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A comparison result for FBSDE with applications to decisions theory

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Abstract. In general, a comparison Lemma for the solutions of Forward-Backward Stochastic Differential Equations (FBSDE) does not hold. Here we prove one for the backward component at the initial time, relying on certain monotonicity conditions on the coefficients of both components. Such a result is useful in applications. Indeed, one can use FBSDE's to define a utility functional able to capture the disappointment-anticipation effect for an agent in an intertemporal setting under risk. Exploiting our comparison result, we prove some "desirable" properties for the utility functional, such as continuity, concavity, monotonicity and risk aversion. Finally, for completeness, in a Markovian setting, we characterize the utility process by means of a degenerate parabolic partial differential equation.

Key words: Forward-Backward SDE's, Comparison Theorem, Utility, Habit

Classification: (AMS 2000) 60H, 90A10 – (JEL 1999) C61, D11, D81

1 Introduction

In recent years the interest for the Forward-Backward SDE's (FBSDE's) of the type

$$X_t = x + \int_0^t b(s, X_s, Y_s)\, ds + \int_0^t \sigma(s, X_s, Y_s) \cdot dW_s, \quad X \in \mathbb{R}^n$$

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$$Y_t = E\left(\Gamma + \int_t^T f(s, X_s, Y_s)\,ds \,|\, \mathscr{F}_t\right), \quad Z \in \mathbb{R}^m,$$

has been steadily growing, due to the mathematical questions they pose, their relation to quasilinear parabolic PDE's and their applications to Finance.

Applications often clash against two intrinsic difficulties of these equations:

- the solution does not necessarily exist over arbitrary time intervals;
- there is no explicit representation formula of the solution (X, Y) for the linear case and, consequently, no general comparison lemma.

The first problem has been addressed in several papers since 1993 ([A], [Ha], [HP], [MPY], [PT], [Hu]). In [A] existence and uniqueness is given for "small" time intervals, in the others over arbitrary ones. Three of them are based on some monotonicity conditions of the coefficients, which unfortunately may limit the models one wants to consider, [MPY] and [Hu] rely instead on the correspondence between FBSDE's and quasilinear parabolic PDE's. To employ this technique efficiently, a nondegeneracy condition of the diffusion coefficient is needed and this is not always verified in applications.

The second problem, to our knowledge, was addressed only in [MY] and in [Wu]. In the first, the authors prove a comparison result in a very particular setting and give a counterexample in the general one. In the second, the author proves a comparison result for the backward component at time 0, exploiting monotonicity conditions very similar to those of [HP] and allowing only for increments of the initial and the final conditions of the FBSDE. Here we prove a slightly more general result, when $n = m = 1$, that allows also increments in the coefficients, based on monotonicity conditions different from Wu's. Indeed, this result is motivated by the *Forward Backward Stochastic Differential Utility* (FBSDU) introduced in [ABM] to represent agent's preferences under risk. The FBSDU, associated with a consumption process c, is defined as the initial solution $(U(\Gamma, c) = V_0)$ of the forward-backward system:

$$H_t = y_0 + \int_0^t [g(s, c_s, V_s) - \alpha_s H_s]\,ds, \tag{1}$$

$$V_t = E\left(\Gamma + \int_t^T [u(s, c_s, H_s) - \beta_s V_s]\,ds \,|\, \mathscr{F}_t\right), \tag{2}$$

where u denotes the istantaneous utility function and c_s the consumption process. In order to establish some useful properties of this functional, as in [DuE] for the pure backward case, a fairly general comparison result is needed. The interest in such a functional lies in the fact that it may model the disappointment-anticipation effect well documented and theorized in the decision theory literature (see [B], [LS], [Lo], [LoP]), due to the fact that agent's tastes are affected by what s/he expects for the future, creating either *disappointment* or *anticipation*. High expectation in the past in terms of utility may induce disappointment towards the current outcome, viewed as not as good as expected, or an "optimistic" anticipation of future utility and therefore a high satisfaction from past consumption. We will extensively discuss the economic interpretation of the above utility functional below.

In the next section we prove the comparison theorem, in section 3 we introduce the FBSDU and, after showing the existence and uniqueness of the solution for (1)–(2), we employ the result with appropriate hypotheses on u and g to explore some "desirable" properties of the utility functional, such as continuity, monotonicity and concavity w.r.t. consumption and the final condition and risk aversion. Finally, in Section 4 employing a by now classical technique, we construct a degenerate parabolic Partial Differential Equation (PDE), whose viscosity solution characterizes the utility process.

2 A comparison result

Let $[0, T]$ be a finite time interval and $(\Omega, \mathscr{F}, P)$ a complete probability space, on which a standard Brownian motion W is defined. We endow the probability space with the filtration $\{\mathscr{F}_t : t \geq 0\}$, generated by W, made right continuous and augmented of the P-null sets, so to verify the "usual hypotheses". Besides we assume $\mathscr{F}_0$ to be trivial.

Let us consider the following Forward-Backward system in $\mathbb{R} \times \mathbb{R}$

$$X_t = x + \int_0^t b(s, X_s, Y_s)\, ds + \int_0^t \sigma(s, X_s, Y_s)\, dW_s \tag{3}$$

$$Y_t = E\left(\Gamma + \int_t^T f(s, X_s, Y_s)\, ds \,|\, \mathscr{F}_t\right), \quad \text{where} \tag{4}$$

(H) $x \in \mathbb{R}$, $\Gamma \in L^2(P)$ is an $\mathscr{F}_T$-measurable random variable, $b, \sigma, f : \Omega \times [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ are adapted, progressively measurable processes globally Lipschitz in the spatial variables with constant k (uniformly in s, ω) and such that

$$E\left(\int_0^T [|f(s,0,0)|^2 + |b(s,0,0)|^2 + |\sigma(s,0,0)|^2]\, ds\right) < +\infty.$$

In the following, $\underline{\underline{S}}^2$ denotes the space of semimartingales such that $E\left(\sup_{t\in[0,T]} |X_t|^2\right) < +\infty$. We assume that on $[0, T]$, for any given x, Γ, b, f, σ satisfying (H), there exists a unique pair of adapted processes in $\underline{\underline{S}}^2 \times \underline{\underline{S}}^2$ (or $L^2(dt \times dP) \times L^2(dt \times dP)$) solution of (3)–(4).

