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Ergodic behavior of a Markov chain model in a stochastic environment

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Abstract. This paper is concerned with the asymptotic behavior of a time dependent Markov model in a stochastic environment, with special relevance to manpower systems. The stochastic concept is established through the notion of optional scenarios applied on the transition process. A theorem is provided for the existence and determination of the limiting structure of the means, variances and covariances of numbers in the classes of the system. It is also proved that, under certain conditions, the rate of convergence is geometric.

Key words: Markov chain models; manpower planning; limiting distribution

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1 Introduction

We consider systems of the following structure: at any time t the members of the system can be classified into k classes on the basis of whatever attributes are relevant for the problem at hand. The classes are assumed to be exclusive and exhaustive while the time scale is discrete. A mobility pattern (model) is sought in order to describe the flows between internal classes and external environment over the course of time.

Several examples of such systems can be found in Manpower Planning, i.e. the discipline of Operations Research concerned with the description and forecasting of the behavior of groups of people. If this is the case, the classes of the system may be formed according to attributes such as grades, length of service, age, economic or social status, occupations, etc. The flows mentioned above are now interpreted as the recruitment of new members (input flows), promotion or demotion (internal flows) and retirement (wastage flows). Researchers used extensively the so called embedded or associated Markov chain model to describe the changing pattern of a manpower planning system [5], [6], [27],

[28]; this is due to the fact that there is an obvious correspondence between the classes of the system and the states of the chain.

Despite the stochastic nature of the process, its analysis has been mainly deterministic in the sense that average values are used in place of random variables. Bartholomew [2], [3], [4], [5] introduced some viewpoints about the consequences of the stochastic environment and his analysis was followed by Abdallaoui [1], Guerry [15] and Gerontidis [13]. More recently, Tsantas and Vassiliou [24], Tsantas [23], Vassiliou [26] and McClean [19] succeeded in constructing a general modelling framework with an inherent stochastic mechanism. Since their motivation came from a series of papers utilizing the well-known non homogeneous Markov system (**NHMS**), established by Vassiliou [25], the new model was called *the non homogeneous Markov system in a stochastic environment* (**S-NHMS**)".

The problem addressed in this paper is concerned with the asymptotic behavior of the S-NHMS. In Section 2 we give a brief review of the model used. In Section 3 we develop an asymptotic analysis for the expectations and the variances-covariances of the class sizes for the S-NHMS under the assumption that the parameters of the system converge to some fixed values. Section 4 examines the rate of convergence to the ergodic distribution and conditions are provided for the convergence to be geometrically fast. In Section 5 our interest is in finding the set of the asymptotic expectations which are possible, provided that we control the limiting recruitment flow to the system. Finally, in Section 6, some numerical examples highlight the practical aspects of the theoretical results. The results proved to be useful from the practical point of view since they provide valuable information about the inherent tendencies in the studied system.

2 The non homogeneous Markov system in a stochastic environment

In this section we define the parameters of the time dependent Markov model using to study a manpower system and we provide the basic recurrence equations describing the progress of its expected structures during the course of time.

Consider a manpower system and let $G = \{1, 2, \ldots, k\}$ be the set of its exclusive and exhaustive internal classes. Establish a discrete time scale $t = 0, 1, 2, \ldots$, and let the size of the j-th class at time t be $N_j(t)$. For $t > 0$ the class sizes are random variables and we shall be concerned mainly with their expectations. Following Bartholomew ([5] p. 51), these will be denoted by placing a bar over the symbol representing the random variable. Thus, at any time t, the row vector of the class expected levels $\overline{\mathbf{N}}(t) = [\overline{N}_1(t), \overline{N}_2(t), \ldots, \overline{N}_k(t)]$ provides a "snapshot" of the system's expected structure ([6] p. 3). Assume an interval of unit length from t to $t+1$; the t-th time interval. It is important to observe that flows relate to an interval and not a point. The internal dynamics of the system regulates a non homogeneous Markov chain with transition probability matrix $\mathbf{P}(t)$, where the element $p_{ij}(t)$ is the probability that a member in class i at the start of the t-th time interval is in class j at the end. Apart from internal mobility, outflows take place towards the external environment denoted by the hypothetical $k+1$ class. The associated vector of the wastage probabilities is $\mathbf{p}_{k+1}(t) = (p_{1,k+1}, p_{2,k+1}, \ldots, p_{k,k+1})$; $p_{i,k+1}$ being the probability that a member of class i at the start has left by the end of the interval. At each time point t the expected total size of the system $T(t) = \overline{\mathbf{N}}(t)\mathbf{1}'$ (prime denotes transposition) is determined in advance; a full discussion of Markov models with given size (expanding or not expanding) is cited on [5] p. 72 and [6] p. 103. Thus, the new members of the system during the t-th time interval, fill vacancies that are created by the wastage rates $\mathbf{p}_{k+1}(t)$ and by the new posts which are represented by $\Delta T(t) = T(t+1) - T(t)$. These new members are allocated to the various classes according to the distribution $\mathbf{p}_0(t) = [p_{0,1}(t), p_{0,2}(t), \ldots, p_{0,k}(t)]$ called the recruitment distribution.

A system like the one just described has been referred to in [25] as *the non homogeneous Markov system* (**NHMS**) and is uniquely determined by the sequences $\{\mathbf{P}(t)\}_{t=0}^{\infty}$, $\{\mathbf{p}_{k+1}(t)\}_{t=0}^{\infty}$, $\{T(t)\}_{t=0}^{\infty}$, $\{\mathbf{p}_0(t)\}_{t=0}^{\infty}$ and the initial structure $\mathbf{N}(0)$. The difference equation

$$\begin{aligned}\overline{\mathbf{N}}(t+1) &= \overline{\mathbf{N}}(t)\mathbf{P}(t) + [\overline{\mathbf{N}}(t)\mathbf{p}'_{k+1}(t) + \Delta T(t)]\mathbf{p}_0(t) \\ &= \overline{\mathbf{N}}(t)\mathbf{Q}(t) + \Delta T(t)\mathbf{p}_0(t) \end{aligned} \tag{1}$$

where $\mathbf{Q}(t) = \mathbf{P}(t) + \mathbf{p}'_{k+1}(t)\mathbf{p}_0(t)$, gives the expected structure of the system at the time $t+1$ as a function of its structure at time t.

