Math Meth Oper Res (2000) 52:57–64

Mathematical Methods
of Operations Research

Generalized vector quasi-equilibrium problems[1]

Jun-Yi Fu

Department of Mathematics, Nanchang University, Nanchang, China 330047

Manuscript received: May 1999/Final version received: February 2000

Abstract. The properly quasi-convexity of multivalued mappings in an ordered vector space is introduced. Existence theorems for generalized vector quasi-equilibrium problems and multivalued vector equilibrium problems are obtained.

Key words. Vector equilibrium problem, multivalued mapping, ordered vector space

1 Introduction

Throughout the paper, let Z be a real topological vector space, $P \subset Z$ a closed, convex, pointed cone. Define a vector ordering on Z by $P : \forall x, y \in Z$, $x \leq y \Leftrightarrow y - x \in P$. Let K be a nonempty subset of a real vector space X, and $\varphi : K \times K \to Z$ such that $\varphi(x, x) \geq 0$ for all $x \in K$. The vector equilibrium problem consists in finding $\bar{x} \in K$ such that

$$\text{(VEP)} \quad \varphi(\bar{x}, y) \geq 0 \quad \forall y \in K.$$

If $Z = \mathbb{R}$ (real numbers) and $P = [0, +\infty)$, then the VEP becomes the scalar equilibrium problem, which has many diverse applications (see Blum and Oettli (1994), and Oettli and Schläger (1997)).

Let D be a nonempty subset of a real vector space Y, and $S : S \rightrightarrows K$ and $A : K \rightrightarrows D$ multivalued mappings. Let $f : K \times D \times K \to Z$ be a given mapping such that $f(x, y, x) \geq 0$ for all $x \in K$ and $y \in A(x)$. The generalized vector quasi-equilibrium problem consists in finding $\bar{x} \in K$ and $\bar{y} \in A(\bar{x})$ such that

$$\text{(GVQEP)} \quad \bar{x} \in S(\bar{x}) \quad \text{and} \quad f(\bar{x}, \bar{y}, x) \geq 0 \quad \forall x \in S(\bar{x}).$$

Let $F : K \times D \rightrightarrows Z$ be given. The multivalued vector equilibrium problem consists in finding $\bar{y} \in D$ such that

$$\text{(MVEP)} \quad F(x, \bar{y}) \subset P \quad \forall x \in K.$$

Until now, only a few papers deal with these problems in the strong sense. The purpose of the paper is to discuss some existence results for these problems. Our results generalize some main results of Chan and Pang (1982) and Parida and Sen (1987), and obtain vector versions of the well-known Walras Excess Demand Theorem and Ky Fan minimax Inequality (see Gwinner (1981)). Now, we recall some notations and preliminary results which will be used throughout the paper.

Let Z^* be the topological dual space of Z, $P^* \subset Z^*$ the polar cone of P, i.e., $P^* = \{z^* \in Z^* : \langle z^*, z\rangle \geq 0 \ \forall z \in P\}$. We assume that P^* has a weak* compact convex base B. This means that $B \subset P^*$ is a weak* compact convex set such that $0 \notin B$ and $P^* = \bigcup_{\lambda \geq 0} \lambda B$ (see [6]).

Lemma 1. (Jeyakumar and Oettli (1993)). *Let B be a weak* compact convex base of P^*, and $z \in Z$. Then*

(i) $z \geq 0 \Leftrightarrow \langle z^*, z\rangle \geq 0 \ \forall z^* \in P^*$;
(ii) $z \geq 0 \Leftrightarrow \langle z^*, z\rangle \geq 0 \ \forall z^* \in B$.

