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Directional derivatives for set-valued mappings and applications*

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Abstract. Set-valued optimisation is an important topic and has wide applications in engineering and game theory. An interesting topic in set-valued optimisation is the appropriate introduction of a derivative concept for set-valued mappings. In this paper, Dini directional derivatives are introduced and investigated for set-valued mappings. A derivative concept of a Jacobificator for set-valued mappings is introduced in terms of the Dini directional derivatives. Applications are given to present optimality conditions and mean value theorems.

Key words: Directional derivative, set-valued mapping, optimality condition, mean value theorem

1 Introduction

There is a lot of interests in the study of set-valued analysis and optimisation. It is well known that set-valued optimisation has important applications in engineering and game theory, see Aubin and Ekeland (1984) and the references cited therein. The set-valued analysis has been presented in Aubin and Frankowska (1990). Systematic study of set-valued optimisation has been presented in Luc (1989) and Aubin and Ekeland (1984). An important aspect in the set-valued optimisation is the study of a derivative or directional derivative concept for a set-valued mapping. This has been initialized to the study of subdifferentials of vector-valued functions, see Zowe (1974), Thibault (1982), and Sawaragi et al (1985), Chen and Craven (1991) and Yang (1992).

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The research of a derivative concept of a set-valued mapping has been advanced by Borwein (1977), Corley (1988) and Luc (1991). In Jahn and Rauh (1997), some studies have been done on the modification of contingent derivatives of a set-valued mapping initially introduced in Aubin and Ekeland (1984) and Luc (1991) to appropriately address optimality conditions of set-valued optimisation. The results in Jahn and Rauh (1997) have been extended in Chen and Jahn (1997) to establish the existence of a contingent derivative.

The Dini directional derivative has played an important role in nonsmooth analysis and optimisation. For example, the Dini directional derivative for a real-valued function has been applied to define various subdifferentials. Recently, the upper and lower Dini directional derivatives for a scalar-valued function have been used to define a convexificator for a continuous function in Jeyakumar and Yang (1997). A mean value theorem is then established for a class of continuous functions. A Dini directional derivative for a vector-valued function is given in Valadir (1979) in terms of infimum. Recently a generalized Dini directional derivative for a vector-valued function is defined using the concept of a minimal element in Yang (1997) and applied to give optimality conditions of a vector optimisation problem.

In this paper, two developments are associated to study set-valued mappings: the first one is the convexificator for a real-valued function given in Jeyakumar and Yang (1997) in terms of Dini directional derivative and the second one is a Dini directional derivative for a vector-valued function introduced in Yang (1997). Set-valued upper and lower Dini directional derivatives for a set-valued mapping are introduced in terms of a minimal element and a maximal element respectively and applied to present optimality conditions for a set-valued optimisation with a convex set constraint. Upper and lower Jacobificators for a set-valued mapping are defined using the set-valued Dini directional derivatives. It is shown that the convex subdifferential of a cone convex function defined in Thibault (1982) is an example of an upper Jacobificator. The optimality conditions obtained in terms of set-valued upper and lower Dini directional derivative are applied to establish a mean value theorem for a set-valued mapping. It will be shown that the conventional mean value theorem can be derived as a special case.

2 Directional derivatives

Let X be a real topological vector space and Y be an ordered vector topology space, in which relations are defined by a closed convex cone P with $\operatorname{int} P \neq \emptyset$. In addition, Y is assumed to be a complete vector lattice, i.e., $\sup\{y_1, y_2\}$ exists for all $y_1, y_2 \in Y$ and every bounded nonempty subset has an infimum and a supremum. Let $K \subset Y$ be a bounded nonempty subset. The infimum and the supremum of K is denoted by $\operatorname{Inf} K$ and $\operatorname{Sup} K$ respectively. The sets of minimal elements and maximal elements of K are defined respectively by

$$\text{V-min} \quad K = \{y \in K \mid (\{y\} - P) \cap K = \{y\}\};$$

$$\text{V-max} \quad K = \{y \in K \mid (\{y\} + P) \cap K = \{y\}\}.$$

Let $L(X, Y)$ be the space of all continuous linear functions from X to Y.

