



Existence and computations of best affine strategies for multilevel reverse Stackelberg games

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Abstract

The multilevel reverse Stackelberg game is considered. In this game, the leader controls the outcome by announcing a strategy as a function of decision variables of the followers to his/her own decision space. Corresponding to the leader's strategy, the player in the next level presents his/her strategy as a function of decision variables of the remaining players. This procedure is repeated until it is the turn of the bottom level player in the hierarchy, who reacts by determining his/her optimal decision variables. The structure of this game can be adopted in decentralized multilevel decision making like resource allocation, energy market pricing, problems with hierarchical controls. In this paper conditions for existence and construction of affine leader reverse Stackelberg strategies are developed for such problems. As an extension to the existing literature, we considered nonconvex sublevel sets of objective functions of followers. Moreover, a method to construct multiple reverse Stackelberg strategies for the leader is also presented.

Keywords Multilevel game · Hierarchical decision · Stackelberg strategy · Reverse Stackelberg strategy · Team solution · Desired equilibrium

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1 Introduction

In control problems where decision making is characterized by a natural hierarchy, sequential control approach can be adopted (Scattolini 2009). Also in model predictive controls of complex dynamical systems composed of subsystems at different layers, hierarchical control approach is essential (Scattolini and Colaneri 2007). In order to deal with such problems, a leader-follower solution concept which was introduced by von Stackelberg (1934) can be used as the framework for the resulting optimization problem. In a Stackelberg game some decision makers are able to act prior to other players which then reacts in a rational manner, i.e. the players in Stackelberg games act in a specific order (Başar and Olsder 1998). Thus, unlike Nash games where players are assumed to act simultaneously, Stackelberg games introduce a sequence of decisions between the players to characterize equilibria. Stackelberg games can be extended to multilevel games in which players are distributed throughout a multilevel hierarchy (Kassa and Kassa 2013, 2016, 2017; Kassa 2018).

Multilevel games are subsets of multilevel hierarchical decision problems that deal with decentralized decision problems involving interacting players that are distributed throughout a n -level hierarchy, $n \geq 2$. The players make their individual decisions in a sequential order, from the top 1st-level leader to the 2nd-level player up to the bottom, n th-level, player with the aim of optimizing their respective objectives. However, the class of reverse Stackelberg strategy is a solution approach where the leader formulates a strategy as a mapping from decision spaces of the followers towards his/her decision space that makes followers to behave as desired (Başar and Selbuz 1978; Groot et al. 2012; Zheng and Başar 1982). Such strategy of the leader induces each of the followers to behave cooperatively in achieving team optimal solution of the next successive level player which eventually coincides with the desired solution of the leader. This strategy is also referred to as equilibrium solution (Başar 1981a, b), incentive strategy (Ehtamo and Hämmäläinen 1989; Zheng and Başar 1982), reverse Stackelberg strategy (Ho et al. 1981), inverse Stackelberg strategy (Olsder 2009).

Several researchers have investigated the existence and construction of reverse Stackelberg strategies for bi-level games (Groot et al. 2012, 2016; Ho et al. 1981; Zheng and Başar 1982; Zheng et al. 1984). Multilevel Stackelberg games have been applied in areas such as resource allocation (Kassa 2018; Mitiku 2007), electricity pricing (Stankova 2009), marketing channel (Groot et al. 2017) and road pricing (Groot et al. 2015). Existence and construction of reverse Stackelberg strategies for trilevel games have been proved in (Başar 1981a, b; Başar et al. 1980) for the case where the objective functions of all the followers are quadratically convex under dynamic information, and in (Mizukami et al. 1989) where the objectives are strictly convex. However, the strict convexity assumption is a strong condition which may not be satisfied by some practical problems. Moreover, the works in (Başar 1981a, b) provide solutions for linear quadratic cost function structures of players, whereas the one in (Mizukami et al. 1989) results in only a single reverse Stackelberg strategy for the leader. However, for some practical problems the leader can have infinitely many possible strategies to achieve his/her desired equilibrium. This article presents existence conditions that are applicable to a more general game setting and formulates a solution method that enables to generate multiple strategies.

In this article, we consider static multilevel reverse Stackelberg games and investigate existence of optimal reverse Stackelberg strategies for each of the players in the game. Existence of affine reverse Stackelberg strategy of the leader (and middle-level players) is established under some mild conditions on the objective functions of the followers at the desired equilibrium. The existence of optimal affine reverse Stackelberg strategies are presented under more relaxed conditions than those in (Başar 1981a, b; Başar et al. 1980; Mizukami et al. 1989). Here, sublevel sets of objective functions of the followers at the desired equilibrium point are allowed to be nonconvex which is a relatively mild condition in comparison to quadratically convex and strictly convex objective functions in the prior works.

The other main contribution of this work is that it proposes existence and construction of an infinite number of affine strategies that can induce the desired behavior of followers. Such multiple optimal affine reverse Stackelberg strategies for the leader are characterized and constructed using free parameters. The construction of multiple strategies for multilevel reverse Stackelberg games enables consideration of additional optimization criterion as a secondary objective (Cansever and Basar 1982). This article also sheds light to the study of constrained version of the problem under consideration.

The paper is organized as follows. The formulation of multilevel reverse Stackelberg game and definition of a corresponding strategy is included in Sect. 2. In Sect. 3, we present the first contribution of this article: the existence conditions for optimal affine reverse Stackelberg strategy that enables the leader to achieve his/her desired solution. The cases for convex sublevel sets and nonconvex sublevel sets are dealt with in Sects. 3.1 and 3.2 respectively. The first part of Sect. 4 contains a method to construct only one optimal affine reverse Stackelberg strategy, with examples, for the leader. The second part starts with an example on trilevel game having infinitely many leader's reverse Stackelberg strategies. In this section, we present the second contribution of this article: Construction of multiple optimal affine reverse Stackelberg strategies of the leader. The insight brought up by the characterization of multiple optimal reverse Stackelberg strategies towards further study of the constrained is given in Sect. 5. Finally, the paper is concluded in Sect. 6 where we discuss concluding remarks and possible future works.

2 Preliminaries and problem formulation

In this Section we shall formulate the problem structure of static multilevel reverse Stackelberg games and present their properties. To make the presentation clear and easier to follow, we use a 3-level hierarchical static game as our main problem. However, it can be easily extended to any n -level hierarchical static game with one decision maker at each level of the hierarchy by repeating the procedure for the middle level decision maker any finite number of times sequentially. That means, we use the structure of trilevel games for development of existence theorems as well as solution procedures in the subsequent sections in order to address a general n -level static game.

