Math Meth Oper Res (2000) 52:195–212

Mathematical Methods
of Operations Research

The efficient frontier for bounded assets*

Michael J. Best[1], **Jaroslava Hlouskova**[2]

[1] Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada (e-mail: mjbest@math.uwaterloo.ca)
[2] Research Associate, Department of Economics and Finance, Institute for Advanced Studies, Stumpergasse 56, A-1060, Vienna, Austria

Manuscript received: October 1999/Final version received: March 2000

Abstract. This paper develops a closed form solution of the mean-variance portfolio selection problem for uncorrelated and bounded assets when an additional technical assumption is satisfied. Although the assumption of uncorrelated assets is unduly restrictive, the explicit determination of the efficient asset holdings in the presence of bound constraints gives insight into the nature of the efficient frontier. The mean-variance portfolio selection problem considered here deals with the budget constraint and lower bounds or the budget constraint and upper bounds. For the mean-variance portfolio selection problem dealing with lower bounds the closed form solution is derived for two cases: a universe of only risky assets and a universe of risky assets plus an additional asset which is risk free. For the mean-variance portfolio selection problem dealing with upper bounds, the results presented are for a universe consisting only of risky assets. In each case, the order in which the assets are driven to their bounds depends on the ordering of their expected returns.

Key words: Parametric quadratic programming, mean-variance portfolio selection, efficient frontier, capital market line.

1 Introduction

In Markowitz (1956, 1959) and Sharpe (1970), the Mean-Variance (M-V) portfolio selection problem is analyzed subject to general linear constraints using quadratic programming and parametric quadratic programming methods. One way to formulate the M-V portfolio selection problem (see Best and Grauer (1990) for example) is:

* This research was supported by the National Science and Engineering Research Council of Canada.

$$\min\{-t\mu'x + \tfrac{1}{2}x'\Sigma x \mid l'x = 1, Ax \le b\}, \tag{1.1}$$

where μ is an n-vector of expected returns, Σ is an (n, n) covariance matrix, x is an n-vector of asset holdings to be determined, l is an n-vector of 1's, A is an (m, n) matrix and b is an m-vector. We use prime ($'$) to denote matrix transposition and adopt the convention that all non-primed vectors are column vectors. The constraint $l'x = 1$ requires the asset holdings to sum to unity and is called the budget constraint. The constraints $Ax \le b$ represent general linear constraints such as non-negativity constraints (precluding short sales), upper bounds on asset holdings, sector constraints and any other linear constraints the investor may wish to impose.

The M-V portfolio selection problem and the related Capital Asset Pricing Model (CAPM) have been studied by many authors under a variety of assumptions. Brennan (1971) addresses the issue of borrowing and lending rates. Turnbull (1977) also considers this along with personal taxation, uncertain inflation and non-market assets. Levy (1983) deals with problems of short sales as does Schnabel (1984).

Let $x(t)$ denote an optimal solution of (1.1) for any fixed t. We refer to the $x(t)$'s as M-V portfolios. Then $\mu_p = \mu'x(t)$ and $\sigma_p^2 = x'(t)\Sigma x(t)$ are its corresponding mean and variance, respectively. We call the plot of all such (σ_p^2, μ_p) the M-V frontier. As t varies from 0 to $+\infty$, the plot of (σ_p^2, μ_p) traces out the efficient frontier. As t varies from 0 to $-\infty$, the lower or inefficient frontier is obtained. Both the efficient and inefficient frontiers can also be viewed in (σ_p, μ_p) space. If the only constraint in (1.1) is the budget constraint, then under the assumption that Σ is positive definite, the solution of (1.1) can be obtained in closed form. See, for example, Best and Grauer (1990). When (1.1) does indeed include inequality constraints, the efficient portfolios are piecewise linear functions of the parameter t. Associated with (1.1) is a set of intervals $0 = t_0 \le t_1$, $t_1 \le t_2, \ldots, t_{v-1} \le t_v$. In each of these v intervals, the efficient portfolios are linear functions of t; i.e., there exist n-vectors h_{0i}, h_{1i}, $i = 1, \ldots, v$ such that

$$x(t) = h_{0i} + th_{1i}, \quad t_{i-1} \le t \le t_i, \tag{1.2}$$

for all $i = 1, \ldots, v$. Determination of all of these intervals and the associated vectors $h_{0i}, h_{1i}, i = 1, \ldots, v$ is a difficult task and a parametric quadratic programming algorithm such as in Perold (1984) or Best (1996) must be used. In this general case, it would be not possible to obtain the efficient portfolios in closed form. Furthermore, the number of parametric intervals, v, is not known *a priori* and may be quite large.

A special case of (1.1), which is usually given in textbooks is

$$\min\{-t(\mu'x + rx_{n+1}) + \tfrac{1}{2}x'\Sigma x \mid l'x + x_{n+1} = 1, x_{n+1} \ge 0\}, \tag{1.3}$$

where x, Σ, μ and l are the n-dimensional quantities previously discussed, x_{n+1} denotes the holdings in the risk free asset and r is the risk free rate. When $t = 0$, the solution of (1.3) is $x = 0$ and $x_{n+1} = 1$. As t is increased, x_{n+1} is eventually reduced to 0 for $t = t_m$ and the corresponding x is the market portfolio x_m. For $0 \le t \le t_m$, the efficient portfolios correspond to the Capital Market Line. As t is increased beyond t_m, the holdings in the risk free asset remain at 0; i.e., the constraint $x_{n+1} \ge 0$ is active for $t \ge t_m$. This corresponds to the familiar efficient frontier for the n risky assets. These two intervals are a special case of (1.2) with $v = 2$, $t_0 = 0$, $t_1 = t_m$ and $t_2 = \infty$. It is because (1.3) has just a single inequality constraint that its solution can be obtained explicitly. If (1.3) were to be augmented with no short sales restrictions; i.e., the constraints $x \geq 0$, then there would be 2^n possibilities for active sets and it would be impossible to obtain a closed form solution in general.

The contribution of this paper is as follows. When the covariance matrix for the risky assets is positive definite, diagonal (i.e., the assets are uncorrelated) and an additional technical assumption is satisfied, we obtain an explicit solution to (1.1) when the linear inequality constraints are lower bounds and $t \geq 0$ (Section 2). The assumption of uncorrelated assets is of course unduly restrictive. However, obtaining an explicit representation of efficient asset holdings subject to bound constraints does give insight into the efficient frontier in the presence of inequality constraints. We show that there are precisely n intervals and the asset holdings are reduced to their lower bounds (and remain there) in the order of smallest expected return to largest expected return. The end of the parametric interval is determined by the asset with smallest expected return, among those still held above their lower bounds, being reduced to its lower bound. This situation is illustrated in Figure 1 for the case of lower bounds of zero.

In Section 3, we examine the analogous case for upper bounds. We do this by first solving the lower bound case for $t \leq 0$; i.e., we determine the results for the inefficient or lower part of the M-V frontier. Having solved this problem the results for the upper bounded problem are obtained by means of a simple transformation.

For the case of lower bounds and $t \geq 0$ in Section 4, we consider the addition of a risk free asset. We show that the Capital Market Line; i.e, the linear part of the efficient frontier in (σ_p, μ_p) space of the problem dealing with risky assets and an additional risk free asset, meets the efficient frontier for the risky assets only with a modified budget constraint in that part of the frontier corresponding to its *first* parametric interval. Section 5 summarizes our results and concludes the paper.

