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Optimal portfolios for exponential Lévy processes

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Abstract. We consider the problem of maximizing the expected utility from consumption or terminal wealth in a market where logarithmic securities prices follow a Lévy process. More specifically, we give explicit solutions for power, logarithmic and exponential utility in terms of the Lévy-Khintchine triplet. In the first two cases, a constant fraction of current wealth should be invested in each of the securities, as is well-known for related discrete-time models and for Brownian motion. The situation is different for exponential utility.

Key words: portfolio optimization, exponential Lévy processes, HARA utility, martingale method

1 Introduction

One of the basic questions in mathematical finance is how to choose an optimal investment strategy in a securities market, or more precisely, how to maximize the expected utility from consumption or terminal wealth (cf. e.g. Korn (1997) for an introduction). This is often called *Merton's problem*, since it was solved by Merton (1969, 1971) in a Markovian Itô-process model. Similar to related work in discrete-time settings (cf. Mossin (1968), Samuelson (1969), Hakansson (1970)), his solution relies crucially on the Hamilton-Jacobi-Bellman equation from stochastic control theory.

An entirely different approach to portfolio optimization is based on *martingale methods*. Harrison & Kreps (1979) and Harrison & Pliska (1981) showed that arbitrage and completeness of securities markets can be expressed in terms of *equivalent martingale measures*. This leads to a two-step procedure for the solution of Merton's problem in complete models (cf. Pliska (1986), Karatzas et al. (1987), and Cox & Huang (1989)): The unique martingale measure yields the optimal terminal payoff, which in turn is used to determine the corresponding portfolio strategy in a second step.

This approach can be transfered to incomplete markets if the unique pricing measure is replaced with an in some sense *least favourable martingale measure* (cf. He & Pearson (1991a,b), Karatzas et al. (1991), Cvitanić & Karatzas (1992), Kramkov & Schachermayer (1999), Schachermayer (1999), Kallsen (1998)). An important early reference for this *martingale or duality method* is Bismut (1975).

It is usually quite hard to compute optimal strategies explicitly unless the market is of a certain simple structure or the logarithm is chosen as utility function. We refer to Hakansson (1971), Merton (1971), Aase (1984), Karatzas et al. (1991), Cvitanić & Karatzas (1992), Goll & Kallsen (1999) for the latter case. In this paper, time-homogeneous models are considered for power, logarithmic, and exponential utility functions. We suppose that logarithmic securities prices follow a process with stationary independent increments.

Our problem has been solved by Merton (1969) for continuous Lévy processes (i.e. Brownian motion with drift). The fraction of current wealth that is invested in each of the securities stays constant over time if power or logarithmic utility is considered. An analogous result has been derived by Mossin (1968), Samuelson (1969) for the discrete-time counterpart of these models. Very recently, Framstad et al. (1999) and Benth et al. (1999) solved the optimal consumption problem for power utility and a quite large class of Lévy processes.

All these papers apply dynamic programming to obtain the explicit solution. By contrast, the duality or martingale approach is used in this paper to derive the optimal portfolio and consumption in terms of the characteristic triplet of the Lévy process. Due to the powerful toolbox of the general theory of stochastic processes, the proofs get much simpler.

The paper is organized as follows. We begin with the problem and our version of the duality link to martingale measures. The explicit solution is given in Section 3. Finally, the appendix contains some auxiliary results from stochastic calculus.

We generally use the notation of Jacod & Shiryaev (1987) and Jacod (1979, 1980). The transposed of a vector or matrix x is denoted as $x^\top$ and its components by superscripts. Stochastic and Stieltjes integrals are written as $\int_0^t H_s\,dX_s = H \cdot X_t$. Increasing processes are identified with their corresponding Lebesgue-Stieltjes measure.

2 Optimal portfolios and martingale measures

Our mathematical framework for a frictionless market model is as follows (cf. Goll & Kallsen (1999)). We work with a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$ in the sense of Jacod & Shiryaev (1987), Definition I.1.2. *Securities* $0,\ldots,d$ are modelled by their *price process* $S := (S^0,\ldots,S^d)$. Security 0 is assumed to be positive and plays a special role. It serves as a *numeraire* by which all other securities are discounted. More specifically, we denote the *discounted price process* as $\hat{S} := \frac{1}{S^0}S := \left(1, \frac{1}{S^0}S^1, \ldots, \frac{1}{S^0}S^d\right)$. We assume that $\hat{S}$ is a $\mathbb{R}^{d+1}$-valued semimartingale. Occasionally, we will identify $\hat{S}$ with the $\mathbb{R}^d$-valued process $(\hat{S}^1,\ldots,\hat{S}^d)$.

We consider an investor (hereafter called "you") who disposes of an *initial endowment* $\varepsilon S_0^0 \in (0, \infty)$. *Trading strategies* are modelled by $\mathbb{R}^{d+1}$-valued, predictable stochastic processes $\varphi = (\varphi^0, \ldots, \varphi^d)$, where φ_t^i denotes the number of shares of security i in your portfolio at time t.

Proposition 2.1. *Assume that S^0 is a semimartingale such that S^0, S_-^0 are positive. Then we have equivalence between*

1. $\varphi \in L(S)$ *and* $\varphi_t^\top S_t = \varphi_0^\top S_0 + \int_0^t \varphi_s^\top \, dS_s$ *for any* $t \in \mathbb{R}_+$,
2. $\varphi \in L(\hat{S})$ *and* $\varphi_t^\top \hat{S}_t = \varphi_0^\top \hat{S}_0 + \int_0^t \varphi_s^\top \, d\hat{S}_s$ *for any* $t \in \mathbb{R}_+$.

(Note that it is not necessary to assume that S^0 is predictable as is – for simplicity – often done in the literature. For the definition of multidimensional integrals cf. Jacod (1980).)

Proof. cf. Goll & Kallsen (1999) □

We call a trading strategy $\varphi \in L(\hat{S})$ with $\varphi_0 = 0$ *self-financing* if $\varphi_t^\top \hat{S}_t = \int_0^t \varphi_s^\top \, d\hat{S}_s$ for any $t \in \mathbb{R}_+$. A self-financing strategy φ belongs to the set $\mathfrak{S}$ of all *admissible* strategies if its *discounted gains process* $\int_0^\cdot \varphi_t^\top \hat{S}_t$ is bounded from below.

Fix a terminal time $T \in \mathbb{R}_+$. We assume that your *discounted consumption* up to time t is of the form $\int_0^t \kappa_s \, dK_s$, where κ denotes the *discounted consumption rate* according to the "clock" K. We assume that K is an increasing function with $K_0 = 0$. Typical choices are $K_t := 1_{[T,\infty)}$ (consumption only at time T), $K_t := t$ (consumption uniformly in time), $K_t := \sum_{s \leq t} 1_{\mathbb{N}}(s)$ (consumption only at integer times). κ is supposed to be an element of the set $\mathfrak{K}$ of all optional processes that are bounded from below and satisfy $\int_0^T |\kappa_s| \, dK_s < \infty$ P-almost surely. For $\kappa \in \mathfrak{K}$, the corresponding *undiscounted consumption rate* at time t is $\kappa_t S_t^0$. Your *discounted wealth* at time t is given by $V_t(\varphi, \kappa) := \varepsilon + \int_0^t \varphi_s^\top \, d\hat{S}_s - \int_0^t \kappa_s \, dK_s$. A pair $(\varphi, \kappa) \in \mathfrak{S} \times \mathfrak{K}$ belongs to the set $\mathfrak{P}$ of *admissible portfolio/consumption pairs* if $V_T(\varphi, \kappa) \geq 0$.

Definition 2.2. A *utility function* is a strictly increasing, differentiable, strictly concave function $u : I \to \mathbb{R}$, where $I = \mathbb{R}$ or $(0, \infty)$.

In the following, u denotes a utility function. Later, we will only consider the cases $u(x) = \dfrac{x^{1-p}}{1-p}$ for $p \in \mathbb{R}_+ \backslash \{0, 1\}$, $u(x) = \log(x)$, and $u(x) = 1 - \dfrac{1}{p} e^{-px}$ for $p \in (0, \infty)$.