Consider a three-player static game with three levels of hierarchy. Let the objective functions of the top level player (leader), the middle level player, and the third level player (follower) be $J_1(u^1, u^2, u^3)$, $J_2(u^1, u^2, u^3)$ and $J_3(u^1, u^2, u^3)$ respec-

tively. The variables $u^1 \in \Omega_1, u^2 \in \Omega_2, u^3 \in \Omega_3$ are decision vectors of the leader, the middle level player and the follower respectively. The strategic representations of objective functions of the leader, the middle level player and the follower are $J_1(\gamma^1, \gamma^2, u^3), J_2(\gamma^1, \gamma^2, u^3)$ and $J_3(\gamma^1, \gamma^2, u^3)$ respectively. The corresponding strategies γ^1 and γ^2 belong to admissible class of strategy spaces Γ^1 and Γ^2 of the leader and the middle level player respectively. For ease of elaboration, we restrict the admissible strategy spaces Γ^1 and Γ^2 to a class of affine strategies. We denote the optimal strategies by $\gamma^{i*}, i = 1, 2$.

Here, the leader is only interested in controlling the state of the system by enforcing his/her optimal decisions to be adopted by the lower level players in the hierarchy. So the first step in determination of the leader’s optimal reverse Stackelberg strategy is to evaluate team (desired) optimal solution of J_1 . We assume that J_1 has a unique team optimal point so that the leader seeks to achieve this unique desired equilibrium. For example, say $(u^{1d}, u^{2d}, u^{3d}) \in \Omega_1 \times \Omega_2 \times \Omega_3$ is the global optimum of J_1 , where Ω_1, Ω_2 and Ω_3 are respectively decision spaces of the leader, the middle level player and the follower. The problem then becomes for the leader to determine an optimal leader function $\gamma^1 : \Omega_2 \times \Omega_3 \rightarrow \Omega_1$ that leads to the desired equilibrium. The optimal reverse Stackelberg strategy of the leader γ^{1*} and the corresponding strategy of middle level player γ^{2*} achieves (u^{1d}, u^{2d}, u^{3d}) if it satisfies

$$(\gamma^{1*}, \gamma^{2*}, u^{3d}) = \underset{(\gamma^1, \gamma^2, u^3) \in \Gamma_1 \times \Gamma_2 \times \Omega_3}{\operatorname{argmin}} J_1(\gamma^1, \gamma^2, u^3), \tag{1}$$

$$(u^{2d}, u^{3d}) = \underset{(u^2, u^3) \in \Omega_2 \times \Omega_3}{\operatorname{argmin}} J_2(\gamma^{1*}, u^2, u^3), \tag{2}$$

$$u^{3d} = \underset{u^3 \in \Omega_3}{\operatorname{argmin}} J_3(\gamma^{1*}, \gamma^{2*}, u^3), \tag{3}$$

where the realization of the strategies are

$$\gamma^{1*}(u^{2d}, u^{3d}) = u^{1d}, \tag{4}$$

$$\gamma^{2*}(u^{3d}) = u^{2d}. \tag{5}$$

If γ^{1*} satisfies Eqs. (2)–(4) and γ^{2*} satisfies Eqs. (3) and (5), then they are called optimal reverse Stackelberg strategies of the leader and the middle level player respectively.

The following proposition presents conditions that should be satisfied by the strategy γ^{1*} so that the leader to achieve the desired solution.

Proposition 1 *A fixed optimal strategy γ^{1*} of the leader satisfying Eqs. (2) and (4) induces the middle level player to form a reverse Stackelberg strategy γ^{2*} to induce the follower.*

Proof In Eq. (2) we see that the leader’s optimal reverse Stackelberg strategy γ^{1*} leads to a team optimal solution of the middle level player as if the follower is cooperating. Thus, the middle level player wants to construct γ^{2*} that satisfies Eqs. (3) and (5) to achieve his/her desired equilibrium (u^{2d}, u^{3d}) . □

When the leader announces his/her optimal reverse Stackelberg strategy $u^1 = \gamma^1(u^2, u^3)$, the trilevel problem reduces to a bilevel problem where the middle level decision maker becomes a new leader. In the resulting two level problem the middle level player (now a leader) constructs his/her optimal reverse Stackelberg strategy $u^2 = \gamma^2(u^3)$ to induce the follower to behave as desired. In this sense, for a general multilevel problem, each step the higher level decision maker announces an optimal reverse Stackelberg strategy, a k level problem becomes $k - 1$ level of the problem until the bottom level follower acts. To illustrate the proposed procedure, we give the following trilevel problem as an example whose detailed solution procedure is found in Example 2.

Example 1 Consider the trilevel problem with objective functions of each hierarchical level is respectively given by

$$\begin{aligned} J_1(u^1, u^2, u^3) &= (u^1 - 2)^2 + (u^2 - 1)^2 + (u^3 - 3)^2, \\ J_2(u^1, u^2, u^3) &= (u^1 - 1)^2 + (u^2)^2 + (u^3)^2, \\ J_3(u^1, u^2, u^3) &= (u^1)^2 + (u^2 - 2)^2 + (u^3)^2. \end{aligned}$$

In this example, the global optimal solution of the leader is $(u^{1d}, u^{2d}, u^{3d}) = (2, 1, 3)$. If the leader chooses a strategy $\gamma^{1*} = -u^2 - 3u^3 + 12$, as a rational decision maker, the middle level player reacts by a strategy $\gamma^{2*} = -u^3 + 4$. The strategy γ^{1*} together with the corresponding strategy γ^{2*} fixes the optimal solution of this trilevel problem to the desired equilibrium of the leader $(2, 1, 3)$. In Sects. 3 and 4, we present the conditions of existence and solution approach of such strategies.

Remark 1 The main goal of the leader is to induce the follower to act in a way that the leader's team optimal solution is achieved. Towards this end, the leader forces or persuades (using incentives and punishment) the middle level player to choose γ^{2*} that satisfies Eqs. (3) and (5) by the announcement of γ^{1*} satisfying Eq. (2). A strategy γ^{1*} that satisfies Eq. (2) modifies J_2 in a way that his/her available optimal solution is (u^{2d}, u^{3d}) . As a result, the middle level player reacts by formulating γ^{2*} that satisfies Eq. (3). The influence of the leader on γ^{2*} can be seen in the relation $\gamma^{1*}(u^2, u^3) = \gamma^{1*}(\gamma^{2*}(u^3), u^3)$.