As mentioned in the introduction, in the literature no explicit representation formula for solutions of linear FBSDE's is given, differently from what happens in either the forward or the backward case. In [MY], the authors characterized those by means of a Riccati type equation, but this is unfortunately still far from an explicit formula. The problem lies in the fact that, when it exists, at each t the solution depends upon its whole trajectory. Therefore any iterative procedure fails and it is hard to derive a version of Gronwall's inequality to employ for a general comparison lemma. In [MY, pag. 24] an example where the comparison result does not hold is provided. Nonetheless, in applications (option pricing, utility theory) it is often the initial value of Y to be of interest. For this, under some conditions, a comparison result can be maintained, as also shown in [Wu]. As we explained before, this result, even though in this line, lacks some flexibility desirable for applications. Hence we prove

Theorem 2.1: *Let $\Gamma_1, x_1, b^1, f^1, \sigma$ and $\Gamma_2, x_2, b^2, f^2, \sigma$ satisfy (H) and be such that*

$$\Gamma_1 \le \Gamma_2, \quad f^1(s,x,y) \le f^2(s,x,y) \text{ a.s. all } s,x,y.$$

Besides a.s. all s, x, y, one of the following holds

(a) $x_1 \le x_2$, $b^1(s,x,y) \le b^2(s,x,y)$, *$f^1$ is increasing in x, b^1 is decreasing in y and σ increasing (decreasing) in x and y;*
(b) $x_1 \ge x_2$, $b^1(s,x,y) \ge b^2(s,x,y)$, *$f^1$ is decreasing in x, b^1 is increasing in y and σ increasing (decreasing) in x and decreasing (increasing) in y.*

If we denote by X^1, Y^1 and X^2, Y^2 the corresponding solutions of (3)–(4), then $Y_0^1 \le Y_0^2$.

Proof: Let $\hat{X} = X^2 - X^1$, $\hat{Y} = Y^2 - Y^1$, then we have

$$\hat{X}_t = \hat{x} + \int_0^t [B_s + \alpha_s \hat{X}_s + \beta_s \hat{Y}_s]\, ds + \int_0^t (\phi_s \hat{X}_s + \psi_s \hat{Y}_s)\, dW_s$$

$$\hat{Y}_t = E\left(\hat{\Gamma} + \int_t^T [F_s + \gamma_s \hat{X}_s + \eta_s \hat{Y}_s]\, ds \,|\, \mathscr{F}_t\right),$$

where

$$B_s = b^2(s, X_s^2, Y_s^2) - b^1(s, X_s^2, Y_s^2) \ge 0,$$

$$F_s = f^2(s, X_s^2, Y_s^2) - f^1(s, X_s^2, Y_s^2) \ge 0$$

$$\alpha_s = \frac{b^1(s, X_s^2, Y_s^2) - b^1(s, X_s^1, Y_s^2)}{X_s^2 - X_s^1} \mathbf{1}_{\{X_s^2 \ne X_s^1\}},$$

$$\beta_s = \frac{b^1(s, X_s^1, Y_s^2) - b^1(s, X_s^1, Y_s^1)}{Y_s^2 - Y_s^1} \mathbf{1}_{\{Y_s^2 \ne Y_s^1\}}$$

$$\gamma_s = \frac{f^1(s, X_s^2, Y_s^2) - f^1(s, X_s^1, Y_s^2)}{X_s^2 - X_s^1} \mathbf{1}_{\{X_s^2 \ne X_s^1\}},$$

$$\eta_s = \frac{f^1(s, X_s^1, Y_s^2) - f^1(s, X_s^1, Y_s^1)}{Y_s^2 - Y_s^1} \mathbf{1}_{\{Y_s^2 \ne Y_s^1\}}$$

$$\phi_s = \frac{\sigma(s, X_s^2, Y_s^2) - \sigma(s, X_s^1, Y_s^2)}{X_s^2 - X_s^1} \mathbf{1}_{\{X_s^2 \ne X_s^1\}},$$

$$\psi_s = \frac{\sigma(s, X_s^1, Y_s^2) - \sigma(s, X_s^1, Y_s^1)}{Y_s^2 - Y_s^1} \mathbf{1}_{\{Y_s^2 \ne Y_s^1\}}.$$

The last six terms are well defined, because of the global Lipschitz property of the coefficients.

We want to show that $\hat{Y}_0 \ge 0$. By contradiction, let us assume the opposite and define the stopping time $\tau = \inf\{t > 0 : \hat{Y}_t \ge 0\} \wedge T$. We supposed $\mathscr{F}_0$ is trivial, thus $P(\tau > 0)$ is either 0 or 1. Since the underlying filtration is generated by the Brownian motion, $\hat{Y}$ is a continuous process with $\hat{Y}_0 < 0$ and this implies that a.s. $\tau > 0$. Again by the continuity of paths, we may conclude that $\hat{Y}_t < 0$ on $[0, \tau)$ and $\hat{Y}_\tau = 0$. The processes are all adapted and square integrable, so $\hat{Y}$ may be written as

$$\hat{Y}_t = E\left(\hat{\Gamma} + \int_0^T [F_s + \gamma_s \hat{X}_s + \eta_s \hat{Y}_s]\, ds \,|\, \mathscr{F}_t\right) - \int_0^t [F_s + \gamma_s \hat{X}_s + \eta_s \hat{Y}_s]\, ds$$

$$= M_t - \int_0^t [F_s + \gamma_s \hat{X}_s + \eta_s \hat{Y}_s]\, ds,$$

where M is a square integrable martingale. Evaluating at the stopping time τ, taking the conditional expectation and recalling $\hat{Y}_\tau = 0$, we have

$$0 > \hat{Y}_t \mathbf{1}_{\{t<\tau\}} = E\left(\int_t^T [F_s + \gamma_s \hat{X}_s + \eta_s \hat{Y}_s] \mathbf{1}_{\{s<\tau\}}\, ds \,|\, \mathscr{F}_t\right) \mathbf{1}_{\{t<\tau\}},$$

which gives (for instance by iterations)

$$\hat{Y}_t \mathbf{1}_{\{t<\tau\}} = E\left(\int_t^T \mathrm{e}^{\int_t^s \eta_r dr} [F_s + \gamma_s \hat{X}_s] \mathbf{1}_{\{s<\tau\}}\, ds \,|\, \mathscr{F}_t\right).$$

