Math Meth Oper Res (1998) 48:229–246

Mathematical Methods
of Operations Research

Cooperative behavior of functions, relations and sets

Jean-Paul Penot

Laboratoire de Mathématiques Appliquées, CNRS UPRES A 5033 Université de Pau,
Av. de l'Université, F-64000 PAU, France
(e-mail: jean-paul.penot@univ-pau.fr)

Received August 1997/Revised version March 1998

Abstract. One of the main difficulties in nonsmooth analysis is to devise calculus rules. It is our purpose here to show that a certain cooperative behavior between functions (resp. sets, resp. multifunctions) yields calculus rules for subdifferentials (resp. normal cones, resp. coderivatives). In previous contributions, the qualification conditions ensuring calculus rules were given in a non symmetric way. The new conditions can be combined easily and encompass various criteria. We also address the important question of the extension of calculus rules from the Lipschitz case to the general case.

Key words: Alliedness, coderivative, homotone, normal, normal compactness, quasi-homotone, subdifferential, synergy

1. Subdifferentials

There are several ways of presenting subdifferentials (see for instance [13], [16], [39], and their references). As noticed by several authors, a unified approach through a set of general properties is convenient: in such a way, specific constructions can be avoided for most developments. In the sequel, we denote by $\mathscr{X}$ a class of normed vector spaces (n.v.s.), for instance the class of all Banach spaces, the class of separable spaces or the class of Asplund spaces. For X in $\mathscr{X}$, $\mathscr{F}(X)$ denotes a subset of the set of lower semicontinuous functions from X to $\mathbb{R}^{\bullet} = \mathbb{R} \cup \{\infty\}$.

We consider a *subdifferential* ∂ associated with the families $\mathscr{X}$, $\mathscr{F}$ as a mapping which associates to any X in $\mathscr{X}$, $f \in \mathscr{F}(X)$, $x \in X$ a subset $\partial f(x)$ of X^* in such a way that the following properties are satisfied (here $L(X, Y)$ denotes the space of linear continuous maps from X to Y, $A \in L(X, Y)$, $b \in Y$, $c \in \mathbb{R}_+$)

(S1) If $f(x) = \infty$ then $\partial f(x)$ is empty;
(S2) if f and g coincide on some neighborhood of x then $\partial f(x) = \partial g(x)$;

(S3) if f is convex then $\partial f(x) = \{x^* : f(\cdot) \geq x^*(\cdot) + f(x) - \langle x^*, x\rangle\}$;
(S4) if f attains at x a local minimum then $0 \in \partial f(x)$;
(S5) if $X = Y \times Z, f(y,z) = g(y) + h(z)$ then $\partial f(y,z) \subset \partial g(y) \times \partial h(z)$;
(S6) if $f(\cdot) = cg(A(\cdot) + b)$, with $A(X) = Y$ then $\partial f(x) \subset c\partial g(A(x) + b) \circ A$.

Related axioms can be given to introduce a notion of *normal cone* to a subset S of X belonging to a certain family $\mathscr{C}(X)$ of subsets C of X. We suppose that for each $f \in \mathscr{F}(X)$ the epigraph $E_f = \operatorname{epi} f$ of f given by $E_f := \{(x,r) \in X \times \mathbb{R} : r \geq f(x)\}$ belongs to $\mathscr{C}(X \times \mathbb{R})$. Given a subdifferential ∂ the normal cone to S at x can be introduced either as $\partial \iota_S(x)$, where ι_S is the indicator function of S ($\iota_S(x) = 0$ for $x \in S$, ∞ for $x \in X \setminus S$) else as the set given by

$$N(S,x) := \mathbb{R}_+ \partial d_S(x)$$

where d_S is the distance function to $S : d_S(x) := d(x,S) := \inf_{y \in S} d(x,y)$. Here we choose the second process as a definition of the normal cone. In turn, starting from a subdifferential ∂ defined on the class of locally Lipschitzian functions, the normal cone concept enables to extend to any function the subdifferential ∂ by setting

$$\partial f(x) := \{x^* : (x^*, -1) \in N(E_f, x_f)\} \quad \text{with } x_f := (x, f(x))$$

when $f(x)$ is finite, the empty set otherwise. Taking the sum norm on products, one shows easily the following result.

Proposition 1.1. *If ∂ satisfies properties (S1)–(S5) on the class of Lipschitzian functions, then its extension defined as above satisfies the same properties on the class of l.s.c. functions.*

Examples. The main examples we have in mind besides the circa-subdifferential of Clarke [6], the moderate subdifferential of Michel-Penot [28] and the b-subdifferential of Treiman [44] are the following ones.

1) *The Fréchet subdifferential* of f at x is the set $\partial^- f(x)$ of all vectors x^* satisfying

$$\liminf_{\|u\| \to 0_+} \|u\|^{-1}(f(x+u) - f(x) - \langle x^*, u\rangle) \geq 0.$$

2) *The Hadamard (or contingent) subdifferential* consists of all x^* satisfying

$$\langle x^*, u\rangle \leq f'(x,u) := \liminf_{(t,v) \to (0_+, u)} t^{-1}(f(x+tv) - f(x)), \quad \forall u \in X.$$

These two preceding examples are particular cases of the notion of subdifferential associated to a bornology $\mathscr{B}$. In the first case one takes for $\mathscr{B}$ the whole family of bounded subsets; in the second case one takes for $\mathscr{B}$ the family of compact subsets.

3) *The viscosity subdifferentials* which are defined with the help of auxiliary smooth functions (see [3], [8], [16]). They are of much use in the study of Hamilton-Jacobi equations.

These three classes of subdifferentials can be considered as forming the group of so-called *elementary subdifferentials*.

Important procedures enable one to get other classes of subdifferentials.

4) *The limiting or stabilization procedure*. Starting with any subdifferential ∂, one defines a new subdifferential $\bar{\partial}$ called the *limiting* or *stabilized* subdifferential associated with ∂ (see [25], [29]–[33] and their references). It is obtained as follows: x^* belongs to $\bar{\partial}f(x)$ if it is a weak* cluster point of a sequence (x_n^*) such that $x_n^* \in \partial f(x_n)$ for each n, where $(x_n) \to x$ and $(f(x_n)) \to f(x)$. A similar definition holds for normal cones.

5) *The approximation and stabilization procedure* (see [13], [16]). Given a Lipschitzian function f with rate c and an element V of the family $\mathscr{V}$ of vector subspaces of X, one sets

$$\partial_V f(x) = \{x^* : x^* \mid V \in \partial(f \mid V)(x)\}$$

and one defines the approximate subdifferential of f at x by

$$\hat{\partial}f(x) = \bigcap_{V \in \mathscr{V}} \limsup_{u \to x} \partial_V f(u) \cap cB_{X*}.$$

In the general case one defines $\hat{\partial}f(x)$ as above, using the normal cone to the epigraph of f.

Let us note the following result whose proof is easy.