The observation in Remark 1 above can be considered, for example, when a policy maker takes the role of the leader. In this context, the policy maker at macro-level may give some incentive mechanisms to influence the firms in the demand-supply chain to act according to the set goal.

3 Existence of optimal affine reverse Stackelberg strategy

For a 3-players hierarchical game given in Eqs. (1)–(5), assume that $(u^{1d}, u^{2d}, u^{3d}) \in \Omega_1 \times \Omega_2 \times \Omega_3$ is a minimizer for the leader's objective function $J_1(u^1, u^2, u^3)$. In reference with this point, let us define the level sets W^{2d} and W^{3d} corresponding to

the objective functions of the followers J_2 and J_3 , respectively, at the desired optimal solution (u^{1d}, u^{2d}, u^{3d}) of the leader as follows.

$$W^{2d} = \{(u^1, u^2, u^3) \in \Omega_1 \times \Omega_2 \times \Omega_3 : J_2(u^1, u^2, u^3) \leq J_2(u^{1d}, u^{2d}, u^{3d})\}. \quad (6)$$

$$W^{3d} = \{(u^1, u^2, u^3) \in \Omega_1 \times \Omega_2 \times \Omega_3 : J_3(u^1, u^2, u^3) \leq J_3(u^{1d}, u^{2d}, u^{3d})\}. \quad (7)$$

The leader should construct the optimal reverse Stackelberg strategy γ^{1*} such that it uniquely intersects with the level set W^{2d} at (u^{1d}, u^{2d}, u^{3d}) . Under the announced leader's strategy γ^{1*} the middle level player constructs a strategy γ^{2*} such that it uniquely intersects W^{3d} at (u^{1d}, u^{2d}, u^{3d}) .

The main results of this section are summarized for the sake of clarity as follows:

Existence of optimal affine reverse strategies that guarantee the achievement of the equilibrium solution at the leader's desired point (u^{1d}, u^{2d}, u^{3d}) is proved,

- *when the level sets Wg^{2d} and W^{3d} are convex and are locally strictly convex at (u^{1d}, u^{2d}, u^{3d}) ; (This is provided in detail in Sect. 3.1, in particular it is summarized in Theorem 11.)*
- *when the level sets W^{2d} and W^{3d} are nonconvex, but their convex hulls are locally strictly convex at (u^{1d}, u^{2d}, u^{3d}) and the point (u^{1d}, u^{2d}, u^{3d}) is an exposed point of the convex hulls of the level sets. (The details of this result is provided in Sect. 3.2.)*

The conditions required in the above two cases are more general assumptions relative to prior works (Başar 1981a, b; Başar et al. 1980; Mizukami et al. 1989) where cost functions of followers are assumed to be strictly convex. We used the results of this Section for the formulation of solution approaches in Sect. 4.

3.1 Convex sublevel sets

In order to prove the existence of affine reverse Stackelberg strategy for the leader, we use geometric properties of convex sublevel sets of objective functions of the middle level player and the follower at the desired solution.

We make the following assumptions on decision spaces, sublevel sets and objective functions of the followers.

Assumption 1 The decision spaces $\Omega_1, \Omega_2, \Omega_3$ are convex and the sublevel sets W^{2d} and W^{3d} are connected.

Assumption 2 The class of admissible strategies Γ^1 and Γ^2 satisfy

$$\Gamma^1 = \left\{ \gamma^1 : \Omega_2 \times \Omega_3 \rightarrow \Omega_1 \mid \gamma^1 \text{ is affine and } \gamma^1(u^{2d}, u^{3d}) = u^{1d} \right\},$$

$$\Gamma^2 = \left\{ \gamma^2 : \Omega_3 \rightarrow \Omega_2 \mid \gamma^2 \text{ is affine and } \gamma^2(u^{3d}) = u^{2d} \right\}.$$

The graph of a function γ^1 , denoted by $\text{Graph}(\gamma^1)$, is a relation that maps points in $\Omega_2 \times \Omega_3$ to Ω_1 and is represented by

$$\text{Graph}(\gamma^1) = \{(u^1, u^2, u^3) : u^3 \in \Omega_3, u^2 \in \Omega_2, u^1 = \gamma^1(u^2, u^3)\}.$$

In order for the leader to be able to force the middle level player's decision to his/her desired equilibrium point, the leader should construct his/her affine optimal reverse Stackelberg strategy whose graph intersects with W^{2d} only at the point (u^{1d}, u^{2d}, u^{3d}) . That is,

$$W^{2d} \cap \text{Graph}(\gamma^{1*}) = (u^{1d}, u^{2d}, u^{3d}).$$

In what follows we consider leader's affine function mapping of dimension m_1

$$\gamma^1 : \Omega_2 \times \Omega_3 \rightarrow \Omega_1 \quad (8)$$

that satisfies Eq. (4).

The leader's affine function γ^1 is constructed in such a way that for all possible pair of actions $(u^2, u^3) \in \Omega_2 \times \Omega_3$ there exists $u^1 \in \Omega_1$ such that $u^1 = \gamma^1(u^2, u^3)$. This can be accomplished by constructing an inverse affine function $\alpha : \Omega_1 \rightarrow \Omega_2 \times \Omega_3$ whose graph passes through (u^{1d}, u^{2d}, u^{3d}) . Towards this, we construct the set of affine relations whose graph passes through (u^{1d}, u^{2d}, u^{3d}) as follows.

Denote the set of all affine relations of dimension $m_2 + m_3$ in $\Omega_2 \times \Omega_3$ whose graph passes through (u^{1d}, u^{2d}, u^{3d}) by \mathcal{A}_1 . An element $\alpha_1 \in \mathcal{A}_1$ satisfies

$$\text{Graph}(\alpha_1) \cap W^{2d} = (u^{1d}, u^{2d}, u^{3d}).$$

Since $\alpha_1 \in \mathcal{A}_1$ is an affine relation having full dimension $m_2 + m_3$, for all $(u^2, u^3) \in \Omega_2 \times \Omega_3$, there exists $u^1 \in \Omega_1$ such that $\alpha_1(u^1) = (u^2, u^3)$. Thus, for every $\alpha_1 \in \mathcal{A}_1$ we have $\alpha_1(\Omega_1) = \Omega_2 \times \Omega_3$. Then the candidate leader function is characterized by $\gamma^1 := (\alpha_1)^{-1}$. Next, let us consider the set of affine relations whose graph passes through (u^{1d}, u^{2d}, u^{3d}) and lie on the supporting hyperplane $\Pi_{W^{2d}}$. This set is denoted by $\mathcal{A}_1^{\Pi_{W^{2d}}}$ and is defined as

$$\mathcal{A}_1^{\Pi_{W^{2d}}} := \{\alpha_1 \in \mathcal{A}_1 : \alpha_1 \subseteq \Pi_{W^{2d}}\}.$$

Now, we state the following two results from (Luenberger 1997) that we shall use them in the subsequent analysis.