The embedded non homogeneous Markov chain is defined by the sequence of the stochastic matrices $\{\mathbf{Q}(t)\}_{t=0}^{\infty}$. The element $q_{ij}(t) = p_{ij}(t) + p_{i,k+1}(t)p_{0,j}(t)$ represents the total probability of transition in the t-th time interval from class i to class j; it can either take place within the system or by loss from class i and replacement to class j ([5] p. 73).

Now, for an NHMS define by $\mathscr{P}$ the set of all possible transition matrices $\{\mathbf{P}(t)\}_{t=0}^{\infty}$ and let $\mathscr{P}_S(t) = \{\mathbf{P}_1(t), \mathbf{P}_2(t), \ldots, \mathbf{P}_Z(t)\}$ be a finite subset such that $[\mathbf{I} - \mathbf{P}_h(t)]\mathbf{1}' = \mathbf{p}'_{k+1}(t)$ for every $h \in S = \{1, 2, \ldots, Z\}$ and $t = 1, 2, \ldots$ The existence of the same number, Z, of transition matrices in every time interval it is not actually a restricted constraint; it does not imply that the pool $\mathscr{P}_S(t)$ should be the same for every t. This number can be easily extended to be a function of time. Assume that at every time interval the NHMS selects a transition matrix from the pool $\mathscr{P}_S(t)$ in the following stochastic way. Let

$$c_{hm}(t) = Prob\{\mathbf{P}(t) = \mathbf{P}_m(t) \mid \mathbf{P}(t-1) = \mathbf{P}_h(t-1)\} \tag{2}$$

$(h, m \in S)$, be the probability that the transition matrix for the time interval $[t, t+1)$ is $\mathbf{P}_m(t)$, given that for the $[t-1, t)$ one was $\mathbf{P}_h(t-1)$, while $\mathbf{P}(0) = \mathbf{P}$ (a known matrix) with probability one. Collect the probabilities (2) in the stochastic matrix $\mathbf{C}(t) = \{c_{hm}(t)\}_{h,m\in S}$. We call the sequence $\{\mathbf{C}(t)\}_{t=1}^{\infty}$ the *compromise non homogeneous Markov chain*, in the sense that is the outcome of the choice of strategy under the various pressures in the environment.

Then, the sequences $\{\mathscr{P}_S(t)\}_{t=1}^{\infty}$, $\{\mathbf{C}(t)\}_{t=1}^{\infty}$, $\{T(t)\}_{t=0}^{\infty}$, $\{\mathbf{p}_{k+1}(t)\}_{t=0}^{\infty}$, $\{\mathbf{p}_0(t)\}_{t=0}^{\infty}$, the initial transition matrix $\mathbf{P}(0) = \mathbf{P}$ and the structure $\mathbf{N}(0)$ uniquely determine a NHMS namely *the non-homogeneous Markov system in a stochastic environment* (**S-NHMS**) [24].

Numerous cases in real life may imply the existence of a pool of transition matrices such as, situations like public opinion variations on various subjects, consumers behavior, desires about the system structure and, mostly, miscellaneous scenarios on the promotion (or any other similar) policy of a company, e.t.c. Then, management or more generally the public sector decision-maker may be interested not only in separate behavior, but may also seek an average description of the *aggregate* mobility in the system. The realization process of the alternative scenarios is implied by the compromise Markov chain.

Since, in a Markov chain model like this, the transition matrix will be selected from the pool $\mathscr{P}_S(t)$ by the stochastic mechanism imposed by the compromise non-homogeneous Markov chain $\mathbf{C}(t)$, we can no longer speak for a specific transition matrix $\mathbf{P}(t)$ for the t-th time interval. This led Tsantas and Vassiliou [24] to introduce the expected transition matrix $E[\mathbf{P}(t)]$, all over the set $\mathscr{P}_S(t) = \{\mathbf{P}_1(t), \mathbf{P}_2(t), \ldots, \mathbf{P}_Z(t)\}$ of the t-th time interval $[t, t+1)$. They also gave the form of this matrix:

$$E[\mathbf{P}(t)] = \sum_{h \in S} \mathbf{e}_h \left[\mathbf{e}_1 \prod_{r=1}^{t} \mathbf{C}(r) \right]' \mathbf{P}_h(t) \quad t = 1, 2, 3, \ldots \tag{3}$$

where $\mathbf{e}_h$ is a $1 \times Z$ row vector with 1 in the hth entry and zero elsewhere. In the S-NHMS model, we can not assume in general, that the matrix $E[\mathbf{P}(t)]$ belongs to the pool $\mathscr{P}_S(t)$, $t = 1, 2, 3, \ldots$ In fact, it should be considered as a measure of tendency for the transition policies available for the system at the t-th time interval.

Under these considerations, the relation

$$\overline{\mathbf{N}}(t+1) = \overline{\mathbf{N}}(t) E[\mathbf{Q}(t)] + \Delta T(t) \mathbf{p}_0(t) \tag{4}$$

where $E[\mathbf{Q}(t)] = E[\mathbf{P}(t)] + \mathbf{p}'_{k+1}(t)\mathbf{p}_0(t)$, not only provides the expected numbers in the classes at the time point $t+1$, but mainly describes the inside of the average aggregate behavior of the system. The sequence of the stochastic matrices $\{E[\mathbf{Q}(t)]\}_{t=1}^{\infty}$ defines what, by analogy with NHMS, we called the *expected embedded non homogeneous Markov chain* for the S-NHMS.

Notice also here that equation (4) is illustrative similar in form to the respective equation of an NHMS (1). However, they are radically different since $E[\mathbf{Q}(t)]$, as a function of $\mathbf{C}(\tau)$ and $\mathscr{P}_S(\tau)$ for $\tau = 1, 2, \ldots, t$ is not directly estimated from the data.

In a model like this, besides the expected numbers, it is of obvious interest to determine the variances and covariances of our predictions. The form of the variance-covariance matrix $\mathbf{V}(t)$ for the S-NHMS has been given by Tsantas [23]:

$$\mathbf{V}(t) = diag\{\overline{\mathbf{N}}(t)\} - \left[\prod_{s=0}^{t-1} E[\mathbf{Q}(s)] \right]' diag\{\overline{\mathbf{N}}(0)\} \left[\prod_{s=0}^{t-1} E[\mathbf{Q}(s)] \right]$$
$$- \sum_{s=0}^{t-1} \left\{ \left[\prod_{\tau=s+1}^{t-1} E[\mathbf{Q}(\tau)] \right]' \Delta T(s) \mathbf{p}'_0(s) \mathbf{p}_0(s) \left[\prod_{\tau=s+1}^{t-1} E[\mathbf{Q}(\tau)] \right] \right\} \tag{5}$$

where for a vector $\mathbf{x} = \{x_i\}_{i \in G}$, $diag\{\mathbf{x}\}$ denotes the matrix $\{\delta_{ij} x_i\}_{i \in G}$, $E[\mathbf{Q}(0)] = \mathbf{Q}(0)$ and the product $\prod_{\tau=s+1}^{t-1}\{\cdot\}$ is defined to be the unit matrix $\mathbf{I}$ for $\tau > (t-1)$. This is a recurrence relation; since the covariances at $t = 0$ are zero, the complete set can be computed from (5).

