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Stability in vector-valued and set-valued optimization[1]

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Abstract. In this paper, we discuss the stability of the sets of efficient points of vector-valued and set-valued optimization problems when the data (E_n, f_n) (resp. (E_n, F_n)) of the approximate problems converge to the data (E, f) (resp. (E, F)) of the original problem in the sense of Painleve-Kuratowski or Mosco. Our results improve and generalize those obtained by Attouch and Riahi in Section 5 in [1].

Key words: Convergence of set sequence, Mosco convergence, Painlevé-Kuratowski convergence, cone extremization, stabililty

1. Introduction

In [1], Attouch and Riahi applied their Theorem 3.3 and established the stability result for the set of efficient points of a multiobjective optimization problem in finite dimensional space R^N under the pareto order. In this paper, we consider the stability of the sets of efficient points of vector-valued and set-valued maps in a Banach space Y under general cone order setting when the data (E_n, f_n) (resp. (E_n, F_n)) of the approximate problems converge to the data (E, f) (resp. (E, F)) of the original problem in the sense of Painleve-Kuratowski and Mosco (for details, see Section 2). Our results improve and generalize those in Section 5 in [1].

This paper is structured as follows. In Section 2, we present some concepts and notations. Section 3 is devoted to the stability results. Section 4 concludes the paper.

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2. Concepts and notations

In this section, we introduce some concepts and notations, which will be used in the sequel.

Throughout this paper, we assume X and Y are both Banach spaces. Y is partially ordered by a nontrival, closed, pointed and convex cone C, i.e., $\forall y_1, y_2 \in Y, y_1 \leq_C y_2$ iff $y_2 - y_1 \in C$. Let $C^* = \{l \in Y^* : l(c) \geq 0, \forall c \in C\}$ denote the positive polar cone of C and $\operatorname{int} C$ denote the interior of C if the interior of C is nonempty.

We first recall the Painlevé-Kuratowski convergence and Mosco convergence of a set sequence.

Definition 2.1. Let Z be a first countable topological space. The Painlevé-Kuratowski convergence of a seqence of subsets $\{D_n : n \in N\}$ of Z to a subset D of Z (i.e., $D_n \xrightarrow{P.K.} D$) means $\limsup_{n\to\infty} D_n \subset D \subset \liminf_{n\to\infty} D_n$ with

$$\liminf_{n\to\infty} D_n = \{x = \lim_{n\to+\infty} x_n : x_n \in D_n, \forall n \in N\}$$

$$\limsup_{n\to\infty} D_n = \{x = \lim_{k\to+\infty} x_{n_k} : x_{n_k} \in D_{n_k}, \forall k, \{n_k\} \text{ a subsequence of } N\}.$$

Definition 2.2. Let Z be a normed space. We say that a sequence of subsets $\{D_n\}$ of Z Mosco converges to $D \subset Z$ if $w - \limsup_{n\to\infty} D_n \subset D \subset \liminf_{n\to\infty} D_n$ with $w - \limsup_{n\to\infty} = \{x = w - \lim_{k\to+\infty} x_{n_k} : x_{n_k} \in D_{n_k}, \forall k, \{n_k\}$ a subsequence of $N\}$, where $x = w - \lim_{k\to+\infty} x_{n_k}$ stands for the weak convergence of x_{n_k} to x.

Definition 2.3. A vector-valued function $f : X \to Y$ is said to be lower semicontinuous (l.s.c.) with respect to (w.r.t.) C if $\forall y \in Y, \{x \in X : f(x) \leq_C y\}$ is closed.

We use $ext_C A$ to denote the set of maximal (efficient) points of A, i.e., $z \in ext_C A$ iff $(z + C) \cap A = \{z\}$.

We introduce a virtual element $+\infty$ in Y meanning that $+\infty - y \in C$, $\forall y \in Y$.