Definition 2.3. 1. We say that $(\varphi, \kappa) \in \mathfrak{P}$ is an *optimal portfolio/consumption pair* if it maximizes $(\tilde{\varphi}, \tilde{\kappa}) \mapsto E(\int_0^T u(\tilde{\kappa}_t) \, dK_t)$ over all $(\tilde{\varphi}, \tilde{\kappa}) \in \mathfrak{P}$.

2. We say that $\varphi \in \mathfrak{S}$ is an *optimal portfolio for terminal wealth* if it maximizes $\tilde{\varphi} \mapsto E(u(\varepsilon + \int_0^T \tilde{\varphi}_t^\top \, d\hat{S}_t))$ over all $\tilde{\varphi} \in \mathfrak{S}$.

Remarks.

1. If we set $K := 1_{[T,\infty)}$, then $\varphi \in \mathfrak{S}$ is an optimal portfolio for terminal wealth if and only if $(\tilde{\varphi}, \tilde{\kappa}) \in \mathfrak{P}$ is an optimal portfolio/consumption pair,

where $\kappa_T := \varepsilon + \int_0^T \varphi_t^\top \, d\hat{S}_t$ and κ_t can be chosen arbitrarily for $t < T$. Therefore the terminal wealth problem can be treated as a special case of maximization of utility from consumption.

2. Let $u(x) = \log(x)$ and suppose that $E(\int_0^T |\log(S_t^0)| \, dK_t) < \infty$. If $(\varphi, \kappa) \in \mathfrak{P}$ is an optimal portfolio/consumption pair, then it maximizes also $(\tilde{\varphi}, \tilde{\kappa}) \mapsto E(\int_0^T u(\tilde{\kappa}_t S_t^0) \, dK_t)$ (i.e., the expected utility of *un*discounted consumption) over all $(\tilde{\varphi}, \tilde{\kappa}) \in \mathfrak{P}$. A similar statement holds for the terminal wealth problem.
3. Let $u(x) = \dfrac{x^{1-p}}{1-p}$ for some $p \in \mathbb{R}_+ \backslash \{0, 1\}$ and suppose that S^0 is deterministic. Define $\tilde{K} := (S^0)^{p-1} K$. Then $(\varphi, \kappa) \in \mathfrak{P}$ is an optimal portfolio/consumption pair if and only if it maximizes $(\tilde{\varphi}, \tilde{\kappa}) \mapsto E(\int_0^T u(\tilde{\kappa}_t S_t^0) \, d\tilde{K}_t)$ over all $(\tilde{\varphi}, \tilde{\kappa}) \in \mathfrak{P}$. A similar statement holds for the terminal wealth problem. In other words: The problem to optimize the utility of undiscounted consumption or terminal wealth can be transformed into an equivalent discounted one, if the consumption clock is appropriately modified.

Lemma 2.4. *Let κ be an optional process. Suppose that there exists a positive martingale Z with the following properties:*

1. $(Z\hat{S})^T$ *is a local martingale,*
2. $Z_t = u'(\kappa_t)$ *for any* $t \in [0, T]$,
3. $E(\int_0^T Z_t \kappa_t \, dK_t) = E(Z_0)\varepsilon$.

Then $E(\int_0^T u(\tilde{\kappa}_t) \, dK_t) \leq E(\int_0^T u(\kappa_t) \, dK_t)$ *for any* $(\tilde{\varphi}, \tilde{\kappa}) \in \mathfrak{P}$.

Proof. We will give a proof that works as well if $T = \infty$, $K_\infty < \infty$ (cf. the remark at the end of this section). Let $(\tilde{\varphi}, \tilde{\kappa}) \in \mathfrak{P}$ be such that $E(\int_0^T u(\tilde{\kappa}_t) \, dK_t)$ is defined. Fix $n \in \mathbb{N}$. Let $P^\star \sim P$ be defined by $\dfrac{dP^\star}{dP} := \dfrac{Z_{T \wedge n}}{E(Z_0)}$. Since $Z^{T \wedge n} \hat{S}^{T \wedge n} = (Z\hat{S})^{T \wedge n}$ is a local martingale, $\hat{S}^{T \wedge n}$ is a $P^\star$-local martingale (cf. Jacod & Shiryaev (1987), II.3.8). We have that

$$
\begin{aligned}
E\left(\int_0^{T \wedge n} Z_t \tilde{\kappa}_t \, dK_t\right) &= E(Z_0) \int_0^{T \wedge n} E\left(\tilde{\kappa}_t E\left(\frac{Z_{T \wedge n}}{E(Z_0)} \middle| \mathscr{F}_t\right)\right) dK_t \\
&= E(Z_0) E_{P^\star}\left(\int_0^{T \wedge n} \tilde{\kappa}_t \, dK_t\right) \\
&\leq E(Z_0) E_{P^\star}(\varepsilon + \tilde{\varphi}^\top \cdot \hat{S}_{T \wedge n} + V^-_{T \wedge n}(\tilde{\varphi}, \tilde{\kappa})).
\end{aligned}
$$

If n is large enough, then $V^-_{T \wedge n}(\tilde{\varphi}, \tilde{\kappa}) = 0$. By Ansel & Stricker (1994), Corollaire 3.5, $\tilde{\varphi}^\top \cdot \hat{S}^{T \wedge n}$ is a $P^\star$-local martingale and hence a $P^\star$-supermartingale. It follows that $E(\int_0^{T \wedge n} Z_t \tilde{\kappa}_t \, dK_t) \leq E(Z_0)\varepsilon$ for n large enough. Monotone convergence yields $\lim_{n \to \infty} E(\int_0^{T \wedge n} Z_t \tilde{\kappa}_t \, dK_t) = \lim_{n \to \infty} (E(\int_0^{T \wedge n} Z_t (\tilde{\kappa}_t + m) \, dK_t) - E(Z_0) m K_{T \wedge n}) = E(\int_0^T Z_t \tilde{\kappa}_t \, dK_t)$, where $-m$ denotes a lower bound of κ. Therefore, we have $E(\int_0^T Z_t \tilde{\kappa}_t \, dK_t) \leq E(Z_0)\varepsilon$. Since u is concave, it follows that

$$E\left(\int_0^T u(\tilde{\kappa}_t)\,dK_t\right) \leq E\left(\int_0^T (u(\kappa_t) + u'(\kappa_t)(\tilde{\kappa}_t - \kappa_t))\,dK_t\right)$$
$$= E\left(\int_0^T u(\kappa_t)\,dK_t\right) + E\left(\int_0^T Z_t\tilde{\kappa}_t\,dK_t\right) - E\left(\int_0^T Z_t\kappa_t\,dK_t\right)$$
$$\leq E\left(\int_0^T u(\kappa_t)\,dK_t\right) \qquad \square$$

Remarks.

1. Lemma 2.4 has a terminal wealth version. If φ is a self-financing strategy and Z a positive martingale such that
   $1'$. $(Z\hat{S})^T$ is a local martingale,
   $2'$. $Z_T = u'(\varepsilon + \int_0^T \varphi_t^\top\,d\hat{S}_t)$,
   $3'$. $E(Z_T \int_0^T \varphi_t^\top\,d\hat{S}_t) = 0$,
   then $E(u(\varepsilon + \int_0^T \tilde{\varphi}_t^\top\,d\hat{S}_t)) \leq E(u(\varepsilon + \int_0^T \varphi_t^\top\,d\hat{S}_t))$ for any $\tilde{\varphi} \in \mathfrak{P}$.
2. Note that $\dfrac{Z_T}{E(Z_0)}$ is the density of a probability measure $P^\star \sim P$. Condition 1 means that $P^\star$ is a *local martingale measure*, i.e. $\hat{S}^T$ is a $P^\star$-local martingale.
3. The above lemma implies that $(\varphi, \kappa) \in \mathfrak{P}$ is optimal if $u'(\kappa) = Z$ for some Z as above. Similarly, $\varphi \in \mathfrak{S}$ is optimal for terminal wealth if $u'(\varepsilon + \int_0^T \varphi_t^\top\,d\hat{S}_t) = Z_T$.
   In Kramkov & Schachermayer (1999), Schachermayer (1999) it is shown that an optimal portfolio for terminal wealth is necessarily of a similar form.
4. Using a different language, a version of the previous lemma can be found in Karatzas et al. (1991), Theorem 9.3. The proof of Lemma 2.4 is essentially classical (cf. e.g. the proof of Theorem 2.0 in Kramkov & Schachermayer (1999)).