2 Lower bounds and the efficient frontier

The problem to be analyzed is the following n-dimensional problem with lower bounds

$$\min\{-t\mu' x + \tfrac{1}{2} x' \Sigma x \mid l' x = d, x \geq e\}, \tag{2.1}$$

where $e = (e_1, e_2, \ldots, e_n)'$ is a vector of lower bounds on asset holdings, l is an n-vector of 1's, d is a scalar, t is a non-negative scalar parameter, $\Sigma = \text{diag}\{\sigma_1, \ldots, \sigma_n\}$ is a diagonal matrix of variances, $\mu = (\mu_1, \ldots, \mu_n)'$ is a vector of expected returns, and x is an n-vector of asset holdings. The budget constraint is usually written as $l'x = 1$; i.e., $d = 1$ in (2.1). It will be convenient in our analysis to allow d to assume any value. Throughout this paper we will refer to $l'x = d$ as the budget constraint.

Our results for (2.1) will require the following assumption to be satisfied.

Assumption 2.1. (a) $\sigma_i > 0$, for $i = 1, \ldots, n$, (b) $\mu_1 < \cdots < \mu_n$.

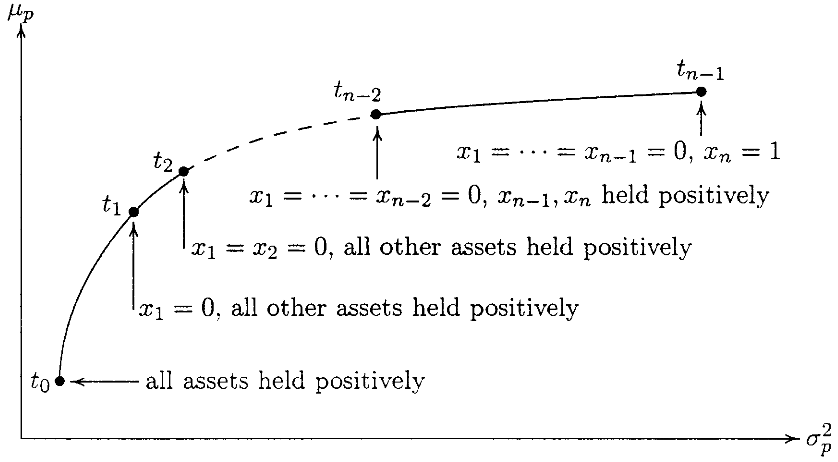

Fig. 1. Efficient Frontier for Risky Assets with No Short Sales

Note that by a suitable re-indexing of the assets, one can always obtain $\mu_1 \le \mu_2 \le \cdots \le \mu_n$. Consequently Assumption 2.1(b) is not restrictive in that if only requires the μ_i's to be distinct.

Throughout this paper, results can be written more concisely if any condition "for $i = 1, \ldots, k-1$" is regarded as vacuous when $k = 1$ and similarly, "the sum from 1 to $k-1$ of some quantities" should be regarded as having value 0 when $k = 1$.

First, for $k = 1, \ldots, n$, we consider a problem with no inequality constraints and which is closely related to (2.1):

$$\min\{-t\mu' x + \tfrac{1}{2} x' \Sigma x \mid l' x = d, x_1 = e_1, \ldots, x_{k-1} = e_{k-1}\}. \tag{2.2}$$

The solution of (2.2) can be formulated concisely in terms of the constants

$$\theta_{1k} = 1/(\sigma_k^{-1} + \cdots + \sigma_n^{-1}), \tag{2.3}$$

$$\theta_{2k} = \theta_{1k}(\mu_k/\sigma_k + \cdots + \mu_n/\sigma_n), \tag{2.4}$$

$$\theta_{3k} = d - (e_1 + \cdots + e_{k-1}), \tag{2.5}$$

for $k = 1, \ldots, n$.

Lemma 2.1. *Let Assumption 2.1(a) be satisfied. Then for $k = 1, \ldots, n$, the optimal solution for (2.2) is*

$$x_i = e_i, \quad i = 1, \ldots, k-1,$$

$$x_i = (\theta_{3k}\theta_{1k} + t(\mu_i - \theta_{2k}))/\sigma_i, \quad i = k, \ldots, n.$$

The multiplier for the budget constraint is $u = -\theta_{3k}\theta_{1k} + t\theta_{2k}$ and the multipliers for the constraints $x_i = e_i, i = 1, \ldots, k-1$, are

$$v_i = e_i\sigma_i - \theta_{3k}\theta_{1k} + t(\theta_{2k} - \mu_i),$$

respectively.

Proof: Let $1 \le k \le n$. From Assumption 2.1(a), the objective function for (2.2) is strictly convex. Furthermore, the constraints of (2.2) are linear. Therefore the Karush-Kuhn-Tucker (KKT) conditions (see Mangasarian (1969)) are both necessary and sufficient for optimality. The dual feasibility portion of the KKT conditions for (2.2) asserts that

$$t\mu_i - \sigma_i x_i = u, \quad i = k, \dots, n.$$

Solving for x_i gives

$$x_i = t\mu_i/\sigma_i - u/\sigma_i, \quad i = k, \dots, n. \tag{2.6}$$

Summing the x_i, using the budget constraint and the constraints $x_i = e_i$ for $i = 1, \dots, k-1$ gives

$$d = e_1 + \cdots + e_{k-1} + t(\mu_k/\sigma_k + \cdots + \mu_n/\sigma_n) - u(1/\sigma_k + \cdots + 1/\sigma_n),$$

from which we obtain

$$u = -\theta_{3k}\theta_{1k} + t\theta_{2k}. \tag{2.7}$$

Substituting for u in (2.6) gives

$$x_i = (\theta_{3k}\theta_{1k} + t(\mu_i - \theta_{2k}))/\sigma_i, \quad i = k, \dots, n. \tag{2.8}$$

Equations (2.7) and (2.8) then verify the first two assertions of the lemma.

The multipliers for the constraints $x_i = e_i, i = 1, \dots, k-1$, are obtained from the remaining portion of the dual feasibility conditions for (2.2)

$$t\mu_i - \sigma_i x_i = u - v_i.$$

Setting $x_i = e_i$ for $i = 1, \dots, k-1$ and substituting u from (2.7) gives

$$v_i = e_i\sigma_i - \theta_{3k}\theta_{1k} + t(\theta_{2k} - \mu_i), \quad i = 1, \dots, k-1,$$

as required. □

Some properties of θ_{1k}, θ_{2k} and θ_{3k} are formulated in the following lemma.

Lemma 2.2. *The following hold:*

(a) $1/\theta_{1,k-1} = 1/\sigma_{k-1} + 1/\theta_{1k}$, $k = 2, \dots, n$,
(b) $\theta_{3k} = \theta_{3,k-1} - e_{k-1}$, $k = 2, \dots, n$,
(c) $\theta_{2k} - \mu_j = \theta_{1k}((\mu_k - \mu_j)/\sigma_k + \cdots + (\mu_n - \mu_j)/\sigma_n)$, $j = 1, \dots, n$, $k = 1, \dots, n$.

Proof: These follow immediately from (2.3), (2.4) and (2.5). □

Our analysis requires the following assumption to be satisfied.

Assumption 2.2. (a) $t \geq 0$, (b) $e_n\sigma_n \leq \cdots \leq e_1\sigma_1$, (c) $d > e_1\sigma_1/\theta_{11}$.