3 Asymptotic behavior of the non-homogeneous Markov system in a stochastic environment

One of the most useful things to know in manpower planning models is the *direction* in which the structure is changing. In this respect, Vassiliou and his associates studied the limiting relative structure [25], [29], [9], [11] and the limiting behavior of variance and covariance [31], [21], [30], [11], [12], [14] of the NHMS model. In the present study our interest concerns the behavior of the structure (4) and of the variance-covariance matrix (5) when t tends to infinity, provided that we control the sequence of vectors of recruitment probabilities (in fact the limit of the sequence).

In what follows we use as norm of a real matrix $\mathbf{U} = \{u_{ij}\}$ the $\sup_i \sum_j |u_{ij}|$ and as a norm of a real vector $\mathbf{x} = \{x_j\}$ the $\max_j |x_j|$. Hence, convergence of matrices and vectors is assumed with respect to these norms. The norms have the additional following properties:

1. $\|\mathbf{AB}\| \leq \|\mathbf{A}\|\,\|\mathbf{B}\|$;
2. $\|\mathbf{uA}\| \leq \|\mathbf{u}\|\,\|\mathbf{A}\|$;
3. $\|\mathbf{P}\| = 1$ for any stochastic matrix $\mathbf{P}$.

To have a common basis we give some definitions and propositions.

Definition 3.1. *If k and m are two positive integers $(k < m)$ and $\{\mathbf{U}(\mathbf{t})\}_{t=0}^{\infty}$ is a sequence of matrices we define $\mathbf{U}(k,m)$ to be the product of the matrices $\mathbf{U}(k)\mathbf{U}(k+1)\cdots\mathbf{U}(m)$.*

Definition 3.2. *We call a stochastic matrix $\mathbf{U}$ regular if and only if it has no eigenvalue $(\neq 1)$ of modules 1; and 1 is a simple root of the characteristic equation* [7].

Proposition 3.1 [17]. *Let $\{\mathbf{U}(t)\}_{t=0}^{\infty}$ be a sequence of stochastic matrices and let also $\lim_{t\to\infty}\mathbf{U}(t) = \mathbf{U}$ with $\mathbf{U}$ being a regular stochastic matrix. Then $\lim_{k\to\infty}\mathbf{U}(m,k) = \mathbf{U}^{\infty}$ for every $m \in \mathbf{N}$, where $\mathbf{U}^{\infty} = \lim_{t\to\infty}\mathbf{U}^{t}$.*

Proposition 3.2 [18]. *Let $\mathbf{U}$ be a regular stochastic matrix. Then, the matrix $\mathbf{U}^{\infty} = \lim_{t\to\infty}\mathbf{U}^{t}$ is a stable stochastic matrix (identical rows).*

Definition 3.3. *Consider a S-NHMS defined by the sequences $\{\mathscr{P}_S(t)\}_{t=1}^{\infty}$, $\{\mathbf{C}(t)\}_{t=1}^{\infty}$, $\{T(t)\}_{t=0}^{\infty}$, $\{\mathbf{p}_{k+1}(t)\}_{t=0}^{\infty}$, an arbitrarily chosen family of input vectors $\{\mathbf{p}_0(t)\}_{t=0}^{\infty}$, as well as, the initial transition matrix $\mathbf{P}(0)$ and the starting structure $\mathbf{N}(0)$. Let $S = \{1, 2, \ldots, Z\}$ be the set of indices of the elements in $\mathscr{P}_S(t)$ and $G = \{1, 2, \ldots, k\}$ the set of the classes of the S-NHMS. The following conditions will be called the limiting conditions of a S-NHMS:*

i) $\lim_{t\to\infty}\mathbf{P}_h(t) = \mathbf{P}_h \;\forall h \in S$;
ii) $\lim_{t\to\infty}\mathbf{C}(t) = \mathbf{C}, \mathbf{C}$ *is a regular stochastic matrix;*
iii) $\lim_{t\to\infty}\mathbf{p}_{k+1}(t) = \mathbf{p}_{k+1}, \mathbf{p}_{k+1}$ *a vector such that* $[\mathbf{I} - \mathbf{P}_h]\mathbf{1}' = \mathbf{p}_{k+1} \;\forall h \in S$;
iv) $\lim_{t\to\infty}\mathbf{p}_0(t) = \mathbf{p}_0$; *and*
v) $\lim_{t\to\infty}T(t) = T < \infty$, *while* $\Delta T(t) \geq 0 \;\forall t$ *(the system is always expanding).*

As a consequence of all our considerations and definitions we can obtain now the form of the limiting structures in a S-NHMS. The corresponding results for the NHMS are scattered in various papers, see for example [31], [29], [30], [11].

Lemma 3.1. *Consider a S-NHMS for which the limiting conditions hold. Then the expected limiting embedded Markov chain matrix has the following form*

$$E[\mathbf{Q}] = \sum_{h \in S} \mathbf{e}_h [\mathbf{e}_1 \mathbf{C}^{\infty}]' \mathbf{P}_h + \mathbf{p}'_{k+1} \mathbf{p}_0 \tag{6}$$

where $\mathbf{C}^{\infty}$ *is the stable stochastic matrix defined as the* $\lim_{t \to \infty} \mathbf{C}^t = \mathbf{C}^{\infty}$.