Proposition 1.2. *If ∂ is a subdifferential satisfying properties (S1)–(S6), then the limiting subdifferential $\bar{\partial}$ satisfies the same properties.*

2 Additional properties of subdifferentials

The following crucial homotonicity (or isotonicity or monotonicity) property corresponds to an antitonicity property of the normal cone associated with the subdifferential. It ensures that the subdifferential has a certain accurateness. On the contrary, a subdifferential which does not satisfy it cannot be very precise.

Definition 2.1. ([39]) *The subdifferential ∂ is said to be* **homotone** *if*

$$f \geq g, \quad f(x_0) = g(x_0) \quad \Rightarrow \quad \partial f(x_0) \supset \partial g(x_0).$$

The elementary subdifferentials are homotone. Although this important property is not shared by other subdifferentials such as the limiting subdifferentials, the Clarke subdifferential, the moderate subdifferential and the approximate subdifferential, one can detect a weaker property which may serve as a substitute in certain situations.

Definition 2.2. *The subdifferential ∂ is said to be* **quasi-homotone** *relatively to a class $\mathscr{N}(X)$ of compatible norms on X if for any $f \in \mathscr{F}(X)$, any subset S of X one has, for the distance function d_S associated with a norm in $\mathscr{N}(X)$,*

$$f \geq d_S, \quad f|_S = 0 \quad \Rightarrow \quad \forall x_0 \in S \quad \partial f(x_0) \supset \partial d_S(x_0)$$

If one can take for $\mathcal{N}(X)$ the class of all compatible norms on X, we simply say that ∂ is quasi-homotone. This property is satisfied for the limiting Fréchet subdifferential in Asplund spaces and for the approximate subdifferential in any Banach space (thanks to the Ioffe's separable reduction principle [14]). It has been recently proved in [41] that it is also satisfied for the Clarke subdifferential. The following two consequences are worth noticing.

Proposition 2.3. *If ∂ is quasi-homotone, then for any subset S of X and any $x_0 \in S$ on has $[0,1]\partial d_S(x_0) = \partial d_S(x_0)$.*

Proof. Given $r \in]0,1]$, taking $f = r^{-1} d_S$ we get that $\partial d_S(x_0) \subset r^{-1}\partial d_S(x_0)$. Since d_S attains its infimum at x_0 we also have $0 \,.\, \partial d_S(x_0) \subset \partial d_S(x_0)$. Thus $[0,1]\partial d_S(x_0) \subset \partial d_S(x_0)$; the reverse inclusion is obvious. $\square$

Proposition 2.4. *If ∂ is quasi-homotone relatively to a class $\mathcal{N}(X)$ of compatible norms on X, then the normal cone associated with ∂ does not depend on the choice of the norm in $\mathcal{N}(X)$.*

Proof. Let $\|\cdot\|'$ be another norm in $\mathcal{N}(X)$: for some $b, c > 0$ one has

$$b\|\cdot\| \leq \|\cdot\|' \leq c\|\cdot\|,$$

so that, for any subset S of X one has for the associated distance function d'_S

$$bd_S \leq d'_S \leq cd_S.$$

Then it follows from the quasi-homotonicity property with $f := c^{-1}d'_S$ that $\partial d'_S(x_0) \supset c\partial d_S(x_0)$. Similarly, assuming that ∂ is quasi-homotone with respect to the norm $\|\cdot\|'$ we get $b\partial d_S(x_0) \supset \partial d'_S(x_0)$. $\square$

Moreover quasi-homotonicity will play a key role when dealing with the following linear estimate (LE) around x_0 for the distance functions associated with the sets of a family $S_1, \ldots, S_k$ with $x_0 \in S := S_1 \cap \cdots \cap S_k$; here $\mathcal{N}(x_0)$ denotes the family of neighborhoods of x_0.

(LE) for some $c > 0$ and some $V \in \mathcal{N}(x_0) : \forall v \in V$

$$d(v, S) \leq c \sum_{i=1}^{k} d(v, S_i).$$

The estimate (LE) and its consequences have been studied by a number of authors including Federer [9], Ioffe [13]–[16], Clarke-Raissi [7], Jourani [20], Jourani-Thibault [23]–[24], Azé-Chou-Penot [1].

Definition 2.5. *The subdifferential ∂ is said to satisfy the* **(exact) sum rule** *on the class $\mathcal{F}$ if for any finite subset $\{f_1, \ldots, f_k\}$ of $\mathcal{F}(X)$ and any $x \in$* dom $f_1 \cap \cdots \cap$ dom f_k

$$\partial(f_1 + \cdots + f_k)(x) \subset \partial f_1(x) + \cdots + \partial f_k(x),$$

provided the following **linear-rate metric qualification condition** *is satisfied:*

$(LRQC)$ *there exist* $c > 0$, $\rho > 0$, $U \in \mathcal{N}(x)$ *such that the inequality*

$$d\left(\left(u, \sum r_i\right), \operatorname{epi} \sum f_i\right) \le c \sum d((u, r_i), \operatorname{epi} f_i)$$

holds for all $u \in U$ *and all* $r_i \in [f_i(x) - \rho, f_i(x) + \rho]$.

This condition is satisfied when all the functions but one at most are Lipschitzian.

The notions we delineated enable us to get an easy proof to the following rule for the calculus of the normal cone to an intersection.

Proposition 2.6. *Suppose* ∂ *is quasi-homotone and satisfies the sum rule on the class of Lipschitzian functions. Let* $S_1, \ldots, S_k$ *be closed subsets of* X *satisfying the linear estimate* (LE) *around* $x \in S := S_1 \cap \cdots \cap S_k$. *Then*

$$N(S, x) \subset N(S_1, x) + \cdots + N(S_k, x).$$

Proof. Let us set $f(u) := c\sum_{i=1}^{k} d(u, S_i)$, so that $f(u) \ge d(u, S)$ for $u \in V$ and $f(u) = 0$ for each $u \in S$. Then, as ∂ is quasi-homotone and satisfies the sum rule for Lipschitzian functions we have

$$\partial d_S(x) \subset \partial f(x) \subset \sum_{i=1}^{k} c\partial d_{S_i}(x).$$

Taking the cones generated by these sets we get the announced inclusion. □

A number of calculus rules can be derived in a simple way from a relation pertaining to the subdifferential of a performance function (see [17]). Recall that the performance function p associated with a function f on the product of two Banach spaces W and X is the function given by

$$p(w) = \min_{x \in X} f(w, x).$$

The following notion (often reproduced under different guises) will be needed.

Definition 2.7. *([35]) Given a multifunction* $S : W \rightrightarrows X$ *and* $w \in W$, $B \subset W$, $C \subset X$, S *is called* **lower semicontinuous** *at* (w, C) *on* B *if for any net* $(w_i)_{i \in I}$ *in* B *converging to* w, *there exists a subnet* $(w_{i(k)})_{k \in K}$ *and a net* $(x_k)_{k \in K}$ *in* D *converging to a certain* $x \in C$ *such that* $x_k \in S(w_{i(k)})$ *for each* $k \in K$.