Definition 1. Let X and Y be two topological spaces, $T : X \rightrightarrows Y$ a multivalued mapping. (i) T is said to be upper semi-continuous (u.s.c.) at $x \in X$ if for each open set V containing $T(x)$, there exists an open set U containing x such that for each $t \in U$, $T(t) \subset V$; T is said to be u.s.c. on X if it is u.s.c. at all $x \in X$. (ii) T is said to be lower semi-continuous (l.s.c.) at $x \in X$ if for any open set V with $T(x) \cap V \neq \emptyset$, there exists an open set containing x such that for each $t \in U$, $T(t) \cap V \neq \emptyset$; T is said to be l.s.c. on X if it is l.s.c. at all $x \in X$. (iii) T is said to be continuous on X if it is at the same time u.s.c. and l.s.c. on X. (iv) T is said to be closed if the graph $G_r(T)$ of T, i.e., $G_r(T) = \{(x, y) : x \in X \text{ and } y \in T(x)\}$, is a closed set in $X \times Y$.

Lemma 2 (Tan (1985)). (i) *T is closed if and only if for any net $\{x_\alpha\}$, $x_\alpha \to x$ and any net $\{y_\alpha\}$, $y_\alpha \in T(x_\alpha)$, $y_\alpha \to y$, one has $y \in T(x)$.*

(ii) *If T is closed and $\overline{T(X)}$ is compact, then T is u.s.c., where $T(X) = \bigcup_{x \in X} T(x)$ and $\bar{E}$ is the closed hull of a set E.*

(iii) *If T is u.s.c. and for each $x \in X$, $T(x)$ is a closed set, then T is closed.*

(iv) *If X is compact and T is u.s.c., and for each $x \in X$, $T(x)$ is compact, then $T(X)$ is compact.*

(v) *T is l.s.c. at $x \in X$ if and only if for any $y \in T(x)$, and any net $\{x_\alpha\}$, $x_\alpha \to x$, there exists a net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ and $y_\alpha \to y$.*

Lemma 3 (Berge (1963)). *Let X and Y be two Hausdorff topological spaces, $T : X \rightrightarrows Y$ a continuous multivalued mapping such that for each $x \in X$, $T(x)$ is a nonempty compact set of Y. Let $\varphi : X \times Y \to \mathbb{R}$ be continuous. Then the function $M(x) = \min_{y \in T(x)} \varphi(x, y)$ is continuous.*

Definition 2 (Ferro (1989)). Let (Z, P) be an ordered topological vector space, and K a nonempty convex subset of a vector space X. Let $f : K \to Z$ be given. (i) f is called convex if for every $x, y \in K$ and $t \in [0, 1]$, we have $f(tx + (1 - t)y) \le tf(x) + (1 - t)f(y)$;

(ii) f is called properly quasi-convex if for every $x, y \in K$ and $t \in [0, 1]$, we have

$$\text{either} \quad f(tx + (1 - t)y) \le f(x)$$

$$\text{or} \quad f(tx + (1 - t)y) \le f(y).$$

Remark 1. A mapping may be convex and not properly quasi-convex, and conversely (see Ferro (1989)). It is easily seen that properly quasi-convexity and quasi-convexity are equivalent to each other in the scalar case ($Z = \mathbb{R}$ and $P = [0, +\infty)$).

The following multivalued version of properly quasi-convexity is new.

Definition 3. Let $F : K \rightrightarrows Z$ be a multivalued mapping. F is said to be properly quasi-convex if for every $x, y \in K$, $t \in [0, 1]$, and $u \in F(x)$, $v \in F(y)$, there exists $z \in F(tx + (1 - t)y)$ such that either $z \le u$ or $z \le v$.

Lemma 3. *Let $F : K \rightrightarrows Z$ be a multivalued mapping. Then F is properly quasi-convex if and only if for any $x_i \in K$, $z_i \in F(x_i)$, $t_i > 0$, $i = 1, \dots, n$, $\sum_{i=1}^{n} t_i = 1$, there exist $z \in F(t_1x_1 + \cdots + t_nx_n)$ and some i such that $z \le z_i$.*