On the other hand, the forward component verifies

$$\hat{X}_t = \hat{x} + \int_0^t [B_s + \alpha_s \hat{X}_s + \beta_s \hat{Y}_s]\, ds + \int_0^t [\phi_s \hat{X}_s + \psi_s \hat{Y}_s]\, dW_s$$

$$= \mathscr{E}(S)_t \left\{\hat{x} + \int_0^t \mathscr{E}(S)_s^{-1} [B_s + (\beta_s - \phi_s \psi_s) \hat{Y}_s]\, ds + \int_0^t \mathscr{E}(S)_s^{-1} \psi_s \hat{Y}_s\, dW_s\right\}$$

$$= \mathscr{E}(S)_t \{\hat{x} + A_t + N_t\}$$

$$dS_t = \alpha_t\, dt + \phi_t\, dW_t,$$

where $\mathscr{E}$ denotes the stochastic exponential of a process and A and N denote the finite variation part and the martingale part (with zero mean) of the process between braces. Substituting in $\hat{Y}$ and applying the optional sampling theorem to the finite variation process $dC_s = \mathrm{e}^{\int_t^s \eta_r dr} \gamma_s \mathscr{E}(S)_s\, ds$, we obtain

$$\hat{Y}_t \mathbf{1}_{\{t<\tau\}} = E\left(\int_t^T \mathrm{e}^{\int_0^s \eta_r dr} [F_s + \gamma_s \mathscr{E}(S)_s \{\hat{x} + A_s + N_s\}] \mathbf{1}_{\{s<\tau\}}\, ds \,|\, \mathscr{F}_t\right)$$

$$= E\left(\int_t^T \mathrm{e}^{\int_t^s \eta_r dr} [F_s + \gamma_s \mathscr{E}(S)_s \{\hat{x} + A_s\}]\, ds \mathbf{1}_{\{s<\tau\}}\, ds \,|\, \mathscr{F}_t\right)$$

$$+ N_t E\left(\int_t^T \mathrm{e}^{\int_t^s \eta_r dr} \gamma_s \mathscr{E}(S)_s \mathbf{1}_{\{s<\tau\}}\, ds \,|\, \mathscr{F}_t\right) \mathbf{1}_{\{t<\tau\}}.$$

If condition (a) holds, then we have that $\hat{x} \geq 0$, $B_s \geq 0$, $\gamma_s \geq 0$, $\beta_s \leq 0$ and $\phi_s \psi_s \geq 0$ a.s. all s, because of the monotonicity of the various coefficients. This implies that on $[0, \tau)$, $B_s + (\beta_s - \phi_s \psi_s)\hat{Y}_s \geq 0$ a.s. and the same happens for A_s because of the positivity of the exponential.

In conclusion, on $[0, \tau)$ the first conditional expectation in the expression of $\hat{Y}$ gives a positive contribution. When considering $\hat{Y}_0$, we have $N_0 = 0$, hence

$$0 > \hat{Y}_0 = E\left(\int_0^T \mathrm{e}^{\int_0^s \eta_r dr} [F_s + \gamma_s \mathscr{E}(S)_s \{\hat{x} + A_s\}] \, ds \mathbf{1}_{\{s<\tau\}} \, ds \right) \geq 0$$

and we arrive at a contradiction on the set of positive probability $\{t < \tau\}$. We can argue similarly if (b) holds and reach the same contradiction. □

3 Forward-backward stochastic differential utility

Let $\mathscr{L}^2 = \{X : X$ is a predictable process such that $E(\int_0^T |X_s|^2 \, ds) < +\infty\}$, and $\mathscr{L}_+^2$ the space of $\mathscr{L}^2$ processes with values in $\mathbb{R}_+ = [0, +\infty)$. Given a consumption process $c \in \mathscr{L}_+^2$, we consider the system:

$$H_t = y + \int_0^t [g(s, c_s, V_s) - \alpha_s H_s] \, ds \tag{5}$$

$$V_t = E\left(\Gamma + \int_t^T [u(s, c_s, H_s) - \beta_s V_s] \, ds \,|\, \mathscr{F}_t \right), \tag{6}$$

otherwise written

$$H_t = \mathrm{e}^{-\int_0^t \alpha_u du} y + \int_0^t \mathrm{e}^{-\int_s^t \alpha_u du} g(s, c_s, V_s) \, ds$$

$$V_t = E\left(\mathrm{e}^{-\int_t^T \beta_u du} \Gamma + \int_t^T \mathrm{e}^{-\int_s^t \beta_u du} u(s, c_s, H_s) \, ds \,|\, \mathscr{F}_t \right).$$

The following Assumption holds.

Assumption 3.1:

A. *α and β are continuous adapted processes, bounded by a constant $M > 0$;*
B. *$u, g : \Omega \times [0, T] \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are adapted processes, so that for some constant $k > 0$*

$$|g(s, c, v_2) - g(s, c, v_1)| \leq k|v_2 - v_1|,$$

$$|u(s, c, h_2) - u(s, c, h_1)| \leq k|h_2 - h_1|$$

for a.e. ω and for all $(s, c) \in [0, T] \times \mathbb{R}_+$, $h_1, h_2, v_2, v_1 \in \mathbb{R}$;

C. $E(\int_0^T (|g(s, c_s, 0)|^2 + |u(s, c_s, 0)|^2)\, ds) < +\infty$, *for any* $c \in \mathscr{L}_+^2$;
D. u *is increasing in* c.

