C(y)$.

Proof. In Corollary 2, let X and Z be endowed with their weak topologies, and let $D = Y = L(X, Z)$ and $f(x, y, t) = (t, y - x)$, $\forall x, y \in K$, $t \in D$. We need to show that $f(x, y, t)$ is weakly continuous in y. Let a net $\{y_\lambda\} \subset K$ be such that $y_\lambda \rightharpoonup y$, where "$\rightharpoonup$" denotes "converges weakly to". Since $t \in L(X, Z)$, t is continuous from the weak topology of X to the weak topology of Z ([7, Chap. 6, Thm 1.1]). We have $f(x, y_\lambda, t) = (t, y_\lambda - x) \rightharpoonup (t, y - x) = f(x, y, t)$. Corollary 2 yields the conclusion. □

Remark 3. Theorem 3.1 in [11] is the special case of $M = K$ in the above corollary 4.

Theorem 4. *Let* X, Y, Z, K, D, C *and* T *be as in Theorem 3, and* $s \in C_+^* \backslash \{0\}$, $H(s) \neq Z$. *Assume that the conditions (i), (iii), (iv) and (v) in Theorem 3 hold, the condition (ii) is replaced with the following*

(ii)′ $f(x, y, t)$ *is* C_y*-concave in* x *and* $H(s)$*-pseudomonotone with respect to* T, *i.e.,* $\forall x, y \in K$, $\forall u \in T(x)$, $v \in T(y)$, $f(x, y, v) \not\subset \operatorname{int} H(s)$ *implies* $f(x, y, u) \not\subset \operatorname{int} H(s)$. *Then (GVEP 1) is solvable.*

Proof. Define $\tilde{f} : K \times K \times D \rightrightarrows \mathbb{R}$ by

$$\tilde{f}(x, y, t) = (s, f(x, y, t)), \quad \forall x, y \in K, t \in D.$$

Since f is $H(s)$-pseudomonotone with respect to T, by Lemma 3, $\forall x, y \in K$, $\forall u \in T(x)$, $v \in T(y)$, $\tilde{f}(x, y, v) \not\subset \operatorname{int} \mathbb{R}_+$ implies $\tilde{f}(x, y, u) \not\subset \operatorname{int} \mathbb{R}_+$. By Corollary 1, $\exists y \in M$ such that $\forall x \in K$, $\exists v \in T(y)$,

$$(s, f(x, y, v)) = \tilde{f}(x, y, v) \not\subset \operatorname{int} \mathbb{R}_+.$$

Hence,

$$f(x, y, v) \not\subset \operatorname{int} H(s).$$

Since $s \in C_+^* \backslash \{0\}$ and $\operatorname{int} H(s) \supset \operatorname{int} C_+ \supset \operatorname{int} C(y)$, we have

$$f(x, y, v) \not\subset \operatorname{int} C(y).$$

Thus y is a solution of (GVEP 1). □

Remark 4. In a like manner, as in Corollaries 3 and 4, we can obtain some results similar to Theorem 6 in [2] and Theorem 4.1 in [11] from the above theorem.

4 Solutions of (GVEP) without monotonicity

Theorem 5. *Let X be a Hausdorff topological vector space, and let Y, Z, K, D, C and T as in Theorem 3. Assume that the following conditions hold:*

(i) *$f : K \times K \times D \rightrightarrows Z$, $\forall x, y \in K$, $t \in D$, $f(x, y, t)$ is a nonempty compact subset of Z, and $0 \in f(x, x, t)$;*
(ii) *for any fixed $y \in K$ and $t \in D$, $f(x, y, t)$ is C_y-concave in x; for any fixed $x \in K$, $f(x, y, t)$ is upper semicontinuous in (y, t);*
(iii) *T is upper semicontinuous and for each $x \in K$, $T(x)$ is a nonempty compact subset of D;*
(iv) *the set-valued mapping $W : K \rightrightarrows Z$ defined by $W(x) := Z \backslash \text{int}\, C(x)$, $\forall x \in K$, is closed;*
(v) *there exist a nonempty compact subset M of K and an $\bar{x} \in M$ such that, $\forall x \in K \backslash M$, $f(\bar{x}, x, T(x)) \subset \text{int}\, C(x)$.*

Then (GVEP 1) is solvable.