In this section we formulate an explicit solution for (2.1) when Assumptions 2.1 and 2.2 are satisfied.

By setting $e = 0$ and $d = 1$ in (2.1) we obtain the portfolio selection problem

$$\min\{-t\mu'x + \tfrac{1}{2}x'\Sigma x \mid l'x = 1, x \geq 0\} \tag{2.9}$$

with no short sales. Since Assumptions 2.2(b) and (c) are automatically satisfied, our results for (2.9) will require only Assumption 2.1 and $t \geq 0$. This will result in the efficient frontier for no short sales. Furthermore, the same conclusion holds when the budget constraint in (2.9) is replaced with a more general budget constraint, $l'x = d$, where d is any strictly positive number.

The principal result for (2.1) is as follows. For $t = 0$, all assets[1] strictly exceed their lower bounds. As t is increased, eventually asset 1 is decreased to e_1 at $t = t_1$. Asset 1 remains at e_1 for all $t \geq t_1$. Assets $2, \ldots, n$ strictly exceed their respective lower bounds $e_2, \ldots, e_n$ for $t \geq t_1$ until asset 2 is reduced to e_2 at $t = t_2$. Now asset 1 remains at e_1 and asset 2 remains at e_2 for all $t \geq t_2$ and assets $3, \ldots, n$ strictly exceed $e_3, \ldots, e_n$, respectively, until asset 3 is decreased to e_3 at $t = t_3$. The process continues in a similar manner and assets $1, 2, \ldots, i-1$ remain at their lower bounds for all $t \geq t_{i-1}$, with asset i being reduced to e_i at t_i and all other assets with higher indices are strictly above their lower bounds. This is illustrated in Figure 1 for the problem (2.9).

For $k = 0, \ldots, n$, define

$$t_k = \begin{cases} 0, & k = 0, \\ (\theta_{3k}\theta_{1k} - e_k\sigma_k)/(\theta_{2k} - \mu_k), & k = 1, \ldots, n-1, \\ \infty, & k = n. \end{cases} \tag{2.10}$$

For $k = 1, \ldots, n$, define

$$\begin{cases} \quad x_k = x_k(t) = ((x_k)_1, (x_k)_2, \ldots, (x_k)_n)' \text{ where,} \\ (x_k)_i = e_i, \quad i = 1, \ldots, k-1, \\ (x_k)_i = (x_k(t))_i = (\theta_{3k}\theta_{1k} + t(\mu_i - \theta_{2k}))/\sigma_i, \quad i = k, \ldots, n, \end{cases} \tag{2.11}$$

$$u_k(t) = -\theta_{3k}\theta_{1k} + t\theta_{2k}, \tag{2.12}$$

$$\begin{cases} \quad v_k = v_k(t) = ((v_k)_1, (v_k)_2, \ldots, (v_k)_n)' \text{ where,} \\ (v_k)_i = (v_k(t))_i \\ \qquad = e_i\sigma_i - \theta_{3k}\theta_{1k} + t(\theta_{2k} - \mu_i), \quad i = 1, \ldots, k-1, \\ (v_k)_i = 0, \quad i = k, \ldots, n, \end{cases} \tag{2.13}$$

with θ_{1k}, θ_{2k} and θ_{3k} being given by (2.3), (2.4) and (2.5), respectively. Then the principal result for (2.1) with $t \geq 0$ is the following theorem.

[1] Technically speaking, we should use "asset holdings" rather than "asset". However, we will use the term asset to mean asset holdings for the sake of brevity.

Theorem 2.1. *Let Assumptions 2.1 and 2.2 be satisfied and let* $t_0, \ldots, t_n$ *be defined by (2.10). Then for* $k = 1, \ldots, n$,

(a) $t_{k-1} < t_k$,
(b) $x(t) = x_k(t)$, *for all* $t \in [t_{k-1}, t_k]$, *is optimal for (2.1) with* $x_k(t)$ *being given by (2.11),*
(c) *the multipliers for the lower bounds are given by* $v(t) = v_k(t)$, *for all* $t \in [t_{k-1}, t_k]$, *where* $v_k(t)$ *is given by (2.13),*
(d) *the multiplier for the budget constraint is given by* $u(t) = u_k(t)$, *for all* $t \in [t_{k-1}, t_k]$, *where* $u_k(t)$ *is given by (2.12).*

Before proceeding with the proof of Theorem 2.1, it is helpful to introduce the following lemma. This lemma will be used first to establish that the constant part of $(x_k)_i$ in (2.11) exceeds its lower bound and second to verify Theorem 2.1(a).

Lemma 2.3. *Let Assumptions 2.1(a) and 2.2(b), (c) be satisfied. Then for* $k = 1, \ldots, n$ *and* $i = 1, \ldots, n$, *the following inequality holds:*

$$e_i < \theta_{3k}\theta_{1k}/\sigma_i.$$

Proof: Using Assumptions 2.2(b), (c) and 2.1(a) we obtain the following for $k = 1, \ldots, n$ and $i = 1, \ldots, n$

$$d > \frac{e_1\sigma_1}{\theta_{11}} = e_1\sigma_1\left(\frac{1}{\sigma_1} + \cdots + \frac{1}{\sigma_n}\right) \geq \left(\frac{e_1\sigma_1}{\sigma_1} + \cdots + \frac{e_{k-1}\sigma_{k-1}}{\sigma_{k-1}}\right) + e_i\sigma_i\left(\frac{1}{\sigma_k} + \cdots + \frac{1}{\sigma_n}\right) = e_1 + \cdots + e_{k-1} + \frac{e_i\sigma_i}{\theta_{1k}}.$$

After re-arranging, the desired inequality is obtained. □

Proof of Theorem 2.1: Let t_k, $k = 0, \ldots, n$, be given by (2.10). Assumptions 2.1(b) and 2.2(c) imply that

$$t_1 = \frac{d\theta_{11} - e_1\sigma_1}{\theta_{21} - \mu_1} > 0 = t_0.$$

For $k = 2, \ldots, n-1$, Lemma 2.2, Assumption 2.2(b) and Lemma 2.3 imply

$$t_{k-1} = \frac{\theta_{3,k-1}\theta_{1,k-1} - e_{k-1}\sigma_{k-1}}{\theta_{2,k-1} - \mu_{k-1}} = \frac{\theta_{3k} - e_{k-1}\sigma_{k-1}/\theta_{1k}}{(\mu_k - \mu_{k-1})/\sigma_k + \cdots + (\mu_n - \mu_{k-1})/\sigma_n}$$
$$= \frac{\theta_{3k}\theta_{1k} - e_{k-1}\sigma_{k-1}}{\theta_{2k} - \mu_{k-1}} < \frac{\theta_{3k}\theta_{1k} - e_{k-1}\sigma_{k-1}}{\theta_{2k} - \mu_k}$$
$$\leq \frac{\theta_{3k}\theta_{1k} - e_k\sigma_k}{\theta_{2k} - \mu_k} = t_k.$$

For $k = n$, part (a) holds trivially and this completes the proof of part (a).