Definition 2.4. We say a seqence of vector-valued functions f_n (defined on X) Painlevé-Kuratowski (P.K. for short) (resp. Mosco (M for short)) converges to a vector-valued function f (defined on X) if

$$epi(f_n) = \{(x, y) : y \in f(x) + C\} \xrightarrow{P.K.} epi(f)$$
$$= \{(x, y) : y \in f(x) + C\}(epi(f_n) \xrightarrow{M} epi(f)).$$

Definition 2.5. We say a sequence of nonempty set-valued maps F_n (defined on X) P.K.(M) converges to a nonempty set-valued map F (defined on X) if

$$epi(F_n) = \{(x, y) : y \in F_n(x) + C\} \xrightarrow{P.K.} epi(F)$$
$$= \{(x, y) : y \in F(x) + C\}(epi(F_n) \xrightarrow{M} epi(F)).$$

Definition 2.6. Let $\{f_n : E_n \to Y, n = 1, 2, \ldots\}$ be a sequence of vector-valued functions and denote by $\{(E_n, f_n) : n = 1, 2, \ldots\}$ the corresponding sequence pairs. $f : E \to Y$. we say (E_n, f_n) P.K.(M) converges to (E, f) if $\overline{f_n} \xrightarrow{P.K.} \bar{f}(\overline{f_n} \xrightarrow{M} \bar{f})$, where

$$\overline{f_n}(x) = \begin{cases} f_n(x), & \text{if } x \in E_n, \\ +\infty, & \text{if } x \in X \backslash E_n; \end{cases}$$

$$\bar{f}(x) = \begin{cases} f(x), & \text{if } x \in E; \\ +\infty, & \text{if } x \in X \backslash E. \end{cases}$$

Definition 2.7. Let $\{F_n : E_n \to 2^Y, n = 1, 2, \ldots\}$ be a sequence of nonempty set-valued maps and denote by $\{(E_n, F_n)\}$ the corresponding pairs. $F : E \to 2^Y$ is a nonempty set-valued map. We say (E_n, F_n) $P.K.(M)$ converges to (E, F) if $\overline{F_n} \xrightarrow{P.K.} \bar{F}$, where

$$\overline{F_n}(x) = \begin{cases} F_n(x), & \text{if } x \in E_n, \\ +\infty, & \text{if } x \in X \backslash E_n; \end{cases}$$

$$\bar{F}(x) = \begin{cases} F(x), & \text{if } x \in E, \\ +\infty, & \text{if } x \in X \backslash E. \end{cases}$$

Definition 2.8. Let $f : X \to Y$ be a vector-valued function. We say f is bounded below if $\exists y_0 \in Y$ such that $f(X) - y_0 \subset C$.

Let $f_n : X \to Y$ be a sequence of vector-valued functions. We say f_n are uniformly bounded below if $\exists y_0 \in Y$ such that $f_n(X) - y_0 \subset C, \forall n \in N$.

Definition 2.9. Let $F : X \to 2^Y$ be a set-valued map. We say that F is bounded below if $\exists y_0 \in Y$ such that $[F(X) - y_0] \subset C$, where $F(x) = \bigcup_{x \in X} F(x)$.

Let $F_n : X \to 2^Y$ be a sequence of set-valued maps. We say that F_n are uniformly bounded below if there exists $y_0 \in Y$ such that $[F_n(X) - y_0] \subset C$ for all n.

3. Stability of the set of efficient (minimal) points

This section presents the main results, which generalize the corresponding results in [1, Section 5]. We shall first state the results and then prove them one by one.

Theorem 3.1. *Assume* $\operatorname{int} C \neq \emptyset, -C \subset \{y \in Y : l(y) + \varepsilon\|y\| \leq 0\}$ *for some* $l \in Y^*$ *and* $\varepsilon > 0$. $(E_n, f_n), (E, f)$ *are as defined in Definition 2.6.* $\forall n \in N, E_n$ *is a nonempty closed subset of X,* f_n *is l.s.c. w.r.t. C.* $E \subset X$ *is nonempty closed.* f *is l.s.c. w.r.t. C.*

In addition,

(a) $\inf_{n \in N} \inf_{x \in E_n} l(f_n(x)) > -\infty$*;*

(b) $(E_n, f_n) \xrightarrow{P.K.} (E, f)$*;*

(c) $\exists$ *a compact subset K of X such that* $E_n \subset K, \forall n \in N$*;*

(d) $\forall \rho > 0, \exists$ *a compact subset* K_ρ *of* Y *such that* $ext_{-C} f_n(E_n) \cap \rho B \subset K_\rho$, *where* B *is the unit ball of* Y.