Proof. According to propositions 3.1 and 3.2 we have that

$$\lim_{t \to \infty} \left\| \prod_{i=1}^{t} \mathbf{C}(i) - \mathbf{C}^{\infty} \right\| = 0. \tag{7}$$

Now, defining

$$E[\mathbf{Q}] = \sum_{h \in S} \mathbf{e}_h [\mathbf{e}_1 \mathbf{C}^{\infty}]' \mathbf{P}_h + \mathbf{p}'_{k+1} \mathbf{p}_0$$

yields

$$\begin{aligned}
\|E[\mathbf{Q}(t)] - E[\mathbf{Q}]\| &= \left\| \sum_{h \in S} \mathbf{e}_h \left\{ \left[\mathbf{e}_1 \prod_{r=1}^{t} \mathbf{C}(r) \right]' \mathbf{P}_h(t) - [\mathbf{e}_1 \mathbf{C}^{\infty}]' \mathbf{P}_h \right\} \right. \\
&\qquad \left. + \{\mathbf{p}'_{k+1}(t) \mathbf{p}_0(t) - \mathbf{p}'_{k+1} \mathbf{p}_0\} \right\| \\
&\le \sum_{h \in S} \|\mathbf{e}_h\| \left\{ \left\| \left[\mathbf{e}_1 \prod_{r=1}^{t} \mathbf{C}(r) \right]' \mathbf{P}_h(t) - [\mathbf{e}_1 \mathbf{C}^{\infty}]' \mathbf{P}_h \right\| \right\} \\
&\qquad + \|\mathbf{p}'_{k+1}(t) \mathbf{p}_0(t) - \mathbf{p}'_{k+1} \mathbf{p}_0\| \\
&\le \sum_{h \in S} \left\{ \left\| \left[\mathbf{e}_1 \prod_{r=1}^{t} \mathbf{C}(r) \right]' - [\mathbf{e}_1 \mathbf{C}^{\infty}]' \right\| \|\mathbf{P}_h(t)\| \right. \\
&\qquad \left. + \|[\mathbf{e}_1 \mathbf{C}^{\infty}]'\| \, \|\mathbf{P}_h(t) - \mathbf{P}_h\| \right\} \\
&\qquad + \|\mathbf{p}'_{k+1}(t) - \mathbf{p}'_{k+1}\| \, \|\mathbf{p}_0(t)\| + \|\mathbf{p}'_{k+1}\| \, \|\mathbf{p}_0(t) - \mathbf{p}_0\|.
\end{aligned}$$

Since $\mathbf{C}^{\infty}$ is a stochastic matrix while the $\mathbf{P}_h(t)$ are sub-stochastic ones, the above inequality leads to

$$\|E[\mathbf{Q}(t)] - E[\mathbf{Q}]\| \leq \left\| \left[\mathbf{e}_1 \prod_{r=1}^{t} \mathbf{C}(r) \right]' - [\mathbf{e}_1 \mathbf{C}^{\infty}]' \right\| + \sum_{h \in S} \|\mathbf{P}_h(t) - \mathbf{P}_h\|$$

$$+ \|\mathbf{p}'_{k+1}(t) - \mathbf{p}'_{k+1}\| + \|\mathbf{p}_0(t) - \mathbf{p}_0\|.$$

Combining this with (7) we get that

$$\lim_{t \to \infty} \|E[\mathbf{Q}(t)] - E[\mathbf{Q}]\| = 0,$$

which completes the proof. ▽

In what remains we will assume that the stochastic matrix $E[\mathbf{Q}]$ defined in (6) is a regular one.

Lemma 3.2. *Consider a S-NHMS for which the limiting conditions hold. Assume that the stochastic matrix $E[\mathbf{Q}]$ defined in (6) is a regular one. Then, the matrix $(E[\mathbf{Q}])^{\infty} = \lim_{t\to\infty}(E[\mathbf{Q}])^{t}$ is a stable stochastic matrix whose identical rows are given by the following expression:*

$$(E[\mathbf{q}])^{*} = \frac{\mathbf{p}_0 \left[\mathbf{I} - \sum_{h \in S} \mathbf{e}_h [\mathbf{e}_1 \mathbf{C}^{\infty}]' \mathbf{P}_h\right]^{-1}}{\mathbf{p}_0 \left[\mathbf{I} - \sum_{h \in S} \mathbf{e}_h [\mathbf{e}_1 \mathbf{C}^{\infty}]' \mathbf{P}_h\right]^{-1} \mathbf{1}'} \tag{8}$$

Proof. Indeed, according to proposition 3.2, the matrix $(E[\mathbf{Q}])^{\infty}$ is a stable stochastic matrix. Moreover, its row $(E[\mathbf{q}])^{*}$ is the left eigenvector of the matrix $E[\mathbf{Q}]$ which corresponds to the eigenvalue 1 [17]. Thus

$$(E[\mathbf{q}])^{*} E[\mathbf{Q}] = (E[\mathbf{q}])^{*} \tag{9}$$

which are the stationary equations of a Markov chain with transition probability matrix $E[\mathbf{Q}]$.

Then, from the equation (9) and lemma 3.1 we obtain

$$(E[\mathbf{q}])^{*} \sum_{h \in S} \mathbf{e}_h [\mathbf{e}_1 \mathbf{C}^{\infty}]' \mathbf{P}_h + (E[\mathbf{q}])^{*} \mathbf{p}'_{k+1} \mathbf{p}_0 = (E[\mathbf{q}])^{*}$$

or

$$(E[\mathbf{q}])^{*} \left[\mathbf{I} - \sum_{h \in S} \mathbf{e}_h [\mathbf{e}_1 \mathbf{C}^{\infty}]' \mathbf{P}_h \right] = (E[\mathbf{q}])^{*} \mathbf{p}'_{k+1} \mathbf{p}_0$$

Since the matrices $\mathbf{P}_h, h \in S$ are sub-stochastic, the same holds for the matrix $\sum_{h \in S} \mathbf{e}_h [\mathbf{e}_1 \mathbf{C}^{\infty}]' \mathbf{P}_h$ too, as a convex combination of them. Hence, the matrix $[\mathbf{I} - \sum_{h \in S} \mathbf{e}_h [\mathbf{e}_1 \mathbf{C}^{\infty}]' \mathbf{P}_h]$ is nonnegatively invertible [20] and thus

$$(E[\mathbf{q}])^{*} = (E[\mathbf{q}])^{*} \mathbf{p}'_{k+1} \mathbf{p}_0 \left[\mathbf{I} - \sum_{h \in S} \mathbf{e}_h [\mathbf{e}_1 \mathbf{C}^{\infty}]' \mathbf{P}_h \right]^{-1}. \tag{10}$$

Multiplying both sides of the above equation by a vector of 1's we get

$$(E[\mathbf{q}])^*\mathbf{p}'_{k+1} = \frac{1}{\mathbf{p}_0[\mathbf{I} - \sum_{h\in S}\mathbf{e}_h[\mathbf{e}_1\mathbf{C}^\infty]'\mathbf{P}_h]^{-1}\mathbf{1}'}$$

which in combination with (10) leads to (8). ▽

Here is our first main result in this paper.