Proof. It is enough to show the necessity. We proceed by induction. When $n = 2$, the conclusion is true. Suppose that for $n = m$, the conclusion is true. If $x_1, \dots, x_{m+1} \in K$, $t_i > 0$, $\sum_{i=1}^{m+1} t_i = 1$, $z_i \in F(x_i)$, $i = 1, \dots, m + 1$, we write $y = \frac{t_m}{t_m + t_{m+1}} x_m + \frac{t_{m+1}}{t_m + t_{m+1}} x_{m+1}$, and $x = \sum_{i=1}^{m+1} t_i x_i$. Then $x = t_1x_1 + \cdots + t_{m-1}x_{m-1} + (t_m + t_{m+1})y$. By the definition, there exists a $\bar{z} \in F(y)$ such that

$$\text{either} \quad \bar{z} \le z_m \quad \text{or} \quad \bar{z} \le z_{m+1}. \tag{1}$$

By the inductive assumption, there exists a $z \in F(x)$ such that either $z \le z_i$ for some i or $z \le \bar{z}$. If $z \le \bar{z}$, by (1) we have either $z \le z_m$ or $z \le z_{m+1}$. The proof is completed.

The classical Knaster-Kuratowski-Mazurkiewicz (in short, KKM) theorem was generalized by Shioji [10]. First we state a generalization of the KKM mapping.

Definition 4. Let X and Y be vector spaces, and K a nonempty convex subset of X. Let $G : K \rightrightarrows Y$ and $T : K \rightrightarrows Y$ be given. G is said to be a T-KKM mapping if for each finite subset $\{x_1, \dots, x_n\}$ of K, $T(\text{co}(x_1, \dots x_n)) \subset \bigcup_{i=1}^{n} G(x_i)$, where $\text{co}(E)$ is the convex hull of a set E.

Lemma 4. (Shioji [10]) *Let X and Y be topological vector spaces, K a compact convex subset of X. Let $G : K \rightrightarrows Y$ and $T : K \rightrightarrows Y$ be multivalued mappings. Assume that* (i) *T is u.s.c. and G is a T-KKM mapping;* (ii) *for each $x \in K$, $T(x)$ is a nonempty convex compact set and $G(x)$ is a closed set. Then* $\bigcap_{x \in K} G(x) \neq \emptyset$.

Definition 5. (Schaefer [9]) A topological vector space X is called quasi-complete if every bounded, closed subset of X is complete.

2 Generalized vector quasi-equilibrium problems

Theorem 1. *Let X, Y and Z be real locally convex Hausdorff topological vector spaces, and let Y be quasi-complete. Let $K \subset X$ be a convex compact set, and $D \subset Y$ a closed convex set, and $P \subset Z$ a closed, convex, pointed cone. Let P^* have a weak* compact convex base B. Let $S : K \rightrightarrows K$ be a continuous mapping such that for each $x \in K$, $S(x)$ is a nonempty closed convex set, and $A : K \rightrightarrows D$ a u.s.c. mapping such that for each $x \in K$, $A(x)$ is a nonempty compact convex set of D. Let $f : K \times D \times K \to Z$ be continuous. Assume that*

(i) *for any $x \in K$ and $y \in A(x)$, $f(x, y, x) \geq 0$;*
(ii) *for any $(x, y) \in K \times D$, $f(x, y, u)$ is properly quasi-convex in u.*

Then there exist $\bar{x} \in K$ and $\bar{y} \in A(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and $f(\bar{x}, \bar{y}, x) \geq 0$ for all $x \in S(\bar{x})$.

Proof. For any fixed $(x, y, u) \in K \times D \times K$, $\langle z^*, f(x, y, u)\rangle$ is weak* continuous on B. Let $g(x, y, u) = \min_{z^* \in B} \langle z^*, f(x, y, u)\rangle$. By Lemma 3, g is continuous on $K \times D \times K$. Define the multivalued mapping $\Phi : K \times D \rightrightarrows K$ by

$$\Phi(x, y) = \{v \in S(x) : g(x, y, v) = \min_{u \in S(x)} g(x, y, u)\}, \quad \forall (x, y) \in K \times D.$$

1) For any fixed $(x, y) \in K \times D$, $\Phi(x, y)$ is a closed subset of $S(x)$.