Then $ext_{-C} f(E)$ *is nonempty and*

$$ext_{-C} f(E) \subset \liminf_{n\to\infty} ext_{-C} f_n(E_n).$$

Theorem 3.2. C *(without the assumption* $\operatorname{int} C \neq \emptyset$*),* f_n, E_n, E, f *are as in Theorem 3.1.* $\forall \lambda \in C^*$, $\lambda(f_n)$ *is l.s.c. on* E_n. $\lambda(f)$ *is l.s.c. on* E.

In addition,

(a) $\inf_{n\in N} \inf_{x\in E_n} l(f_n(x)) > -\infty$;

(b) $(E_n, f_n) \xrightarrow{P.K.} (E, f)$;

(c) $\exists$ *a compact subset* K *of* X *such that* $E_n \subset K, \forall n \in N$;

(d) $\forall \rho > 0, \exists$ *a compact subset* K_ρ *of* Y *such that* $ext_{-C} f_n(E_n) \cap \rho B \subset K_\rho$, *where* B *is the unit ball of* Y.

Then $ext_{-C} f(E)$ *is nonempty and*

$$ext_{-C} f(E) \subset \liminf_{n\to\infty} ext_{-C} f_n(E_n).$$

Theorem 3.3. X *and* Y *are reflexive Banach spaces.* C, E_n, f_n, E, f *are as in Theorem 3.1.* f_n, f *are l.s.c. (with respect to the weak topology of* X*) w.r.t.* C.

In addition,

(a) $\inf_{n\in N} \inf_{x\in E_n} l(f_n(x)) > -\infty$;

(b) $(E_n, f_n) \xrightarrow{M} (E, f)$;

(c) $\exists$ *a bounded closed subset* K *of* X *such that* $E_n \subset K, \forall n \in N$;

(d) $\forall \rho > 0, \exists$ *a compact subset* K_ρ *of* Y *such that* $ext_{-C} f_n(E_n) \cap \rho B \subset K_\rho$, *where* B *is the unit ball of* Y.

Then $ext_{-C} f(E)$ *is nonempty and*

$$ext_{-C} f(E) \subset \liminf_{n\to\infty} ext_{-C} f_n(E_n).$$

Theorem 3.4. X, Y, C *(without the assumption* $\operatorname{int} C \neq \emptyset$*),* E_n, E, f_n, f *are as in Theorem 3.3* $\forall \lambda \in C^*, \lambda(f_n), \lambda(f)$ *are l.s.c. (with respect to the weak topology of* X*).*

In addition,

(a) $\inf_{n\in N} \inf_{x\in E_n} l(f_n(x)) > -\infty$;

(b) $(E_n, f_n) \xrightarrow{M} (E, f)$;

(c) $\exists$ *a bounded closed subset* K *of* X *such that* $E_n \subset K, \forall n \in N$;

(d) $\forall \rho > 0, \exists$ *a compact subset* K_ρ *of* Y *such that* $ext_{-C} f_n(E_n) \cap \rho B \subset K_\rho$, *where* B *is the unit ball of* Y.

Then $ext_{-C} f(E)$ *is nonempty and*

$$ext_{-C} f(E) \subset \liminf_{n\to\infty} ext_{-C} f_n(E_n).$$

Remark 3.1.