Theorem 3.1. *Consider a S-NHMS for which the limiting conditions are true. Then the limiting expected structure and the limiting variance-covariance matrix exist and are of the following form*

$$\overline{\mathbf{N}}(\infty) = \lim_{t\to\infty}\overline{\mathbf{N}}(t) = T\{(E[\mathbf{q}])^*\} \tag{11}$$

$$\mathbf{V}(\infty) = \lim_{t\to\infty}\mathbf{V}(t) = T\{diag\{(E[\mathbf{q}])^*\} - [(E[\mathbf{q}])^*]'[(E[\mathbf{q}])^*]\} \tag{12}$$

Proof. Using (4) recursively we obtain

$$\overline{\mathbf{N}}(t) = \overline{\mathbf{N}}(0)\prod_{r=0}^{t-1}E[\mathbf{Q}(r)] + \sum_{\tau=1}^{t}\Delta T(\tau-1)\mathbf{p}_0(\tau-1)\prod_{h=\tau}^{t-1}E[\mathbf{Q}(h)] \tag{13}$$

where the product $\prod_{h=\tau}^{t-1}\{\cdot\}$ is defined to be the unit matrix $\mathbf{I}$ for $h > (t-1)$.
Let further be

$$A(t) = \sum_{\tau=1}^{t}\Delta T(\tau-1)\mathbf{p}_0(\tau-1)\prod_{h=\tau}^{t-1}E[\mathbf{Q}(h)]$$

It follows that

$$\begin{aligned}
&\|A(t) - [T(t) - T(0)]\mathbf{p}_0(E[\mathbf{Q}])^\infty\| \\
&\quad = \left\|\sum_{\tau=1}^{t}\Delta T(\tau-1)\mathbf{p}_0(\tau-1)\prod_{h=\tau}^{t-1}E[\mathbf{Q}(h)] - \sum_{\tau=1}^{t}\Delta T(\tau-1)\mathbf{p}_0(E[\mathbf{Q}])^\infty\right\| \\
&\quad \le \sum_{\tau=1}^{t}\Delta T(\tau-1)\left\|\mathbf{p}_0(\tau-1)\prod_{h=\tau}^{t-1}E[\mathbf{Q}(h)] - \mathbf{p}_0(E[\mathbf{Q}])^\infty\right\|
\end{aligned} \tag{14}$$

where the matrix $(E[\mathbf{Q}])^\infty$ was defined in lemma 3.2 as the $\lim_{t\to\infty}(E[\mathbf{Q}])^t$. This matrix is a stable one and its identical rows is represented by $(E[\mathbf{q}])^*$. Hence

$$\mathbf{p}_0(\tau-1)(E[\mathbf{Q}])^\infty = \mathbf{p}_0(E[\mathbf{Q}])^\infty = (E[\mathbf{q}])^* \tag{15}$$

and (14) gives

$$\|A(t) - [T(t) - T(0)]\mathbf{p}_0(E[\mathbf{Q}])^{\infty}\|$$

$$\leq \sum_{\tau=1}^{t} \Delta T(\tau-1)\left\|\mathbf{p}_0(\tau-1)\left\{\prod_{h=\tau}^{t-1} E[\mathbf{Q}(h)] - (E[\mathbf{Q}])^{\infty}\right\}\right\|$$

$$\leq \sum_{\tau=1}^{t} \Delta T(\tau-1)\left\|\prod_{h=\tau}^{t-1} E[\mathbf{Q}(h)] - (E[\mathbf{Q}])^{\infty}\right\|. \qquad (16)$$

Since $\lim_{t\to\infty} T(t) = T$, applying proposition 3.1 to (16), yields immediately

$$\lim_{t\to\infty} A(t) = [T - T(0)]\mathbf{p}_0(E[\mathbf{Q}])^{\infty}.$$

Then, from equation (13) and proposition 3.1 we obtain the desired result

$$\begin{aligned}\overline{\mathbf{N}}(\infty) = \lim_{t\to\infty} \overline{\mathbf{N}}(t) &= \mathbf{N}(0)(E[\mathbf{Q}])^{\infty} + [T - T(0)]\mathbf{p}_0(E[\mathbf{Q}])^{\infty} \\ &= \mathbf{N}(0)\mathbf{1}'(E[\mathbf{q}])^{*} + T(E[\mathbf{q}])^{*} - T(0)(E[\mathbf{q}])^{*} \\ &= T(E[\mathbf{q}])^{*}. \end{aligned} \qquad (17)$$

Part 2 of the theorem is proved similarly. Since the stochastic matrix $E[\mathbf{Q}]$ is regular, by proposition 3.1 we get that

$$\lim_{t\to\infty} \mathbf{N}(0)\prod_{\tau=0}^{t-1} E[\mathbf{Q}(\tau)] = \mathbf{N}(0)(E[\mathbf{Q}])^{\infty} = T(0)(E[\mathbf{q}])^{*}$$

and thus

$$\begin{aligned}&\lim_{t\to\infty}\left[\prod_{\tau=0}^{t-1} E[\mathbf{Q}(\tau)]\right]' \mathit{diag}\{\mathbf{N}(0)\}\left[\prod_{\tau=0}^{t-1} E[\mathbf{Q}(\tau)]\right] \\ &= [\mathbf{1}'(E[\mathbf{q}])^{*}]'\,\mathit{diag}\{\mathbf{N}(0)\}[\mathbf{1}'(E[\mathbf{q}])^{*}] \\ &= T(0)[(E[\mathbf{q}])^{*}]'[(E[\mathbf{q}])^{*}] \end{aligned} \qquad (18)$$

On the other hand,

$$\lim_{t\to\infty} \sum_{s=0}^{t-1} \Delta T(s)\mathbf{p}_0(s) \prod_{\tau=s+1}^{t-1} E[\mathbf{Q}(\tau)] = [T - T(0)](E[\mathbf{q}])^{*}$$

and hence

$$\lim_{t\to\infty}\sum_{s=0}^{t-1}\left\{\left[\prod_{\tau=s+1}^{t-1}E[\mathbf{Q}(\tau)]\right]'\Delta T(s)\mathbf{p}_0'(s)\mathbf{p}_0(s)\left[\prod_{\tau=s+1}^{t-1}E[\mathbf{Q}(\tau)]\right]\right\}$$
$$= [T - T(0)][(E[\mathbf{q}])^*]'[(E[\mathbf{q}])^*] \tag{19}$$