Let $1 \le k \le n$ and x_k, u_k, v_k be as in the statement of Theorem 2.1. With Assumption 2.1(a), the KKT conditions are both necessary and sufficient for optimality (see Mangasarian (1969)). These conditions are

$$\begin{cases} x \ge e, \quad l'x = d, \\ t\mu - \Sigma x = ul - v, \quad v \ge 0, \\ v'(x - e) = 0. \end{cases} \tag{2.14}$$

It follows directly from Lemma 2.1 that $l'x_k = d$ and $t\mu - \Sigma x_k = u_k l - v_k$. Furthermore, the definitions of x_k, v_k imply that $v_k'(x_k - e) = 0$. In order to show that (2.14) is satisfied, it remains to show that for all t with $t_{k-1} \le t \le t_k$ the following inequalities are satisfied:

$$(x_k)_i \ge e_i, \quad i = k, \ldots, n \text{ and} \tag{2.15}$$

$$(v_k)_i \ge 0, \quad i = 1, \ldots, k-1. \tag{2.16}$$

We first verify (2.15). The definition of x_k given by (2.11) and Lemma 2.3 imply that the constant part of x_k exceeds e. When $k = n$, the coefficient of t vanishes. Thus, (2.15) holds for $k = n$. Now let k be such that $1 \le k \le n - 1$. Since the budget constraint $l'x_k = d$ is satisfied for any value of the parameter t, this implies that the sum of the coefficients of t in x_k equals zero; i.e.,

$$(\mu_k - \theta_{2k})/\sigma_k + \cdots + (\mu_n - \theta_{2k})/\sigma_n = 0. \tag{2.17}$$

Assumption 2.1(b) and (2.17) imply the existence of an integer ρ_k with $k \le \rho_k \le n$ such that

$$\mu_i - \theta_{2k} < 0, \quad i = k, \ldots, \rho_k \text{ and} \tag{2.18}$$

$$\mu_i - \theta_{2k} \ge 0, \quad i = \rho_k + 1, \ldots, n. \tag{2.19}$$

From Lemma 2.3, Assumption 2.2(a), (2.11) and (2.19) it follows that $(x_k)_i > e_i$ for $i = \rho_k + 1, \ldots, n$ and

$$t \ge 0. \tag{2.20}$$

In order for $(x_k)_i$ to also satisfy the lower bounds for i with $k \le i \le \rho_k$, it follows from (2.11) and (2.18) that t must satisfy

$$t \le \min\left\{ \frac{\theta_{3k}\theta_{1k} - e_i\sigma_i}{\theta_{2k} - \mu_i} \,\middle|\, i = k, \ldots, \rho_k \right\}. \tag{2.21}$$

Assumption 2.2(b) implies that

$$\theta_{3k}\theta_{1k} - e_k\sigma_k \le \theta_{3k}\theta_{1k} - e_i\sigma_i \quad i = k+1, \ldots, n \tag{2.22}$$

and from Assumption 2.1(b),

$$\theta_{2k} - \mu_k > \theta_{2k} - \mu_i, \quad i = k+1, \ldots, n. \tag{2.23}$$

From (2.18), $\theta_{2k} - \mu_i > 0$ for $i = k, \ldots, \rho_k$ and from Lemma 2.3, $\theta_{3k}\theta_{1k} - e_i\sigma_i > 0$ for $i = 1, \ldots, n$. It now follows from (2.22) and (2.23) that

$$\frac{\theta_{3k}\theta_{1k} - e_k\sigma_k}{\theta_{2k} - \mu_k} < \frac{\theta_{3k}\theta_{1k} - e_i\sigma_i}{\theta_{2k} - \mu_i}, \quad i = k+1, \ldots, \rho_k. \tag{2.24}$$

Inequality (2.24) then implies

$$\min\left\{\frac{\theta_{3k}\theta_{1k} - e_i\sigma_i}{\theta_{2k} - \mu_i} \,\middle|\, i = k, \ldots, \rho_k\right\} = \frac{\theta_{3k}\theta_{1k} - e_k\sigma_k}{\theta_{2k} - \mu_k}.$$

The quantity on the right-hand side of this last equation is precisely t_k. This, (2.20) and (2.21) imply that

(2.15) is satisfied for all t with $0 \le t \le t_k$. (2.25)

To verify (2.16), first observe that from (2.10) and (2.13)

$$(v_k(t_{k-1}))_{k-1} = e_{k-1}\sigma_{k-1} - \theta_{3k}\theta_{1k} + \frac{\theta_{3,k-1}\theta_{1,k-1} - e_{k-1}\sigma_{k-1}}{\theta_{2,k-1} - \mu_{k-1}}(\theta_{2k} - \mu_{k-1}).$$

Using Lemma 2.2(b) and (c) gives

$$\begin{aligned}(v_k(t_{k-1}))_{k-1} &= e_{k-1}(\sigma_{k-1} + \theta_{1k}) - \theta_{3,k-1}\theta_{1k} \\ &\quad + \frac{\theta_{3,k-1} - e_{k-1}\sigma_{k-1}/\theta_{1,k-1}}{\frac{\mu_k - \mu_{k-1}}{\sigma_k} + \cdots + \frac{\mu_n - \mu_{k-1}}{\sigma_n}}\theta_{1k} \\ &\quad \times \left(\frac{\mu_k - \mu_{k-1}}{\sigma_k} + \cdots + \frac{\mu_n - \mu_{k-1}}{\sigma_n}\right).\end{aligned}$$

Re-arranging and then applying Lemma 2.2(a) gives

$$\begin{aligned}(v_k(t_{k-1}))_{k-1} &= e_{k-1}(\sigma_{k-1} + \theta_{1k} - \sigma_{k-1}\theta_{1k}/\theta_{1,k-1}), \\ &= e_{k-1}(\sigma_{k-1} + \theta_{1k} - \theta_{1k} - \sigma_{k-1}), \\ &= 0.\end{aligned} \tag{2.26}$$

Next observe that from Assumptions 2.1(b), 2.2(a), (b) and (2.13)

$$(v_k(t))_{i-1} \ge (v_k(t))_i, \quad 2 \le i \le k-1, \ t \ge 0. \tag{2.27}$$

By definition of v_k, its first $(k-1)$ components are strictly increasing functions of t. This with (2.26) and (2.27) implies that

(2.16) is satisfied for all t with $t \ge t_{k-1}$. (2.28)

Thus, (2.25) together with (2.28) imply that (2.15) and (2.16) are satisfied simultaneously for $t_{k-1} \le t \le t_k$ which completes the proof of the theorem. □

3 Upper bounds and the efficient frontier

In this section we will solve (2.1) for $t \leq 0$. In addition to Assumption 2.1 we require the analog of Assumption 2.2, namely

Assumption 3.1. (a) $t \leq 0$, (b) $e_1\sigma_1 \leq \cdots \leq e_n\sigma_n$, (c) $d > e_n\sigma_n/\vartheta_{11}$.

Throughout this paper, results can be written more concisely if any condition "for $i = n - k + 1, \ldots, n$" should be regarded as vacuous when $k = 0$ and similarly, "the sum from $n - k + 1$ to n of some quantities" should be regarded as having value 0 when $k = 0$.

Analogous to the discussion following Assumption 2.2, when $e = 0$ and d takes on any positive value, Assumptions 3.1(b) and (c) are satisfied. Thus when $e = 0$, $d > 0$ and $t \leq 0$ only Assumption 2.1 is required for (2.1) to obtain the results in this section.

Remark 3.1. The problem considered in this section has the following financial interpretation. Let $y = -x$ and $s = -t$ in (2.1) with $t \leq 0$. This gives the problem

$$\min\{-s\mu' y + \tfrac{1}{2} y' \Sigma y \mid l' y = -d, y \leq -e\}. \tag{3.1}$$

The problem (3.1) is the portfolio selection problem with n risky assets, for which the covariance matrix Σ is diagonal, the budget constraint is $l'y = -d$ and where an upper bound of $-e_i$ is imposed on each asset i. Thus, solving (2.1) with lower bounds and $t \leq 0$ is equivalent to solving (3.1) with upper bounds and $s \geq 0$.