Lemma 1 (Geometric Hahn-Banach Theorem, Luenberger 1969) *Let K be a convex set having a nonempty interior in a real normed linear vector space X . Suppose that V is a linear variety in X containing no interior points of K . Then there is a closed hyperplane in X containing V but containing no interior points of K .*

Proof See Theorem 1 in (Luenberger 1997). □

Lemma 2 (Support Theorem, Luenberger 1969) *If x is not an interior point of a convex set K which contains interior points, there is a closed hyperplane Π containing x such that K lies on one side of Π .*

Proof See Theorem 2 in (Luenberger 1997). □

To utilize these concepts in our context we state the following corollaries, that follow from Lemmas 1 and 2, which can be used in the subsequent analysis.

Corollary 2 *Assume that W^{2d} is convex and locally strictly convex at the point (u^{1d}, u^{2d}, u^{3d}) . Let $\Omega_1 = \mathbb{R}^{m_1}$, $\Omega_2 = \mathbb{R}^{m_2}$, $\Omega_3 = \mathbb{R}^{m_3}$ and let $\alpha_1 \in \mathcal{A}_1$ be any affine function such that $\text{Graph}(\alpha_1) \cap W^{2d} = (u^{1d}, u^{2d}, u^{3d})$. Then α_1 lies on the hyperplane $\Pi_{W^{2d}}$ supporting W^{2d} at the point (u^{1d}, u^{2d}, u^{3d}) .*

Proof Product of Euclidean spaces $\Omega_1 = \mathbb{R}^{m_1}$, $\Omega_2 = \mathbb{R}^{m_2}$, $\Omega_3 = \mathbb{R}^{m_3}$ is a normed linear space. Hence, $\Omega_1 \times \Omega_2 \times \Omega_3$ is a normed linear space and the sublevel set $W^{2d} \subset \Omega_1 \times \Omega_2 \times \Omega_3$. Moreover, as $\text{Graph}(\alpha_1) \cap W^{2d} = (u^{1d}, u^{2d}, u^{3d})$, then α_1 does not contain interior point of W^{2d} . Then by Lemma 1 there exists a hyperplane $\Pi_{W^{2d}}$ through (u^{1d}, u^{2d}, u^{3d}) containing affine (linear variety) α_1 . Since W^{2d} is locally strictly convex at (u^{1d}, u^{2d}, u^{3d}) , then $W^{2d} \cap \Pi_{W^{2d}}(u^{1d}, u^{2d}, u^{3d}) = (u^{1d}, u^{2d}, u^{3d})$. Hence, $\Pi_{W^{2d}}$ supports W^{2d} at (u^{1d}, u^{2d}, u^{3d}) follows from the definition of the supporting hyperplane and Lemma 2. □

Corollary 3 *Assume that W^{2d} is convex and locally strictly convex at the point (u^{1d}, u^{2d}, u^{3d}) . Let $\Omega_1 = \mathbb{R}^{m_1}$, $\Omega_2 = \mathbb{R}^{m_2}$, $\Omega_3 = \mathbb{R}^{m_3}$. Then a supporting hyperplane $\Pi_{W^{2d}}(u^{1d}, u^{2d}, u^{3d})$ intersects with W^{2d} only at the point (u^{1d}, u^{2d}, u^{3d}) .*

Proof We have proved in Corollary 2 the existence of the supporting hyperplane $\Pi_{W^{2d}}(u^{1d}, u^{2d}, u^{3d})$ to W^{2d} at (u^{1d}, u^{2d}, u^{3d}) . Then by the definition of a supporting hyperplane, (u^{1d}, u^{2d}, u^{3d}) is not in an interior point of W^{2d} . Since W^{2d} is locally strictly convex at (u^{1d}, u^{2d}, u^{3d}) , applying Lemma 2 we see that $\Pi_{W^{2d}}$ supports W^{2d} only at (u^{1d}, u^{2d}, u^{3d}) . □

In the following theorem we prove the existence of leader's affine reverse Stackelberg strategy provided the assumptions are satisfied. To consider the trilevel problem in its entirety (Eqs. (1)–(5)) we make the following assumption.

Assumption 3 *If the mapping (8) is announced by the leader as an optimal reverse Stackelberg strategy to the followers, then the middle level player can find a strategy $\gamma^2 \in \Gamma^2$ that satisfies Eq. (2).*

This assumption states that in the trilevel hierarchical Stackelberg game, for each action chosen by the leader, the middle level player always has a room to respond optimally in the required direction. The conditions for which this assumption is satisfied are given in Theorem 9. But first we shall show in the next arguments that the leader can achieve his/her desired equilibrium solution if he/she announces γ^{1*} as optimal reverse Stackelberg strategy.

Theorem 4 *Suppose that W^{2d} is convex and locally strictly convex at the point (u^{1d}, u^{2d}, u^{3d}) , Assumption 3 is satisfied, $J_2(u^1, u^2, u^3)$ is Fréchet differentiable at (u^{1d}, u^{2d}, u^{3d}) and $\Omega_1 = \mathbb{R}^{m_1}, \Omega_2 = \mathbb{R}^{m_2}, \Omega_3 = \mathbb{R}^{m_3}$. If $\nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}) \neq 0$, then, there exists leader's affine map $\gamma^1 : \Omega_2 \times \Omega_3 \rightarrow \Omega_1$ that realizes the desired equilibrium (u^{1d}, u^{2d}, u^{3d}) .*

Proof Since the mapping γ^1 belongs to Γ^1 , it satisfies Eq. (4). In view of Assumption 3 the problem faced by the leader is to find γ^1 that satisfies Eq. (2). From Corollary 2 and Corollary 3, it follows that there exists an affine mapping

$$\alpha_1^{\Pi_{W^{2d}}} \in \mathcal{A}_1^{\Pi_{W^{2d}}} \tag{9}$$

such that

$$\text{Graph}(\alpha_1^{\Pi_{W^{2d}}}) \cap W^{2d} = (u^{1d}, u^{2d}, u^{3d}).$$