Let $F : X \to 2^Y$ be a set-valued mapping, i.e., for each $x \in X$, $F(x)$ is a subset of Y. A function $f : X \to Y$ is said to be a continuous selection of F if f is continuous and $f(x) \in F(x)$, $\forall x \in X$. See Ding et al (1992). Denote by $CS(F)$ the set of all continuous selections of F. Assume that $CS(F) \neq \emptyset$.

Definition 2.1. *Given x, $d \in X$, $\mu \in F(x)$, define the limiting set of F at x in the direction d with respect to μ as*

$$Y_F^{\mu}(x;d) = \left\{ z \,\middle|\, z = \lim_{t_i \downarrow 0} \frac{f(x + t_i d) - \mu}{t_i} \text{ for some } f \in CS(F), f(x) = \mu \right\}. \tag{1}$$

If $F = f$ is a **single-valued function**, *then the following limiting set of f at x in the direction d is defined*

$$Y_f(x;d) = \left\{ z \,\middle|\, z = \lim_{t_i \downarrow 0} \frac{f(x + t_i d) - f(x)}{t_i} \right\}. \tag{2}$$

For our approach in this paper, we need:

Assumption 2.1. *The subset $Y_F^{\mu}(x, d)$ (and $Y_f(x, d)$) has a minimal element and a maximal element.*

See Jahn (1986) for conditions on the existence of a minimal element.

Definition 2.2. *Let $F : X \to 2^Y$ be a set-valued mapping. Let x, $d \in X$ and $\mu \in F(x)$. The* **upper and lower Dini-directional derivatives** *of F at x in the direction d with respect to μ are defined respectively by*

$$F_+^{\mu}(x;d) = \text{V-max } Y_F^{\mu}(x;d), \quad F_-^{\mu}(x;d) = \text{V-min } Y_F^{\mu}(x;d).$$

It is clear that

$$F_+^{\mu}(x;d) \cup F_-^{\mu}(x;d) \subset Y_F^{\mu}(x;d).$$

Proposition 2.1. *If Assumption 2.1 holds, then*

(i) $F_+^{\mu}(x;d) \neq \emptyset$ *and* $F_-^{\mu}(x;d) \neq \emptyset$;
(ii) $F_+^{\mu}(x;d)$ *and* $F_-^{\mu}(x;d)$ *as mappings of d are positively homogeneous.*

Proof: (i) follows from Assumption 2.1 and (ii) follows from Definitions 2.1 and 2.2. □

Remark 2.1. Let $F = f : X \to Y$ be a single-valued function. In Yang (1997), the subset $Y_f(x;d)$ in (2) was defined and the upper and lower Dini-directional derivatives were defined respectively by

$$f_+^d(x;d) = \text{V-max } Y_f(x,d), \quad f_-^d(x;d) = \text{V-min } Y_f(x,d).$$

Remark 2.2. Recall that Thibault (1982) defined the following subset for a single-valued Lipschitz function $f : X \to Y$ in the sense of Clarke

$$D_f(x, d) = \left\{ z \,\middle|\, z = \lim_{x_i \to x, t_i \downarrow 0} \frac{f(x_i + t_i d) - f(x_i)}{t_i} \right\}.$$

It is clear that $Y_f(x, d) \subset D_f(x, d)$. When f is a P-convex function (see Definition 3.1), Valadir (1972) defined the directional derivative

$$f'_-(x; d) = \operatorname*{Inf}_{t>0} \frac{f(x + td) - f(x)}{t},$$

and the subdifferential

$$\partial_c f(x) = \{T \in L(X, Y) \mid T(d) \leq f'_-(x; d), \forall d \in X\}.$$

It follows (Thibault (1982)) that

$$f'_-(x; d) = \operatorname{Sup}\{T(d) \mid T \in \partial_c f(x)\}, \tag{3}$$

if $f'_-(x; d)$ is continuous as a function of d.

3 Optimality conditions

In this section we apply the directional derivatives defined in the last section to characterise optimality conditions for a set-valued optimization problem. This optimality result will be applied to establish a mean value theorem for a set-valued mapping in Section 5. We begin by presenting a characterisation of the convexity of a set-valued mapping.

In the following, denote $\lambda y + (1 - \lambda)x$ by $y\lambda x$.