The solution of (2.1) for $t \geq 0$ given in Theorem 2.1 requires the critical parameter values $t_0 < t_1 < \cdots < t_n$. The solution of (2.1) for $t \leq 0$ also requires $n + 1$ critical parameter values. In order to distinguish the critical parameter values for the lower part of the M-V frontier from those for the upper, we will use a negative subscript and the analogs of the former critical parameters will be $t_0 > t_{-1} > \cdots > t_{-n}$. The remaining quantities will be indexed in a similar manner.

Analogous to (2.2) for $t \geq 0$, for the case of $t \leq 0$ we consider the following problem for $k = 0, \ldots, n - 1$:

$$\min\{-t\mu' x + \tfrac{1}{2} x' \Sigma x \mid l' x = d, x_{n-k+1} = e_{n-k+1}, \ldots, x_n = e_n\}. \tag{3.2}$$

The solution of (3.2) can be formulated concisely in terms of the constants

$$\vartheta_{1,k+1} = 1/(\sigma_1^{-1} + \cdots + \sigma_{n-k}^{-1}), \tag{3.3}$$

$$\vartheta_{2,k+1} = \vartheta_{1,k+1}(\mu_1/\sigma_1 + \cdots + \mu_{n-k}/\sigma_{n-k}), \tag{3.4}$$

$$\vartheta_{3,k+1} = d - (e_{n-k+1} + \cdots + e_n), \tag{3.5}$$

for $k = 0, \ldots, n - 1$.

The following is a corollary to Lemma 2.1.

Corollary 3.1. *Let Assumption 2.1(a) be satisfied. Then for* $k = 0, \ldots, n-1$, *the optimal solution for (3.2) is*

$$x_i = (\vartheta_{3,k+1}\vartheta_{1,k+1} + t(\mu_i - \vartheta_{2,k+1}))/\sigma_i, \quad i = 1, \ldots, n-k,$$

$$x_i = e_i, \quad i = n-k+1, \ldots, n.$$

The multiplier for the budget constraint is $u = -\vartheta_{3,k+1}\vartheta_{1,k+1} + t\vartheta_{2,k+1}$ *and the multipliers for the constraints* $x_i = e_i, i = n-k+1, \ldots, n$ *are*

$$v_i = e_i\sigma_i - \vartheta_{3,k+1}\vartheta_{1,k+1} + t(\vartheta_{2,k+1} - \mu_i),$$

respectively.

Proof: This follows from Lemma 2.1 by re-indexing the assets. □

The principal result for (2.1) for $t \leq 0$ is as follows. For all t with $t_{-n} \leq t \leq t_{-(n-1)}$, asset 1 (i.e., the asset with the smallest expected return) remains at ϑ_{3n} while each other asset i remains at its lower bound e_i for $i = 2, \ldots, n$. As t increases beyond $t_{-(n-1)}$, the first asset strictly exceeds e_1, the second asset strictly exceeds e_2 and each other asset i remains at e_i. As t is increased further, assets 1, 2 and 3 strictly exceed their respective lower bounds while the other assets remain at their bounds. The process continues until t is increased to a critical value $t_{-1} < 0$. For $t_{-1} < t \leq 0$, all assets are held strictly above their lower bounds.

We next give a precise statement of the solution of (2.1) with $t \leq 0$. For $k = -1, 0, \ldots, n-1$, define

$$t_{-(k+1)} = \begin{cases} 0, \quad k = -1, \\ (\vartheta_{3,k+1}\vartheta_{1,k+1} - e_{n-k}\sigma_{n-k})/(\vartheta_{2,k+1} - \mu_{n-k}), \\ k = 0, \ldots, n-2, \\ -\infty, \quad k = n-1. \end{cases} \tag{3.6}$$

For $k = 0, \ldots, n-1$, define

$$\begin{cases} \quad x_{-k} = x_{-k}(t) = ((x_{-k})_1, (x_{-k})_2, \ldots, (x_{-k})_n)' \text{ where,} \\ (x_{-k})_i = (x_{-k}(t))_i \\ (x_{-k})_i = (\vartheta_{3,k+1}\vartheta_{1,k+1} + t(\mu_i - \vartheta_{2,k+1}))/\sigma_i, \quad i = 1, \ldots, n-k, \\ (x_{-k})_i = e_i, \quad i = n-k+1, \ldots, n, \end{cases} \tag{3.7}$$

$$u_{-k}(t) = -\vartheta_{3,k+1}\vartheta_{1,k+1} + t\vartheta_{2,k+1}, \tag{3.8}$$

$$\begin{cases} \quad v_{-k} = v_{-k}(t) = ((v_{-k})_1, (v_{-k})_2, \ldots, (v_{-k})_n)' \text{ where,} \\ (v_{-k})_i = 0, \quad i = 1, \ldots, n-k, \\ (v_{-k})_i = (v_{-k}(t))_i \\ \qquad = e_i\sigma_i - \vartheta_{3,k+1}\vartheta_{1,k+1} + t(\vartheta_{2,k+1} - \mu_i), \quad i = n-k+1, \ldots, n, \end{cases} \tag{3.9}$$

with $\vartheta_{1,k+1}$, $\vartheta_{2,k+1}$ and $\vartheta_{3,k+1}$ being given by (3.3), (3.4) and (3.5). The principal result for (2.1) with $t \leq 0$ is the following theorem.

Theorem 3.1. *Let Assumptions 2.1 and 3.1 be satisfied and let* $t_0, t_{-1}, \ldots, t_{-n}$ *be defined by (3.6). Then for* $k = 0, \ldots, n-1$,

(a) $t_{-(k+1)} < t_{-k}$,
(b) $x(t) = x_{-k}(t)$, *for all* $t \in [t_{-(k+1)}, t_{-k}]$, *is optimal for (2.1) with* $x_{-k}(t)$ *being given by (3.7),*
(c) *the multipliers for lower bounds are given by* $v(t) = v_{-k}(t)$, *for all* $t \in [t_{-(k+1)}, t_{-k}]$, *where* $v_{-k}(t)$ *is given by (3.9),*
(d) *the multiplier for the budget constraint is given by* $u(t) = u_{-k}(t)$, *for all* $t \in [t_{-(k+1)}, t_{-k}]$, *where* $u_{-k}(t)$ *is given by (3.8).*

Proof: The proof is similar to the proof of Theorem 2.1 and uses Lemma 3.1 below. □

Note that all bound constraints are inactive at $x_0(t)$ for $t_{-1} < t < t_1$ and x_0 defined by (3.7) is identical to x_0 defined by (2.11) although their domains of definition differ.

The following lemma is used first to establish that the constant part of $(x_{-k})_i$ in (3.7) exceeds its lower bound and second to verify Theorem 3.1(a).