Now, we need to show that $\alpha_1^{\Pi_{W^{2d}}}(\Omega_1) = \Omega_2 \times \Omega_3$. Since J_2 is Fréchet differentiable at (u^{1d}, u^{2d}, u^{3d}) the normal vector to W^{2d} at (u^{1d}, u^{2d}, u^{3d}) exists, is unique and equal to $\nabla J_2(u^{1d}, u^{2d}, u^{3d})$, in which case its supporting hyperplane $\Pi_{W^{2d}}(u^{1d}, u^{2d}, u^{3d})$ is given by

$$\begin{aligned} \langle \nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}), u^1 - u^{1d} \rangle + \langle \nabla_{u^2} J_2(u^{1d}, u^{2d}, u^{3d}), \\ u^2 - u^{2d} \rangle + \langle \nabla_{u^3} J_2(u^{1d}, u^{2d}, u^{3d}), u^3 - u^{3d} \rangle = 0. \end{aligned} \tag{10}$$

If $\nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}) \neq 0$, then the normal vector defining the hyperplane $\Pi_{W^{2d}}(u^{1d}, u^{2d}, u^{3d})$ is not orthogonal to the decision space Ω_1 , that means,

$$\text{Proj}_{\Omega_1}^{\Pi_{W^{2d}}(u^{1d}, u^{2d}, u^{3d})} \neq \{0\}.$$

Then it follows that the hyperplane is not orthogonal to $\{0\}^{m_1} \times \Omega_2 \times \Omega_3$. Thus,

$$\text{Proj}_{\Omega_2 \times \Omega_3}^{\Pi_{W^{2d}}(u^{1d}, u^{2d}, u^{3d})} = \Omega_2 \times \Omega_3.$$

Hence, for all $(u^2, u^3) \in \Omega_2 \times \Omega_3$ there exists $u^1 \in \Omega_1$ such that $(u^1, u^2, u^3) \in \Pi_{W^{2d}}(u^{1d}, u^{2d}, u^{3d})$. Thus, the affine map $\alpha_1^{\Pi_{W^{2d}}}$ whose existence is verified in Eq. (9) is full dimensional and fulfills the condition that

$$\alpha_1^{\Pi_{W^{2d}}}(\Omega_1) = \Omega_2 \times \Omega_3.$$

The leader's strategy given by $\gamma^1 := (\alpha_1^{\Pi_{W^{2d}}})^{-1}$ restricts the optimization of the middle level player to set of points determined by the map γ^1 . Since graph of γ^1 intersects the level set W^{2d} only at (u^{1d}, u^{2d}, u^{3d}) , the minimum of $J_2(\gamma^{1*}, u^2, u^3)$ over $\Omega_2 \times \Omega_3$ is obtained at $(\gamma^{1*}(u^{2d}, u^{3d}), u^{2d}, u^{3d})$.

□

Remark 2 If $\nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}) = 0$, then J_2 is not sensitive to the change of leader’s control variable u^1 . In this case, the leader can not directly influence the decision of the middle level player. On the other hand, if $\nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}) \neq 0$ and $\nabla_{u^1} J_3(u^{1d}, u^{2d}, u^{3d}) = 0$ the leader can still achieve the desired equilibrium under the conditions in Theorem 9.

Theorem 5 Suppose that W^{2d} is convex and is locally strictly convex at the point (u^{1d}, u^{2d}, u^{3d}) , Assumption 3 is satisfied, $J_2(u^1, u^2, u^3)$ is Fréchet differentiable at (u^{1d}, u^{2d}, u^{3d}) and $\Omega_1 = \mathbb{R}^{m_1}, \Omega_2 = \mathbb{R}^{m_2}, \Omega_3 = \mathbb{R}^{m_3}$. If the leader’s affine map $\gamma^1 : \Omega_2 \times \Omega_3 \rightarrow \Omega_1$ realizes the desired equilibrium (u^{1d}, u^{2d}, u^{3d}) , then

$$\nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}) \neq 0.$$

Proof Since J_2 is differentiable at (u^{1d}, u^{2d}, u^{3d}) , the normal vector to W^{2d} at (u^{1d}, u^{2d}, u^{3d}) exists, is unique and is equal to $\nabla J_2(u^{1d}, u^{2d}, u^{3d})$ in which case the supporting hyperplane $\Pi_{W^{2d}}(u^{1d}, u^{2d}, u^{3d})$ is given by Eq. (10).

The proof is carried out by contrapositive arguments. Suppose that,

$$\nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}) = 0.$$

Then it follows from Eq. (10) that

$$\langle \nabla_{u^2} J_2(u^{1d}, u^{2d}, u^{3d}), u^2 - u^{2d} \rangle + \langle \nabla_{u^3} J_2(u^{1d}, u^{2d}, u^{3d}), u^3 - u^{3d} \rangle = 0.$$

Then the normal vector defining the hyperplane $\Pi_{W^{2d}}(u^{1d}, u^{2d}, u^{3d})$ is

$$v = (0, \nabla_{u^2} J_2(u^{1d}, u^{2d}, u^{3d}), \nabla_{u^3} J_2(u^{1d}, u^{2d}, u^{3d})).$$

Since W^{2d} is locally strictly convex at (u^{1d}, u^{2d}, u^{3d}) , the hyperplane $\Pi_{W^{2d}}(u^{1d}, u^{2d}, u^{3d})$ is orthogonal to $\{0\}^{m_1} \times \Omega_2 \times \Omega_3$. That means,

$$\Omega_2 \times \Omega_3 \not\subseteq \text{Proj}_{\Omega_2 \times \Omega_3}^{\Pi_{W^{2d}}(u^{1d}, u^{2d}, u^{3d})}.$$

Hence, $\Pi_{W^{2d}}(u^{1d}, u^{2d}, u^{3d})$ does not include any element $(u^1, u^2, u^3) \in \Omega_1 \times (\Omega_2 \times \Omega_3 \setminus \{(u^{2d}, u^{3d})\})$, which implies that $\alpha_1^{\Pi_{W^{2d}}(\Omega_1)} \not\subseteq \Omega_2 \times \Omega_3$. That means, $\gamma^1 = (\alpha_1^{\Pi_{W^{2d}}})^{-1}$ does not hold. As a result the affine map γ^1 can not be defined. □

Proposition 6 Suppose that all the conditions of Theorem 4 are satisfied to guarantee the existence of γ^1 . Then for all $(u^1, u^2, u^3) \in \text{Graph}(\gamma^1)$, we have

$$J_2(u^1, u^2, u^3) \geq J_2(u^{1d}, u^{2d}, u^{3d}), \tag{11}$$

where equality can only be achieved at the equilibrium point (u^{1d}, u^{2d}, u^{3d}) , which is desired by the leader.