Definition 3.1. *Let C be a convex subset of X and $F : X \to 2^Y$. F is said to be* **P-convex** *on C if*

$$\lambda F(y) + (1 - \lambda)F(x) \subset F(y\lambda x) + P,$$

for any $x, y \in C$, $\lambda \in (0, 1)$. In particular, a single-valued function $f : X \to Y$ is said to be P-convex on C if

$$\lambda f(y) + (1 - \lambda)f(x) \in f(y\lambda x) + P,$$

for any $x, y \in C$, $\lambda \in (0, 1)$.

We need the following assumptions.

Assumption 3.1. *Let $Y_F^\mu(x; d)$ be defined as in (1). The domination property is said to hold for $Y_F^\mu(x; d)$ if*

$$Y_F^\mu(x; d) \subset (\text{V-min } Y_F^\mu(x; d) + P) \cap (\text{V-max } Y_F^\mu(x; d) - P).$$

Assumption 3.2. *Let* $x, y \in C$. *If* $z \in F(y\lambda x) + P$, $\forall \lambda \in (0, 1)$, *then there exists* $f \in CS(F)$ *such that* $z \in f(y\lambda x) + P$, $\forall \lambda \in (0, 1)$.

Note that Assumption 3.2 holds if $F(x) = f(x) + P$, $\forall x \in C$ and f is P-convex.

Proposition 3.1. *Let* C *be a convex subset of* X *and* $F : X \to 2^Y$. *If Assumption 3.2, and the mapping* F *is* P*-convex on* C, *then for any* $x, y \in C$ *and* $\mu \in F(x)$,

$$F(y) - \mu \subset Y_F^{\mu}(x; y - x) + P.$$

If, in addition, Assumption 3.1 holds, then

$$F(y) - \mu \subset F_-^{\mu}(x; y - x) + P.$$

Proof: For any $\mu \in F(x)$, $\lambda \in (0, 1)$,

$$\lambda F(y) + (1 - \lambda)\mu \subset F(y\lambda x) + P.$$

For any $w \in F(y)$, by Assumption 3.2, there exists $f \in CS(F)$ such that

$$\lambda w + (1 - \lambda)\mu \in f(y\lambda x) + P,$$

i.e.,

$$w - \mu \in \frac{f(x + \lambda(y - x)) - \mu}{\lambda} + P.$$

Thus

$$w - \mu \in Y_F^{\mu}(x; y - x) + P.$$

Then

$$F(y) - \mu \subset Y_F^{\mu}(x; y - x) + P.$$

Furthermore, it follows from Assumptions 3.1 that

$$Y_F^{\mu}(x; y - x) \subset \text{V-min } Y_F^{\mu}(x; y - x) + P.$$

Thus

$$F(y) - \mu \subset F_-^{\mu}(x; y - x) + P. \qquad \square$$

Consider the set-valued optimization problem (**P**)

$$\text{Weak} - \min\{F(x) \mid x \in C\},$$

where C is a subset of X and $F : X \to 2^Y$.

Definition 3.2. *(i) The point* $(x_0, \mu) \in C \times Y$ *is said to be* **a weak minimiser** *of (P) if* $\mu \in F(x_0)$*, and*

$$(F(x) - \mu) \cap -\text{int}\, P = \emptyset, \quad \forall x \in C. \tag{4}$$

(ii) The point $(x_0, \mu) \in C \times Y$ *is said to be a* **weak maximiser** *of (P) if* $\mu \in F(x_0)$*, and*

$$(F(x) - \mu) \cap \text{int}\, P = \emptyset, \quad \forall x \in C.$$

An equivalent definition for a weak minimiser is given as follows: (x_0, μ) is a weak minimiser of (P) if and only if $\mu \in F(x_0)$, and

$$(\{\mu\} - \text{int}\, P) \cap F(C) = \emptyset,$$

where $F(C) = \bigcup_{x \in C} F(x)$. See Luc (1989) and Chen and Jahn (1997).