When C is a singleton $\{x\}$, this definition corresponds to the usual lower semicontinuity of S at (w, x). Another extreme case is when $C = X$. Intermediate cases such as $C = S(w)$ may also be of an interest. It will be convenient to adopt the following terminology.

Definition 2.8. *Let us say that* ∂ **satisfies property** (P) *or is* **performable** *if whenever* f *and* p *are as above and the solution mapping* S *given by* $S(u) =$

argmin$f(u,\cdot)$ *is lower semicontinuous at* (w,C) *for some* $C \subset S(w)$ *one has*

$$w^* \in \partial p(w) \quad \Rightarrow \quad (w^*,0) \in \bigcup_{x\in C} \partial f(w,x).$$

We say that ∂ satisfies property (N) if for any subsets, E, F of X and W respectively, any $w \in C$, $C \subset E$ and any surjective continuous linear map $A : X \to W$ such that $(A|E)^{-1}$ is lower semicontinuous on F at (w,C) one has

$$A^T(N(F,w)) \subset \bigcup_{e\in C} N(E,e).$$

Proposition 2.9. *([17]) Any elementary subdifferential (Hadamard, Fréchet or viscosity)* ∂ *satisfies properties* (N) *and* (P). *The limiting Fréchet subdifferential and the approximate subdifferential satisfy property* (P).

Proposition 2.10. *([43]) Suppose* ∂ *satisfies property* (P) *and the exact sum rule on the class of Lipschitzian functions. Then if* $A : X \to Y$ *is linear and continuous and if* g *is Lipschitzian on* Y, *for* $f = g \circ A$ *one has*

$$\partial f(x) \subset A^T(\partial g(Ax)).$$

The following result extends [16] Proposition 6.1 and Corollary 6.2 to a general subdifferential.

Proposition 2.11. *([43]) Suppose* ∂ *is quasi-homotone, satisfies property* (P) *and the exact sum rule on the class of Lipschitzian functions. Then it satisfies the sum rule on the class of l.s.c. functions.*

The following notion represents a variant of the sum rule which corresponds exactly to our needs in the next section. Thus it is a variant of the notion of trustworthiness due to A. D. Ioffe [13]. Moreover, we incorporate in it a quantitative concern. The idea of taking just a part N_u of the normal cone N instead of the whole normal cone has already been used by Ioffe ([16]) and Jourani and Thibault ([23], [24]); however, the estimates we get below using such a concept seem to be new.

Definition 2.12. *A subdifferential* ∂ *is said to be* **amiable** *(resp. metrically amiable) with respect to a subset* S *of a Banach space* X *if for any* $\varepsilon > 0$, *and any convex function* f *on* X *which is Lipschitzian with rate 1 and attains its infimum on* S *at some* $x \in S$ *there exist* $y \in S \cap B(x,\varepsilon)$, $z \in B(x,\varepsilon)$, $y^* \in N(S,y)$ (resp. $y^* \in N_u(S,y) := \partial d_S(y)$), $z^* \in \partial f(z)$ *such that* $\|y^* + z^*\| \le \varepsilon$.

If one can take $\varepsilon = 0$ in what precedes, ∂ is said to be *exactly amiable* or *exactly metrically amiable* respectively. It is said to be amiable on X with respect to a class $\mathscr{C}(X)$ of subsets of X if it is amiable with respect to S for each S in the class $\mathscr{C}(X)$. In particular, if $\mathscr{C}(X)$ is the whole family of closed subsets of X, it is said to be amiable on X. It is said to be amiable on a family of Banach spaces $\mathscr{X}$ if for any X in $\mathscr{X}$ it is amiable on X. Clearly, if ∂ satisfies the sum rule it is exactly amiable. If moreover $\partial f(x) \subset B_{X^*}$ whenever f is Lipschitzian with rate 1 and $x \in X$, then ∂ is exactly metrically amiable. A similar observation holds when ∂ satisfies a fuzzy sum rule.

3 Allied and synergetic sets

The following two symmetric properties represent a kind of cooperative behavior which will be at the heart of our study of subdifferential calculus and codifferential calculus. They are closely related. The simplest one (which is also the strongest one) receives the simplest terminology (borrowed from Jameson in his study of cones). It emerged from the works of A. D. Ioffe ([13]–[14] and latter contributions), Jourani ([20]), Jourani and Thibault ([23], [24]) and the author ([36]–[40]). These two definitions can be extended to an arbitrary number of subsets; we only consider two subsets for notational convenience and simplicity. Here we denote weak* convergence by $\overset{*}{\to}$ and $(x_n) \overset{F}{\to} z_0$ means $(x_n) \to z_0$ and $x_n \in F$ for each n.

Definition 3.1. *([40]) Given a member X of $\mathcal{X}$, a subdifferential ∂ and the normal cone notion N associated with it, two elements F, G of the family $\mathscr{C}(X)$ are said to be* **allied** *at $(x_0, y_0) \in F \times G$ if whenever $(x_n) \overset{F}{\to} x_0$, $(y_n) \overset{G}{\to} y_0$, $x_n^* \in N(F, x_n)$, $y_n^* \in N(G, y_n)$, the relation $(\|x_n^* + y_n^*\|) \to 0$ implies $(\|x_n^*\|) \to 0$, $(\|y_n^*\|) \to 0$.*

Definition 3.2. *Given X, ∂, N, F, G as above, F and G are said to be* **synergetic** *at $(x_0, y_0) \in F \times G$ if whenever $(x_n) \overset{F}{\to} x_0$, $(y_n) \overset{G}{\to} y_0$, $x_n^* \in N(F, x_n)$, $y_n^* \in N(G, y_n)$, the relations $(\|x_n^* + y_n^*\|) \to 0$, $(x_n^*) \overset{*}{\to} 0$, $(y_n^*) \overset{*}{\to} 0$ imply $(\|x_n^*\|) \to 0$, $(\|y_n^*\|) \to 0$.*

Usually one takes $x_0 = y_0 = z_0$ for some point $z_0 \in F \cap G$ and one says that F and G are allied (resp. synergetic) at z_0. If in the preceding definitions one imposes $x_n^* \in N_u(F, x_n) := \partial d_F(x_n)$ (resp. $y_n^* \in N_u(G, y_n) := \partial d_G(y_n)$) instead of $x_n^* \in N(F, x_n)$ (resp. $y_n^* \in N(G, y_n)$) one says that F and G are *metrically allied* or *metrically synergetic* respectively. We observe that in a finite dimensional space X any pair of subsets F, G of X is always synergetic (but not always allied). The relationships between these two notions are clarified in the following statement.