The component H_t is the agent's *habit process*, with y as the initial standard of living, while V_t is the utility process, with Γ representing utility at time T. We refer to $U(\Gamma, c) = V_0$ as the *Forward-Backward Stochastic Differential Utility*. If the above utility functional represents the agent's preferences over consumption streams, then U can be interpreted as a measure of the level of satisfaction for the agent.

The representation of agents' preferences under risk is an active field of research in economic theory. The reference framework is provided by the *additive expected utility*: if the agent's tastes do not change over time and they satisfy some axioms, then preferences can be represented as the expectation of a (utility) functional of consumption. This approach was challenged by other theories of decision making under risk, as some experimental results showed that agents do not behave according to the above hypothesis (e.g. the Allais and the Ellsberg paradoxes). For a survey on these topics we refer the reader to [E, C].

In an intertemporal setting, one of the key points is that history affects agent's tastes. This fact was modeled in many ways; among them we recall the habit persistence effect analyzed in [Con, DZ1]. In this case (which means g independent of h in (5)–(6)), istantaneous utility is increasing in consumption and decreasing in a process (the habit) often described as a time average of past consumption. As a consequence, the higher the standard of living is, the lower the istantaneous utility from consumption results. In some works it was recognized that agent's tastes are affected also by what s/he expects for the future in two different and opposite directions: *disappointment* or *anticipation*. In [B, LS], the authors point out that an agent may experience disappointment-elation comparing an outcome with his past expectation on it: if the expectation was high, then s/he will be disappointed when the outcome is not as good as expected, the opposite happens when the outcome is better than expected. In [Lo, LoP], the authors remark that agents may instead anticipate future consumption-utility: the expectation of future utility positively affects current utility. The FBSDU captures these features in a setting well suited for asset pricing applications (continuous time with a flow of information described by a Brownian motion).

We remark that our functional does not belong to the class of the stochastic differential utility, introduced in [DuE]. The coefficient g represents the contribution of consumption and expected utility on the habit. If g is independent of V and increasing in c, then the system is decoupled and H reduces to the classical habit process (see [Con], [DZ1]). Allowing the habit to depend on the utility process implies that the standard of living is influenced by the past experienced expected utility. If g is increasing in v and u is decreasing in h, then the agent's instantaneous utility is negatively affected by what s/he expected in the past about the future, capturing a disappointment effect. Instead, u increasing in y models anticipation. High expected utility in the past generates a "positive" expectation for the future and the agent is inclined to appreciate the actual consumption rate. This interpretation is made clear by the following example.

Example 3.2: *Consider the Additive Expected Utility (AEU)* $\hat{U}(c) = E(\int_0^T e^{-\beta s} u(c_s)\, ds)$ *with instantaneous utility* u *and the following binary choice*

$$c_s^a = \underline{c}\ \forall s \in [0,T], \quad c_s^b = \begin{cases} 0 & s < t \\ \bar{c} & t \le s \le T \text{ with probability } \pi \\ 0 & t \le s \le T \text{ with probability } 1-\pi \end{cases}$$

for some fixed t and constants $\bar{c}$ and $\underline{c}$. We assume $u(0) = 0$ and we define $\underline{c}$ and $\bar{c}$ so that c^a and c^b are ordinally equivalent under the AEU, that is

$$\int_0^T e^{-\beta s} u(\underline{c})\, ds = \pi \int_t^T e^{-\beta s} u(\bar{c})\, ds. \tag{7}$$

A disappointment effect would imply c^a is better than c^b, anticipation the opposite. One can represent these rankings by considering a linear FBSDU with the same u and $y = 0$:

$$V_t = E\left(\int_t^T [u(c_s) - \gamma H_s - V_s]\, ds \,|\, \mathscr{F}_t \right), \quad \gamma \in (-1, 1)$$

$$H_t = \int_0^t [V_s - H_s]\, ds.$$

We can show that $\gamma > 0$ models a disappointment effect and $\gamma < 0$ an anticipation effect.

One can show (see [ABM]) that when the solution exists, then

$$U(c) = V_0 = E\left(\int_0^T \frac{e_{11}^{As} e_{22}^{AT} - e_{12}^{AT} e_{21}^{As}}{e_{22}^{AT}} u(c_s)\, ds \right) = E\left(\int_0^T \lambda_s u(c_s)\, ds \right), \tag{8}$$

where $A = \begin{pmatrix} -1 & -\gamma \\ -1 & 1 \end{pmatrix}$ with real eigenvalues $\pm\sqrt{1+\gamma}$ and e_{ij}^{As} is the ij-th element of the exponential of A. Consequently $\lambda_s = \dfrac{\sqrt{1+\gamma}\cosh(\sqrt{1+\gamma}(T-s)) + \sinh(\sqrt{1+\gamma}(T-s))}{\sqrt{1+\gamma}\cosh(\sqrt{1+\gamma}T) + \sinh(\sqrt{1+\gamma}T)}$. Note that $\lambda_s < e^{-s} \Leftrightarrow \gamma > 0$. It can be shown that

$$\int_0^T \lambda_s u(\underline{c})\, ds > \pi \int_t^T \lambda_s u(\bar{c})\, ds \Leftrightarrow \gamma > 0 \quad (\text{equality for } \gamma = 0), \tag{9}$$

by dividing (9) by (7) and proving analitically that $\int_0^T \lambda_s\, ds (e^{-t} - e^{-T}) > \int_t^T \lambda_s\, ds (1 - e^{-T})$.

Even though c^a and c^b are equivalent under the AEU, they no longer are under the FBSDU: c^a is better than c^b if and only if $\gamma > 0$. Note that this preferences order cannot be obtained with the classical habit formation process.

As well known, Lipschitz coefficients are not sufficient to have the existence and uniqueness of the solution of FBSDE's, unless the system is decoupled (g independent of v). To ensure these for (5)–(6), we use the results in [A], based on a restriction of the time interval.