Let the cone of feasible directions of C at x be defined by

$$S(x_0, C) = \{d \in X \mid \exists t_0 > 0,\ x_0 + td \in C,\ \forall t \in [0, t_0]\}.$$

Theorem 3.1. *Consider the set-valued optimization problem (P).*

(i) If $(x_0, \mu) \in C \times Y$ *is a weak minimiser of (P), then*

$$Y_F^{\mu}(x_0; d) \cap -\text{int}\, P = \emptyset, \quad \forall d \in S(x_0, C). \tag{5}$$

In particular,

$$F_{-}^{\mu}(x_0; d) \cap -\text{int}\, P = \emptyset, \quad \forall d \in S(x_0, C).$$

(ii) Assume that C is convex and F is P-convex. If Assumption 3.2 holds, (5) holds and $x_0 \in C$*,* $\mu \in F(x_0)$*, then* (x_0, μ) *is a weak minimiser of (P).*

Proof: (i) Since $(x_0, \mu) \subset C \times Y$ is a weak minimiser of (P), we have from (4)

$$(F(x) - \mu) \cap -\text{int}\, P = \emptyset, \quad \forall x \in C.$$

Thus for any $d \in S(x_0, C)$, there exists $t_0 > 0$ such that

$$(F(x_0 + td) - \mu) \cap -\text{int}\, P = \emptyset, \quad \forall t \in [0, t_0].$$

As $\text{int}\, P$ is an open set, we have

$$Y_F^{\mu}(x_0; d) \cap -\text{int}\, P = \emptyset, \quad \forall d \in S(x_0, C).$$

(ii) If $(x_0, \mu) \in C \times Y$ is not a weak minimiser of (P), there exists $x \in C$, $\xi \in F(x)$ such that

$$\xi - \mu \in -\text{int}\, P.$$

Since $x - x_0 \in S(x_0, C)$, we have

$$Y_F^{\mu}(x_0; x - x_0) \cap -\mathrm{int}\, P = \emptyset.$$

It follows from the P-convexity of F and Proposition 3.1 that

$$F(x) - \mu \subset Y_F^{\mu}(x_0; x - x_0) + P.$$

Thus

$$\xi - \mu \in Y_F^{\mu}(x_0; x - x_0) + P.$$

There exists $\eta \in Y_F^{\mu}(x_0; x - x_0)$, such that $\xi - \mu \in \eta + P$. Then $\eta \in -\mathrm{int}\, P$, a contradiction to (5). □

Remark 3.1. Similarly, if $(x_0, \mu) \in C \times Y$ is a weak maximiser of (P), then

$$Y_F^{\mu}(x_0; d) \cap \mathrm{int}\, P = \emptyset, \quad \forall d \in S(x_0, C). \tag{6}$$

In particular,

$$F_+^{\mu}(x_0; d) \cap \mathrm{int}\, P = \emptyset, \quad \forall d \in S(x_0, C).$$

Consider the following optimization problem (P_1)

$$\text{Weak} - \min \{ f(x) \mid x \in C_1, A(x) = b \},$$

where C_1 is a convex subset of X, $f : X \to Y$, $A : X \to Z$ and Z is a real topology vector space. Now let $C = \{x \in X \mid x \in C_1, A(x) = b\}$. Assume that $x_0 \in C$ is a weak minimizer of (P_1). From Yang (1992), there exists a linear operator $T : X \to Y$ such that

$$f(x) + T(A(x) - b) - f(x_0) \notin -\mathrm{int}\, P, \quad \forall x \in C_1.$$

Without any difficulty under the current setting that X is a real topology vector space and Y is an ordered vector topology space, one can establish the continuity of T, see Wang (1986).

Hitherto, assume that T is continuous. For any $d \in S(x_0, C)$, there exists $t_0 > 0$,

$$f(x_0 + td) + T(A(x_0 + td) - b) - f(x_0) \notin -\mathrm{int}\, P, \quad t \in [0, t_0].$$

From the continuity of T, we have

$$f'_-(x_0; d) + (T \circ A)(d) \notin -\mathrm{int}\, P.$$

Assume that $f'_-(x_0; d)$ is continuous as a function of d. From (3), for each $d \in S(x_0, C)$, there exists $T_d \in \partial_c f(x_0)$ such that

$$T_d(d) + (T \circ A)(d) \notin -\mathrm{int}\, P.$$

4 Jacobificators

Recall that $g : X \to \mathbb{R}$ is a continuous real-valued function. The dual space of X is denoted by X^* and it is equipped with the weak* topology. Let $x \in X$ at which g is finite. The lower and upper Dini directional derivative of g at x in the direction v are defined respectively by