Proof. Define $F : K \rightrightarrows K$ by

$$F(x) = \{y \in K : f(x, y, T(y)) \cap (Z \backslash \text{int}\, C(y)) \neq \varnothing\}, \quad \forall x \in K.$$

(i) $\forall x \in K$, $F(x)$ is closed in K;

In fact, let a net $\{y_\lambda\} \subset F(x)$ be such that $y_\lambda \to y \in K$. We need to show $y \in F(x)$. Since $y_\lambda \in F(x)$, we have

$$f(x, y_\lambda, T(y_\lambda)) \cap (Z \backslash \text{int}\, C(y_\lambda)) \neq \varnothing.$$

Then for each λ, $\exists v_\lambda \in T(y_\lambda)$ such that

$$f(x, y_\lambda, v_\lambda) \cap (Z \backslash \text{int}\, C(y_\lambda)) \neq \varnothing.$$

Therefore, for each λ, $\exists w_\lambda \in f(x, y_\lambda, v_\lambda)$ such that $w_\lambda \in Z \backslash \text{int}\, C(y_\lambda)$. Since T is upper semicontinuous and $v_\lambda \in T(y_\lambda)$, by Lemma 2, there exist $v \in T(y)$ and a subnet $\{v_{\lambda'}\}$ of $\{v_\lambda\}$ such that $v_{\lambda'} \to v$. Since $f(x, y, v)$ is upper semicontinuous in (y, v), by Lemma 2, there exist $w \in f(x, y, v)$ and a subnet $\{w_{\lambda''}\}$ of $\{w_{\lambda'}\}$ such that $w_{\lambda''} \to w$. Hence $w \in f(x, y, T(y))$. Since $G_r(W)$ is closed and $(y_{\lambda''}, w_{\lambda''}) \to (y, w)$, we have $w \in Z \backslash \text{int}\, C(y)$, i.e., $f(x, y, T(y)) \cap (Z \backslash \text{int}\, C(y)) \neq \varnothing$. Thus $y \in F(x)$.

(ii) F is a KKM-map;

If it is false, then there exist $x_1, \ldots, x_n \in K$ and $\alpha_i > 0$, $\sum_{i=1}^{n} \alpha_i = 1$, $\hat{x} = \sum_{i=1}^{n} \alpha_i x_i$ such that $\hat{x} \notin \bigcup_{i=1}^{n} F(x_i)$. Then, $\forall i$, $\hat{x} \notin F(x_i)$, i.e.,

$$f(x_i, \hat{x}, T(\hat{x})) \subset \text{int}\, C(\hat{x}), \quad i = 1, 2, \ldots, n.$$

For each $t \in T(\hat{x})$,

$$f(x_i, \hat{x}, t) \subset \operatorname{int} C(\hat{x}), \quad i = 1, \ldots, n. \tag{4}$$

Since $f(x, y, t)$ is C_y-concave in x and $\operatorname{int} C(x)$ is convex, by (4),

$$0 \in f(\hat{x}, \hat{x}, t) \subset \sum_{i=1}^{n} \alpha_i f(x_i, \hat{x}, t) + C(\hat{x}) \subset \operatorname{int} C(\hat{x}) + C(\hat{x}) \subset \operatorname{int} C(\hat{x}),$$

a contradiction to $C(\hat{x}) \neq Z$.