Then (5), together with (17), (18) and (19) leads to

$$\mathbf{V}(\infty) = \lim_{t\to\infty}\mathbf{V}(t) = diag\{T[(E[\mathbf{q}])^*]\} - T(0)[(E[\mathbf{q}])^*]'[(E[\mathbf{q}])^*]$$
$$- [T - T(0)][(E[\mathbf{q}])^*]'[(E[\mathbf{q}])^*]$$
$$= T[diag\{(E[\mathbf{q}])^*\} - [(E[\mathbf{q}])^*]'[(E[\mathbf{q}])^*]] \qquad \triangledown$$

Remark 3.1. Taking into account that $(E[\mathbf{q}])^*$ is the left eigenvector of the matrix $E[\mathbf{Q}]$ which corresponds to the eigenvalue 1, equation (17) can be re-written equivalently in the form

$$\overline{\mathbf{N}}(\infty) = T(E[\mathbf{q}])^* = T(E[\mathbf{q}])^*E[\mathbf{Q}] = \overline{\mathbf{N}}(\infty)E[\mathbf{Q}]. \tag{20}$$

These are the stationary equations of a homogeneous S-NHMS with transition matrix $E[\mathbf{Q}]$.

Remark 3.2. Due to equation (17) the asymptotic structure of a S-NHMS is of multinomial type with size T and probabilities $(E[\mathbf{q}])^*$ (see also [11], [30]).

4 Rate of convergence of a non-homogeneous Markov system in a stochastic environment to its asymptotic structure

In this section we will investigate the rate at which the S-NHMS converges to its limit. Rates of convergence for the first and second central moments of the class sizes in a NHMS has been studied extensively in [22] and [29].

Following [16], we say that a sequence of matrices $\{\mathbf{U}_n\}_{n=0}^{\infty}$ converges with geometrical rate to a matrix $\mathbf{U}$, if there exist constants $c > 0$ and $0 < b < 1$ such that

$$\|\mathbf{U}_n - \mathbf{U}\| \leq cb^n \quad n = 0, 1, 2, \ldots$$

We now state our second main result in this paper.

Theorem 4.1. *Consider a S-NHMS for which the limiting conditions are true. If the rate of convergence in all of these conditions is geometric, then*

1. *the sequence of the S-NHMS' expected structures $\{\overline{\mathbf{N}}(t)\}_{t=0}^{\infty}$ converges to its limit geometrically fast.*

2. *the sequence of the S-NHMS' variance-covariance matrices* $\{\mathbf{V}(t)\}_{t=0}^{\infty}$ *converges to its limit geometrically fast.*

Proof. It is sufficient to prove that there are constants $c > 0$ and $0 < b < 1$ such that

$$\|\overline{\mathbf{N}}(t) - \overline{\mathbf{N}}(\infty)\| \le cb^t \quad \text{for } t = 0, 1, 2, \dots \tag{21}$$

where $\overline{\mathbf{N}}(\infty) = T\mathbf{p}_0(E[\mathbf{Q}])^{\infty}$ (see equations 15 and 17).

From the recurrence relation (13) we have

$$\begin{aligned}
\|\overline{\mathbf{N}}(t) - \overline{\mathbf{N}}(\infty)\| &= \Bigg\| \left\{ \mathbf{N}(0) \prod_{\tau=0}^{t-1} E[\mathbf{Q}(\tau)] + \sum_{\tau=1}^{t} \Delta T(\tau-1)\mathbf{p}_0(\tau-1) \prod_{h=\tau}^{t-1} E[\mathbf{Q}(h)] \right\} \\
&\qquad - T\mathbf{p}_0(E[\mathbf{Q}])^{\infty} \Bigg\| \\
&\le \left\| \mathbf{N}(0) \prod_{\tau=0}^{t-1} E[\mathbf{Q}(\tau)] - \mathbf{N}(0)(E[\mathbf{Q}])^{\infty} \right\| \\
&\quad + \Bigg\| \sum_{\tau=1}^{t} \Delta T(\tau-1)\mathbf{p}_0(\tau-1) \prod_{h=\tau}^{t-1} E[\mathbf{Q}(h)] \\
&\qquad - \{T - T(0)\}\mathbf{p}_0(E[\mathbf{Q}])^{\infty} \Bigg\|.
\end{aligned} \tag{22}$$

The matrix $(E[\mathbf{Q}])$ is regular and thus, the sequence of matrices $\{\prod_{\tau=t}^{t+n} E[\mathbf{Q}(\tau)]\}_{n=0}^{\infty}$ converges to $(E[\mathbf{Q}])^{\infty}$ geometrically fast, uniformly in t [16]. Hence there are constants $c_1 > 0$ and $0 < b_1 < 1$ such that

$$\left\| \prod_{\tau=t}^{t+n} E[\mathbf{Q}(\tau)] - (E[\mathbf{Q}])^{\infty} \right\| \le c_1 b_1^n \quad \text{for every } t, n = 0, 1, 2, \dots$$

Similar, there are constants $c_2 > 0$ and $0 < b_2 < 1$ such that $\|T(t) - T\| \le c_2 b_2^t$ for every $t = 0, 1, 2, \dots$

Choose now $0 < b < 1$ such that $b > \max\{b_1, b_2\}$. Then for every $t = 1, 2, \dots$ we have

$$\begin{aligned}
\left\| \mathbf{N}(0) \prod_{\tau=0}^{t-1} E[\mathbf{Q}(\tau)] - \mathbf{N}(0)(E[\mathbf{Q}])^{\infty} \right\| &\le T(0) \left\| \prod_{\tau=0}^{t-1} E[\mathbf{Q}(\tau)] - (E[\mathbf{Q}])^{\infty} \right\| \\
&\le T(0) c_1 b_1^t \le c_3 b^t,
\end{aligned} \tag{23}$$

with some suitable constant c_3.