Proposition 3.3: *Let Assumption 3.1 hold and set* $K = \max\{k, M\}$.

If $\sqrt{8}KT < 1$*, then there exists a unique pair* (H, V) *in* $\underline{\underline{S}}^2 \times \underline{\underline{S}}^2$ *satisfying (5)–(6).*

Proof: Under Assumption 3.1, the operator

$$L\begin{pmatrix} H_t \\ V_t \end{pmatrix} = \begin{pmatrix} y_0 + \int_0^t [g(s, c_s, V_s) - \alpha_s H_s]\, ds \\ E\left(\Gamma + \int_t^T [u(s, c_s, H_s) - \beta_s V_s]\, ds \,|\, \mathscr{F}_t\right) \end{pmatrix}$$

goes from $\underline{\underline{S}}^2 \times \underline{\underline{S}}^2$ into itself. If $\sqrt{8}KT < 1$, L acts as a contraction on $\underline{\underline{S}}^2 \times \underline{\underline{S}}^2$, which is a Banach space and hence it identifies a unique fixed point. For more details we refer the reader to [A]. □

Remark 3.4: *The time restriction invoked in Proposition 3.3 is certainly an undesirable feature, but it does not seem possible to circumvent it at least in this context. For examples explaining the necessity of this, we refer the reader to [A]. Methods to avoid this restriction have been devised in [MPY], [Hu], [HP], [Ha] and [PT], that are either based on the degeneracy of the diffusion coefficient of the forward components or on various monotonicity conditions of all the coefficients. Unfortunately the nondegeneracy condition is not fulfilled in our model and the monotonicity conditions required for the existence of the solution and for the comparison result in [Wu] do not match ours and, for instance, exclude some linear cases that may be interesting in our setting.*

Once established the existence of the utility process, we want to show some "desirable" properties for $U(\Gamma, c) = V_0$. Following [DuE], we focus on

- continuity with respect to Γ and c;
- concavity with respect to c;
- monotonicity in Γ and c;
- risk aversion.

We complete Assumption 3.1 with the following

Assumption 3.5:

(i) *u and g are continuous in c and such that for a.e.* ω *and all* s, h, v

$$|u(s, c, h)|,\ |g(s, c, v)| \leq k(1 + |c|).$$

(ii) *One of the following holds:*
   a. *u is increasing in h and concave in the couple, while g is decreasing in v and concave in the couple, a.s. for all c and s,*
   b. *u is decreasing in h and concave in the couple, while g is increasing in v and convex in the couple, a.s. for all c and s.*

We remark that also in the proof of the first proposition, the restriction of the time interval plays a fundamental role. The other propositions are proved by using the result of the previous section.

Proposition 3.6: *Let Assumptions 3.1 and 3.5 (i) hold. The Forward-Backward Utility functional* $U : L^2 \times \mathbb{D} \to \mathbb{R}$, *where* $\mathbb{D}$ *denotes the space of càdlàg processes, is continuous.*

Proof: Consider the two solutions (H^1, V^1) and (H^2, V^2) of (5)–(6) associated respectively with (Γ_1, c^1) and $(\Gamma_2, c^2) \in L^2 \times \mathscr{L}^2_+$. Evaluating the differences, we obtain

$$|H_t^2 - H_t^1| \leq \int_0^t [|g(s, c_s^2, V_s^2) - g(s, c_s^1, V_s^1)| + |\alpha_s|\, |H_s^2 - H_s^1|]\, ds$$

$$|V_t^2 - V_t^1| \leq E\Bigg(|\Gamma_2 - \Gamma_1| + \int_t^T [|u(s, c_s^2, H_s^2) - u(s, c_s^1, H_s^1)| + |\beta_s|\, |V_s^2 - V_s^1|]\, ds \,|\, \mathscr{F}_t\Bigg).$$

The Lipschitz hypotheses on the coefficients and the definition of K imply

$$\begin{aligned}
&|H_t^2 - H_t^1| + |V_t^2 - V_t^1| \\
&\quad \leq E\Bigg(|\Gamma_2 - \Gamma_1| + K\int_0^T (|H_s^2 - H_s^1| + |V_s^2 - V_s^1|)\, ds \\
&\qquad + \int_0^T \{|g(s, c_s^2, V_s^1) - g(s, c_s^1, V_s^1)| + |u(s, c_s^2, H_s^1) \\
&\qquad - u(s, c_s^1, H_s^1)|\}\, ds \,|\, \mathscr{F}_t\Bigg).
\end{aligned}$$

By Cauchy-Schwarz inequality and Doob's martingale inequality we may conclude

$$\begin{aligned}
&E\Big(\sup_{0 \leq t \leq T} (|H_t^2 - H_t^1| + |V_t^2 - V_t^1|)^2\Big) \\
&\quad \leq 8K^2T^2 E\Big(\sup_{0 \leq t \leq T} (|H_t^2 - H_t^1| + |V_t^2 - V_t^1|)^2\Big) \\
&\qquad + 8E\Bigg([|\Gamma_2 - \Gamma_1| + \int_0^T (|g(s, c_s^2, V_s^1) - g(s, c_s^1, V_s^1)| + |u(s, c_s^2, H_s^1) \\
&\qquad - u(s, c_s^1, H_s^1)|)\, ds]^2\Bigg)
\end{aligned}$$

and recalling that $\sqrt{8}KT < 1$ we have

$$\|(H^2, V^2) - (H^1, V^1)\|^2_{\underline{\underline{S}}^2 \times \underline{\underline{S}}^2} \leq \frac{24}{1 - 8K^2T^2} \|\Gamma_2 - \Gamma_1\|_{L^2} + \frac{24T}{1 - 8K^2T^2} E \times \left(\int_0^T [|u(s, c_s^2, H_s^1) - u(s, c_s^1, H_s^1)|^2 + |g(s, c_s^2, V_s^1) - g(s, c_s^1, V_s^1)|^2] \, ds \right).$$