The following lemma adresses the uniqueness of optimal portfolio/consumption pairs.

Lemma 2.5. *Let (φ, κ) and $(\tilde{\varphi}, \tilde{\kappa})$ be optimal portfolio/consumption pairs with finite expected utility $E(\int_0^T u(\kappa_t)\,dK_t)$. Then $\kappa = \tilde{\kappa}$ holds $(P \otimes K)$-almost everywhere on $\Omega \times [0, T]$. Moreover, $\int_0^t \varphi_s^\top\,d\hat{S}_s = \int_0^t \tilde{\varphi}_s^\top\,d\hat{S}_s$ and hence $V_t(\varphi, \kappa) = V_t(\tilde{\varphi}, \tilde{\kappa})$ for all t with $K_{t-} < K_T$. In particular, the discounted wealth processes coincide up to indistinguishability. An analogous statement holds for optimal portfolios for terminal wealth.*

Proof. Let $(\varphi, \kappa), (\tilde{\varphi}, \tilde{\kappa}) \in \mathfrak{P}$ be optimal portfolio/consumption pairs.

First step: Define $\hat{\varphi} := \frac{1}{2}(\varphi + \tilde{\varphi})$, $\hat{\kappa} := \frac{1}{2}(\kappa + \tilde{\kappa})$. Obviously, $(\hat{\varphi}, \hat{\kappa}) \in \mathfrak{P}$. By optimality of $(\varphi, \kappa), (\tilde{\varphi}, \tilde{\kappa})$, we have $\int_{\Omega\times[0,T]}(u(\hat{\kappa}_t) - \frac{1}{2}(u(\kappa_t) + u(\tilde{\kappa}_t)))d(P \otimes K) = \int_{\Omega\times[0,T]} u(\hat{\kappa}_t)d(P \otimes K) - \int_{\Omega\times[0,T]} u(\kappa_t)d(P \otimes K) \leq 0$. Since u is concave, the integrand $u(\hat{\kappa}_t) - \frac{1}{2}(u(\kappa_t) + u(\tilde{\kappa}_t))$ is non-negative, which implies that it is 0 $(P \otimes K)$-almost everywhere. Therefore $\tilde{\kappa} = \kappa$ $(P \otimes K)$-almost everywhere because u is strictly concave.

Second step: Let $t_0 \in [0, T]$ with $K_{t_0-} < K_T$, moreover $A := \{V_{t_0}(\varphi, \kappa) < V_{t_0}(\tilde{\varphi}, \tilde{\kappa})\} \in \mathscr{F}_{t_0}$ and $D := 1_A(V_{t_0}(\tilde{\varphi}, \tilde{\kappa}) - V_{t_0}(\varphi, \kappa)) \geq 0$. Define a new portfolio/consumption pair $(\bar{\varphi}, \bar{\kappa})$ by

$$\bar{\varphi}_t(\omega) := \begin{cases} \tilde{\varphi}_t(\omega) & \text{if } t \leq t_0 \text{ or } \omega \in A^C \\ \varphi_t(\omega) & \text{if } t > t_0 \text{ and } \omega \in A, \end{cases}$$

$$\bar{\kappa}_t := \begin{cases} \kappa_t & \text{for } t < t_0 \\ \kappa_t + \dfrac{D}{K_T - K_{t_0}} & \text{for } t \geq t_0. \end{cases}$$

More precisely, let $\bar{\varphi}^0_t := \varphi^0_t + D$ for $t > t_0$ so that $\bar{\varphi}$ is a self-financing strategy. Since $\kappa = \tilde{\kappa}$, we have $\varphi^\top \cdot S_{t_0} < \tilde{\varphi}^\top \cdot S_{t_0}$ on A. This implies that $\bar{\varphi}$ is admissible. Moreover, we have $V_t(\bar{\varphi}, \bar{\kappa}) = V_t(\varphi, \kappa) + D - D\dfrac{K_t - K_{t_0-}}{K_T - K_{t_0-}} \geq V_t(\varphi, \kappa)$ for $t \geq t_0$, which implies that $(\bar{\varphi}, \bar{\kappa}) \in \mathfrak{P}$. Obviously, $\bar{\kappa} > \kappa$ on $A \times [t_0, T]$. In view of the first step, this is only possible if $P(A) = 0$. □

Remark. If $K_\infty := \lim_{t \to \infty} K_t < \infty$, then one can extend Definition 2.3 to include the case $T = \infty$ as well. To this end, we define the set $\mathfrak{P}$ of admissible portfolio/consumption pairs as follows: $(\varphi, \kappa) \in \mathfrak{P}$ if there exists some $t_0 \in \mathbb{R}_+$ such that $V_t(\varphi, \kappa) \geq 0$ for any $t \in [t_0 \wedge T, \infty)$. Note that this coincides with the old definition if T is finite. With this notion of admissibility, Lemmas 2.4 and 2.5 hold for $T = \infty$ as well. We do not want to consider terminal wealth for $T = \infty$, since the limit $\int_0^\infty \varphi_t^\top \, d\hat{S}_t$ is usually non-existent.

3 Solution in terms of triplets

We turn now to the explicit solution of the utility maximization problem. We assume that $\hat{S}^1, \ldots, \hat{S}^d$ are positive processes of the form

$$\hat{S}^i = \hat{S}^i_0 \mathscr{E}(L^i), \tag{3.1}$$

where L is a $\mathbb{R}^d$-valued Lévy process with characteristic triplet (b, c, F) relative to some truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$ (i.e., a PIIS in the sense of Jacod & Shiryaev (1987), Definition II.4.1). By Lemma 4.2, these processes coincide with those of the form $\hat{S}^i = \hat{S}^i_0 \exp(\tilde{L}^i)$ for $\mathbb{R}^d$-valued Lévy processes $\tilde{L}$. If the undiscounted price process S is given in the form (3.1), then $\hat{S}$ is of the form (3.1) as well, but for a different Lévy process whose characteristics are obtained with the help of Lemma 4.3.

In the last couple of years, processes of the above type have become popular for securities models, since they are mathematically tractable and provide a good fit to real data (cf. Eberlein & Keller (1995), Eberlein et al. (1998), Madan & Senata (1990), Barndorff-Nielsen (1998)).

Theorem 3.1 (Logarithmic utility). *Let* $u(x) = \log(x)$. *Assume that there exists some* $\gamma \in \mathbb{R}^d$ *such that*

1. $F(\{x \in \mathbb{R}^d : 1 + \gamma^\top x \le 0\}) = 0$
2. $\int \left| \frac{x}{1 + \gamma^\top x} - h(x) \right| F(dx) < \infty$
3. $$b - c\gamma + \int \left(\frac{x}{1 + \gamma^\top x} - h(x) \right) F(dx) = 0.$$

Let

$$\kappa_t := \frac{\varepsilon}{K_T} \mathscr{E}(\gamma^\top L)_t,$$

$$V_t := \kappa_t (K_T - K_t),$$

$$\varphi_t^i := \frac{\gamma^i}{\hat{S}_{t-}^i} V_{t-} \quad \text{for } i = 1, \dots, d, \quad \varphi_t^0 := \int_0^t \varphi_s^\top \, d\hat{S}_s - \sum_{i=1}^d \varphi_t^i \hat{S}_t^i$$

for $t \in [0, T]$, *where we set* $V_{0-} := 0$. *Then* $(\varphi, \kappa) \in \mathfrak{P}$ *is an optimal portfolio/consumption pair with discounted wealth process* V.