(i) The condition $-C \subset \{y \in Y : l(y) + \varepsilon\|y\| \le 0\}$ for some $l \in Y^*$ and $\varepsilon > 0$ is equivalent to the statement that $\exists l \in C^* \backslash \{0\}$ and ε such that $l(c) \ge \varepsilon$ for any $c \in C$ with $\|c\| = 1$. The latter is fulfilled when Y is a finite dimensional space and C is a nontrivial, pointed, closed and convex cone. Note that Attouch and Riahi considered only the special case when $Y = R^N$ and $C = R^N_+$. On the other hand, the following example shows that the condition $-C \subset \{y \in Y : l(y) + \varepsilon\|y\| \le 0\}$ for some $l \in Y^*$ and $\varepsilon > 0$ can be satisfied when Y is infinite dimensional, which illustrates the fact that Theorem 3.1 (Theorem 3.3) does generalize the corresponding results in [1, Section 5].

Example 3.1. Let $Y = l^1 = \{a = (a_1, \ldots, a_n, \ldots) : a_i \in R^1, \sum_{i=1}^{\infty} |a_i| < +\infty\}$, $C = \{a = (a_1, \ldots, a_n, \ldots) : a_i \ge 0, i = 1, 2, \ldots\}$, $l = (1, 1, \ldots) \in C^*$. Then $l(c) = \sum_{i=1}^{\infty} c_i = 1$ for any $c = (c_1, \ldots, c_n, \ldots) \in C$ such that $\|c\|_{l_1} = 1$. It follows that $-C \subset \{y \in Y : l(y) + \varepsilon\|y\| \le 0\}$.

(ii) It is clear that if $\overline{f_n}$ are uniformly bounded below (i.e., there exists $y_0 \in Y$ such that $f_n(x) - y_0 \in C, \forall x \in E_n, \forall n$), then (a) holds automatically.

(iii) Let $Y = R^N$, $C = R^N_+$. It is not hard to see that the conditions of Theorem 5.2 in [1] imply all the conditions in our Theorem 3.1. However, the Definition 5.1 in [1] is a stronger version of convergence than (b) in our Theorem 3.1. Therefore, our Theorem 3.1 improves Theorem 5.2 in [1].

We need the following lemmas to prove the theorems above.

Lemma 3.1. *Under the assumptions of Theorem 3.1 (or Theorem 3.2), we have*

$$(E_n, f_n) \xrightarrow{P.K.} (E, f) \Rightarrow f_n(E_n) + C \xrightarrow{P.K.} f(E) + C.$$

Proof. Firstly, we prove

$$f(E) + C \subset \liminf_{n\to\infty} f_n(E_n) + C \tag{1}$$

$\forall y \in f(E)$ with $y = f(x), \forall c \in C$, then $(x, y + c) \in epi(\bar{f})$. Since $\overline{f_n} \xrightarrow{P.K.} \bar{f}$, we have $epi(\bar{f}) \subset \liminf_{n\to\infty} epi(\overline{f_n})$.

So $\exists (x_n, y_n) \in epi(\overline{f_n})$ with $y_n = \overline{f_n}(x_n) + c_n$ such that $(x_n, y_n) \to (x, y + c)$. Thus $y_n \to y + C$.

Obviously, $x_n \in E_n$ when n is sufficiently large. Hence, $y_n = \overline{f_n}(x_n) + c_n = f(x_n) + c_n \in f_n(E_n) + C$, when n is sufficiently large, implying (1).

Secondly, We prove $\limsup_{n\to\infty}(f_n(E_n) + C) \subset f(E) + C$.

For any $x_{n_k} \in E_{n_k}, c_{n_k} \in C$ with $f_{n_k}(x_{n_k}) + c_{n_k} \in f_{n_k}(E_{n_k}) + C$ such that $f_{n_k}(x_{n_k}) + c_{n_k} \to y$.

Now that $\{x_{n_k}\} \subset K$ and K is a compact subset of X, we deduce that there exist a subsequence $\{x_{n_{k_l}}\}$ and $x \in K$ such that $x_{n_{k_l}} \to x$. Noticing that $y \ne +\infty$, we obtain $x \in E$. So $(x, y) \in epi(f)$, i.e., $y \in f(E) + C$. The proof is complete.

Similarly, we can prove the following Lemma 3.2.