Of course, the above inequality holds also for $|V_0^2 - V_0^1|$. Chosen a sequence of $\mathscr{F}_T$-measurable random variables $\{\Gamma_n\}_n$ converging to Γ in L^2 and a sequence of processes $\{c^n\}_n$ converging to c in $\mathscr{L}_+^2$, from the above inequality we get

$$|U^n(\Gamma_n, c^n) - U(\Gamma, c)|^2 \leq \frac{24}{1 - 8K^2T^2} \|\Gamma_n - \Gamma\|_{L^2} + \frac{24T}{1 - 8K^2T^2} E \times \left(\int_0^T [|u(s, c_s^n, H_s) - u(s, c_s, H_s)|^2 + |g(s, c_s^n, V_s) - g(s, c_s, V_s)|^2] \, ds \right).$$

Convergence in $\mathbb{D}$ implies, along some subsequence of every subsequence, pointwise convergence a.e. on $\Omega \times [0, T]$, hence by the continuity of u and g with respect to Γ and c, we have the thesis, applying the dominated convergence theorem. □

To address optimal consumption problems and to carry out an equilibrium analysis with a FBSDU, it is useful to check the concavity of $U(\Gamma, c)$, with respect to the consumption process c. When considering the Backward SDU, this is readily obtained by assuming that u is concave in c (see [DuE]), the situation is more complex for a FBSDU and the second part of Assumption 3.5 allows us to employ Theorem 2.1.

Proposition 3.7: *Let assumptions 3.1 and 3.5 hold. Then for any $c^1, c^2 \in \mathscr{L}_+^2$ and any constant $\lambda \in [0, 1]$, we have $U(\cdot, \lambda c^1 + (1 - \lambda)c^2) \geq \lambda U(\cdot, c^1) + (1 - \lambda) U(\cdot, c^2)$.*

Proof: For (Γ, c^1), $(\Gamma, c^2) \in L^2(P) \times \mathscr{L}^2$, let (H^i, V^i), $i = 1, 2$, be the corresponding solutions given by Proposition 3.3 and let us set

$$c_t^\lambda = \lambda c_t^1 + (1 - \lambda) c_t^2, \quad H_t^\lambda = \lambda H_t^1 + (1 - \lambda) H_t^2,$$

$$V_t^\lambda = \lambda V_t^1 + (1 - \lambda) V_t^2,$$

$$f^1(s, y, v) = \lambda u(s, c_s^1, (1 - \lambda)(H_s^1 - H_s^2) + y) + (1 - \lambda) u(s, c_s^2, \lambda(H_s^2 - H_s^1) + y) - \beta_s v,$$

$$b^1(s,y,v) = \lambda g(s, c_s^1, (1-\lambda)(V_s^1 - V_s^2) + v) + (1-\lambda) g(s, c_s^2, \lambda(V_s^2 - V_s^1) + v) - \alpha_s y,$$

$$f^2(s,y,v) = u(s, c_s^\lambda, y) - \beta_s v, \quad b^2(s,y,v) = g(s, c_s^\lambda, v) - \alpha_s y.$$

With this notation, the couple (H^λ, V^λ) verifies the system

$$H_t^\lambda = y + \int_0^t b^1(s, H_s^\lambda, V_s^\lambda)\, ds$$

$$V_t^\lambda = E\left(\Gamma + \int_t^T f^1(s, H_s^\lambda, V_s^\lambda)\, ds \,|\, \mathscr{F}_t\right).$$

On the other hand, for T small enough, associated with (Γ, c^λ), there exists a unique pair of processes (X^λ, U^λ) solution of

$$X_t^\lambda = y + \int_0^t [g(s, c_s^\lambda, U_s^\lambda) - \alpha_s X_s^\lambda]\, ds = y + \int_0^t b^2(s, X_s^\lambda, U_s^\lambda)\, ds$$

$$U_t^\lambda = E\left(\Gamma + \int_t^T [u(s, c_s^\lambda, X_s^\lambda) - \beta U_s^\lambda]\, ds \,|\, \mathscr{F}_t\right)$$

$$= E\left(\Gamma + \int_t^T f^2(s, X_s^\lambda, U_s^\lambda)\, ds \,|\, \mathscr{F}_t\right).$$

We want to show that $V_0^\lambda \le U_0^\lambda$. We have that $f^1(s,y,v) \le f^2(s,y,v)$ a.s. all s, y, v, because of the concavity hypotheses on u. If part (ii) a. of assumption 3.5 holds, it is easy to verify that the processes $b^1(s,y,v), b^2(s,y,v)$ respond to assumption (a) of theorem 2.1. If instead part (ii) b. is valid then the above processes verify (b). □

At this point we can easily get the monotonicity in Γ and, with an extra hypotheses, in c and the risk aversion property (in the case of deterministic coefficients).

Proposition 3.8: *Let assumptions 3.1 and 3.5 hold. Then for any $\Gamma_1, \Gamma_2 \in L^2$ such that $\Gamma_2 \ge \Gamma_1$ a.e., we have $U(\Gamma_2, \cdot) \ge U(\Gamma_1, \cdot)$. Moreover if g is decreasing in c when u is decreasing in h or g is increasing in c when u is increasing in h then U is increasing w.r.t. the process c.*

Proof: We remark that here concavity/convexity of the coefficients is not needed, but it is only their monotonicity that is important.

Chosen $\Gamma_1, \Gamma_2 \in L^2$ and $c^1, c^2 \in \mathscr{L}_+^2$ such that $\Gamma_1 \le \Gamma_2$, $c_s^2 \le c_s^2$ a.s. all s, it suffices to apply theorem 2.1 to the random coefficients

$$f^1(s,y,v) = u(s, c_s^1, y) - \beta_s v, \quad f^2(s,y,v) = u(s, c_s^2, y) - \beta_s v,$$

$$b^1(s,y,v) = g(s, c_s^1, v) - \alpha_s y, \quad b^2(s,y,v) = g(s, c_s^2, v) - \alpha_s y.$$ □

We follow [DuE] defining an agent to be risk averse if s/he prefers to consume the expectation of a consumption process rather than the process. The following Proposition holds.