Proof. Set $p := 1$ in the proof of Theorem 3.2 below. □

Theorem 3.2 (Power utility). *Let* $u(x) = \frac{x^{1-p}}{1-p}$ *for some* $p \in \mathbb{R}_+ \backslash \{0, 1\}$. *Assume that there exists some* $\gamma \in \mathbb{R}^d$ *such that*

1. $F(\{x \in \mathbb{R}^d : 1 + \gamma^\top x \le 0\}) = 0$
2. $\int \left| \frac{x}{(1 + \gamma^\top x)^p} - h(x) \right| F(dx) < \infty$
3. $$b - pc\gamma + \int \left(\frac{x}{(1 + \gamma^\top x)^p} - h(x) \right) F(dx) = 0. \tag{3.2}$$

Let

$$\alpha := \frac{1-p}{2} \gamma^\top c \gamma + \frac{1}{p} \int \left(\frac{1 + p\gamma^\top x}{(1 + \gamma^\top x)^p} - 1 \right) F(dx), \quad A_t := t,$$

$$\kappa_t := \frac{\varepsilon}{\int_0^T e^{\alpha s} \, dK_s} \mathscr{E}(\gamma^\top L + \alpha A)_t,$$

$$V_t := \kappa_t \int_0^T 1_{[0,t]^C}(s) e^{\alpha(s-t)} \, dK_s,$$

$$\varphi_t^i := \frac{\gamma^i}{\hat{S}_{t-}^i} V_{t-} \quad \text{for } i = 1, \dots, d, \quad \varphi_t^0 := \int_0^t \varphi_s^\top \, d\hat{S}_s - \sum_{i=1}^d \varphi_t^i \hat{S}_t^i$$

for $t \in [0, T]$, where we set $V_{0-} := 0$. Then $(\varphi, \kappa) \in \mathfrak{P}$ is an optimal portfolio/consumption pair with discounted wealth process V.

Proof. First step: If $(\tilde{b}, \tilde{c}, \tilde{F})$ denotes the triplet of the Lévy process $\gamma^\top L$ relative to some truncation function $\tilde{h} : \mathbb{R} \to \mathbb{R}$, then we have $\tilde{F}(G) = \int 1_G(\gamma^\top x)F(dx)$ for $G \in \mathscr{B}$ and $\int(|x|^2 1_{[0,\delta]}(|x|) + 1_{[0,\delta]^c}(|x|))\tilde{F}(dx) < \infty$ for any $\delta > 0$. Since $\gamma^\top h(x) - \tilde{h}(\gamma^\top x)$ is bounded and 0 in a neighbourhood of 0, it follows that $\int |\gamma^\top h(x) - \tilde{h}(\gamma^\top x)|F(dx) < \infty$. Moreover, $\int \left|\frac{x}{(1+x)^p}\right| 1_{[0,\delta]^c}(|x|) \cdot \tilde{F}(dx) < \infty$ by Condition 2. A second order Taylor expansion yields that $\frac{1+px}{(1+x)^p} - 1 = O(x^2)$ for $x \to 0$. Together, it follows that $\int \left| \frac{1+p\gamma^\top x}{(1+\gamma^\top x)^p} - 1 \right| F(dx) = \int \left| \frac{1+px}{(1+x)^p} - 1 \right| \tilde{F}(dx) < \infty$. Hence, α is well defined.

Second step: We have $E(\sum_{t \le T} 1_{(-\infty,0]}(1 + \gamma^\top \Delta L_t)) = E(1_{(-\infty,0]}(1 + \gamma^\top x) * \mu_t^L) = E(1_{(-\infty,0]}(1 + \gamma^\top x) * \nu_t^L) = 0$ by Condition 1. Therefore, $P(\text{Ex. } t \in [0, T] \text{ with } \Delta(\gamma^\top L + \alpha A)_t \le -1) = 0$. By Jacod & Shiryaev (1987), I.4.64 and I.4.61c, this implies that $\kappa = \kappa_0 \mathscr{E}(\gamma^\top L + \alpha A)$ is positive on $[0, T]$.

Third step: Define $Z := \kappa^{-p}$ and $N := -p\gamma^\top L^c + ((1 + \gamma^\top x)^{-p} - 1) * (\mu^L - \nu^L)$. We will show that $Z = \kappa_0^{-p}\mathscr{E}(N)$. Note that $\langle \kappa^c, \kappa^c \rangle_t = (\kappa_-^2 \gamma^\top c\gamma) \cdot A$ and $\mu^\kappa([0, t] \times G) = 1_G(\kappa_- \gamma^\top x) * \mu_t^L$ for $t \in [0, T]$, $G \in \mathscr{B}$. An application of Itô's formula (cf. Lemma 4.1) yields that $Z = Z_0 + \kappa_-^{-p} \cdot (-p\gamma^\top L - p\alpha A + \frac{p(1+p)}{2}\gamma^\top c\gamma A + ((1 + \gamma^\top x)^{-p} - 1 + p\gamma^\top x) * \mu^L)$. It remains to show that

$$
\begin{aligned}
&-p\gamma^\top L - p\alpha A + \frac{p(1+p)}{2}\gamma^\top c\gamma A + ((1 + \gamma^\top x)^{-p} - 1 + p\gamma^\top x) * \mu^L \\
&\quad = -p\gamma^\top L^c + ((1 + \gamma^\top x)^{-p} - 1) * (\mu^L - \nu^L). \qquad (3.3)
\end{aligned}
$$

Note that $L = L^c + h(x) * (\mu^L - \nu^L) + (x - h(x)) * \mu^L + bA$ and $-p\gamma^\top bA - p\alpha A = -\frac{p(1+p)}{2}\gamma^\top c\gamma A - ((1 + \gamma^\top x)^{-p} - 1 + p\gamma^\top h(x)) * \nu^L$ by Equation (3.2). Summing up the terms on the left-hand side of Equation (3.3), we obtain $-p\gamma^\top L^c - p\gamma^\top h(x) * (\mu^L - \nu^L) + ((1 + \gamma^\top x)^{-p} - 1 + p\gamma^\top h(x)) * \mu^L - ((1 + \gamma^\top x)^{-p} - 1 + p\gamma^\top h(x)) * \nu^L$, which equals the right-hand side of Equation (3.3).

Fourth step: Since N is a local martingale, Z is a positive local martingale. N is obviously a Lévy process. By Lemmas 4.2 and 4.4, Z is even a martingale.

Fifth step: Fix $i \in \{1, \ldots, d\}$. Equation (3.2) yields $pc^{i\cdot}\gamma A = b^i A + \left(\frac{x^i}{(1+\gamma^\top x)^p} - h^i(x)\right) * \nu^L$. Therefore,

$$
\begin{aligned}
[L^i, N] &= \langle L^{i,c}, N^c \rangle + \sum_{s \le \cdot} \Delta L_s^i \Delta N_s \\
&= -p \sum_{j=1}^d \gamma^j \langle L^{i,c}, L^{j,c} \rangle + \sum_{s \le \cdot} \Delta L_s^i \left(\frac{1}{(1 + \gamma^\top \Delta L_s)^p} - 1 \right)
\end{aligned}
$$

$$= -b^i A - \left(\frac{x^i}{(1+\gamma^\top x)^p} - h^i(x) \right) * \nu^L + \left(\frac{x^i}{(1+\gamma^\top x)^p} - x^i \right) * \mu^L$$

$$= -b^i A + \left(\frac{x^i}{(1+\gamma^\top x)^p} - h^i(x) \right) * (\mu^L - \nu^L) + \left(h^i(x) - x^i \right) * \mu^L.$$

In view of the canonical representation of the semimartingale L (cf. Jacod & Shiryaev (1987), II.2.34), this implies that $L^i + [L^i, N]$ is a local martingale. Hence, $\hat{S}^i Z = \hat{S}^i_0 Z_0 + (Z\hat{S}^i)_- \cdot L^i + (\hat{S}^i Z)_- \cdot N + (\hat{S}^i Z)_- \cdot [L^i, N]$ is a local martingale as well.