$$g^-(x, v) := \liminf_{t \downarrow 0} \frac{g(x + tv) - g(x)}{t}$$

$$g^+(x, v) := \limsup_{t \downarrow 0} \frac{g(x + tv) - g(x)}{t}.$$

The function $g : X \to \mathbb{R}$ is said to have a convexificator of g at x if there exists a weak* compact convex subset $\partial^{dj} g(x)$ of the dual space X^* satisfying, for all $v \in X$,

$$g^-(x, v) \leq \max_{x^* \in \partial^{dj} g(x)} x^*(v),$$

$$g^+(x, v) \geq \max_{x^* \in \partial^{dj} g(x)} x^*(v).$$

See Jeyakumar and Yang (1997) and references cited therein for details. One advantage of the introduction of a convexificator is that a Mean Value Theorem can be established for a class of continuous functions.

In this section, we extend the above concept of a convexificator for a real-valued function and introduce a Jacobificator of a set-valued mapping. We show that for a P-convex function, the subdifferential $\partial_c f(x_0)$ is an example of an upper Jacobificator.

Definition 4.1. *Let $F : X \to 2^Y$ be a set-valued mapping. Let $x \in X$ and $\mu \in F(x)$. F is said to admit an* **upper Jacobificator** *at x with respect to μ if there is a compact convex subset $\partial_d^+ F(x)$ of $L(X, Y)$ such that for any $d \in X$,*

$$\mathrm{Sup}\langle \partial_d^+ F(x), d \rangle \in F_-^{\mu}(x; d) + P,$$

where the set $\langle \partial_d^+ F(x), d \rangle := \{T(d) \mid T \in \partial_d^+ F(x)\}$.

Definition 4.2. *Let $F : X \to 2^Y$ be a set-valued mapping. Let $x \in X$ and $\mu \in F(x)$. F is said to admit a* **lower Jacobificator** *at x with respect to μ if there is a compact convex subset $\partial_d^- F(x)$ of $L(X, Y)$ such that for any $d \in X$,*

$$\mathrm{Inf}\langle \partial_d^- F(x), d \rangle \in F_+^{\mu}(x; d) - P,$$

where the set $\langle \partial_d^- F(x), d \rangle := \{T(d) \mid T \in \partial_d^- F(x)\}$.

Definition 4.3. *Let $F : X \to 2^Y$ be a set-valued mapping. Let $x \in X$ and $\mu \in F(x)$. F is said to admit a* **Jacobificator** *$\partial_d F(x)$ at x with respect to μ if $\partial_d F(x)$ is both upper and lower Jacobificator of F at x with respect to μ.*

Remark 4.1. (i) Assume that $f : X \to Y$ is a P-convex function on X and that $f'_-(x_0; d)$ is continuous as a function of d. Then from (3)

$$\text{Sup}\langle \partial_c f(x), d\rangle = f'_-(x; d),$$

where $\langle \partial_c f(x), d\rangle = \{T(d) \mid T \in \partial_c f(x)\}$. Thus $\partial_c f(x)$ is an upper Jacobificator of f at x.

(ii) If $Y = R$, $F = g : X \to R$, $P = \mathbb{R}_+$, then $\partial_d F(x) = \partial^{dj} g(x)$.

5 Mean value theorem

In this section, mean value theorems are derived for set-valued mappings using the Jacobificator.

Lemma 5.1. *Assume that (i) $F : X \to 2^Y$ is a set-valued mapping, (ii) for each $x \in X$, $F(x)$ is compact, (iii) $F(a)$ and $F(b)$ are singleton and (iv) Assumption 2.1 holds. Define $H : [0,1] \to 2^Y$ by*

$$H(t) = F(a + t(b-a)) - F(a) + t(F(b) - F(a)), \quad t \in [0,1].$$

Let $\gamma \in (0,1)$, $\mu \in H(\gamma)$ and $c = a + \gamma(b-a)$. Then there exists $\mu^1 \in F(c)$ such that for any $v \in \mathbb{R}$,