Let a net $\{v_\alpha\} \subset \Phi(x, y)$, $v_\alpha \to v$ be arbitrarily given. Since $v_\alpha \in S(x)$ and $g(x, y, v_\alpha) = \min_{u \in S(x)} g(x, y, u)$, for any $u \in S(x)$, we have $g(x, y, v_\alpha) \leq g(x, y, u)$. Since g is continuous and $S(x)$ is closed, we have $v \in S(x)$ and

$$g(x, y, v) = \lim_\alpha g(x, y, v_\alpha) \leq g(x, y, u).$$

Hence, $g(x, y, v) = \min_{u \in S(x)} g(x, y, u)$ and $v \in \Phi(x, y)$.

2) For any fixed $(x, y) \in K \times D$, $\Phi(x, y)$ is a convex subset of $S(x)$.

We proceed by contradiction. Suppose that there exists some $(x, y) \in K \times D$ such that $\Phi(x, y)$ is not convex. Then there exist $v_1, v_2 \in \Phi(x, y)$ and a $t \in (0, 1)$ such that $tv_1 + (1 - t)v_2 \notin \Phi(x, y)$. Since $S(x)$ is convex, $tv_1 + (1 - t)v_2 \in S(x)$. Observe that $g(x, y, v_i) = \min_{u \in S(x)} g(x, y, u)$, $i = 1, 2$, and

$$g(x,y,v_i) < g(x,y,tv_1 + (1-t)v_2), \quad i = 1,2. \tag{2}$$

By the definition of g, there exist $z_1^*, z_2^* \in B$ such that $g(x,y,v_i) = \langle z_i^*, f(x,y,v_i)\rangle$, $i = 1,2$. Since $f(x,y,u)$ is properly quasi-convex in u, we have

$$\text{either} \quad f(x,y,tv_1 + (1-t)v_2) \leq f(x,y,v_1) \tag{3}$$

$$\text{or} \quad f(x,y,tv_1 + (1-t)v_2) \leq f(x,y,v_2). \tag{4}$$

If (3) holds, then

$$\begin{aligned} g(x,y,tv_1 + (1-t)v_2) &\leq \langle z_1^*, f(x,y,tv_1 + (1-t)v_2)\rangle \\ &\leq \langle z_1^*, f(x,y,v_1)\rangle = g(x,y,v_1). \end{aligned} \tag{5}$$

By (2) and (5), we have a contradiction. If (4) holds, then

$$\begin{aligned} g(x,y,tv_1 + (1-t)v_2) &\leq \langle z_2^*, f(x,y,tv_1 + (1-t)v_2)\rangle \\ &\leq \langle z_2^*, f(x,y,v_2)\rangle = g(x,y,v_2). \end{aligned} \tag{6}$$

Also, a contradiction.

3) Φ is u.s.c.

Observe that K is compact. By Lemma 2(ii), we need only to show that Φ is closed. Let a net $(x_\alpha, y_\alpha) \in K \times D$ be given such that $(x_\alpha, y_\alpha) \to (x,y) \in K \times D$, and Let a net $v_\alpha \in \Phi(x_\alpha, y_\alpha)$ be given such that $v_\alpha \to v$. We shall show $v \in \Phi(x,y)$. Since S is u.s.c. and for each $x \in K$, $S(x)$ is a closed set, by Lemma 2(iii), S is closed. It follows from $v_\alpha \in S(x_\alpha)$ and $x_\alpha \to x$, $v_\alpha \to v$ that $v \in S(x)$. Since S is l.s.c., by Lemma 2(v), for any $u \in S(x)$, there exists a net $u_\alpha \in S(x_\alpha)$ such that $u_\alpha \to u$. Since $v_\alpha \in \Phi(x_\alpha, y_\alpha)$, we have

$$g(x_\alpha, y_\alpha, v_\alpha) = \min_{u \in S(x_\alpha)} g(x_\alpha, y_\alpha, u) \leq g(x_\alpha, y_\alpha, u_\alpha).$$

It follows from the continuity of g that

$$g(x,y,v) = \lim_\alpha g(x_\alpha, y_\alpha, v_\alpha) \leq \lim_\alpha g(x_\alpha, y_\alpha, u_\alpha) = g(x,y,u), \quad \forall u \in S(x).$$

Thus, $g(x,y,v) = \min_{u \in S(x)} g(x,y,u)$. This means that $v \in \Phi(x,y)$ and Φ is u.s.c.