Sixth step: Define a probability measure $P^\star \sim P$ by $\frac{dP^\star}{dP} := \frac{Z_T}{Z_0}$. If we set $\beta := -p\gamma$ and $Y(x) := (1+\gamma^\top x)^{-p}$, then $\langle Z^c, L^{i,c} \rangle = Z_- \cdot \langle N^c, L^{i,c} \rangle = (\sum_{j=1}^d c^{ij} \beta^j Z_-) \cdot A$ and $Z_t = Z_{t-} Y(x)$ for μ^L-almost all (t, x), which implies $YZ_- = M^P_{\mu^L}(Z|\tilde{\mathscr{P}})$ in the sense of Jacod & Shiryaev (1987), III.3.15/16. Since L is a Lévy process and β, Y are deterministic, it follows from Girsanov's theorem for semimartingales (cf. Jacod & Shiryaev (1987), III.3.24) that L is a $P^\star$-Lévy process as well. The calculation in the fifth step shows that $L^i Z = Z_- \cdot L^i + Z_- \cdot N + Z_- \cdot [L^i, N]$ is a local martingale for $i = 1, \ldots, d$. Therefore, $\gamma^\top L$ is a $P^\star$-local martingale (cf. Jacod & Shiryaev (1987), III.3.8b). Note that $\mathscr{E}(\gamma^\top L) = \frac{1}{\kappa_0} e^{-\alpha A} \kappa > 0$. Using Lemmas 4.2 and 4.4, we conclude that $\mathscr{E}(\gamma^\top L)$ is a $P^\star$-martingale, and hence $E_{P^\star}(\kappa_t) = \kappa_0 e^{\alpha t}$ for $t \in [0, T]$. It follows that $E(\int_0^T Z_t \kappa_t \, dK_t) = Z_0 \int_0^T E_{P^\star}(\kappa_t) \, dK_t = Z_0 \kappa_0 \int_0^T e^{\alpha t} \, dK_t = Z_0 \varepsilon$.

Seventh step: We have

$$\begin{aligned} V &= \kappa e^{-\alpha A}(e^{\alpha A} \cdot K_T - e^{\alpha A} \cdot K) \\ &= V_0 + (e^{-\alpha A}(e^{\alpha A} \cdot K_T - e^{\alpha A} \cdot K_-)) \cdot \kappa + \kappa \cdot (e^{-\alpha A}(e^{\alpha A} \cdot K_T - e^{\alpha A} \cdot K)) \\ &= \varepsilon + V_- \cdot (\gamma^\top L + \alpha A) - \kappa \cdot K - (V\alpha) \cdot A \\ &= \varepsilon + \varphi^\top \cdot \hat{S} - \kappa \cdot K, \end{aligned}$$

where we used partial integration in the sense of Jacod & Shiryaev (1987), I.4.49 and the fact that $(V_-\alpha) \cdot A = (V\alpha) \cdot A$ because $A_t = t$ is continuous. Moreover, κ and hence V are non-negative. Together, it follows that (φ, κ) is an admissible portfolio/consumption pair with wealth process V. Note that φ^0 is well-defined, since $\varphi^\top \cdot \hat{S} = (\varphi^1, \ldots \varphi^d)^\top \cdot (\hat{S}^1, \ldots, \hat{S}^d)$. In view of Lemma 2.4, the proof is complete. □

Remarks.

1. If Conditions 1–3 in Theorem 3.1 resp. 3.2 are met and $\varphi \in \mathfrak{S}$ is defined by

$$\varphi^i_t := \frac{\gamma^i}{\hat{S}^i_{t-}} \varepsilon \mathscr{E}(\gamma^\top L)_{t-} \quad \text{for } i = 1, \ldots, d, \quad \varphi^0_t := \int_0^t \varphi^\top_s \, d\hat{S}_s - \sum_{i=1}^d \varphi^i_t \hat{S}^i_t$$

for $t \in (0, T]$, then φ is an optimal portfolio for terminal wealth and its discounted wealth process equals $\varepsilon\mathscr{E}(\gamma^\top L)$.
2. In the framework of the remark at the end of Section 2, Theorems 3.1, 3.2 hold for $T = \infty$ as well. Only the proof of the sixth step has to be slightly modified.

Theorem 3.3 (Exponential utility). *Let* $u(x) = 1 - \frac{1}{p} e^{-px}$ *for some* $p \in (0, \infty)$. *Suppose that there exists some* $\gamma \in [0, \infty)^d$ *such that*

1. $\int |xe^{-p\gamma^\top x} - h(x)| F(dx) < \infty$
2. $$b - pc\gamma + \int \Big(xe^{-p\gamma^\top x} - h(x) \Big) F(dx) = 0. \tag{3.4}$$

Let

$$\alpha := -\frac{p}{2}\gamma^\top c\gamma - \frac{1}{p}\int e^{-p\gamma^\top x}(e^{p\gamma^\top x} - 1 - p\gamma^\top x) F(dx).$$

For any $n \in \mathbb{N}$ *define a stopping time*

$$T_n := \inf\left\{ t \in \mathbb{R}_+ : \min_{i=1,\dots,d} L^i_t \leq -n \text{ or } \int_0^t (K_T - K_{s-})\gamma^\top \, dL_s \leq -n \right\}$$

and processes $\kappa^{(n)}$, $V^{(n)}$, $\varphi^{(n)}$ *by*

$$\begin{aligned}
\kappa^{(n)}_t &:= \frac{\varepsilon - \alpha \int_0^T s \, dK_s}{K_T} + \gamma^\top L^{T_n}_t + \alpha t, \\
V^{(n)}_t &:= \kappa^{(n)}_t (K_T - K_t) + \alpha \int_0^T 1_{[0,t]^C}(s)(s - t)\, dK_s \\
\varphi^{(n),i}_t &:= \frac{\gamma^i}{\hat{S}^i_{t-}}(K_T - K_{t-}) 1_{[0,T_n]}(t) \quad \text{for } i = 1, \dots, d, \\
\varphi^{(n),0}_t &:= \int_0^t (\varphi^{(n)}_s)^\top \, d\hat{S}_s - \sum_{i=1}^d \varphi^{(n),i}_t \hat{S}^i_t
\end{aligned} \tag{3.5}$$

for $t \in [0, T]$. *Then* $(\varphi^{(n)}, \kappa^{(n)}) \in \mathfrak{P}$ *has wealth process* $V^{(n)}$ *for any* $n \in \mathbb{N}$ *and we have*

$$\lim_{n\to\infty} E\left(\int_0^T u(\kappa^{(n)}_t)\, dK_t \right) = \sup_{(\varphi,\kappa)\in\mathfrak{P}} E\left(\int_0^T u(\kappa_t)\, dK_t \right).$$

Proof. First step: If $(\tilde{b}, \tilde{c}, \tilde{F})$ denotes the triplet of the Lévy process $\gamma^\top L$ relative to some truncation function $\tilde{h} : \mathbb{R} \to \mathbb{R}$, then we have $\tilde{F}(G) = \int 1_G(\gamma^\top x) F(dx)$

for $G \in \mathscr{B}$ and $\int (|x|^2 1_{[0,\delta]}(|x|) + 1_{[0,\delta]^c}(|x|))\tilde{F}(dx) < \infty$ for any $\delta > 0$. Since $\gamma^\top h(x) - \tilde{h}(\gamma^\top x)$ is bounded and 0 in a neighbourhood of 0, it follows that $\int |\gamma^\top h(x) - \tilde{h}(\gamma^\top x)| F(dx) < \infty$. Moreover, $\int |xe^{-px}| 1_{[0,\delta]^c}(|x|)\tilde{F}(dx) < \infty$ by Condition 1, and therefore $\int |e^{-px}(e^{px} - 1 - px)| 1_{[0,\delta]^c}(|x|)\tilde{F}(dx) = \int |1 - (1+px)e^{-px}| 1_{[0,\delta]^c}(|x|)\tilde{F}(dx) < \infty$. A second order Taylor expansion yields $e^{px} - 1 - px = O(x^2)$ and hence $e^{-px}(e^{px} - 1 - px) = O(x^2)$ for $x \to 0$. Together, it follows that $\int |e^{-p\gamma^\top x}(e^{p\gamma^\top x} - 1 - p\gamma^\top x)| F(dx) = \int |e^{-px}(e^{px} - 1 - px)| \tilde{F}(dx) < \infty$. Hence, α is well defined.