$$Y_H^{\mu}(\gamma; v) = Y_F^{\mu^1}(c; v(b-a)) + v(F(a) - F(b)). \tag{7}$$

Proof: Since $\mu \in H(\gamma)$, there exists $\mu^1 \in F(c)$ such that

$$\mu = \mu^1 - F(a) + \gamma(F(b) - F(a)).$$

We have

$$\frac{H(\gamma + t_i v) - \mu}{t_i}$$

$$= \frac{F(a + (\gamma + t_i v)(b-a)) + t_i v(F(a) - F(b)) - \mu^1}{t_i}$$

$$= \frac{F(a + \gamma(b-a) + t_i v(b-a)) - \mu^1 + t_i v(F(a) - F(b))}{t_i}$$

$$= \frac{F(c + t_i v(b-a)) - \mu^1}{t_i} + v(F(a) - F(b)).$$

Then (7) holds. □

Theorem 5.1. *Assume that (i) $F : X \to 2^Y$ is a set-valued mapping, and a, $b \in X$, (ii) for each $x \in (a,b)$, $F(x)$ is compact, $F(a)$ and $F(b)$ are singleton, (iii) F admits a Jacobificator at every point on the interval (a,b), and (iv)*

Assumption 2.1 holds, then there exists $c \in (a, b)$ such that

$$F(b) - F(a) \notin (\mathrm{Sup}\langle \partial_d F(c), b-a\rangle + \mathrm{int}\, P) \cup (\mathrm{Inf}\langle \partial_d F(c), b-a\rangle - \mathrm{int}\, P). \tag{8}$$

Proof: Define

$$H(t) = F(a + t(b-a)) - F(a) + t(F(b) - F(a)), \quad t \in [0, 1].$$

Then

$$H(0) = H(1) = 0.$$

Since $F(x)$ is compact for every $x \in (a, b)$ and $F(a)$ and $F(b)$ are singleton, it is clear that $H(t)$ is compact for every $t \in [0, 1]$. So $\cup\{H(t) | t \in [0, 1]\}$ is compact. Thus H attains a weak minimiser or maximiser at some $\gamma \in (0, 1)$. Assume first that H attains a weak minimiser at some $\gamma \in (0, 1)$. From Theorem 3.1 (i), there exists $\mu \in H(\gamma)$ such that

$$Y_H^{\mu}(\gamma; v) \cap -\mathrm{int}\, P = \varnothing, \quad \forall v \in \mathbb{R}.$$

Then from Lemma 5.1,

$$(Y_F^{\mu^1}(c; v(b-a)) = v(F(a) - F(b))) \cap -\mathrm{int}\, P = \varnothing, \quad \forall v \in \mathbb{R},$$

where $c = a + \gamma(b-a)$ and $\mu^1 \in F(c)$. Consequently, from Assumption 2.1, $F_-^{\mu^1}(c; v(b-a)) \neq \varnothing$, and

$$(F_-^{\mu^1}(c; v(b-a)) + v(F(a) - F(b))) \cap -\mathrm{int}\, P = \varnothing, \quad \forall v \in \mathbb{R}.$$

Let $v = 1$. Then

$$(F_-^{\mu^1}(c; b-a) + F(a) - F(b)) \cap -\mathrm{int}\, P = \varnothing. \tag{9}$$

From Definition 4.1,

$$\mathrm{Sup}\langle \partial_d F(c), b-a\rangle \in F_-^{\mu^1}(c; b-a) + P, \quad \forall d \in X. \tag{10}$$

Then, from (9) and (10),

$$F(b) - F(a) \notin \mathrm{Sup}\langle \partial_d F(c), b-a\rangle + \mathrm{int}\, P. \tag{11}$$

Let $v = -1$. Then

$$(F_-^{\mu^1}(c; a-b) + F(b) - F(a)) \cap -\mathrm{int}\, P = \varnothing. \tag{12}$$

From Definition 4.1,

$$\mathrm{Sup}\langle \partial_d F(c), a-b\rangle \in F_-^{\mu^1}(c; a-b) + P. \tag{13}$$

Then from (12) and (13),

$$F(b) - F(a) \notin -\mathrm{Sup}\langle \partial_s F(c), a-b\rangle - \mathrm{int}\, P = \mathrm{Inf}\langle \partial_s F(c), b-a\rangle - \mathrm{int}\, P. \tag{14}$$

Combining (11) and (14), we have

$$F(b) - F(a) \notin (\mathrm{Sup}\langle \partial_d F(c), b-a\rangle + \mathrm{int}\, P) \cup (\mathrm{Inf}\langle \partial_d F(c), b-a\rangle - \mathrm{int}\, P).$$