Lemma 3.1. *Let Assumptions 2.1(a), 3.1(b), (c) be satisfied. Then for* $k = 0, \ldots, n-1$ *and* $i = 1, \ldots, n$, *the following inequality holds:*

$$e_i < \vartheta_{3,k+1}\vartheta_{1,k+1}/\sigma_i.$$

Proof: The proof is similar to the proof of Lemma 2.3. □

We next use Theorem 3.1 to provide an optimal solution for the upper bounded problem

$$\min\{-t\mu' x + \tfrac{1}{2} x'\Sigma x \mid l'x = d, x \le e\} \tag{3.10}$$

for $t \ge 0$. In doing so, it is helpful to introduce the following notation. Let $x(t,d,e)$, $v(t,d,e)$ and $u(t,d,e)$ denote the optimal solution, the vector of multipliers for the upper bounds and the multiplier for the budget constraint for (3.10), respectively, and $\hat{x}(t,d,e)$, $\hat{v}(t,d,e)$ and $\hat{u}(t,d,e)$ denote the optimal solution, the vector of multipliers for the lower bounds and the multiplier for the budget constraint for (2.1), respectively, as explicit functions of their problem data t, d and e, respectively. The solution for (3.10) is formulated in the following theorem.

Theorem 3.2. *Let Assumptions 2.1, 2.2(a), (b) be satisfied and assume* $d < e_n\sigma_n/\vartheta_{11}$ *in (3.10). Then* $x(t,d,e) = -\hat{x}(-t,-d,-e)$, $v(t,d,e) = \hat{v}(-t,-d,-e)$ *and* $u(t,d,e) = -\hat{u}(-t,-d,-e)$.

Proof: As in Remark 3.1, we utilize the transformation $\hat{x} = -x$ and $s = -t$ in (3.10) giving

$$\min\{-s\mu'\hat{x} + \tfrac{1}{2}\hat{x}'\Sigma\hat{x} \mid l'\hat{x} = -d, \hat{x} \ge -e\} \tag{3.11}$$

with $s \le 0$. For the data of (3.11), Assumption 2.1 is satisfied. Assumption

2.2(b) implies $-e_1\sigma_1 \leq \cdots \leq -e_n\sigma_n$ so that Assumption 3.1(b) for (3.11) is satisfied. Furthermore, from the assumption in the statement of the theorem, $-d > -e_n\sigma_n/\vartheta_{11}$ and thus Assumption 3.1(c) is satisfied. Thus, Theorem 3.1 may be applied to (3.11).

From Theorem 3.1 and the notation just introduced, the optimal solution for (3.11) is $\hat{x}(-t, -d, -e)$, with the vector of multipliers for the lower bounds and the multiplier for the budget constraint being given by $\hat{v}(-t, -d, -e)$ and $\hat{u}(-t, -d, -e)$. The assertion in the statement of the theorem now follows from comparing the KKT conditions for (3.10) and (3.11). □

4 Lower bounds with a risk free asset

In this section we consider the following $(n+1)$-dimensional problem with lower bounds

$$\min\{-t\mu_0 x_0 - t\mu' x + \tfrac{1}{2}x'\Sigma x \mid l'x + x_0 = d, x_0 \geq e_0, x \geq e\}, \tag{4.1}$$

where e_0, μ_0, x_0 are the lower bound, the expected return and the holdings of the risk free asset 0, respectively, and l, e, d, t, Σ, μ, x have the same meaning as in Section 2. The quantity μ_0 is usually called the risk free rate and is sometimes denoted by r_f. In what follows it is sometimes convenient to represent the risk free holdings and the risky holdings as a single vector. Thus for example, we represent the entire holdings vector for the k-th interval as the $(n+1)$-dimensional vector $x_k(t)$ where component 0 of $x_k(t)$ denotes the holdings for the risk free asset. The use of this notation should be clear from context.

We will consider (4.1) under Assumption 2.1(a), $t \geq 0$ and the following assumption:

Assumption 4.1. (a) $\mu_0 < \mu_1 < \cdots < \mu_n$, (b) $e_n\sigma_n \leq \cdots \leq e_1\sigma_1 \leq 0$, (c) $d > e_0$.

Note that Assumption 4.1(b) with Assumption 2.1(a) imply $e_i \leq 0$, $i = 1, \ldots, n$.

First we consider a problem with no inequality constraints that is closely related to (4.1):

$$\min\{-t\mu_0 x_0 - t\mu' x + \tfrac{1}{2}x'\Sigma x \mid l'x + x_0 = d\}. \tag{4.2}$$

The optimal solution for (4.2) is formulated in the following lemma.

Lemma 4.1. *Let Assumption 2.1(a) be satisfied. The optimal solution for (4.2) is*

$$x_0 = d - t(\theta_{21} - \mu_0)/\theta_{11}, \quad x_i = t(\mu_i - \mu_0)/\sigma_i, \quad i = 1, \ldots, n,$$

where θ_{11} and θ_{21} are defined as in (2.3) and (2.4) for $k = 1$ and the multiplier for the budget constraint is $u = t\mu_0$.

Proof: The proof follows directly from the KKT conditions for (4.2). □

The principal result for (4.1) is as follows. For $t = 0$, only the risk free asset 0 is held; i.e., $x_0(0) = d$ and $x_i(0) = 0$ for $i = 1, \dots, n$. As t is increased, the risk free asset 0 is reduced and all risky assets are increased from zero. At $t = t_0$, the risk free asset is reduced to its lower bound e_0 and remains there for all $t \geq t_0$. Furthermore, at $t = t_0$, all of the risky assets strictly exceed their lower bounds. As t is increased beyond t_0, the process continues precisely as described by Theorem 2.1 with the $t_0 = 0$ of Theorem 2.1 replaced by the t_0 just described and the right-hand side of the budget constraint in (2.1) replaced with $d - e_0$. Thus, as t is increased from zero in (4.1), the risk free asset is reduced to its lower bound first, then the first risky asset, then the second risky asset and so on. For $t_{-1} \leq t \leq t_0$, the first piece of the efficient frontier for (4.1) in (σ_p, μ_p) space is a straight line, namely the Capital Market Line (CML). The remainder of the efficient frontier is piece-wise hyperbolic. The CML meets the efficient frontier for the risky assets (with the budget constraint now being $l'x = d - e_0$) at some point at that part for the frontier corresponding to its first parametric interval, where all risky assets strictly exceed their lower bounds. This is illustrated in Figure 2 for $e_i = 0$, $i = 0, \dots, n$ and $d = 1$.

For $k = -1, \dots, n$, define

$$t_k = \begin{cases} 0, \quad k = -1, \\ \theta_{11}(d - e_0)/(\theta_{21} - \mu_0), \quad k = 0, \\ ((\theta_{3k} - e_0)\theta_{1k} - e_k\sigma_k)/(\theta_{2k} - \mu_k), \quad k = 1, \dots, n-1, \\ \infty, \quad k = n. \end{cases} \tag{4.3}$$