Second step: We set $\kappa := (\varepsilon - \alpha \int_0^T s\, dK_s)K_T^{-1} + \gamma^\top L + \alpha A$, where $A_t := t$. Define $Z := e^{-p\kappa}$ and $N := -p\gamma^\top L^c + (e^{-p\gamma^\top x} - 1) * (\mu^L - \nu^L)$. We will show that $Z = e^{-p\kappa_0}\mathscr{E}(N)$. An application of Itô's formula (cf. Lemma 4.1) yields that $Z = Z_0 + e^{-p\kappa_-} \cdot \left(-p\gamma^\top L - p\alpha A + \frac{p^2}{2}\gamma^\top c\gamma A + (e^{-p\gamma^\top x} - 1 + p\gamma^\top x) * \mu^L\right)$. It remains to show that

$$-p\gamma^\top L - p\alpha A + \frac{p^2}{2}\gamma^\top c\gamma A + (e^{-p\gamma^\top x} - 1 + p\gamma^\top x) * \mu^L$$
$$= -p\gamma^\top L^c + (e^{-p\gamma^\top x} - 1) * (\mu^L - \nu^L). \tag{3.6}$$

Note that

$$L = L^c + h(x) * (\mu^L - \nu^L) + (x - h(x)) * \mu^L + bA \tag{3.7}$$

and $-p\gamma^\top bA - p\alpha A = -\frac{p^2}{2}\gamma^\top c\gamma A - (e^{-p\gamma^\top x} - 1 + p\gamma^\top h(x)) * \nu^L$ by Equation (3.4). Summing up the terms on the left-hand side of Equation (3.6), we obtain $-p\gamma^\top L^c - p\gamma^\top h(x) * (\mu^L - \nu^L) + (e^{-p\gamma^\top x} - 1 + p\gamma^\top h(x)) * \mu^L - (e^{-p\gamma^\top x} - 1 + p\gamma^\top h(x)) * \nu^L$, which equals the right-hand side of Equation (3.6).

Third step: Since N is a local martingale, Z is a positive local martingale. By Lemma 4.4, Z is even a martingale.

Fourth step: Fix $i \in \{1, \ldots, d\}$. Equation (3.4) yields $pc^{i\cdot}\gamma A = b^i A + (x^i e^{-p\gamma^\top x} - h^i(x)) * \nu^L$. Therefore,

$$[L^i, N] = \langle L^{i,c}, N^c\rangle + \sum_{s \le \cdot} \Delta L^i_s \Delta N_s$$
$$= -p\sum_{j=1}^d \gamma^j \langle L^{i,c}, L^{j,c}\rangle + \sum_{s \le \cdot} \Delta L^i_s (e^{-p\gamma^\top \Delta L_s} - 1)$$
$$= -b^i A - (x^i e^{-p\gamma^\top x} - h^i(x)) * \nu^L + (x^i e^{-p\gamma^\top x} - x^i) * \mu^L$$
$$= -b^i A + (x^i e^{-p\gamma^\top x} - h^i(x)) * (\mu^L - \nu^L) + (h^i(x) - x^i) * \mu^L.$$

In view of the canonical representation of the semimartingale L (cf. Equation (3.7)), this implies that $L^i + [L^i, N]$ is a local martingale. Hence, $\hat{S}^i Z = \hat{S}^i_0 Z_0 + (Z\hat{S}^i)_- \cdot L^i + (\hat{S}^i Z)_- \cdot N + (\hat{S}^i Z)_- \cdot [L^i, N]$ is a local martingale as well.

Fifth step: Define a probability measure $P^\star \sim P$ by $\frac{dP^\star}{dP} := \frac{Z_T}{Z_0}$. If we set $\beta := -p\gamma$ and $Y(x) := e^{-p\gamma^\top x}$, then $\langle Z^c, L^{i,c}\rangle = Z_- \cdot \langle N^c, L^{i,c}\rangle = (\sum_{j=1}^d c^{ij}\beta^j Z_-)\cdot A$ and $Z_t = Z_{t-}Y(x)$ for μ^L-almost all (t,x), which implies $YZ_- = M^P_{\mu^L}(Z|\tilde{\mathscr{P}})$ in the sense of Jacod & Shiryaev (1987), III.3.15/16. Since L is a Lévy process and β, Y are deterministic, it follows from Girsanov's theorem for semimartingales (cf. Jacod & Shiryaev (1987), III.3.24) that L is a $P^\star$-Lévy process as well. The calculation in the fifth step shows that $L^iZ = Z_- \cdot L^i + Z_- \cdot N + Z_- \cdot [L^i, N]$ is a local martingale for $i = 1, \dots, d$. Therefore, $\gamma^\top L$ is a $P^\star$-local martingale (cf. Jacod & Shiryaev (1987), III.3.8b). Since it is a $P^\star$-Lévy process, it is even a $P^\star$-martingale (cf. Lemma 4.4). Hence $E_{P^\star}(\kappa_t) = \kappa_0 + \alpha t$. It follows that $E(\int_0^T Z_t\kappa_t\, dK_t) = Z_0 \int_0^T E_{P^\star}(\kappa_t)\, dK_t = Z_0(\kappa_0 K_T + \alpha\int_0^T t\, dK_t) = Z_0\varepsilon$.

Sixth step: Fix $n \in \mathbb{N}$. Obviously, $(\varphi^{(n)})^\top \cdot \hat{S}_t \geq -n$ for $t < T_n$ and $\Delta((\varphi^{(n)})^\top \cdot \hat{S})_{T_n} \geq -K_T \sum_{i=1}^d \gamma^i$ because $\Delta L^i > -1$ for $i = 1, \dots, d$. Therefore, $(\varphi^{(n)})^\top \cdot \hat{S}$ is bounded from below. Similarly, one shows that $\kappa^{(n)}$ is bounded from below.

Seventh step: We have

$$\begin{aligned} V^{(n)} &= \kappa^{(n)}(K_T - K) + \alpha(A\cdot K_T - A\cdot K - A(K_T - K)) \\ &= V_0^{(n)} + (K_T - K_-)\cdot\kappa^{(n)} - \kappa^{(n)}\cdot K + \alpha A\cdot(K_T - K) - \alpha A(K_T - K) \\ &= \varepsilon + (\varphi^{(n)})^\top\cdot\hat{S} - \kappa^{(n)}\cdot K, \end{aligned}$$

where we used partial integration in the sense of Jacod & Shiryaev (1987), I.4.49 and the fact that $A(K_T - K) = A\cdot(K_T - K) + (K_T - K_-)\cdot A$. Since $V_T^{(n)} = 0$, it follows that $(\varphi^{(n)}, \kappa^{(n)})$ is an admissible portfolio/consumption pair with wealth process $V^{(n)}$.

Eighth step: Since $e^{-p\kappa} = Z$ is a martingale, it is of class (D) on $[0, T]$. A simple calculation shows that the family $(e^{-p\kappa^{T_n}})_{n\in\mathbb{N}}$ and hence also $(e^{-p\kappa^{(n)}})_{n\in\mathbb{N}}$ of real-valued measurable functions on $\Omega\times[0,T]$ is uniformly integrable with respect to the finite measure $P\otimes K$. For $n\to\infty$, we have $\kappa^{(n)}\to\kappa$ $(P\otimes K)$-almost everywhere, which implies that $\int e^{-p\kappa^{(n)}}d(P\otimes K) \to \int e^{-p\kappa}d(P\otimes K)$ for $n\to\infty$. Therefore, $\lim_{n\to\infty} E(\int_0^T u(\kappa_t^{(n)})\, dK_t) = E(\int_0^T u(\kappa_t)\, dK_t)$. In view of Lemma 2.4, the proof is complete. □

Remarks.

1. By considering the case $K = 1_{[T,\infty)}$, we obtain the corresonding statement for the terminal wealth problem.
2. Intuitively, one may regard the limiting portfolio/consumption pair $(\varphi^{(\infty)}, \kappa^{(\infty)})$ as optimal for exponential utility – even if the constraint $\gamma\in[0,\infty)^d$ is not met. However, the trading strategy $\varphi^{(\infty)}$ is not admissible because its gain process is not bounded from below. Our way out is to approximate $\varphi^{(\infty)}$ by stopping. This works well if all components of γ are non-negative or if the jump measure F has compact support (i.e., $\hat{S}$ is locally bounded). But this method fails if some of the assets with unbounded jumps are sold short.