Proposition 3.3. *Alliedness implies synergy. Conversely, under the following qualification condition in which $\overline{N}$ is the limiting normal cone associated with N, synergy of F, G at z_0 implies alliedness of F, G at z_0*

$$\overline{N}(F, z_0) \cap (-\overline{N}(G, z_0)) = \{0\}.$$

Proof. The first assertion is obvious. Suppose the qualification condition holds and F and G are not allied at z_0. Then there exist $c > 0$ and sequences $(x_n) \overset{F}{\to} z_0$, $(y_n) \overset{G}{\to} z_0$, $x_n^* \in N(F, x_n)$, $y_n^* \in N(G, y_n)$, such that $(\|x_n^* + y_n^*\|) \to 0$ and $r_n := \max(\|x_n^*\|, \|y_n^*\|) \geq c$ for each n. Dividing x_n^* and y_n^* by r_n we may suppose x_n^* and y_n^* belong to the closed unit ball B_{X^*} of X^* for each n. In view of our qualification condition any weak* cluster point u^* of (x_n^*) must be 0 as $-u^*$ is a cluster point of (y_n^*). Thus, by the weak* compactness of B_{X^*}, we get $(x_n^*) \overset{*}{\to} 0$, $(y_n^*) \overset{*}{\to} 0$ and F and G are not synergetic at z_0. □

In order to present examples, let us recall the following notion which has been introduced (in terms of nets) in the convex case in [36] and in the nonconvex case in [38]. The influence of properties detected in [27] has been decisive for the emergence of this concept, in conjunction with the notion of operator of type (S_+) due to F. E. Browder ([4], [5]; see also [2] for uses in optimization theory). The sequential form we adopt here arouse from discussions with A. Ioffe to whom other variants [17] and their characterizations are due [16].

Definition 3.4. *A set F in $\mathscr{C}(X)$ is said to be (sequentially)* **normally compact** *at $x_0 \in F$ if for any sequences $(x_n) \xrightarrow{F} x_0$, $(x_n^*) \xrightarrow{*} 0$ in X^* such that $x_n^* \in N(F, x_n)$ for each n, one has $(x_n^*) \to 0$.*

The relationships of this notion with synergy are clear and simple.

Proposition 3.5. *If F is normally compact at z_0, then for any subset G the sets F, G are synergetic at z_0.*

Proof. The result is immediate since for any sequences as in Definition 3.2 one has $(\|y_n^*\|) \to 0$ whenever $(\|x_n^*\|) \to 0$. $\square$

Thus, the following examples provide important instances in which normal compactness is satisfied.

Example. Any finite codimensional C^1-submanifold of X is normally compact.

Example. Any convex subset with a nonempty interior is normally compact (see [36] Lemma 2.13).

Example. Any Loewen set around z_0 is normally compact around z_0 (see [27], [38], [16]). Here the set F is said to be a *Loewen set* (for ∂ or N) around $z_0 \in F$ if

(LC) there exist a neighborhood V of z_0 and a weak* closed, weak* locally compact cone C in X^* such that $N(F, x) \subset C$ for each $x \in V \cap F$.

Example. ([27] for the Fréchet normal cone, [19]–[22], [16] for the case of the approximate normal cone). The set F is a Loewen set whenever F is *compactly epi-Lipschitzian around* z_0 in the sense: there exist $\tau > 0$, a compact set K and $\rho > 0$ such that for each $t \in (0, \tau)$

$$F \cap (z_0 + \rho B_X) + t\rho B_X \subset F - tK.$$

The situation in Proposition 3.5 and in the subsequent examples is not symmetric; however, there are cases in which both subsets have to play a role.

Example. ([36]) Two closed convex subsets C and D of X are synergetic at $z_0 \in C \cap D$ whenever they are tranverse (see [34]) in the sense that $\mathbb{R}_+(C - z_0) - \mathbb{R}_+(D - z_0) = X$.

Synergy is satisfied in cases more general than the cases of a Loewen set or of a normally compact set, since in the following statement one can suppose F_1 and G_2 are Loewen sets and F_2, G_1 are arbitrary.

Proposition 3.6. *Suppose F_i and G_i are allied (resp. synergetic) at $(x_i, y_i) \in F_i \times G_i$ for $i = 1, 2$. Then $F_1 \times F_2$ and $G_1 \times G_2$ are allied (resp. synergetic) at $((x_1, x_2), (y_1, y_1))$.*

Proof. It is an easy consequence of the definitions and of property (S5) of subdifferentials. □

Thus the alliedness and the synergy properties can be easily combined.

The following result is the main motivation for the introduction of the alliedness notion (see also [14]).

Theorem 3.7. *Suppose ∂ is amiable (resp. metrically amiable). Then alliedness (resp. metric alliedness) implies the linear estimate* (LE).

Theorem 3.7 is a consequence of the following two more precise results. We begin with the metric version; the other one, treated in Proposition 3.9, is similar. The proof we give has many similarities with several previous results (see [16], Prop. 6.3 and its references), but is also has its specificities.

Proposition 3.8. *Let $S_1, \dots, S_k$ be closed subsets of the Banach space X. Suppose ∂ is metrically amiable for $X \times S_1 \times \cdots \times S_k$ or exactly metrically amiable for $S_1 \times \cdots \times S_k$ and there exist $a > 0$, $r > 0$, $t \in]0, 1[$, $x_0 \in S := \bigcap_{j=1}^k S_j$ such that for any $x_j \in S_j \cap B(x_0, r)$, $x_j^* \in N_u(S_j, x_j)$ one has*

$$\max_{1 \le j \le k} \|x_j^*\| \ge t \quad \Rightarrow \quad \left\| \sum_{j=1}^k x_j^* \right\| \ge a^{-1} t. \tag{1}$$

Then there exists $s > 0$ depending on a, r only such that for each $x \in V := B(x_0, s)$ one has

$$d(x, S) \le a t^{-1} \sum_{j=1}^k d(x, S_j). \tag{2}$$

Proof. Let us first consider the case ∂ is metrically exactly amiable on $S_1 \times \cdots \times S_k$. Let $s > 0$ be such that $s < (a^{-1} + 2)^{-1} r$. Let us show that for each $c > a$ and each $x \in V := B(x_0, s)$ the relation

$$d(x, S) \le c t^{-1} \sum_{j=1}^k d(x, S_j) \tag{3}$$

holds. Taking the infimum on c, the result will follow.

Suppose on the contrary that there exists some $u \in V$ such that

$$d(u, S) > c t^{-1} \sum_{j=1}^k d(u, S_j). \tag{4}$$

Then there exist some $u_j \in S_j$ satisfying

$$d(u, S) > c t^{-1} \sum_{j=1}^k \|u - u_j\|. \tag{5}$$

Let $b > 0$ be so small that

$$(1+b)^{-1}(1-b) > t, \quad c^{-1} < a^{-1}(1+b) - kbt^{-1}.$$

Applying Ekeland's theorem to f given by $f(x, x_1, \ldots, x_k) := \sum_{j=1}^{k} \|x - x_j\|$ on $Z := X \times S_1 \times \cdots \times S_k$ endowed with the metric given by

$$d((x, x_1, \ldots, x_k), (x', x'_1, \ldots, x'_k)) = c^{-1}t\|x - x'\| + b\sum_{j=1}^{k} \|x_j - x'_j\|$$

we can find $(v, v_1, \ldots, v_k) \in Z$ such that

$$\sum_{j=1}^{k} \|v - v_j\| + c^{-1}t\|u - v\| + b\sum_{j=1}^{k} \|u_j - v_j\| \leq \sum_{j=1}^{k} \|u - u_j\| \tag{6}$$

$$\sum_{j=1}^{k} \|v - v_j\| \leq \sum_{j=1}^{k} \|x - x_j\| + c^{-1}t\|x - v\| + b\sum_{j=1}^{k} \|x_j - v_j\| \tag{7}$$

for each $(x, x_1, \ldots, x_k) \in Z$.