Proposition 3.9: *Let u, g, α, β be deterministic and let assumptions 3.1 and 3.5 hold. For any $c \in \mathscr{L}^2_+$, let us denote by $\bar{c}_t = E(c_t)$, then we have $U(\Gamma, c) \leq U(E(\Gamma), \bar{c})$.*

Proof: We set $\bar{c}_t = E(c_t)$, and denote by $(\overline{H}_t, \overline{V}_t)$ the solution associated with $\bar{c}$ and $E(\Gamma)$. This turns out to be deterministic, since no source of randomness occurs (another way to realize it is by the PDE representation we prove in Section 4). Taking the expectation of (H, V) we have

$$E(H_t) = y + \int_0^t [E(g(s, c_s, V_s - E(V_s) + E(V_s)) - \alpha_s E(H_s)] \, ds$$

$$E(V_t) = E(\Gamma) + \int_t^T [E(u(s, c_s, H_s - E(H_s) + E(H_s)) - \beta_s E(V_s)] \, ds.$$

Thus applying the main theorem to the functions

$$f^1(s, y, v) = E(u(s, c_s, H_s - E(H_s) + y)) - \beta_s v,$$

$$f^2(s, y, v) = u(s, \bar{c}_s, y) - \beta_s v$$

$$b^1(s, y, v) = E(g(s, c_s, V_s - E(V_s) + v)) - \alpha_s y,$$

$$b^2(s, y, v) = g(s, \bar{c}_s, v) - \alpha_s y$$

we obtain the statement. □

4 A partial differential equation characterization

As mentioned before, the literature does not provide any explicit formula for the solution of FBSDE's, but one can obtain a representation of the utility process by means of the viscosity solution of a PDE associated to the FBSDE, following the by now classical approach developed by [CM], [MPY], [PT] and others for the FBSDE case, and by [DuL] or [ElPQ] for the Backward case. For completeness we sketch it here in our case.

We assume $\Gamma = 0$ and that the consumption process c has dynamics

$$dc_t = \mu(t, c_t) \, dt + \sigma(t, c_t) \, dW_t, \quad c_0 = \gamma_0 > 0, \tag{10}$$

with μ and σ deterministic, continuous in t and globally Lipschitz in x with constant k_1. By Picard's iterations it is standard to prove the existence of a unique solution $c \in \underline{\underline{S}}^2 \subseteq \mathscr{L}^2$.

We need to specify a little further our hypotheses, so in place of Assumption 3.1 we use

Assumption 4.1:

(i) *β, α are deterministic, positive functions uniformly bounded by M.*

(ii) *The functions* $\mu, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ *are differentiable, with derivatives uniformly bounded by a constant* k_1*;*
(iii) u, g *are deterministic and Lipschitz with constant* k_2 *in the first variable and with constant* k *in the second, uniformly in* s.

As before, K denotes the $\max(k, M)$. First we show that the solution of (10), (5) and (6) exhibits continuous dependence on the parameters. To do so, we extend u and g to all $[0, T] \times \mathbb{R} \times \mathbb{R}$ by continuity and we take t, x, y varying in $[0, T] \times \mathbb{R} \times \mathbb{R}$. We consider the following flows associated with our equations

$$c_s^{t,x} = x + \int_t^s \mu(r, c_r^{t,x})\, dr + \int_0^t \sigma(r, c_r^{t,x})\, dW_r, \quad c_t^{t,x} = x \tag{11}$$

$$H_s^{t,x,y} = y + \int_t^s [g(r, c_r^{t,x}, V_r^{t,x,y}) - \alpha_r H_r^{t,x,y}]\, dr, \quad H_t^{t,x,y} = y \tag{12}$$

$$V_s^{t,x,y} = E\left(\int_s^T [u(r, c_r^{t,x}, H_r^{t,x,y} \right) - \beta_r V_r^{t,x,y}]\, dr + |\mathscr{F}_s). \tag{13}$$

For any fixed $t_1, t_2 \in [0, T]$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we denote

$$c_\cdot^i = c_{\cdot \vee t_i}^{t_i, x_i}, \; H_\cdot^i = H_{\cdot \vee t_i}^{t_i, x_i, y_i}, \; V_\cdot^i = V_{\cdot \vee t_i}^{t_i, x_i, y_i}, \quad i = 1, 2, \quad s \vee t = \max(s, t).$$

Proposition 4.2: *Under Assumption 4.1, the above flows are continuous in* t, x, y*. More specifically, for given* t_1 *and* x_1*, there exists a constant* C_1 *depending only on* k_1, T, t_1, x_1*,* $\mu(r, 0)$*,* $\sigma(r, 0)$*, such that*

$$E\left(\sup_{s \in [0,T]} |c_s^2 - c_s^1|^2 \right) \le C_1(|x_2 - x_1|^2 + |t_2 - t_1|). \tag{14}$$

Moreover, for given t_1, x_1 *and* y_1*, provided that* $\sqrt{8}K(K+1)T < 1$*, there exists a constant* C_2*, depending only on* $k, k_1, k_2, T, t_1, x_1, y_1$ *such that*

$$E\left(\sup_{s \in [0,T]} [|H_s^2 - H_s^1| + |V_s^2 - V_s^1|]^2 \right)$$
$$\le \frac{C_2(|x_2 - x_1|^2 + |y_2 - y_1|^2 + |t_2 - t_1|)}{1 - \sqrt{8}(K^2 + K)T}. \tag{15}$$

Proof: Assume $t_1 < t_2$, by the Lipschitz property of $\mu, \sigma(r, c_r^1)$, taking expectations and exploiting Doob's inequality we get

$$\|c^2 - c^1\|^2_{\underline{\underline{S}}^2_{[0,t]}} \le 5|x_2 - x_1|^2 + 5k_1^2(|t - t_2| + 1) \int_{t_2}^{t \vee t_2} E\left(\sup_{0 \le s \le r} |c_s^2 - c_s^1|^2 \right) dr$$
$$+ 5|t_2 - t_1|(1 + |t_2 - t_1|)$$
$$\times \left[\|c^1\|^2_{\underline{\underline{S}}^2_{[t_1,t_2]}} + \max_{0 \le r \le T} (|\mu(r, 0)|^2 + |\sigma(r, 0)|^2) \right].$$