It is possible to get around this problem by defining admissibility differently. However, the reasonable choice of the set of trading strategies is a very delicate point, as it should be at the same time economically meaningful and allow for elegant mathematical theorems. In our case, a convenient set of portfolios would contain $\varphi^{(\infty)}$ without affecting the validity of Lemma 2.4.

We have not taken this path in order to avoid an ad-hoc definition which is only suitable to this particular setting. Nevertheless, we feel that the question of admissibility deserves some attention and is still wanting a satisfactory answer in this context.

3. One can observe some qualitative differences between the power and logarithmic case on the one hand and exponential utility one the other hand. Let us consider the terminal wealth problem for simplicity. For power and logarithmic utility, a *constant fraction of wealth* is invested in each of the assets, whereas a *constant amount of money* is assigned to each security in the exponential case. Moreover, the discounted wealth process is an exponential Lévy process for power and logarithmic utility and a Lévy process for exponential utility.
4. In his pioneering paper, Merton (1969) obtained optimal portfolio/consumption pairs for continuous Lévy processes, i.e. $F = 0$. Framstad et al. (1999) treated the case $F(\mathbb{R}^d) < \infty$ for power utility. This result was extended to a larger class of jump measures by Benth et al. (1999).
5. As noted in the introduction, utility maximazation problems are linked by duality to the choice of an equivalent local martingale measure. In this paper, this *dual measure* appears in the application of Lemma 2.4. Piecing together results from He & Pearson (1991a,b), Karatzas et al. (1991), Kramkov & Schachermayer (1999), Bellini & Frittelli (1997), Schachermayer (1999), this measure also minimizes a certain distance functional. For $u(x) = 1 - e^{-x}$, this is the *relative entropy*. It gives rise to the *minimum entropy martingale measure* which has been determined by Miyahara (1999) and Chan (1999) in an exponential Lévy process setting.

Theorem 3.4 (Exponential Utility, $T = \infty$). *Suppose that, in addition to the conditions in Theorem 3.3, we have $\int_0^\infty t\,dK_t < \infty$. Let $\delta > 0$. There exists some $T' \in \mathbb{R}_+$ such that if $T_n, \kappa^{(n)}, V^{(n)}, \varphi^{(n)}$ are defined as above but relative to T' instead of T, then $(\varphi^{(n)}, \kappa^{(n)}) \in \mathfrak{P}$ has wealth process $V^{(n)}$ and for large n*

$$E\left(\int_0^\infty u(\kappa_t^{(n)})\,dK_t\right) \geq \sup_{(\varphi,\kappa)\in\mathfrak{P}} E\left(\int_0^\infty u(\kappa_t)\,dK_t\right) - \delta.$$

Proof. Define κ as in the proof of Theorem 3.3. For fixed $t_0 \in [1, \infty)$ define $P^\star \sim P$ by $\frac{P^\star}{P} := \frac{Z_{t_0}}{Z_0}$. The fifth step in the proof of Theorem 3.3 shows that κ is a $P^\star$-Lévy process and $\kappa - \alpha A$ is a $P^\star$-martingale on $[0, t_0]$. One easily shows that $E(Z_t|\kappa_t|) = Z_0 E_{P^\star}(|\kappa_t|) \leq Z_0(t+1)\sup_{s\in[0,1]} E_{P^\star}(|\kappa_s|) \leq (t+1)\cdot(E(Z_1|\gamma^\top L_1|) + |\alpha|)$ and $E(Z_t\kappa_t) = Z_0 E_{P^\star}(\kappa_t) = Z_0(\kappa_0 + \alpha t)$ for $t \in [0, t_0]$. Therefore the equation $E(\int_0^T Z_t\kappa_t\,dK_t) = \int_0^T E(Z_t\kappa_t)\,dK_t = Z_0(\kappa_0 K_T + \alpha\int_0^T t\,dK_t) = Z_0\varepsilon$ holds for $T = \infty$ as well. By Lemma 2.4, it follows that $\sup_{(\tilde\varphi,\tilde\kappa)\in\mathfrak{P}} E(\int_0^T u(\tilde\kappa_t)\,dK_t) \leq E(\int_0^T u(\kappa_t)\,dK_t)$.

Choose T' so large that $K_\infty - K_{T'} < \frac{\delta}{3} p$ and $\exp\left(\frac{p\alpha}{K_T}\int_{T'}^\infty t\,dK_t\right)$ $E(-\int_0^{T'}\exp(-p\kappa_t)\,dK_t) \geq E(-\int_0^\infty \exp(-p\kappa_t)\,dK_t) - \frac{\delta}{3}p$. Define $\tilde{\kappa}$ in the same way as κ, but relative to T' instead of T and 0 on (T', ∞). Since $\alpha \leq 0$, we have $\tilde{\kappa}_t \geq \kappa_t + \frac{\alpha}{K_T}\int_{T'}^\infty s\,dK_s$ for $t \in [0, T']$ and hence $E(-\int_0^{T'}\exp(-p\tilde{\kappa}_t)\,dK_t) \geq \exp\left(\frac{p\alpha}{K_T}\int_{T'}^\infty t\,dK_t\right)E(-\int_0^{T'}\exp(-p\kappa_t)\,dK_t)$. The claim follows now as in the last three steps of the above proof, applied to T' instead of $T = \infty$. □

4 Appendix

In this appendix, we summarize results from stochastic calculus that are needed in the previous section. Truncation functions h, h_d, h_{d+1} on $\mathbb{R}, \mathbb{R}^d, \mathbb{R}^{d+1}$, respectively, are supposed to be fixed. We begin with a simple reformulation of Itô's formula.

Lemma 4.1 (Itô's formula). *Let U be an open subset of $\mathbb{R}^d$ and X a U-valued semimartingale such that X_- is U-valued as well. Moreover, let $f : U \to \mathbb{R}$ be a function of class C^2. Then $f(X)$ is a semimartingale, and we have*

$$f(X_t) = f(X_0) + \int_0^t Df(X_{s-})^\top\,dX_s + \frac{1}{2}\sum_{i,j=1}^d \int_0^t D_{ij}^2 f(X_{s-})d\langle X^{i,c}, X^{j,c}\rangle_s$$

$$+ \int_{[0,t]\times\mathbb{R}^d} (f(X_{s-}+x) - f(X_{s-}) - Df(X_{s-})^\top x)\mu^X(ds, dx) \tag{4.8}$$

for any $t \in \mathbb{R}_+$. Here, $Df = (D_1 f, \ldots, D_d f)$ and $(D_{ij}^2 f)_{i,j=1,\ldots,d}$ denote the first and second derivatives of f, respectively.

Proof. This follows immediately from Jacod (1979), (2.54). Note that $\bigcup_{n\in\mathbb{N}}[0, R_n] = \mathbb{R}_+$ if X_- is U-valued. □

The following lemma shows how stochastic and usual exponentials of Lévy processes relate to each other. A proof can be found in Goll & Kallsen (1999), Lemma 5.7.

Lemma 4.2 (Exponential Lévy processes). 1. *Let $\tilde{L}$ be a real-valued Lévy process with characteristic triplet $(\tilde{b}, \tilde{c}, \tilde{F})$. Then the process $Z := e^{\tilde{L}}$ is of the form $\mathscr{E}(L)$ for some Lévy process L whose triplet (b, c, F) is given by*

$$b = \tilde{b} + \frac{\tilde{c}}{2} + \int (h(e^x - 1) - h(x))\tilde{F}(dx),$$

$$c = \tilde{c},$$

$$F(G) = \int 1_G(e^x - 1)\tilde{F}(dx) \quad \textit{for } G \in \mathscr{B}.$$

2. *Let L be a real-valued Lévy process with characteristic triplet (b, c, F). Suppose that $Z := \mathscr{E}(L)$ is positive. Then $Z = e^{\tilde{L}}$ for some Lévy process $\tilde{L}$ whose triplet $(\tilde{b}, \tilde{c}, \tilde{F})$ is given by*

$$\tilde{b} = b - \frac{c}{2} + \int (h(\log(1+x)) - h(x))F(dx),$$

$$\tilde{c} = c,$$

$$\tilde{F}(G) = \int 1_G(\log(1+x))F(dx) \quad \textit{for } G \in \mathscr{B}.$$

The effect of discounting on the triplet of a Lévy process is considered below. Note that many terms vanish if $\bar{L}^0$ is very simple (e.g. $\bar{L}^0_t = rt$ for $r \in \mathbb{R}$).