Assume now that H attains a weak maximiser at some $\gamma \in (0,1)$. From (6), there exists $\mu \in H(\gamma)$ such that

$$Y_H^{\mu}(\gamma; v) \cap \mathrm{int}\, P = \emptyset, \quad \forall v \in \mathbb{R}.$$

Then from Lemma 5.1,

$$(Y_F^{\mu^1}(c; v(b-a)) + v(F(a) - F(b))) \cap \mathrm{int}\, P = \emptyset, \quad \forall v \in \mathbb{R},$$

where $c = a + \gamma(b-a)$ and $\mu^1 \in F(c)$. Consequently,

$$(F_+^{\mu^1}(c; v(b-a)) + v(F(a) - F(b))) \cap \mathrm{int}\, P = \emptyset, \quad \forall v \in \mathbb{R}.$$

As before, let $v = 1, -1$ and apply Definition 4.1, we have

$$F(b) - F(a) \notin \mathrm{Inf}\langle \partial_d F(c), b-a\rangle - \mathrm{int}\, P.$$

$$F(b) - F(a) \notin \mathrm{Sup}\langle \partial_d F(c), b-a\rangle + \mathrm{int}\, P.$$

Therefore (8) holds. □

Corollary 5.1. *If $f: X \to Y$ is differentiable and $a, b \in X$, then*

$$f(b) - f(a) \notin (\nabla f(c)(b-a) + \mathrm{int}\, P) \cup (\nabla f(c)(b-a) - \mathrm{int}\, P),$$

where $c \in (a, b)$.

Corollary 5.2. *If $f: X \to Y$ is P-convex, and a, $b \in X$ then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) \notin (\mathrm{Sup}\langle \partial_c f(c), b-a\rangle + \mathrm{int}\, P) \cup (\mathrm{Inf}\langle \partial_c f(c), b-a\rangle - \mathrm{int}\, P). \tag{15}$$

Proof: Define

$$h(t) = f(a + t(b-a)) - f(a) + t(f(b) - f(a)), \quad t \in [0,1].$$

Then

$$h(0) = h(1) = 0.$$

Since f is P-convex, it is clear that h attains a minimal at some $\gamma \in (0, 1)$. From Theorem 3.1 (i), we have

$$Y_h(\gamma; v) \cap -\text{int}\, P = \emptyset, \quad \forall v \in \mathbb{R}.$$

Then from Lemma 5.1,

$$(Y_f(c; v(b-a)) + v(f(a) - f(b))) \cap -\text{int}\, P = \emptyset, \quad \forall v \in \mathbb{R},$$

where $c = a + \gamma(b-a)$. Consequently,

$$(f_-(c; v(b-a)) + v(f(a) - f(b))) \cap -\text{int}\, P = \emptyset, \quad \forall v \in \mathbb{R}.$$

Then the rest of the proof is similar to that of Theorem 5.1 and is omitted. Then, (15) holds. □

Next result shows that for a real-valued function, Theorem 5.1 is reduced to a Mean Value Theorem in Jeyakumar and Yang (1997).

Corollary 5.3. *Let a, $b \in X$ and let $g : X \to \mathbb{R}$ be a continuous function. Assume, that, for each $x \in (a, b)$, $\partial^{dj} g(x)$ is a convexificators of g at x. Then, there exists $c \in (a, b)$ and $x^* \in \partial^{dj} g(c)$ such that*

$$g(b) - g(a) = x^*(b-a). \tag{16}$$

Proof: It follows from Remark 4.1 (ii) and Theorem 5.1 that

$$g(b) - g(a) \notin (\text{Sup}\langle \partial^{dj} g(c), b-a \rangle + \text{int}\, P) \cup (\text{Inf}\langle \partial^{dj} g(c), b-a \rangle - \text{int}\, P).$$

where $P = \mathbb{R}_+$. Thus

$$\text{Inf}\langle \partial^{dj} g(c), b-a \rangle \leq g(b) - g(a) \leq \text{Sup}\langle \partial^{dj} g(c), b-a \rangle.$$

Since $\partial^{dj} g(x)$ is compact and convex for each x, there exist $c \in (a, b)$ and $x^* \in \partial^{dj} g(c)$ such that (16) holds. □

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