Lemma 3.2. *Under the assumptions of Theorem 3.3 (or Theorem 3.4),*

$$(E_n, f_n) \xrightarrow{M} (E, f) \Rightarrow f_n(E_n) + C \xrightarrow{M} f(E) + C.$$

Lemma 3.3. *Let $E \subset X$ be nonempty compact. $f : E \to Y$ is l.s.c. w.r.t. C on E. Then $f(E) + C$ is nonempty closed and $ext_{-C}[f(E) + C] = ext_{-C}f(E)$.*

Proof. For any $f(x_n) + c_n \in f(E) + C$ with $x_n \in E$, $c_n \in C$ such that

$$f(x_n) + c_n \to y \tag{2}$$

By the compactness of E, we get a subseqence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in E$ such that $x_{n_k} \to x$. This combined with (2) yields $f(x_{n_k}) + c_{n_k} \to y$.

Arbitarily fix an $e \in \operatorname{int} C$, then $\forall \varepsilon > 0, \exists K_0$, when $k \geq K_0$, $f(x_{n_k}) + c_{n_k} \leq_C y + \varepsilon e$. So $f(x_{n_k}) \leq_C y + \varepsilon e$. By the l.s.c. of f w.r.t. C, we know that $f(x) \leq_C y + \varepsilon e$, i.e., $f(x) - y - \varepsilon e \in -C$. Letting $\varepsilon \to 0$, we have $f(x) - y \in -C$ (since $-C$ is closed). Hence $y \in f(x) + C \subset f(E) + C$.

It is obvious that the relation $ext_{-C}[f(E) + C] = ext_{-C}f(E)$ holds.

Lemma 3.4. *Let $E \subset X$ be nonempty compact. Let $f : X \to Y$ be such that $\forall \lambda \in C^*$, $\lambda(f)$ is l.s.c. on E. Then $f(E) + C$ is nonempty closed and $ext_{-C}[f(E) + C] = ext_{-C}[f(E)]$.*

Proof. We only show that $f(E) + C$ is closed.

For any $f(x_n) + c_n \in f(E) + C$ with $x_n \in E, c_n \in C$ such that $f(x_n) + c_n \to y$. By the compactness of E, we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in E$ such that $x_{n_k} \to x$. Moreover, $f(x_{n_k}) + c_{n_k} \to y$. Hence $\lambda(f(x_{n_k}) + \lambda(c_{n_k}) \to \lambda(y), \forall \lambda \in C^*$, implying $\lambda(f(x)) \leq \liminf\limits_{n\to\infty} \lambda(f(x_{n_k}) \leq \lambda(y) \forall \lambda \in C^*$. Thus $f(x) \leq_C y$, implying $y \in f(x) + C \subset f(E) + C$.

Proof of Theorem 3.1. We simply apply Theorem 3.3 in [1] with C replaced by $-C$, $D_n = f_n(E_n) + C, D = f(E) + C$. By Lemma 3.1 and Lemma 3.3, we know that D_n, D are nonempty closed and $D_n \xrightarrow{P.K.} D$. In addition, $\inf\limits_{n\in N} \inf\limits_{y \in D_n} l(y) \geq \inf\limits_{n \in N} \inf\limits_{x \in E_n} > -\infty$. Moreover, $\forall \rho > 0$, $(ext_{-C}\, D_n) \cap \rho B = (ext_{-C}\, f_n(E_n)) \cap \rho B \subset K_\rho$. So all the conditions of Theorem 3.3 in [1] hold, hence, $ext_{-C}f(E) = ext_{-C}\, D \neq \emptyset$ and $ext_{-C}f(E) = ext_{-C}\, D \subset \liminf\limits_{n\to\infty}\, ext_{-C}\, D_n = \liminf\limits_{n\to\infty}\, ext_{-C}f(E_n)$. The proof is completed.

Theorem 3.2 can be similarly proved.