Lemma 4.3 (Discounting). *Let S be a $\mathbb{R}^{d+1}$-valued semimartingale of the form $S^i = S^i_0 \mathscr{E}(\bar{L}^i)$ for some $\mathbb{R}^{d+1}$-valued Lévy process with characteristic triplet $(\bar{b}, \bar{c}, \bar{F})$. Then the discounted process $(\hat{S}^1, \ldots, \hat{S}^d) = \left(\frac{S^1}{S^0}, \ldots, \frac{S^d}{S^0}\right)$ is of the form $\hat{S}^i = \hat{S}^i_0 \mathscr{E}(L^i)$ for an $\mathbb{R}^d$-valued Lévy process L with triplet (b, c, F) given by*

$$b^i = \bar{b}^i - \bar{b}^0 - \bar{c}^{0i} + \bar{c}^{00} + \int_{\mathbb{R}^{d+1}} \left(h^i_d\left(\frac{1+x^1}{1+x^0} - 1, \ldots, \frac{1+x^d}{1+x^0} - 1\right) - h^i_{d+1}(x^0, \ldots, x^d) + h^0_{d+1}(x^0, \ldots, x^d)\right) \bar{F}_t(d(x^0, \ldots, x^d)), \tag{4.9}$$

$$c^{ij} = \bar{c}^{ij} - \bar{c}^{0i} - \bar{c}^{0j} + \bar{c}^{00}, \tag{4.10}$$

$$F(G) = \int_G \left(\frac{1+x^1}{1+x^0} - 1, \ldots, \frac{1+x^d}{1+x^0} - 1\right) \bar{F}(dx) \tag{4.11}$$

for $i, j \in \{1, \ldots, d\}$, $t \in \mathbb{R}_+$, $G \in \mathscr{B}^d$.

Proof. Applying Itô's formula to $f(x^0, \ldots, x^d) \mapsto \frac{x^i}{x^0}$ yields

$$\begin{aligned}
\hat{S}^i = \hat{S}^i_0 &+ \left(\frac{1}{S^0_-} S^i_-\right) \cdot \bar{L}^i - \left(\frac{S^i_-}{(S^0_-)^2} S^0_-\right) \cdot \bar{L}^0 - \left(\frac{1}{(S^0_-)^2} S^i_- S^0_- \bar{c}^{i0}\right) \cdot A \\
&+ \left(\frac{S^i_-}{(S^0_-)^3} (S^0_-)^2 \bar{c}^{00}\right) \cdot A \\
&+ \left(\frac{S^i_-(1+x^i)}{S^0_-(1+x^0)} - \frac{S^i_-}{S^0_-} - \frac{1}{S^0_-} S^i_- x^i + \frac{S^i_-}{(S^0_-)^2} S^0_- x^0\right) * \mu^{\bar{L}} \\
= \hat{S}^i_0 &+ \hat{S}^i_-\left(\bar{L}^i - \bar{L}^0 + (\bar{c}^{00} - \bar{c}^{i0}) \cdot A + \left(\frac{1+x^i}{1+x^0} - 1 - x^i + x^0\right) * \mu^{\bar{L}}\right),
\end{aligned}$$

where $A_t := t$. So, $\hat{S}^i = \hat{S}^i_0 \mathscr{E}(L^i)$ for $L^i = \bar{L}^i - \bar{L}^0 + (\bar{c}^{00} - \bar{c}^{i0}) \cdot A + \left(\frac{1+x^i}{1+x^0} - 1 - x^i + x^0\right) * \mu^{\bar{L}}$. Since $\Delta L^i = \Delta \bar{L}^i - \Delta \bar{L}^0 + \frac{1+\Delta L^i}{1+\Delta L^0} - 1 - \Delta L^i + \Delta L^0 = \frac{1+\Delta L^i}{1+\Delta L^0} - 1$, we have $\nu^L([0,t] \times G) = 1_G\left(\frac{1+x^1}{1+x^0} - 1, \ldots, \frac{1+x^d}{1+x^0} - 1\right) * \nu^{\bar{L}}_t$ for $t \in \mathbb{R}_+$, $G \in \mathscr{B}^d$ and hence $\nu^L = \lambda \otimes F$, where F is as in Equation (4.11). Moreover, we have $\langle L^{i,c}, L^{j,c} \rangle = \langle \bar{L}^{i,c}, \bar{L}^{j,c} \rangle - \langle \bar{L}^{i,c}, \bar{L}^{0,c} \rangle - \langle \bar{L}^{0,c}, \bar{L}^{j,c} \rangle + \langle \bar{L}^{0,c}, \bar{L}^{0,c} \rangle = c^{ij} \cdot A$ with c^{ij} as in Equation (4.10). From the canonical semimartingale representation of $\bar{L}$ (cf. Jacod & Shiryaev (1987), II.2.34), we have that $L^i = \bar{L}^{i,c} + \bar{L}^{0,c} + (\bar{b}^i - \bar{b}^0 + \bar{c}^{00} - \bar{c}^{i0}) \cdot A + (h^i_{d+1}(x) - h^0_{d+1}(x)) * (\mu^{\bar{L}} - \nu^{\bar{L}}) + \left(\frac{1+x^i}{1+x^0} - 1 - h^i_{d+1}(x) + h^0_{d+1}(x)\right) * \mu^{\bar{L}}$. On the other hand, the canonical representation of L yields

$$L^i = L^{i,c} + B^i + h^i_d\left(\frac{1+x^1}{1+x^0} - 1, \ldots, \frac{1+x^d}{1+x^0} - 1\right) * (\mu^{\bar{L}} - \nu^{\bar{L}})$$
$$+ \left(\frac{1+x^i}{1+x^0} - 1 - h^i_d\left(\frac{1+x^1}{1+x^0} - 1, \ldots, \frac{1+x^d}{1+x^0} - 1\right)\right) * \mu^{\bar{L}}.$$

By Goll & Kallsen (1999), Proposition 5.3, this implies $B^i = (\bar{b}^i - \bar{b}^0 + \bar{c}^{00} - \bar{c}^{i0}) \cdot A + \left(h^i_d\left(\frac{1+x^1}{1+x^0} - 1, \ldots, \frac{1+x^d}{1+x^0} - 1\right) - h^i_{d+1}(x) + h^0_{d+1}(x)\right) * \nu^{\bar{L}} = b^i \cdot A$, where b^i is as in Equation (4.9). The assertion follows from Jacod & Shiryaev (1987), II.4.19. □

Finally, we cite some statements concerning the integrability of Lévy processes.

Lemma 4.4. *Let L be a real-valued Lévy process.*

1. *If L is a local martingale, then it is a martingale.*
2. *If $E(e^{L_{t_0}}) < \infty$ for some $t_0 \in (0, \infty)$, then e^L is of class (D) on any interval $[0, t]$, $t \in \mathbb{R}_+$.*
3. *If e^L is a local martingale, then it is a martingale.*

Proof.

1. cf. Sidibé (1979)
2. Let $a := \frac{1}{t_0}\log(E(e^{L_{t_0}}))$. It is easy to show that $E(e^{L_t}) = e^{at}$ for any $t \in \mathbb{R}_+$ and that $(e^{L_t - at})_{t \in \mathbb{R}_+}$ is a martingale. Therefore, it is of class (D) on any interval $[0, t]$. Since $(e^{at})_{t \in \mathbb{R}_+}$ is bounded on any interval $[0, t]$, it follows that e^L is of class (D) on $[0, t]$ as well.
3. A positive local martingale is a supermartingale (cf. Jacod (1979), (5.17)), which implies $E(e^{L_1}) < \infty$. In view of Statement 2 and Jacod & Shiryaev (1987), I.1.47c, the claim follows. □

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