Since $d(u, S) \leq d(u, x_0) \leq s$, relations (6) and (5) ensure that

$$\|u - v\| \leq ct^{-1}\sum_{j=1}^{k} \|u - u_j\| < d(u, S) \leq s,$$

$$\|v_j - x_0\| \leq \|v_j - v\| + \|v - u\| + \|u - x_0\| \leq (c^{-1} + 2)s < r.$$

Setting

$$g(x, x_1, \ldots, x_k) = \sum_{j=1}^{k} \|x - x_j\| + c^{-1}t\|x - v\| + b\sum_{j=1}^{k} \|x_j - v_j\|,$$

$$p(x_1, \ldots, x_k) = \inf_{x \in X} g(x, x_1, \ldots, x_k),$$

we get from (7) that for any $(x_1, \ldots, x_k) \in S_1 \times \cdots \times S_k$ we have

$$p(v_1, \ldots, v_k) \leq \sum_{j=1}^{k} \|v_j - v\| \leq p(x_1, \ldots, x_k),$$

and equality holds for $(x_1, \ldots, x_k) = (v_1, \ldots, v_k)$.

As g is Lipschitzian with rate $(1 + b)$ with respect to $(x_1, \ldots, x_k)$ uniformly in x, the performance function p is Lipschitzian with rate $(1 + b)$. Since ∂ is exactly metrically amiable for $S_1 \times \cdots \times S_k$ we can find $(v_1^*, \ldots, v_k^*) \in \partial p(v_1, \ldots, v_k)$ such that

$$-(1+b)^{-1}(v_1^*, \ldots, v_k^*) \in N_u(S_1 \times \cdots \times S_k, (v_1, \ldots, v_k)). \tag{8}$$

As $g(\cdot, v_1, \ldots, v_k)$ attains its infimum on X at $x = v$, a well known rule of subdifferential calculus in convex analysis yields

$$(0, v_1^*, \ldots, v_k^*) \in \partial g(v, v_1, \ldots, v_k).$$

Now, as

$$g(x, x_1, \ldots, x_k) = h(A(x, x_1, \ldots, x_k)) + c^{-1}t\|x - v\| + b\sum_{j=1}^{k} \|x_j - v_j\|$$

with $A(x, x_1, \ldots, x_k) := (x_1 - x, \ldots, x_k - x)$, $h(y_1, \ldots, y_k) := \sum_{j=1}^{k} \|y_j\|$, there exist $y_j^* \in \partial\| \cdot \|(v_j - v)$ for $j = 1, \ldots, k$ such that

$$(0, v_1^*, \ldots, v_k^*) = \left(-\sum_{j=1}^{k} y_j^*, y_1^*, \ldots, y_k^*\right) + c^{-1}tB_{X^*} \times bB_{X^*} \times \cdots \times bB_{X^*}. \tag{9}$$

We cannot have $v_j = v$ for each j since otherwise we would have $v \in S$ and, by (5), (6),

$$\sum_{j=1}^{k} \|u - u_j\| < c^{-1}t\, d(u, S) \le c^{-1}t\|u - v\| \le \sum_{j=1}^{k} \|u - u_j\|,$$

a contradiction. Therefore, $\max_{1 \le j \le k} \|y_j^*\| = 1$. Since we take the sum norm on a product, it follows from (S5) that $(1 + b)^{-1}v_j^* \in N_u(S_j, v_j)$ with $v_j \in S_j \cap B(x_0, r)$ for $j = 1, \ldots, k$, and with (9) we get

$$\max_{1 \le j \le k} (1 + b)^{-1}\|v_j^*\| \ge \max_{1 \le j \le k} (1 + b)^{-1}(\|y_j^*\| - b) = (1 + b)^{-1}(1 - b) > t,$$

$$\begin{aligned}\left\|\sum_{j=1}^{k}(1 + b)^{-1}v_j^*\right\| &\le (1 + b)^{-1}\left(\left\|\sum_{j=1}^{k} y_j^*\right\| + kb\right)\\ &\le (1 + b)^{-1}(c^{-1}t + kb) < a^{-1}t,\end{aligned}$$

a contradiction.

Now let us consider the case ∂ is metrically amiable on $X \times S_1 \times \cdots \times S_k$. Let us take s, u, u_j, v, v_j, f, g, b as above, let us set $m := k + c^{-1}t$,

$$\bar{g}(x, x_1, \ldots, x_k) = g(m^{-1}x, x_1, \ldots, x_k)$$

and let us choose $\varepsilon > 0$ so small that

$$\|v_j - x_0\| + \varepsilon < r, \quad \varepsilon < \frac{1}{2}\max_j \|v - v_j\|,$$

$$(1 + b)^{-1}(1 - b) - \varepsilon > t, \quad c^{-1} + kbt^{-1} + (1 + b)(k + m)t^{-1}\varepsilon < a^{-1}(1 + b).$$

Thus $\bar{g}$ attains its infimum on $Z := X \times S_1 \times \cdots \times S_k$ at $(mv, v_1, \ldots, v_k)$ and is Lipschitzian with rate $1 + b$. Since ∂ is metrically amiable with respect to Z we can find

$$(w, w_1, \ldots, w_k) \in B((mv, v_1, \ldots, v_k), \varepsilon),$$

$$(z, z_1, \ldots, z_k) \in Z \cap B((mv, v_1, \ldots, v_k), \varepsilon),$$

$$(w^*, w_1^*, \ldots, w_k^*) \in (1+b)^{-1} \partial \bar{g}(w, w_1, \ldots, w_k),$$

$$(z^*, z_1^*, \ldots, z_k^*) \in N_u(Z, (z, z_1, \ldots, z_k))$$

such that $\|z^* + w^*\| < \varepsilon$, $\max \|z_j^* + w_j^*\| < \varepsilon$. Then $z^* = 0$, $z_j^* \in N_u(S_j, z_j)$, $(mw^*, w_1^*, \ldots, w_k^*) \in (1+b)^{-1} \partial g(m^{-1}w, w_1, \ldots, w_k)$ and there exist $y_j^* \in \partial \| \cdot \| (w_j - m^{-1}w)$ for $j = 1, \ldots, k$ such that

$$(1+b)(mw^*, w_1^*, \ldots, w_k^*) = \left(-\sum_{j=1}^{k} y_j^*, y_1^*, \ldots y_k^* \right) + c^{-1} t B_{X^*} \times b B_{X^*} \times \cdots \times b B_{X^*}.$$