4) Let $A(K) = \bigcup_{x \in K} A(x)$. By Lemma 2(iv), $A(K)$ is compact. Let $L = \overline{\text{co}}(A(K))$. Since Y is quasi-complete, it follows that L is a compact convex set of Y (see Schaefer (1980)). Define the multivalued mapping $F : K \times L \rightrightarrows K \times L$ by $F(x,y) = (\Phi(x,y), A(x)), \forall (x,y) \in K \times L$. Then F is u.s.c, and for every $(x,y) \in K \times L$, $F(x,y)$ is a nonempty convex compact set of $K \times L$. By the Kakutani-Fan-Gliksberg fixed point theorem, there exists a point $(\bar{x}, \bar{y}) \in K \times L$ such that $(\bar{x}, \bar{y}) \in F(\bar{x}, \bar{y})$, i.e., $\bar{x} \in \Phi(\bar{x}, \bar{y})$ and $\bar{y} \in A(\bar{x})$. This means that $\bar{x} \in S(\bar{x})$, $\bar{y} \in A(\bar{x})$ and $g(\bar{x}, \bar{y}, x) \geq g(\bar{x}, \bar{y}, \bar{x}) \geq 0$, $\forall x \in S(\bar{x})$. By Lemma 1, we get the conclusion. The proof is completed.

Remark 2. When $Z = \mathbb{R}$ and $P = [0, +\infty)$, Theorem 1 contains as special cases Theorem 1 of Chan and Pang (1982) and Theorem 1 of Parida and Sen (1987).

Corollary 1. *Let K, D, P, P^*, S and A be as in Theorem 1. Let $f : K \times D \times K \to Z$ be continuous. Assume that*

(i) *there exists a $c \in Z$ such that for any $x \in K$ and $y \in A(x), f(x, y, x) \geq c$;*
(ii) *for any fixed $(x, y) \in K \times D, f(x, y, u)$ is properly quasi-convex in u.*

Then there exist $\bar{x} \in K$ and $\bar{y} \in A(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and $f(\bar{x}, \bar{y}, x) \geq c$ for all $x \in S(\bar{x})$.

The following corollary is a vector version of the Walras Excess Demand Theorem.

Corollary 2. *Let K, D, P, P^* and A be as in Theorem 1. Let $\varphi : K \times D \to Z$ be continuous. Assume that*

(i) *there exists $c \in Z$ such that for any $x \in K$ and $y \in A(x)$, $\varphi(x, y) \geq c$;*
(ii) *for any fixed $y \in D$, $\varphi(u, y)$ is properly quasi-convex in u.*

Then there exist $\bar{x} \in K$ and $\bar{y} \in A(\bar{x})$ such that $\varphi(x, \bar{y}) \geq c$ for all $x \in K$.

Proof. In Corollary 1, define the multivalued mapping $S : K \rightrightarrows K$ by $S(x) = K$ for all $x \in K$. Then S is continuous. For every $(x, y, u) \in K \times D \times K$, let $f(x, y, u) = \varphi(u, y)$. Corollary 1 yields the conclusion.

3 Multivalued vector equilibrium problems

Theorem 2. *Let Z be an ordered topological vector space with the vector ordering induced by a closed, convex, pointed cone P. Let X and Y be topological vector spaces, $K \subset X$ a nonempty compact convex set, and $D \subset Y$ a nonempty closed convex set. Let $T : K \rightrightarrows D$ be a u.s.c. multivalued mapping such that for each $x \in K$, $T(x)$ is a nonempty compact convex set. Let $F : K \times D \rightrightarrows Z$ be a multivalued mapping. Assume that*

(i) *for any $x \in K$ and $y \in T(x)$, $F(x, y) \subset P$;*
(ii) *for any $x \in K$, $\{y \in D : F(x, y) \subset P\}$ is closed;*
(iii) *for any $y \in D$, $F(x, y)$ is properly quasi-convex in x.*

Then there exists $\bar{y} \in D$ such that $F(x, \bar{y}) \subset P$ for all $x \in K$.