Proof Suppose $(u^1, u^2, u^3) \in \text{Graph}(\gamma^1)$, then by definition we have $(u^1, u^2, u^3) \in \Pi_{W^{2d}}$. Since $\Pi_{W^{2d}}$ supports W^{2d} at (u^{1d}, u^{2d}, u^{3d}) , W^{2d} lies on one side of $\Pi_{W^{2d}}$. Hence for all $(u^1, u^2, u^3) \in \Pi_{W^{2d}}$, we have

$$J_2(u^1, u^2, u^3) \geq J_2(u^{1d}, u^{2d}, u^{3d}) \tag{12}$$

Since $\text{Graph}(\gamma^1)$ lies on $\Pi_{W^{2d}}$, the result follows as well in $\text{Graph}(\gamma^1)$. Then, it follows from Eqs. (6) and (12) that equality is achieved at the equilibrium point (u^{1d}, u^{2d}, u^{3d}) . \square

Therefore, it is the best (rational) interest of the middle level player to find a reverse Stackelberg strategy γ^2 to induce the follower’s decision to the desired value $u^3 = u^{3d}$.

Next, we analyze the existence of a reverse Stackelberg strategy $\gamma^2 \in \Gamma^2$ which satisfies Assumption 3. The following proposition guarantees the existence of a supporting hyperplane to the sublevel set W^{3d} of J_3 at (u^{1d}, u^{2d}, u^{3d}) .

Proposition 7 *If W^{3d} is convex and locally strictly convex at (u^{1d}, u^{2d}, u^{3d}) , $J_3(u^1, u^2, u^3)$ is Fréchet differentiable at (u^{1d}, u^{2d}, u^{3d}) and $\Omega_1 = \mathbb{R}^{m_1}$, $\Omega_2 = \mathbb{R}^{m_2}$, $\Omega_3 = \mathbb{R}^{m_3}$ then there exists a supporting hyperplane $\Pi_{W^{3d}}$ that intersects W^{3d} uniquely at the point (u^{1d}, u^{2d}, u^{3d}) .*

Proof The result follows automatically from Corollary 2 and Corollary 3. \square

In consideration of the case where $\nabla_{u^1} J_3(u^{1d}, u^{2d}, u^{3d}) \neq 0$ we then analyze the influence of γ^1 on J_3 .

Since $J_3(u^1, u^2, u^3)$ is Fréchet differentiable at (u^{1d}, u^{2d}, u^{3d}) the equation of the hyperplane $\Pi_{W^{3d}}$ can be written as

$$\begin{aligned} &\langle \nabla_{u^1} J_3(u^{1d}, u^{2d}, u^{3d}), u^1 - u^{1d} \rangle + \langle \nabla_{u^2} J_3(u^{1d}, u^{2d}, u^{3d}), \\ &u^2 - u^{2d} \rangle + \langle \nabla_{u^3} J_3(u^{1d}, u^{2d}, u^{3d}), u^3 - u^{3d} \rangle = 0. \end{aligned} \tag{13}$$

By virtue of Theorem 4, if $\nabla_{u^1} J_3(u^{1d}, u^{2d}, u^{3d}) \neq 0$, there exists an affine leader function

$$\gamma^{1'} : \Omega_2 \times \Omega_3 \rightarrow \Omega_1 \tag{14}$$

that lies on the supporting hyperplane $\Pi_{W^{3d}}$ to the sublevel set W^{3d} .

Note that both the affine relations γ^1 and $\gamma^{1'}$ intersect the sublevel sets W^{2d} and W^{3d} at the point (u^{1d}, u^{2d}, u^{3d}) and lie on the supporting hyperplanes $\Pi_{W^{2d}}$ and $\Pi_{W^{3d}}$ respectively.

The two supporting hyperplanes $\Pi_{W^{2d}}$ and $\Pi_{W^{3d}}$ passing through (u^{1d}, u^{2d}, u^{3d}) have common points. Therefore, there exists a set containing points that satisfy both Eqs. (10) and (13). Define this set Φ as

$$\Phi = \{(u^1, u^2, u^3) \in \Omega_1 \times \Omega_2 \times \Omega_3 | (u^1, u^2, u^3) \text{ satisfies Eqs. (10) and (13)}\}. \tag{15}$$

However, the main results of this article follow regardless of the value of $\nabla_{u^1} J_3(u^{1d}, u^{2d}, u^{3d})$. That means, the leader can acquire the desired solution even when $\nabla_{u^1} J_3(u^{1d}, u^{2d}, u^{3d}) = 0$ by declaring a strategy γ^1 in conformity with Eq. (3).

Proposition 8 *Suppose that all the conditions of Theorem 4 are satisfied. Then for $(u^1, u^2, u^3) \in \Phi$, we have*

$$J_3(u^1, u^2, u^3) \geq J_3(u^{1d}, u^{2d}, u^{3d}), \quad (16)$$

where equality can only be achieved at the desired equilibrium point (u^{1d}, u^{2d}, u^{3d}) .

Proof If Φ is a singleton, then $\Phi = \{(u^{1d}, u^{2d}, u^{3d})\}$ and the result follows automatically. Suppose Φ is not a singleton. Let $(u^1, u^2, u^3) \in \Phi$ such that $(u^1, u^2, u^3) \neq (u^{1d}, u^{2d}, u^{3d})$. Then by definition we have $(u^1, u^2, u^3) \in \Pi_{W^{3d}}$. Since $\Pi_{W^{3d}}$ supports W^{3d} at (u^{1d}, u^{2d}, u^{3d}) , W^{3d} lies on one side of $\Pi_{W^{3d}}$. Hence for all $(u^1, u^2, u^3) \in \Pi_{W^{3d}}$, we have

$$J_3(u^1, u^2, u^3) \geq J_3(u^{1d}, u^{2d}, u^{3d}). \quad (17)$$

Then, it follows from Eqs. (7) and (17) that equality is achieved at the equilibrium point (u^{1d}, u^{2d}, u^{3d}) . \square

Therefore, if the middle level player can find a strategy $\gamma^2 \in \Gamma^2$, corresponding to leader's strategy γ^1 , to constrain the follower's decision space to Φ , the optimal performance available to the follower can only be achieved at the point (u^{1d}, u^{2d}, u^{3d}) . That also means that the problem faced by the middle level player is to find γ^2 that satisfies Eq. (3).