Finally Gronwall's inequality gives (14). Using the martingale representation theorem and conditional expectation, similarly we can show, by means of Cauchy-Schwarz and Doob's inequalities, that for some constant C depending only on T, K, k_2,

$$\| |H^2 - H^1| + |V^2 - V^1| \|_{\underline{\underline{S}}^2}^2$$

$$\leq \frac{C^2}{1 - 8T^2K^2} \left\{ |y_2 - y_1|^2 + E\left(\int_{t_2}^{T} |c_r^2 - c_r^1|^2 \, dr \right) + |t_2 - t_1| E \right.$$

$$\left. \times \left(\int_{t_1}^{t_2} [|H_r^1|^2 + |V_r^1|^2 + |c_r^1|^2 + |u(r,0,0)|^2 + |g(r,0,0)|^2] \, dr \right) \right\}$$

which gives our thesis, by virtue of (14). □

The coefficients occurring in the previous equations are deterministic and differentiable. By the standard technique of time shift and Blumenthal's 0-1 law, one can show that

$$\gamma(t,x) = c_t^{t,x}, \quad \phi(t,x,y) = H_t^{t,x,y}, \quad \theta(t,x,y) = V_t^{t,x,y}$$

are all deterministic functions. Proposition 4.2 implies that these functions are locally Lipschitz in x, y and Hölder of order $\frac{1}{2}$ in t, consequently their derivatives are defined a.s. and bounded on compacts.

The solution of (5)–(6) may be characterized through the solution of a nonlinear degenerate parabolic PDE associated with the FBSDE (for instance see [MPY]).

We need to consider viscosity solutions instead of the classical ones, since the PDE is degenerate and we have no a-priori information on the regularity of the solution. For the definition of viscosity solution we refer the reader to [FL].

Theorem 4.3: *Under Assumptions 4.1, $\theta(t,x,y)$ is a viscosity solution of the PDE problem in $[0,T] \times \mathbb{R} \times \mathbb{R}$,*

$$\begin{cases} \dfrac{\partial \theta}{\partial t} + \dfrac{\sigma^2(t,x)}{2} \dfrac{\partial^2 \theta}{\partial x^2} + \mu(t,x) \dfrac{\partial \theta}{\partial x} + (g(t,x,\theta) - \alpha_t y) \dfrac{\partial \theta}{\partial y} \\ \qquad - u(t,x,y) + \beta_t \theta = 0 \\ \theta(T,x,y) = 0. \end{cases} \tag{16}$$

Proof: By construction, the processes $c_s^{t,x}, H_s^{t,x,y}$ and $V_s^{t,x,y}$ have continuous paths and are adapted with respect to the filtration generated by the Brownian motion. Therefore by the Markov property and the pathwise uniqueness of the solution, it is possible to show that actually $V_s^{t,x,y} = \theta(s, c_s^{t,x}, H_s^{t,x,y})$ a.s..

We need to show that θ is both a sub and a super-solution of (16). We show only the sub-solution inequality, since the proof of the other follows same lines.

Consider a point $(t, x, y) \in [0, T] \times \mathbb{R}^2 = \mathcal{O}$ and $\varphi \in \mathcal{C}^{1,2}(\bar{\mathcal{O}})$ such that $0 = \theta(t, x, y) - \varphi(t, x, y)$ is a global maximum for $\theta - \varphi$ (without loss of generality we can assume this maximum to be zero). This means that for any stopping time, necessarily

$$\theta(\tau, c_\tau^{t,x}, H_\tau^{t,x,y}) - \varphi(\tau, c_\tau^{t,x}, H_\tau^{t,x,y}) \leq 0. \tag{17}$$

Following the same lines as in [MY] we arrive at the inequality

$$E\left(\int_t^\tau \Sigma(r, c_r, H_r)\, dr\right) \leq 0,$$

$$\text{where } \Sigma(\cdot,\cdot,\cdot) = -\frac{\partial \varphi}{\partial t} + L(\cdot,\cdot,\cdot,\theta(\cdot,\cdot,\cdot),\varphi(\cdot,\cdot,\cdot)), \tag{18}$$

$$\begin{aligned} L(t, x, y, \theta(t, x, y), \varphi(t, x, y)) = {} & \frac{1}{2}\sigma^2(t, x)\frac{\partial^2 \varphi}{\partial x^2}(t, x, y) + \mu(t, x)\frac{\partial \varphi}{\partial x}(t, x, y) \\ & + (g(t, x, \theta(t, x, y)) - \alpha_t y)\frac{\partial \varphi}{\partial y}(t, x, y) \\ & - u(t, x, y) + \beta_t \theta(t, x, y). \end{aligned}$$

To affirm that θ is a subsolution of (16) we must verify that $\Sigma(t, x, y) \leq 0$. By contradiction, we assume there exists an $\varepsilon_0 > 0$ such that $\Sigma(t, x, y) > \varepsilon_0$ and we define the stopping time $\tau_1 = \inf\left\{s > t : \Sigma(s, c_s, H_s) \leq \frac{\varepsilon_0}{2}\right\} \wedge T$. Since $\Sigma(t, x, y) > \varepsilon_0$, we have $\tau_1 > t$ a.s. Inequality (18) holds for any stopping time, therefore also for τ_1 and we have

$$0 < \frac{\varepsilon_0}{2}(\tau_1 - t) < E\left(\int_t^{\tau_1} \Sigma(s, c_s, H_s)\, ds\right) \leq 0$$

which is a clear contradiction, hence we proved that θ is a subsolution of (16).

It is interesting to remark that this generalized Feynman-Kaĉ formula permits to affirm the existence of the viscosity solution of quasilinear parabolic PDE's of the type (16), at least for small time intervals.

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