We cannot have $w_j = m^{-1}w$ for each j since $\|w_j - v_j\| < \varepsilon$, $\|m^{-1}w - v\| \leq m^{-1}\varepsilon < \varepsilon$, $2\varepsilon < \max_j \|v - v_j\|$. Therefore, we have $\max_{1 \leq j \leq k} \|y_j^*\| = 1$. Then

$$\max_{1 \leq j \leq k} \|z_j^*\| \geq (1+b)^{-1}(1-b) - \varepsilon > t,$$

$$\begin{aligned} \left\| \sum_{j=1}^{k} z_j^* \right\| &\leq \left\| \sum_{j=1}^{k} w_j^* \right\| + k\varepsilon \\ &\leq (1+b)^{-1} \left(\left\| \sum_{j=1}^{k} y_j^* \right\| + kb \right) + k\varepsilon \\ &\leq \|mw^*\| + (1+b)^{-1}(c^{-1}t + kb) + k\varepsilon \\ &\leq m\varepsilon + (1+b)^{-1}(c^{-1}t + kb) + k\varepsilon < a^{-1}t \end{aligned}$$

and since $z_j \in B(v_j, \varepsilon) \cap S_j \subset B(x_0, r) \cap S_j$, we get a contradiction. $\square$

Proposition 3.9. *Suppose ∂ is amiable for $X \times S_1 \times \cdots \times S_k$ or exactly amiable for $S_1 \times \cdots \times S_k$. Suppose there exist $a > 0$, $r > 0$, $x_0 \in S := \bigcap_{j=1}^{k} S_j$ such that for any $x_j \in S_j \cap B(x_0, r)$, $x_j^* \in N(S_j, x_j)$ one has*

$$\max_{1 \leq j \leq k} \|x_j^*\| \geq 1 \quad \Rightarrow \quad \left\| \sum_{j=1}^{k} x_j^* \right\| \geq a^{-1}. \tag{10}$$

Then there exist $s > 0$ depending on a, r only such that for each $x \in V := B(x_0, s)$ one has

$$d(x, S) \leq a \sum_{j=1}^{k} d(x, S_j). \tag{11}$$

Proof. Since for each $t \in (0,1)$ we have, for any $x_j \in S_j \cap B(x_0, r)$, $x_j^* \in N(S_j, x_j)$,

$$\max_{1 \le j \le k} \|x_j^*\| \ge t \quad \Rightarrow \quad \max_{1 \le j \le k} \|t^{-1} x_j^*\| \ge 1 \quad \Rightarrow \quad \left\| \sum_{j=1}^{k} x_j^* \right\| \ge a^{-1} t,$$

an inspection of the preceding proof in which we replace relation (8) by

$$-(1+b)^{-1}(v_1^*, \ldots, v_k^*) \in N(S_1 \times \cdots \times S_k, (v_1, \ldots, v_k)).$$

shows that

$$d(x, S) \le a t^{-1} \sum_{j=1}^{k} d(x, S_j)$$

for each $x \in B(x_0, s)$, with $s < (a^{-1} + 2)^{-1} r$. Taking the supremum over t we get the result in the case ∂ is exactly amiable for $S_1 \times \cdots \times S_k$. The case ∂ is amiable for $X \times S_1 \times \cdots \times S_k$ is similar. □

In the following corollary we make use of an obvious extension of Proposition 3.3 to a finite family of sets.

Corollary 3.10. *In order that the sets $S_1, \ldots, S_k$ satisfy the linear estimate (LE) it suffices that they are metrically synergetic with respect to an amiable subdifferential and satisfy the following pointwise qualification condition for the limiting normal cone associated with it:*

$$x_i^* \in \overline{N}(S_i, x_0), \quad i = 1, \ldots, k \quad \sum_{i=1}^{k} x_i^* = 0 \quad \Rightarrow \quad x_1^* = \cdots = x_k^* = 0.$$

4 Allied and synergetic functions

We devote the present section to a functional version of the preceding notion of cooperative behavior. Since our approach essentially consists in applying the preceding concepts to the epigraphs of the functions, we will only sketch the results.

Definition 4.1. *Two l.s.c. functions $f, g : X \to \mathbb{R} \cup \{\infty\}$ are said to be* **allied** *(resp. synergetic) at $\bar{z} \in dom f \cap dom\, g$ if $(E_f - (\bar{z}, f(\bar{z})))$ and $(E_g - (\bar{z}, g(\bar{z})))$ are allied (synergetic) at* $(0,0)$.

Example. If f is l.s.c. and if g is locally Lipschitzian around $\bar{z} \in dom f$, then f and g are synergetic at $\bar{z}$.

Example. ([36]) Two convex l.s.c. functions f and g are synergetic at $\bar{z} \in dom f \cap dom\, g$ when

$$\mathbb{R}_+(dom f - \bar{z}) - \mathbb{R}_+(dom\, g - \bar{z}) = X.$$

As observed in ([34], [36]) for instance, such a condition is independent of $\bar{z}$ as it can be written

$$\mathbb{R}_+(dom f - dom\, g) = X.$$

The subsets F and G of X are synergetic at $\bar{z}$ iff their indicator functions i_F and i_G are synergetic at $\bar{z}$.

In the following result we use the asymptotic subdifferential given by

$$\bar{\partial}_\infty f(\bar{z}) := \{\bar{z}^* \in X^* : (\bar{z}^*, 0) \in \overline{N}(E_f, \bar{z}_f)\}.$$

Theorem 4.2. *([40]) Suppose ∂ is either the Fréchet subdifferential and X is an Asplund space or ∂ is the viscosity subdifferential associated to a bornology $\mathscr{B}$ and X is $\mathscr{B}$-smooth with B_{X^*} sequentially weak* compact. Let f and g be l.s.c. on X, be synergetic at $\bar{z} \in X$ and such that*

$$\bar{\partial}_\infty f(\bar{z}) \cap (-\bar{\partial}_\infty g(\bar{z})) = \{0\}.$$

Then

$$\bar{\partial}(f+g)(\bar{z}) \subset \bar{\partial} f(\bar{z}) + \bar{\partial} g(\bar{z}).$$

Corollary 4.3. *Suppose ∂ and X are as above, F, G of X are synergetic at $e \in E := F \cap G$ and*

$$\overline{N}(F, e) \cap (-\overline{N}(G, e)) = \{0\}.$$

Then

$$\overline{N}(E, e) \subset \overline{N}(F, e) + \overline{N}(G, e).$$

Conversely, it can be shown that a result bearing on the calculus of the normal cone to an intersection can be transfered into a sum rule for functions.

5 Allied and synergetic multimappings

Coderivatives are the appropriate tools for the infinitesimal study of multimappings (or correspondences or set-valued mappings). Since multimappings appear naturally in a number of problems, this tool is important, as noticed by those who promoted it, among whom Aubin, Borwein, Ioffe, Mordukhovich, Pshenichnii, Rockafellar played a prominent role.