Substituting the fixed leader's optimal reverse Stackelberg strategy γ^{1*} , we denote $J_3(\gamma^{1*}, u^2, u^3)$ by $\bar{J}_3(u^2, u^3)$. For such a case, the sublevel set of J_3 at (u^{1d}, u^{2d}, u^{3d}) is denoted and defined as

$$\bar{W}^{3d} = \{(u^2, u^3) \in \Omega_2 \times \Omega_3 \mid \bar{J}_3(u^2, u^3) \leq \bar{J}_3(u^{2d}, u^{3d})\}. \quad (18)$$

Remark 3 Since \bar{W}^{3d} is the outcome of W^{3d} for fixed affine leader's strategy γ^{1*} , it is convex and locally strictly convex at (u^{2d}, u^{3d}) .

Substituting the fixed leader's optimal affine reverse Stackelberg strategy $\gamma^{1*}(u^1, u^2, u^3)$ in Eq. (13) and using Assumption 2 we have

$$\langle \nabla_{u^2} \bar{J}_3(u^{2d}, u^{3d}), u^2 - u^{2d} \rangle + \langle \nabla_{u^3} \bar{J}_3(u^{2d}, u^{3d}), u^3 - u^{3d} \rangle = 0 \quad (19)$$

which is a hyperplane orthogonal to the decision space $\Omega_2 \times \Omega_3$. The middle level player can retain his/her desired solution (u^{2d}, u^{3d}) by an affine strategy $\gamma^2 : \Omega_3 \rightarrow \Omega_2$ whose graph intersects \bar{W}^{3d} only at (u^{2d}, u^{3d}) . This idea will be proved in Theorem 9 below.

Denote the set of all affine relations of dimension m_3 in Ω_3 whose graph passes through (u^{2d}, u^{3d}) by \mathcal{A}_2 . Then an element $\alpha_2 \in \mathcal{A}_2$ satisfies

$$\text{Graph}(\alpha_2) \cap \bar{W}^{3d} = (u^{2d}, u^{3d}).$$

Since $\alpha_2 \in \mathcal{A}_2$ is an affine relation having full dimension m_3 , for all $u^3 \in \Omega_3$, there exists $u^2 \in \Omega_2$ such that $\alpha_2(u^2) = u^3$. Thus, for every $\alpha_2 \in \mathcal{A}_2$ we have $\alpha_2(\Omega_2) = \Omega_3$. Then the candidate for the leader’s function is characterized by $\gamma^2 := (\alpha_2)^{-1}$. Next, let us consider the set of affine relations whose graph passes through (u^{2d}, u^{3d}) and lie on the supporting hyperplane $\Pi_{\bar{W}^{3d}}$. This set is denoted by $\mathcal{A}_2^{\Pi_{\bar{W}^{3d}}}$ and is defined as

$$\mathcal{A}_2^{\Pi_{\bar{W}^{3d}}} := \{\alpha_2 \in \mathcal{A}_2 : \alpha_2 \subseteq \Pi_{\bar{W}^{3d}}\}.$$

Theorem 9 *Suppose that \bar{W}^{3d} is convex and locally strictly convex at the point (u^{2d}, u^{3d}) , $\bar{J}_3(u^2, u^3)$ is Fréchet differentiable at (u^{2d}, u^{3d}) and $\Omega_2 = \mathbb{R}^{m_2}$, $\Omega_3 = \mathbb{R}^{m_3}$. Under the announced leader’s optimal reverse Stackelberg strategy γ^{1*} , if $\nabla_{u^2} \bar{J}_3(u^{2d}, u^{3d}) \neq 0$, there exists middle level player’s optimal strategy $\gamma^{2*} : \Omega_3 \rightarrow \Omega_2$ that satisfies Eq. (3).*

Proof Since the mapping γ^2 is assumed to satisfy Eq. (5) by Assumption 2, the problem faced by the middle level player is to find γ^2 that satisfies Eq. (3). From Corollary 2 and Corollary 3, it follows that there exists an affine mapping

$$\alpha_2^{\Pi_{\bar{W}^{3d}}} \in \mathcal{A}_2^{\Pi_{\bar{W}^{3d}}} \tag{20}$$

such that

$$\text{Graph}(\alpha_2^{\Pi_{\bar{W}^{3d}}}) \cap \bar{W}^{3d} = (u^{2d}, u^{3d}).$$

Now, we need to show that $\alpha_2^{\Pi_{\bar{W}^{3d}}}(\Omega_2) = \Omega_3$. Since \bar{J}_3 is Fréchet differentiable at (u^{2d}, u^{3d}) the normal vector to \bar{W}^{3d} at (u^{2d}, u^{3d}) exists, is unique and equal to $\nabla \bar{J}_2(u^{2d}, u^{3d})$, in which case its supporting hyperplane $\Pi_{\bar{W}^{3d}}(u^{2d}, u^{3d})$ is determined by the relation in Eq. (19).

If $\nabla_{u^2} \bar{J}_2(u^{2d}, u^{3d}) \neq 0$, then the normal vector defining the hyperplane $\Pi_{\bar{W}^{3d}}(u^{2d}, u^{3d})$ is not orthogonal to the decision space Ω_2 , that means,

$$\text{Proj}_{\Omega_2}^{\Pi_{\bar{W}^{3d}}(u^{2d}, u^{3d})} \neq \{0\}.$$

Then it follows that the hyperplane is not orthogonal to $\{0\}^{m_2} \times \Omega_3$. Thus,

$$\text{Proj}_{\Omega_3}^{\Pi_{\bar{W}^{3d}}(u^{2d}, u^{3d})} = \Omega_3.$$

Hence, for all $u^3 \in \Omega_3$ there exists $u^2 \in \Omega_2$ such that $(u^2, u^3) \in \Pi_{\bar{W}^{3d}}(u^{2d}, u^{3d})$. Thus, the affine map $\alpha_2^{\Pi_{\bar{W}^{3d}}}$ whose existence is verified in Eq. (20) is full dimensional and fulfills the relation:

$$\alpha_2^{\Pi_{\bar{W}^{3d}}}(\Omega_2) = \Omega_3.$$

The middle level player’s strategy given by $\gamma^2 := (\alpha_2^{\Pi_{\bar{W}^{3d}}})^{-1}$ restricts the optimization of the follower to set of points determined by the map γ^1 and γ^2 . Since the graph of γ^2 intersects the level set \bar{W}^{3d} only at (u^{2d}, u^{3d}) , the minimum of $\bar{J}_3(\gamma^{1*}, \gamma^2, u^3)$ is obtained at u^{3d} . □