Lemma 3.5. *Let X be a reflexive Banach space and* $\operatorname{int} C \neq \emptyset$. *If $E \subset X$ is nonempty, closed and bounded. $f : E \to Y$ is l.s.c. (with respect to the weak topology of X) on E w.r.t. C. Then $f(E) + C$ is a nonempty closed set and $ext_{-C}[f(E) + C] = ext_{-C}f(E)$.*

Proof. Since X is reflexive, we know that E is a weakly compact subset of X.

$\forall f(x_n) + c_n \in f(E) + C$ with $x_n \in E, c_n \in C$ such that $f(x_n) + c_n \to y$.

By the weak compactness of E, we obtain a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in E$ such that $x_{n_k} \xrightarrow{w} x$. Arbitarily fix an $e \in \operatorname{int} C$, $\forall \varepsilon > 0$, $\exists K_0 > 0$, when $k \geq K_0$, we have $f(x_{n_k}) + c_{n_k} \leq_C y + \varepsilon e$. Thus $f(x_{n_k}) \leq_C y + \varepsilon e$. By the l.s.c. of f (w.r.t. C and the weak topology of X), we have $f(x) \leq_C y + \varepsilon e$. Hence, $f(x) \leq_C y + \varepsilon e$, i.e., $y \in f(x) + C \subset f(E) + C$.

Similar to the proof of Lemma 3.4 (applying the weak topology of X), we can prove Lemma 3.6.

Lemma 3.6. *Let X be a reflexive Banach space. If $E \subset X$ is nonempty, closed and bounded, $f : E \to Y$ is such that $\forall \lambda \in C^*$, $\lambda(f)$ is l.s.c. (with respect to the weak topology of X) on E, then $f(E) + C$ is nonempty closed and $ext_{-C}[f(E) + C] = ext_{-C} f(E)$.*

Applying our Lemma 3.2, Lemma 3.5 and Theorem 3.5 in [1], we can easily prove Theorem 3.3.

Applying our Lemma 3.2, Lemma 3.6 and Theorem 3.5 in [1], we can also prove Theorem 3.4.

Theorem 3.5. *Let $C \subset \{y \in Y : l(y) + \varepsilon\|y\| \leq 0\}$ for some $l \in Y^*, \varepsilon > 0$. $\forall n \in N, E_n$ is a nonempty closed subset of X, $F_n : X \to 2^Y$ is u.s.c. nonempty compact-valued. E is a nonempty closed subset of X, $F : X \to 2^Y$ is u.s.c. nonempty compact-valued. In addition,*

(a) $\inf_{n \in N} \inf_{x \in E_n} \inf_{y \in F_n(x)} l(y) > -\infty$;

(b) $(E_n, F_n) \xrightarrow{M} (E, F)$;

(c) *$\exists$ a bounded closed subset K of X such that $E_n \subset K, \forall n \in N$;*

(d) *$\forall \rho > 0, \exists$ a compact subset K_ρ of Y such that $ext_{-C} F_n(E_n) \cap \rho B \subset K_\rho$, where B is the unit ball of Y.*

Then $ext_{-C} F(E)$ is nonempty and $ext_{-C} F(E) \subset \liminf_{n\to\infty} ext_{-C} F_n(E_n)$.

Theorem 3.6. *Let X, Y be reflexive Banach spaces. C, E_n, F_n, E, F are as in Theorem 3.5. F_n, F are u.s.c. (with respect to the weak topology of X). In addition,*

(a) $\inf_{n \in N} \inf_{x \in E_n} \inf_{y \in F_n(x)} l(y) > -\infty$;

(b) $(E_n, F_n) \xrightarrow{M} (E, F)$;

(c) *$\exists$ a bounded closed subset K of X such that $E_n \subset K, \forall n \in N$;*

(d) *$\forall \rho > 0, \exists$ a compact subset K_ρ of Y such that $ext_{-C} F_n(E_n) \cap \rho B \subset K_\rho$, where B is the unit ball of Y.*

Then $ext_{-C} F(E)$ is nonempty and $ext_{-C} F(E) \subset \liminf_{n\to\infty} ext_{-C} F_n(E_n)$.