Definition 5.1. *The* **coderivative (associated with ∂)** *of a multifunction $F : X \rightrightarrows Y$ at $(x, y) \in F$ is the multifunction $D^*F(x, y) : Y^* \rightrightarrows X^*$ defined by*

$$D^*F(x, y)(y^*) = \{x^* : (x^*, -y^*) \in N(\mathrm{Graph}\, F, (x, y))\}.$$

Coderivatives establish a link between normal cones and subdifferentials: for a function $f \in \mathscr{F}(X)$ with epigraph E_f considered as a multifunction from

X to $\mathbb{R}$ one has

$$\partial f(x) = D^* E_f(x, f(x))(1).$$

Calculus rules for coderivatives can be established under linear-rate metric qualification conditions. We limit our study here to a chain rule, referring to [17], [23], [24], [32] for other rules.

Theorem 5.2. *Let ∂ satisfy property* (N), *be quasi-homotone and satisfy the exact sum rule. Let $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ be multifunctions with closed graphs. Let $H := G \circ F$, $\bar{z} \in H(\bar{x})$. Set $R(x, z) = F(x) \cap G^{-1}(z)$. Assume that for some $C \subset R(\bar{x}, \bar{z})$*

(a) the resultant multifunction R is lower semicontinuous at $(\bar{x}, \bar{z}, C)$ on Graph H;
(b) for any $\bar{y} \in R(\bar{x}, \bar{z})$ there is a $c > 0$ such that for all (x, y, z) of a neighborhood of $(\bar{x}, \bar{y}, \bar{z})$ one has

$$(LEC) \quad d((x, z, y), R) \le cd((x, y), F) + cd((y, z), G).$$

Then for all $\bar{z}^* \in Z^*$

$$D^* H(\bar{x}, \bar{z})(\bar{z}^*) \subset \bigcup_{\bar{y} \in C} (D^* F(\bar{x}, \bar{y})) \circ (D^* G(\bar{y}, \bar{z}))(\bar{z}^*)$$

Proof. This result is an easy consequence of the calculus of the normal cone to an intersection, owing to the facts that

$$R' := \{(x, y, z) : y \in R(x, z)\} = F \times Z \cap X \cap G$$

and that H is the projection of R' onto $X \times Z$. $\square$

Coderivative criteria for obtaining the linear estimate for composition (LEC) can be given using the following notion.

Definition 5.3. *The multifunctions $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$ are said to be* **allied** *at (x, y, z) if $y \in R(x, z) := F(x) \cap G^{-1}(z)$ and if for any sequences $(x_n, y_n) \xrightarrow{F} (x, y)$, $(w_n, z_n) \xrightarrow{G} (y, z)$, $(x_n^*) \to 0$, $(z_n^*) \to 0$ with $(w_n^* - y_n^*) \to 0$, $x_n^* \in D^* F(x_n, y_n)(y_n^*)$, $w_n^* \in D^* G(y_n, z_n)(z_n^*)$ one has $(y_n^*) \to 0$. They are said to be* **synergetic** *at (x, y, z) if the conditions $(y_n^*) \xrightarrow{*} 0$, $(w_n^*) \xrightarrow{*} 0$ are added in the assumptions of the preceding definition.*

Again, if Y is finite dimensional, any pair of multifunctions $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$ is synergetic. Clearly, (F, G) is allied (resp. synergetic) iff the sets $F \times Z$, $X \times G$ are allied (resp. synergetic) so that we get the following results.

Proposition 5.4. *If the multifunctions $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$ are allied at (x, y, z) and if the subdifferential ∂ is amiable then the condition (LEC) of the preceding proposition is satisfied.*

Proposition 5.5. *If the multifunctions $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$ are synergetic at (x, y, z) and if the following qualification condition (QC) holds then they are* *allied at* (x, y, z):

$$(QC) \quad (\bar{D}^*F(x,y))^{-1}(0) \cap \bar{D}^*G(y,z)(0) = \{0\}.$$

Here $\bar{D}^*$ denotes the coderivative associated with the limiting normal cone $\bar{N}$.

Corollary 5.6. *If ∂ is quasi-homotone and satisfies (N) and the exact sum rule, if the multifunctions F, G are synergetic at (x, y, z) and satisfy condition (QC), and if the resultant multifunction R is lower semicontinuous at (x, z, y) then for each $z^* \in Z^*$ one has*

$$D^*(G \circ F)(x,z)(z^*) \subset (D^*F(x,y)) \circ (D^*G(y,z))(z^*)$$

Example 5.7. If F (resp. G^{-1}) is sequentially coderivatively compact then for any multifunction G (resp. F) the pair (F, G) is synergetic. Here a multifunction F is said to be *sequentially coderivatively compact* at $(\bar{x}, \bar{y}) \in \operatorname{Graph} F$ if for any sequence (x_n, y_n, x_n^*, y_n^*) such that $y_n \in F(x_n)$, $x_n^* \in D^*F(x_n, y_n)(y_n^*)$, $(x_n, y_n) \to (\bar{x}, \bar{y})$, $\|x_n^*\| \to 0$, the sequence (y_n^*) norm converges to 0, provided it weak* converges to 0. This property (introduced in terms of nets in [38]) is a weakening of the notion of partial normal compactness of a set-valued mapping given in the preprint [32] which requires that (x_n, y_n) is an arbitrary sequence of some neighborhood of $(\bar{x}, \bar{y})$ and is formulated as follows:

There exists a weak-star closed subspace $L^* \subset Y^*$ of finite codimension, positive numbers γ and σ as well as a compact set $S \subset Y$ such that

$$\|x^*\| + \max_{s \in S} |\langle y^*, s \rangle| \geq \sigma$$

for any $(x^*, y^*) \in N(F, (x, y))$ with $(x, y) \in F \cap B((\bar{x}, \bar{y}), \gamma)$, $\|y^*\| = 1$ and $d(y^*, L^*) \leq \gamma$.

The notion of coderivative compactness has been suggested to us by the notion of normal compactness of a set. It appeared for the first time in [37], [38] and, slightly later on, in [24] of which we borrow the terminology. Its sequential variant stems from discussions with A. D. Ioffe during the summer of 1995. A deep characterization of it is given in [16] Theorem 1, along with a complete analysis; see also [24], [32] and [38]. It is obviously satisfied if the graph of F is compactly epi-Lipschitz at $(\bar{x}, \bar{y})$ or if the graph of F is normally compact at $(\bar{x}, \bar{y})$.

Corollary 5.8. *Let X, Y, Z be Banach spaces, let F and G be set-valued mappings from X into Y and from Y into Z respectively, and let ∂ be a quasi-homotone subdifferential satisfying the sum rule. Set as above $R(x, z) = F(x) \cap G^{-1}(z)$ and assume that $(\bar{x}, \bar{y}, \bar{z}) \in \operatorname{Graph} H$. Suppose that the following two conditions are satisfied:*

(a) either G is sequentially coderivatively compact at $(\bar{y}, \bar{z})$ or F^{-1} is sequentially coderivatively compact at $(\bar{y}, \bar{x})$;
(b) $0 \in D^*F(\bar{x}, \bar{y})(y^*)$ & $y^* \in D^*G(\bar{y}, \bar{z})(0) \Rightarrow y^* = 0$.