Proof. Define $G : K \rightrightarrows D$ by

$$G(x) = \{y \in D : F(x, y) \subset P\}, \quad \forall x \in K.$$

By condition (ii), $G(x)$ is closed. We need to show that G is a T-KKM mapping. Suppose to the contrary that $T(\bar{x}) \not\subset \bigcup_{i=1}^{n} G(x_i)$ for some

$x_1, \ldots, x_\eta \in K$ and some $\bar{x} \in \mathrm{co}(\{x_1, \ldots, x_n\})$. Then there exists a $\bar{y} \in T(\bar{x})$ such that $\bar{y} \notin \bigcup_{i=1}^{n} G(x_i)$, i.e., $F(x_i, \bar{y}) \not\subset P$, $i = 1, \ldots, n$. Hence, for each i, there exists a $z_i \in F(x_i, \bar{y})$ such that

$$z_i \notin P, \quad i = 1, \ldots, n. \tag{7}$$

Since $F(x, \bar{y})$ is properly quasi-convex in x, by Lemma 3, there exist a $\bar{z} \in F(\bar{x}, \bar{y}) \subset P$ and some i such that

$$0 \leq \bar{z} \leq z_i. \tag{8}$$

By (7) and (8), we get a contradiction. Lemma 4 yields that $\bigcap_{x \in K} G(x) \neq \emptyset$, i.e., there exists a $\hat{y} \in D$ such that $F(x, \hat{y}) \subset P$ for all $x \in K$.

Remark 3. If for any $x \in K$, $F(x, y)$ is l.s.c. in y, then by Lemma 2(v), it is easy to check that $\{y \in D : F(x, y) \subset P\}$ is closed.

The following is a vector version of the *Ky* Fan minimax inequality.

Corollary 3. *Let Z, P and K be as in Theorem 2, and $\varphi : K \times K \to Z$ a single-valued mapping. Assume that*

(i) *for any $x \in K$, $\varphi(x, x) \geq 0$;*
(ii) *for any $x \in K$, $\{y \in K : \varphi(x, y) \geq 0\}$ is closed;*
(iii) *for any $y \in K$, $\varphi(x, y)$ is properly quasi-convex in x.*

Then there exists $\bar{y} \in K$ such that $\varphi(x, \bar{y}) \geq 0$ for all $x \in K$.

Proof. In Theorem 2, let $D = K$, $T = I$ (identity mapping) and $F = \varphi$. Then Theorem 2 yields the conclusion.

Acknowledgment. The author would like to thank Professor W. Oettli for providing copies of References [2] and [7], and an anonymous referee for helpful comments. This work was supported by the Natural Science Foundation of Jiangxi Province, China.

References

1. Berge C (1963) Topological spaces. Oliver & Boyd LTD, Edinburgh and London.
2. Blum E, Oettli W (1994) From optimization and variational inequalities to equilibrium problems. The Math. Student. 63:123–145
3. Chan D, Pang JS (1982) The generalized quasi-variational inequality problem. Math. Oper. Res. 7:211–222
4. Ferro F (1989) A minimax theorem for vector-valued functions. J. Optim. theory and Appl. 60:19–31
5. Gwinner J (1981) On fixed points and variational inequalities – A circular tour. 5:565–583
6. Jeyakumar V, Oettli W, Natividad M (1993) A solvability theorem for a class of quasi-convex mappings with applications to optimization. J. Math. Anal. Appl. 197:537–546
7. Oettli W, Schläger D (1997) Generalized vectorial equilibria and generalized monotonicity. (Preprint)

8. Parida J, Sen A (1987) A variational-like inequality for multi-functions with applications. J. Math. Anal. Appl. 124:73–81
9. Schaefer HH (1980) Topological vector spaces. Springer-Verlag, New York, Berlin.
10. Shioji N (1991) A further generalization of the Knaster-Kuratowski-Mazurkiewicz theorem. Proc. Amer. Math. Soc. 111:187–195
11. Tan NX (1985) Quasi-variational inequalities in topological linear locally convex Hausdorff spaces. Math. Nachr. 122:231–245