Remark 3.2. If $\overline{F_n}$ are uniformly bounded below (i.e. there exists $y_0 \in Y$ such that $[F_n(E_n) - y_0] \subset C, \forall n$), then (a) in Theorems 3.5 and 3.6 holds automatically.

The following lemmas are needed to prove Theorems 3.5, 3.6.

Lemma 3.7. *Under the assumptions of Theorem 3.5, $(E_n, F_n) \xrightarrow{P.K.} (E, F)$ implies $F_n(E_n) + C \xrightarrow{P.K.} F(E) + C$.*

Lemma 3.8. *Under the assumptions of Theorem 3.6, $(E_n, F_n) \xrightarrow{M} (E, F)$ implies $F_n(E_n) + C \xrightarrow{M} F(E) + C$.*

The proofs of Lemma 3.7, Lemma 3.8 are similar to those of Lemma 3.1, Lemma 3.2, respectively, we omit it.

Lemma 3.9. *Let $E \subset X$ be nonempty compact. Let $F : E \to 2^Y$ be u.s.c. nonempty compact-valued, then $F(E) + C$ is nonempty closed and $ext_{-C}F(E) = ext_{-C}[F(E) + C]$.*

Proof. We only need to show that $F(E) + C$ is closed.

$\forall y_n + c_n \in F(x_n) + C$ with $y_n \in F(x_n)(x_n \in E), c_n \in C$ and $y_n + c_n \to y$. We prove that $y \in F(E) + C$ in the following two cases, respectively. (i) $x_n \equiv x$ for some $x \in X$ when n is sufficiently large.

Then $y_n \in F(x)$, when n is sufficiently large. By the compactness of $F(x)$, we get a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and $y' \in F(x)$ such that $y_{n_k} \to y'$. However, $y_{n_k} + c_{n_k} \to y$, so $c_{n_k} \to y - y' \in C$, i.e., $y \in F(x) + C \subset F(E) + C$.

(ii) $\exists$ a subsequence $\{x_{n_k}\}$ whose elements are different from one another such that $y_{n_k} \in F(x_{n_k})$.

By the compactness of E, we obtain a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ and $x \in E$ such that $x_{n_{k_l}} \to x$. By the u.s.c. of F and the compactness of $F(x)$, we have a subsequence $\{y_{n_{k_{l_t}}}\}$ of $\{y_{n_{k_l}}\}$ and $y' \in F(x)$ such that $y_{n_{k_{l_t}}} \to y'$. So $c_{n_{k_{l_t}}} \to y - y' \in C$, i.e., $y \in F(x) + C \subset F(E) + C$. The proof is completed.

Lemma 3.10. *X is a reflexive space, E is a nonempty closed bounded subset of X. $F : E \to 2^Y$ is an u.s.c. (w.r.t. the weak topology of X) nonempty compact-valued map, then $F(E) + C$ is nonempty closed and $ext_{-C}F(E) = ext_{-C}[F(E) + C]$.*

The proof of Lemma 3.10 is almost the same as that of Lemma 3.9, the only difference being that the weak topology of X should be applied.

The combination of our Lemma 3.7, Lemma 3.9 and Theorem 3.3 in [1] completes the proof of our Theorem 3.5.

The combination of our Lemma 3.8, Lemma 3.10 and Theorem 3.5 in [1] completes the proof of our Theorem 3.6.

4. Conclusions

This paper considered the stability of vector-valued and set-valued optimization problems based on the concepts of Painleve-kuratowski and Mosco convergence of sets. The results generalized the corresponding results of Attouch and Riahi in [1, Section 5]. The generalization is threefold: the objective space Y is extended from finite dimensional to infinite dimensional; the dominating cone C is extended from R^N_+ to a general ordering cone; the objective functions are extended to general vector-valued functions or set-valued maps. Further research, for example, the stability of the set of weakly efficient points (when the dominating cone C has nonempty interior) based on the concepts of Painleve-Kuratowski and Mosco convergence of sets can be expected.

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