For $k = 0, \dots, n$, let

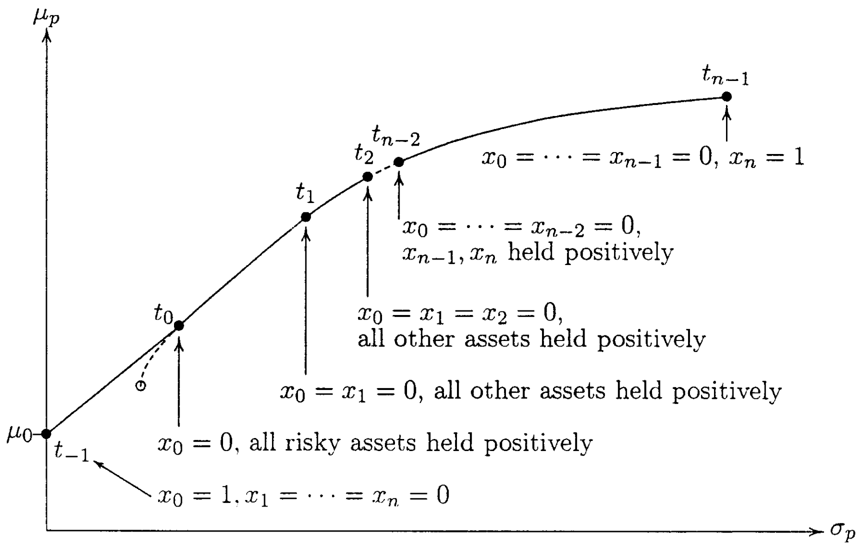

Fig. 2. Efficient Frontier for Risky/Risk Free Assets with No Short Sales

$$x_k = x_k(t) = ((x_k)_0, (x_k)_1, \ldots, (x_k)_n),$$

$$u_k = u_k(t),$$

$$v_k = v_k(t) = ((v_k)_0, (v_k)_1, \ldots, (v_k)_n).$$

For $k = 0$, define

$$(x_0)_i = \begin{cases} d - t(\theta_{21} - \mu_0)/\theta_{11}, & i = 0, \\ t(\mu_i - \mu_0)/\sigma_i, & i = 1, \ldots, n, \end{cases} \tag{4.4}$$

$$u_0 = t\mu_0, \tag{4.5}$$

$$v_0 = 0, \tag{4.6}$$

and for $k = 1, \ldots, n$, define

$$\begin{cases} (x_k)_i = e_i, \quad i = 0, \ldots, k-1, \\ (x_k)_i = (x_k(t))_i \\ \qquad = ((\theta_{3k} - e_0)\theta_{1k} + t(\mu_i - \theta_{2k}))/\sigma_i, \quad i = k, \ldots, n, \end{cases} \tag{4.7}$$

$$u_k(t) = (e_0 - \theta_{3k})\theta_{1k} + t\theta_{2k}, \tag{4.8}$$

$$\begin{cases} (v_k)_i = (v_k(t))_i \\ \qquad = e_i\sigma_i + (e_0 - \theta_{3k})\theta_{1k} + t(\theta_{2k} - \mu_i), \quad i = 0, \ldots, k-1, \\ (v_k)_i = 0, \quad i = k, \ldots, n, \end{cases} \tag{4.9}$$

where $\theta_{1k}, \theta_{2k}, \theta_{3k}$ are given by (2.3), (2.4), (2.5) for $k = 1, \ldots, n$ and $\sigma_0 = 0$. The principal result for (4.1) with $t \geq 0$ is the following theorem.

Theorem 4.1. *Let Assumptions 2.1(a) and 4.1 be satisfied and let $t_{-1}, t_0, \ldots, t_n$ be defined by (4.3) with $t \geq t_{-1}$. Then*

(a) *$t_{k-1} < t_k$, for $k = 0, \ldots, n$,*
(b) *$x(t) = x_k(t)$, for all $t \in [t_{k-1}, t_k]$, is optimal for (4.1) with $x_0(t)$ being given by (4.4) and $x_k(t)$ being given by (4.7) for $k = 1, \ldots, n$,*
(c) *the multipliers for lower bounds are given by $v(t) = v_k(t)$, for all $t \in [t_{k-1}, t_k]$, where $v_0(t)$ is given by (4.6) and $v_k(t)$ is given by (4.9) for $k = 1, \ldots, n$,*
(d) *the multiplier for the budget constraint is given by $u(t) = u_k(t)$, for all $t \in [t_{k-1}, t_k]$, where $u_0(t)$ is given by (4.5) and $u_k(t)$ is given by (4.8) for $k = 1, \ldots, n$.*

Proof: Let t_k be defined by (4.3) for $k = -1, \ldots, n$. Assumption 4.1 implies

$$t_{-1} = 0 < t_0 = \frac{(d - e_0)\theta_{11}}{\theta_{21} - \mu_0} < \frac{(d - e_0)\theta_{11}}{\theta_{21} - \mu_1} \leq \frac{(d - e_0)\theta_{11} - e_1\sigma_1}{\theta_{21} - \mu_1} = t_1;$$

i.e.,

$$t_{-1} < t_0 < t_1. \tag{4.10}$$

Because of (4.10), the proof of the theorem can now proceed according to the two cases $0 \le t \le t_0$ and $t_0 \le t \le t_n$.

Let x_0, u_0 and v_0 be as in the statement of Theorem 4.1 and let $t \in [0, t_0]$. For $t = 0$, $(x_0)_0 = d$ and $(x_0)_i = 0$ for $i = 1, \ldots, n$. According to Lemma 4.1, x_0 is the solution and u_0 is the multiplier for the budget constraint of (4.2) where no inequality constraints are active. Assumptions 4.1(b) and (c) imply that x_0 satisfies the lower bounds for $t = 0$ and thus x_0 is also the optimal solution of (4.1) for $t = 0$. The same argument verifies that x_0 is the optimal solution of (4.1) with the multiplier for the budget constraint given by u_0, for all $t \ge 0$ such that $(x_0)_0 \ge e_0$. Similar to Lemma 2.2(c), for $k = 1, \ldots, n$,

$$\theta_{2k} - \mu_0 = \theta_{1k}((\mu_k - \mu_0)/\sigma_k + \cdots + (\mu_n - \mu_0)/\sigma_n), \tag{4.11}$$

so that from Assumption 4.1(a) and (4.11), $\theta_{21} - \mu_0 > 0$. Thus from (4.4) $(x_0)_0$ is a decreasing function of t. Furthermore, from (4.4) and Assumption 2.1 it follows that $(x_0)_i$, $i = 1, \ldots, n$ are increasing functions of t. Since $(x_0(t_0))_0 = e_0$, it follows that $x_0(t)$ is indeed optimal for $0 \le t \le t_0$.

Let $1 \le k \le n$ and x_k, u_k, v_k be as in the statement of Theorem 4.1 and $t \in [t_{k-1}, t_k]$. The KKT conditions for (4.1) are

$$x_0 \ge e_0, \quad x \ge e, \quad l'x = d - x_0, \tag{4.12}$$

$$t\mu - \Sigma x = ul - v, \quad v \ge 0, \tag{4.13}$$

$$t\mu_0 = u - v_0, \quad v_0 \ge 0, \tag{4.14}$$

$$v_0(x_0 - e_0) = 0, \quad v'(x - e) = 0, \tag{4.15}$$

where v_0 is the multiplier for the constraint $x_0 \ge e_0$ and v is the n-vector of multipliers corresponding to the lower bounds $x_i \ge e_i$ for $i = 1, \ldots, n$.

Consider the following problem dealing only with the n risky assets:

$$\min\{-t\mu'x + \tfrac{1}{2}x'\Sigma x \mid l'x = d - e_0, x \ge e\}. \tag{4.16}$$

We want to apply Theorem 2.1 to (4.16) for $t \ge t_0$. From Assumptions 4.1(b) and (c), $d > e_0 \ge e_0 + e_1\sigma_1/\theta_{11}$ which implies that $(d - e_0) > e_1\sigma_1/\theta_{11}$. This implies that Assumption 2.2(c) is satisfied for (4.16) with the right-hand side of the budget constraint being replaced with $d - e_0$. Thus Assumptions 2.1 and 2.2 are satisfied for (4.16). Note that the definition of t_k in Theorem 2.1 is defined in terms of θ_{3k}. In the present context d in the definition of θ_{3k} must be replaced by $d - e_0$. Doing so and using the t_k's from Theorem 2.1 gives the t_k's defined by (4.3). Part (a) of the present theorem then follows from (4.10) and Theorem 2.1(a). Furthermore, it also follows from Theorem 2.1 that the vector of the last n components of x_k is optimal for (4.16) with the vector of multipliers for the lower bounds being the last n components of the $(n+1)$-vector v_k and the multiplier for the budget constraint being u_k for all t with $t_{k-1} \le t \le t_k$ and $k = 1, \ldots, n$. Thus, the KKT conditions (4.12), (4.13) and (4.15) are satisfied.