On the other hand

$$\left\|\sum_{\tau=1}^{t}\Delta T(\tau-1)\mathbf{p}_0(\tau-1)\prod_{h=\tau}^{t-1}E[\mathbf{Q}(h)]-\{T-T(0)\}\mathbf{p}_0(E[\mathbf{Q}])^{\infty}\right\|$$

$$\leq\left\|\sum_{\tau=1}^{t}\Delta T(\tau-1)\mathbf{p}_0(\tau-1)\prod_{h=\tau}^{t-1}E[\mathbf{Q}(h)]-\{T(t)-T(0)\}\mathbf{p}_0(E[\mathbf{Q}])^{\infty}\right\|$$

$$+|T-T(t)|$$

$$\leq\sum_{\tau=1}^{t}\Delta T(\tau-1)\left\|\prod_{h=\tau}^{t-1}E[\mathbf{Q}(h)]-(E[\mathbf{Q}])^{\infty}\right\|+|T-T(t)|$$

$$\leq\sum_{\tau=1}^{t-1}\{T-T(\tau-1)\}c_1b_1^{t-1-\tau}+c_2b_2^t$$

$$\leq\sum_{\tau=1}^{t-1}c_2b_2^{\tau-1}c_1b^{t-1-\tau}+c_2b^t$$

$$=c_1c_2b^{t-2}\sum_{\tau=1}^{t-1}\left(\frac{b_2}{b}\right)^{\tau-1}+c_2b^t$$

$$\leq c_4b^t \tag{24}$$

where c_4 is some suitable constant.

Applying the results (23) and (24) to (22) we obtain

$$\|\overline{\mathbf{N}}(t)-\overline{\mathbf{N}}(\infty)\|\leq cb^t\quad\text{for } t=0,1,2,\ldots \tag{25}$$

where $c=c_3+c_4$. Thus, the sequence $\{\overline{\mathbf{N}}(t)\}_{t=0}^{\infty}$ converges geometrically fast to $\overline{\mathbf{N}}(\infty)$.

Part 2 of the theorem is proved similarly. ▽

5 Asymptotically attainable structures in non-homogeneous Markov system in a stochastic environment

The control of structures, i.e. the way to attain and/or maintain a desired structure, has been a major area of research in manpower systems; list of references as well as an extended discussion on related material can be found in [5] and [6]. Vassiliou and his associates obtain results for the expected numbers in the classes of a NHMS under recruitment control (see for example [32], [33], [29], [30], [10], [8]). In the present section we will study the problem of finding which expected structures are possible as limiting/attaining ones provided that we control the sequence of recruitment probabilities (in fact the limit of the sequence).

Definition 5.1. *We say that an S-NHMS has an asymptotically attainable expected structure* $\overline{\mathbf{N}}(t)$ *under asymptotic recruitment control, if there exists a sequence of recruitment vectors* $\{\mathbf{p}_0(t)\}_{t=0}^{\infty}$ *with* $\lim_{t\to\infty}\mathbf{p}_0(t)=\mathbf{p}_0$ *such that* $\lim_{t\to\infty}\overline{\mathbf{N}}(t)=\overline{\mathbf{N}}(\infty)$.

Theorem 5.1. *Consider a S-NHMS for which the limiting conditions are true. Then the set* $\mathscr{A}^{\infty}$ *of the asymptotically attainable expected structures under asymptotic recruitment control is the convex hull of the points*

$$\mathbf{z}_i = T\lambda_i^{-1}\left\{\mathbf{e}_i\left[\mathbf{I}-\sum_{h\in S}\mathbf{e}_h[\mathbf{e}_1\mathbf{C}^{\infty}]'\mathbf{P}_h\right]^{-1}\right\} \quad i=1,2,\ldots,k \tag{26}$$

where λ_i *is the sum of the elements of the i-th row of the matrix* $[\mathbf{I}-\sum_{h\in S}\mathbf{e}_h[\mathbf{e}_1\mathbf{C}^{\infty}]'\mathbf{P}_h]^{-1}$.

Proof. Taking into account lemma 3.2, equation (17) can be re-written in the form

$$\overline{\mathbf{N}}(\infty) = T\sum_{i=1}^{k}\lambda_i p_{0i}\frac{1}{\sum_{j=1}^{k}\lambda_j p_{0j}}\lambda_i^{-1}\left\{\mathbf{e}_i\left[\mathbf{I}-\sum_{h\in S}\mathbf{e}_h[\mathbf{e}_1\mathbf{C}^{\infty}]'\mathbf{P}_h\right]^{-1}\right\},$$

from which the proof of the theorem follows directly. ▽

Remark 5.1. Inherently, it is proved that it is the limit vector $\mathbf{p}_0$ of the sequence of input probabilities that controls the asymptotic structure and not the corresponding sequence; this determines the stationary distribution of the expected embedded non homogeneous Markov chain $E[\mathbf{Q}]$.

Remark 5.2. Tsantas and Vassiliou [24] proved that if for a structure $\mathbf{N}(t)$ holds

$$N_j(t+1) > \sum_{h\in S}\mathbf{e}_h\left[\mathbf{e}_1\prod_{r=1}^{t}\mathbf{C}(r)\right]^T\sum_{i=1}^{k}N_i(t)p_{h,ij}(t)$$

the probability to maintain it tends to 1. Because of (20) this seems to be true for all the structures of the set $\mathscr{A}^{\infty}$. Clearly this is an expected result.

6 Illustration

In this section we illustrate the previous results with an example typical in the literature on manpower planning.

Consider a firm with three grades. Let $t=0$ for the last year of records and suppose that

$$\mathbf{P}(0)=\begin{pmatrix}0.80314 & 0.08836 & 0.00000\\ 0.00000 & 0.90670 & 0.03700\\ 0.00000 & 0.00000 & 0.78942\end{pmatrix}.$$

Assume that the sets $\mathscr{P}_S(t)$ for $t = 1, 2, \ldots$ consist of the following transition matrices:

$$\mathbf{P}_1(t) = \begin{pmatrix} 0.77120 - \dfrac{1}{(t+1)^5} & 0.12818 - \dfrac{1}{(t+1)^5} & 0.00000 \\ 0.00000 & 0.78766 - \dfrac{1}{(t+1)^5} & 0.08232 - \dfrac{1}{(t+1)^5} \\ 0.00000 & 0.00000 & 0.88406 - \dfrac{1}{(t+1)^5} \end{pmatrix}$$

$$\mathbf{P}_2(t) = \begin{pmatrix} 0.66295 - \dfrac{1}{(t+1)^5} & 0.23643 - \dfrac{1}{(t+1)^5} & 0.00000 \\ 0.00000 & 0.71888 - \dfrac{1}{(t+1)^5} & 0.15110 - \dfrac{1}{(t+1)^5} \\ 0.00000 & 0.00000 & 0.88406 - \dfrac{1}{(t+1)^5} \end{pmatrix}$$

$$\mathbf{P}_3(t) = \begin{pmatrix} 0.69619 - \dfrac{1}{(t+1)^5} & 0.20319 - \dfrac{1}{(t+1)^5} & 0.00000 \\ 0.00000 & 0.75768 - \dfrac{1}{(t+1)^5} & 0.11230 - \dfrac{1}{(t+1)^5} \\ 0.00000 & 0.00000 & 0.88406 - \dfrac{1}{(t+1)^5} \end{pmatrix},$$

i.e. $\mathscr{P}_S(t) = \{\mathbf{P}_1(t), \mathbf{P}_2(t), \mathbf{P}_3(t)\}$; the figures in these matrices reflect the kind of condition which one might find in a typical management hierarchy.