Then there is a $c > 0$ such that (LEC) holds for all (x, y, z) of a neighborhood of $(\bar{x}, \bar{y}, \bar{z})$.

Acknowledgements. The author thanks C. Zălinescu and two anonymous referees for their careful reading and their criticisms.

References

[1] Azé D, Chou CC, Penot J-P (1993) Subtraction theorems and approximate openness for multifunction: topological and infinitesimal viewpoints. To appear in J. Math. Anal. Appl.
[2] Barbet L (1992) Etude de sensibilité différentielle dans un problème d'optimisation paramétré avec contraintes en dimension infinie. Thesis, Univ. Poitiers
[3] Borwein JM, Zhu QJ (1996) Viscosity solutions and viscosity subderivatives in smooth Banach spaces with applications to metric regularity. SIAM J. Control Optim. 34:1576–1591
[4] Browder FE (1970) Existence theorems for nonlinear partial differential equations. In: Global analysis, Proc. Symp. Pure Math. 16, Amer. Math. Soc., Providence pp. 1–60
[5] Browder FE (1976) Nonlinear operators and nonlinear equations of evolution. Proc. Symp. Pure Math. 18, Amer. Math. Soc., Providence
[6] Clarke FH (1983) Optimization and nonsmooth analysis. Wiley, New York
[7] Clarke FH, Raïssi N (1990) Formules d'intersection en analyse non-lisse. Ann. Sci. Math. Québec 14:121–129
[8] El Haddad E, Deville R (1996) The viscosity subdifferential of the sum of two functions in Banach spaces I: the first order case. J. Convex Anal. 3 (2):295–306
[9] Federer H (1959) Curvature measures. Trans. Amer. Math. Soc. 93:418–491
[10] Ioffe A (1983) On subdifferentiability spaces. Ann. N.Y. Acad. Sci. 410:107–119
[11] Ioffe AD (1984) Calculus of Dini subdifferentials of functions and contingent coderivatives of set-valued maps. Nonlinear Anal. Theory. Meth. Appl. 8 (5):517–539
[12] Ioffe AD (1987) On the local surjection property. Nonlinear Anal. Theory Meth. Appl. 11 (5):565–592
[13] Ioffe A (1989) Approximate subdifferentials and applications 3. The metric theory. Mathematika 36:1–38
[14] Ioffe A (1996) Separable reduction theorem for approximate subdifferentials. C.R. Acad. Sci. Paris, 323, série I:107–12
[15] Ioffe A (1995) Directional compactness, scalarization and nonsmooth semi-Fredholm mappings. Preprint Univ. of Pau
[16] Ioffe A (1996) Codirectional compactness, metric regularity and subdifferential calculus. Preprint, Technion, Haifa
[17] Ioffe A, Penot J-P (1996) Subdifferentials of performance functions and calculus of coderivatives of set-valued mappings. Serdica Math. J. 22:359–384
[18] Jourani A (1993) Formules d'intersection dans un espace de Banach. C.R. Acad. Sci. Paris, 317:825–828
[19] Jourani A (1994) Qualification conditions for multivalued functions in Banach spaces with application to nonsmooth vector optimization problems. Math. Programming 66:1–23
[20] Jourani A (1995) Intersection formulae and marginal functions in Banach spaces. J. Math. Anal. Appl. 192:867–891
[21] Jourani A (1995) Compactly epi-Lipschitzian sets and A-subdifferentials in WT-spaces. Optimization 34:1–17
[22] Jourani A, Thibault L Metric regularity for strongly compactly Lipschitzian mappings. Nonlinear Anal. Theory. Meth. Appl.
[23] Jourani A, Thibault L Chain rules for coderivatives of multivalued mappings in Banach spaces. Proc. Amer. Math. Soc., to appear.
[24] Jourani A, Thibault L (1995) Coderivatives of multivalued mappings, locally compact cones and metric regularity. Preprint Univ. of Dijon and Montpellier
[25] Kruger AY (1985) Properties of generalized differentials. Siberian Math. J. 26:822–832
[26] Loewen Ph (1989) Methods of dynamic and nonsmooth optimization. CBMS-NSF Regional Conference series in applied math. 57, SIAM, Philadelphia
[27] Loewen Ph (1992) Limits of Fréchet normals in nonsmooth analysis. In: Ioffe A, Marcus M, Reich S (eds.) Optimization and Nonlinear Analysis, Longman, Essex, pp. 178–188
[28] Michel Ph, Penot J-P (1992) A generalized derivative for calm and stable functions. Diff. Integ. Equations 5 (2):433–454

[29] Mordukhovich B (1994) Generalized differential calculus for nonsmooth and set-valued mappings. J. Math. Anal. Appl., 183:250–288
[30] Mordukhovich B, Shao Y (1996) Nonsmooth sequential analysis in Asplund spaces. Trans. Amer. Math. Soc. 348:1235–1280
[31] Mordukhovich B, Shao Y (1995) Stability of set-valued mappings in infinite dimensions: point criteria and applications. Preprint, Wayne State Univ. Detroit, to appear SIAM J. Control and Opt.
[32] Mordukhovich B, Shao Y (1995) Nonconvex differential calculus for infinite dimensional multifunctions. Preprint Wayne State Univ., to appear in Set-Valued Analysis
[33] Mordukhovich B, Shao Y (1995) Fuzzy calculus for coderivatives of multifunctions, to appear in Nonlinear Anal. Theory. Meth. Appl.
[34] Penot J-P (1982) On regularity conditions in mathematical programming. Math. Program. Study 19:167–199
[35] Penot J-P (1983) Continuity of performance functions. In: Hiriart-Urruty J-B, Oettli W, Stoer J (eds.) Optimization, Theory and Algorithms, Lecture notes in pure and applied Math. 86, Dekker, New York, pp. 77–90
[36] Penot J-P (1996) Subdifferential calculus without qualification assumptions. J. of Convex Anal. 3 (2):1–13
[37] Penot J-P (1995) Views on nonsmooth analysis (appendix). Unpublished notes for the June 1995 meeting on nonsmooth analysis, Pau
[38] Penot J-P (1995) Compactness properties, openness criteria and coderivatives. Preprint Univ. of Pau
[39] Penot J-P (1995) Subdifferential calculus and subdifferential compactness. In: Font-Romeu-Odeillo, Sofonea M, Corvellec J-N (eds.) Proc. of the Second Catalan Days on Applied Mathematics, Presses Universitaires de Perpignan, pp. 209–226
[40] Penot J-P (1996) Synergetic sets, synergetic functions and subdifferential calculus. Preprint, Univ. of Pau
[41] Penot J-P (1997) The compatibility with order of some subdifferentials. Preprint, Univ. of Pau
[42] Penot J-P Synergetic correspondences and codifferential calculus. In preparation
[43] Penot J-P Nonsmooth analysis: from subdifferential calculus to codifferential calculus, in preparation
[44] Treiman JS (1991) Shrinking generalized gradients. Nonlin. Anal., Th., Methods, Appl. 68 (3):563–582