(iii) By the condition (v), $F(\bar{x}) \subset M$. Since $F(\bar{x})$ is closed and M is compact, $F(\bar{x})$ is compact. It follows from Theorem 2 that $\bigcap_{x \in K} F(x) \neq \emptyset$. If $\bar{y} \in \bigcap_{x \in K} F(x)$, then $\bar{y} \in K$ such that, $\forall x \in K$, $f(x, \bar{y}, T(\bar{y})) \cap (Z \backslash \operatorname{int} C(\bar{y})) \neq \emptyset$. Therefore, $\exists v \in T(\bar{y})$ such that $f(x, \bar{y}, v) \not\subset \operatorname{int} C(\bar{y})$, i.e., $\bar{y}$ is a solution of (GVEP 1). □

Corollary 5. *Let $f : K \times K \times D \to Z$ be a gingle valued mapping in Theorem 5. Assume that conditions (iii), (iv) and (v) in Theorem 5 hold. The conditions (i) and (ii) in Theorem 5 are replaced with the following*

(i)′ $\forall x \in K$, $t \in D$, $f(x, x, t) = 0$;
(ii)′ *for any fixed $y \in K$ and $t \in D$, $f(x, y, t)$ is C_y-concave in x; for any fixed $x \in K$, $f(x, y, t)$ is continuous in (y, t);*

Then there is a $y \in K$ such that, $\forall x \in K$, $\exists v \in T(y)$, $f(x, y, v) \notin \operatorname{int} C(y)$.

Corollary 6. *Let X and Z be real Banach spaces, and K be a nonempty convex subset of X. Let $C : K \rightrightarrows Z$ be as in Theorem 5, and $T : K \rightrightarrows L(X, Z)$ be a set-valued mapping with nonempty compact values. Assume that the following conditions hold:*

(i) *T is upper semicontinuous;*
(ii) *the set-valued mapping $W : K \rightrightarrows Z$ defined by $W(x) := Z \backslash \operatorname{int} C(x)$, $\forall x \in K$, has a weakly closed graph $G_r(W)$ in $X \times Z$;*
(iii) *there exist a nonempty, weakly compact subset M of K and an $\bar{x} \in M$ such that, $\forall x \in K \backslash M$, $\bigcup_{t \in T(x)} (t, x - \bar{x}) = (T(x), x - \bar{x}) \subset \operatorname{int} C(x)$.*

Then there is $y \in K$ such that, $\forall x \in K$, $\exists t \in T(y)$, $(t, y - x) \notin \operatorname{int} C(y)$.

Proof. In Corollary 5, let $Y = D = L(X, Z)$ and $f(x, y, t) = (t, y - x)$, $\forall x$, $y \in K$, $t \in D$. Let X and Z be endowed with their weak topologies. We shall show that $f(x, y, t)$ is weakly continuous in (y, t).

Indeed, let a net $\{y_\lambda\} \subset K$ and a net $\{t_\lambda\} \subset D$ be such that $y_\lambda \rightharpoonup y \in K$ and $t_\lambda \to t \in D$. We need to show $f(x, y_\lambda, t_\lambda) \rightharpoonup f(x, y, t)$. Since $f(x, y_\lambda, t_\lambda) = (t_\lambda, y_\lambda - x) = (t_\lambda - t, y_\lambda - x) + (t, y_\lambda - x)$ and $\{y_\lambda\}$ is bounded in the norm topoloty of X, we have

$$\|(t_\lambda - t, y_\lambda - x)\| \leq \|t_\lambda - t\| \cdot \|y_\lambda - x\| \to 0,$$

$$\text{i.e., } (t_\lambda - t, y_\lambda - x) \to 0.$$

Since $t \in L(X,Z)$ and t is continuous from the weak topology of X to the weak topology of Z, we have

$$(t, y_\lambda - x) \rightharpoonup (t, y - x).$$

Hence,

$$f(x, y_\lambda, t_\lambda) = (t_\lambda - t, y_\lambda - x) + (t, y_\lambda - x) \rightharpoonup (t, y - x) = f(x, y, t).$$

Corollary 5 yields the desired result. □

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