The sequence of the total members of the system was supposed to be T(0) = 1066 while the expansion sequence determined by corresponding to $\Delta T(t) = 43 * (0.6)^t$.

Obviously in our data, there exists a time point t_0 such that for $t \geq t_0$ the sets $\mathscr{P}_S(t)$ converge to the set $\mathscr{P}_S(\infty) = \{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}$ where

$$\mathbf{P}_1 = \lim_{t\to\infty} \mathbf{P}_1(t) = \begin{pmatrix} 0.77120 & 0.12818 & 0.00000 \\ 0.00000 & 0.78766 & 0.08232 \\ 0.00000 & 0.00000 & 0.88406 \end{pmatrix}$$

$$\mathbf{P}_2 = \lim_{t\to\infty} \mathbf{P}_2(t) = \begin{pmatrix} 0.66295 & 0.23643 & 0.00000 \\ 0.00000 & 0.71888 & 0.15110 \\ 0.00000 & 0.00000 & 0.88406 \end{pmatrix}$$

$$\mathbf{P}_3 = \lim_{t\to\infty} \mathbf{P}_3(t) = \begin{pmatrix} 0.69619 & 0.20319 & 0.00000 \\ 0.00000 & 0.75768 & 0.11230 \\ 0.00000 & 0.00000 & 0.88406 \end{pmatrix}.$$

Assume that after this $t \geq t_0$, a simulation program generates a sequence of 1000 instants of these $\mathbf{P}_h$ ($h \in S = \{1, 2, 3\}$) transition matrices:

$$\mathbf{P}_1 \to \mathbf{P}_2 \to \mathbf{P}_2 \to \mathbf{P}_1 \to \mathbf{P}_1 \to \mathbf{P}_1 \to \mathbf{P}_2 \to \mathbf{P}_3 \to \mathbf{P}_2 \to \mathbf{P}_3 \to \mathbf{P}_3 \to \cdots \tag{27}$$

Observing the succession we were able to estimate the (limiting) elements for the compromise non homogeneous Markov chain:

$$\mathbf{C} = \begin{pmatrix} 0.33693 & 0.23693 & 0.42614 \\ 0.31493 & 0.31514 & 0.36993 \\ 0.34714 & 0.40493 & 0.24793 \end{pmatrix}$$

($= \lim_{t\to\infty} \mathbf{C}(t)$). We have also that $T(\infty) = \lim_{t\to\infty} T(t) = 1131$. Then, for asymptotic recruitment vector equal to $\mathbf{p}_0 = (1/3, 1/3, 1/3)$ (equal recruitment), it follows from Lemma 3.1 that the expected limiting embedded Markov chain matrix will be

$$E[\mathbf{Q}] = \begin{pmatrix} 0.74713 & 0.21933 & 0.03354 \\ 0.04334 & 0.80193 & 0.15473 \\ 0.03865 & 0.03865 & 0.92271 \end{pmatrix}$$

which gives

$$(E[\mathbf{Q}])^{\infty} = \begin{pmatrix} 0.13689 & 0.26776 & 0.59536 \\ 0.13689 & 0.26776 & 0.59536 \\ 0.13689 & 0.26776 & 0.59536 \end{pmatrix}.$$

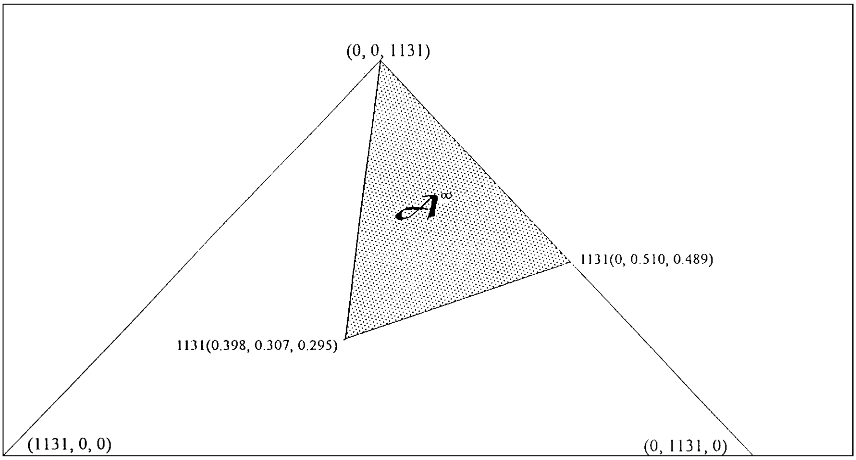
Whatever the initial distribution is, the limiting numbers in the classes of our system are of multinomial type (Remark 3.2) with size 1131 and probabilities $(0.13689, 0.26776, 0.59536)$. To be more precise, we apply Theorem 3.1. Then

$$\overline{\mathbf{N}}(\infty) = (154.754, 302.703, 673.054)$$

and

$$\mathbf{V}(\infty) = \begin{pmatrix} 133.56858 & -41.09367 & -92.13575 \\ -41.09367 & 221.64583 & -180.21300 \\ -92.13575 & -180.21300 & 272.34876 \end{pmatrix}.$$

In order to find the set $\mathscr{A}^{\infty}$ of all the asymptotically attainable expected structures we employ Theorem 5.1. Since $\sum_{i=1}^{3} \overline{N}_i(\infty) = 1131$, all the $\overline{\mathbf{N}}(\infty)$'s lies on the hyper-plane defined by the points $1131(1, 0, 0)$, $1131(0, 1, 0)$ and $1131(0, 0, 1)$. The convex region $\mathscr{A}^{\infty}$, is a subset of this hyper-plane; equations



(26) determine its vertices:

$$1131(0.3985644 \quad 0.3067367 \quad 0.2946989),$$

$$1131(0.0000000 \quad 0.5100075 \quad 0.4899925),$$

$$1131(0.0000000 \quad 0.0000000 \quad 1.0000000).$$

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