From (4.14), the multiplier for the constraint $(x_k)_0 \ge e_0$ is

$$(v_k)_0 = u_k - t\mu_0$$

for all $t \geq t_0$. Substitution of u_k from (4.8) gives

$$(v_k)_0 = (e_0 - \theta_{3k})\theta_{1k} + t(\theta_{2k} - \mu_0),$$

in agreement with the statement of Theorem 4.1. It remains to show that

$$(v_k)_0 \geq 0 \quad \text{for all } t \geq t_0. \tag{4.17}$$

To verify (4.17), first observe that from (4.9) and (4.11), $(v_k)_0$ is increasing in t for any interval $[t_{k-1}, t_k]$. Consequently, (4.17) will be established by verifying that $(v_k(t_{k-1}))_0 \geq 0$. Although we have used Σ to denote the covariance matrix, in the following we will use $\sum$ to denote summation. Observe first that from (4.3) and (4.9)

$$(v_k(t_{k-1}))_0 = (e_0 - \theta_{3k})\theta_{1k} + \frac{(\theta_{3,k-1} - e_0)\theta_{1,k-1} - e_{k-1}\sigma_{k-1}}{\theta_{2,k-1} - \mu_{k-1}}(\theta_{2k} - \mu_0).$$

Using Lemma 2.2(b), (c) and re-arranging gives

$$(v_k(t_{k-1}))_0 = e_{k-1}\theta_{1k} + (e_0 - \theta_{3,k-1})\theta_{1k} + \frac{\theta_{3,k-1} - e_0 - e_{k-1}\sigma_{k-1}/\theta_{1,k-1}}{\sum_{j=k}^{n}(\mu_j - \mu_{k-1})/\sigma_j}\theta_{1k}\sum_{j=k}^{n}\frac{\mu_j - \mu_0}{\sigma_j}.$$

Further re-arranging and applying Lemma 2.2(a) leads to

$$\begin{aligned}(v_k(t_{k-1}))_0 &= e_{k-1}\theta_{1k} + \frac{(e_0 - \theta_{3,k-1})\sum_{j=k}^{n}(\mu_j - \mu_{k-1})/\sigma_j}{\sum_{j=k}^{n}(\mu_j - \mu_{k-1})/\sigma_j}\theta_{1k} \\ &\quad + \frac{(\theta_{3,k-1} - e_0)\sum_{j=k}^{n}(\mu_j - \mu_0)/\sigma_j}{\sum_{j=k}^{n}(\mu_j - \mu_{k-1})/\sigma_j}\theta_{1k} \\ &\quad - e_{k-1}\sigma_{k-1}\frac{\sum_{j=k}^{n}(\mu_j - \mu_0)/\sigma_j}{\sum_{j=k}^{n}(\mu_j - \mu_{k-1})/\sigma_j}\frac{\theta_{1k}}{\theta_{1,k-1}}, \\ &= e_{k-1}\theta_{1k} + \frac{(\theta_{3,k-1} - e_0)(\mu_{k-1} - \mu_0)}{\sum_{j=k}^{n}(\mu_j - \mu_{k-1})/\sigma_j} \\ &\quad - e_{k-1}\theta_{1k}\frac{\theta_{2k} - \mu_0}{\theta_{2k} - \mu_{k-1}} - e_{k-1}\sigma_{k-1}\frac{\theta_{2k} - \mu_0}{\theta_{2k} - \mu_{k-1}}, \\ &= -e_{k-1}\theta_{1k}\frac{\mu_{k-1} - \mu_0}{\theta_{2k} - \mu_{k-1}} + \frac{(\theta_{3,k-1} - e_0)(\mu_{k-1} - \mu_0)}{\sum_{j=k}^{n}(\mu_j - \mu_{k-1})/\sigma_j} \\ &\quad - e_{k-1}\sigma_{k-1}\frac{\theta_{2k} - \mu_0}{\theta_{2k} - \mu_{k-1}}, \\ &\geq 0,\end{aligned}$$

where the last inequality follows from Assumption 4.1, Lemma 2.2(c) and (4.11). Thus, all of the KKT conditions for (4.1) are satisfied and the proof is complete. □

5 Conclusion

We considered a portfolio selection problem of risky, uncorrelated assets subject to lower bounds on all asset holdings. Under a technical assumption we formulated a closed form solution for all portfolios corresponding to the efficient frontier. We showed that as an investor's aversion to risk decreases (i.e., t increases from zero), the risky asset holdings were reduced to their lower bounds (and remained there) in the order of smallest expected return to largest expected return. We also considered the case when this problem was augmented by a risk free asset. Using the results for the all risky asset case, we obtained a closed form solution for the risk free asset problem. We showed that in (σ_p, μ_p) space, the CML meets the efficient frontier for the risky assets only with a modified budget constraint in the part for that frontier corresponding to its first parametric interval.

We also considered a portfolio selection problem with risky assets similar to the previous, but with upper bounds on asset holdings, rather than lower. Under a technical assumption we developed a closed form solution for all portfolios corresponding to the efficient frontier. We formulated a result showing that as an investor's aversion to risk decreases, the risky asset holdings were increased to their upper bounds (and remained there) in the order of largest expected return to smallest expected return.

References

Best MJ (1996) An algorithm for the solution of the parametric quadratic programming problem. In Applied Mathematics and Parallel Computing – Festschrift for Klaus Ritter, H. Fischer, B. Riedmüller and S. Schäffler (editors), Physica-Verlag, Heidelberg, 57–76

Best MJ, Grauer RR (1990) The efficient set mathematics when mean-variance problems are subject to general linear constraints. Journal of Economics and Business, Vol. 42, 105–120

Brennan MJ (1971) Capital market equilibrium with divergent borrowing and lending rates. Journal of Financial and Quantitative Analysis, 1197–1205

Levy H (1983) The capital asset pricing model: theory and empiricism. The Economic Journal, Vol. 93, 145–165

Mangasarian OL (1969) Nonlinear Programming. McGraw-Hill, New York

Markowitz H (1956) The optimization of a quadratic function subject to linear constraints. Naval Research Logistics Quarterly, Vol. March–June, 111–133

Markowitz H (1959) Portfolio Selection: Efficient Diversification of Investments. John Wiley, New York; Yale University Press, New Haven

Perold AF (1984) Large-scale portfolio optimization. Management Science, Vol. 30, No. 10, 1143–1160

Schnabel JA (1984) Short sales restrictions and the security market line. Journal of Business Research, Vol. 12, 87–96

Sharpe WF (1970) Portfolio Theory and Capital Markets. McGraw-Hill, New York

Turnbull SM (1977) Market imperfections and the capital asset pricing model. Journal of Business Finance & Accounting, Vol. 4, 3, 327–337