The supporting hyperplanes $\Pi_{W^{2d}}$ and $\Pi_{\bar{W}^{3d}}$ through the point (u^{1d}, u^{2d}, u^{3d}) have common points. Therefore, there exists a set containing points that satisfy both Eqs. (10) and (19). Define this set Ψ as

$$\Psi = \{(u^{1d}, u^2, u^3) \in \Omega_1 \times \Omega_2 \times \Omega_3 | (u^{1d}, u^2, u^3) \text{ satisfies Eqs. (10) and (19)}\} \tag{21}$$

Proposition 10 *Suppose that the conditions of Theorems 4 and 5 are satisfied so that the existence of γ^1 and γ^2 are guaranteed. Then for $(u^{1d}, u^2, u^3) \in \Psi$, we have*

$$J_3(u^{1d}, u^2, u^3) \geq J_3(u^{1d}, u^{2d}, u^{3d}), \tag{22}$$

where equality can only be achieved at the desired equilibrium point (u^{1d}, u^{2d}, u^{3d}) .

Proof Follows a similar argument as in the proof of Proposition 8. □

Lemma 3 *Suppose that the conditions of Theorems 4 and 5 are satisfied. Then for optimal reverse Stackelberg strategies γ^{1*} and γ^{2*} , the sets Φ and Ψ are equal.*

Proof Suppose that $(u^1, u^2, u^3) \in \Phi$, then Eq. (10) follows automatically and Eq. (19) follows by Eq. (4). Therefore, $\Phi \subseteq \Psi$.

Conversely if $(u^{1d}, u^2, u^3) \in \Psi$, then we have Eq. (19) which is equal to Eq. (13) for fixed γ^{1*} . And Eq. (10) follows automatically. Hence, $\Psi \subseteq \Phi$. □

The optimal reverse Stackelberg strategies γ^{1*} and γ^{2*} lie on the supporting hyperplanes $\Pi_{W^{2d}}$ and $\Pi_{\bar{W}^{3d}}$ respectively. Therefore, results of Propositions 8, 10 and Lemma 3 holds for the set of points determined by the affine reverse Stackelberg strategies γ^{1*} and γ^{2*} .

We conclude the discussion about the existence of optimal reverse Stackelberg strategies γ^{1*} and γ^{2*} with the following theorem.

Theorem 11 *Suppose that W^{2d} and \bar{W}^{3d} are convex and locally strictly convex at the point (u^{1d}, u^{2d}, u^{3d}) and (u^{2d}, u^{3d}) , respectively, $J_2(u^1, u^2, u^3)$ and $\bar{J}_3(u^2, u^3)$ are Fréchet differentiable at (u^{1d}, u^{2d}, u^{3d}) and (u^{2d}, u^{3d}) respectively and $\Omega_1 = \mathbb{R}^{m_1}$, $\Omega_2 = \mathbb{R}^{m_2}$, $\Omega_3 = \mathbb{R}^{m_3}$. If $\nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}) \neq 0$, $\nabla_{u^2} \bar{J}_3(u^{2d}, u^{3d}) \neq 0$, then there exist optimal reverse Stackelberg strategies γ^{1*} and γ^{2*} , that guarantee the achievement of the equilibrium solution (u^{1d}, u^{2d}, u^{3d}) .*

Proof The proof follows from Theorems 4 and 9. □

3.2 Nonconvex sublevel sets

In this Section, we consider nonconvex sublevel sets W^{2d} and W^{3d} . By taking the convex hulls of W^{2d} and W^{3d} , we develop conditions under which results for the convex case can be applied.

When the sublevel sets W^{2d} and W^{3d} are allowed to be nonconvex, the desired solution (u^{1d}, u^{2d}, u^{3d}) of the leader may not be at the boundary point of the sublevel sets. So, the requirement $\text{Graph}(\alpha_1) \cap W^{2d} = (u^{1d}, u^{2d}, u^{3d})$ fails to hold. Consequently, the results in Theorems 4 and 9 can not be applied directly. In the following propositions, we present conditions under which results of Theorems 4 and 9 can be applied to nonconvex sublevel sets W^{2d} and W^{3d} . In what follows, we denote the convex hull of a set A by $\text{conv}(A)$.

Proposition 12 *Let $\text{conv}(W^{2d})$ be locally strictly convex at (u^{1d}, u^{2d}, u^{3d}) and assume that $J_2(u^1, u^2, u^3)$ is Fréchet differentiable at (u^{1d}, u^{2d}, u^{3d}) and $\Omega_1 = \mathbb{R}^{m_1}$, $\Omega_2 = \mathbb{R}^{m_2}$, $\Omega_3 = \mathbb{R}^{m_3}$. If (u^{1d}, u^{2d}, u^{3d}) is an exposed point of $\text{conv}(W^{2d})$ and $\nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}) \neq 0$, then there exists leader's affine map $\gamma^1 : \Omega_2 \times \Omega_3 \rightarrow \Omega_1$ that realizes the desired equilibrium (u^{1d}, u^{2d}, u^{3d}) .*

Proof As (u^{1d}, u^{2d}, u^{3d}) is an exposed point of $\text{conv}(W^{2d})$, the existence of a hyperplane $\Pi_{\text{conv}(W^{2d})}$ that supports $\text{conv}(W^{2d})$ at (u^{1d}, u^{2d}, u^{3d}) follows by definition. Furthermore, we have $\Pi_{\text{conv}(W^{2d})}(u^{1d}, u^{2d}, u^{3d}) \cap \text{conv}(W^{2d}) = (u^{1d}, u^{2d}, u^{3d})$. Noting that the desired equilibrium point (u^{1d}, u^{2d}, u^{3d}) is in W^{2d} and $W^{2d} \subset \text{conv}(W^{2d})$, the hyperplane $\Pi_{\text{conv}(W^{2d})}(u^{1d}, u^{2d}, u^{3d})$ supports W^{2d} at (u^{1d}, u^{2d}, u^{3d}) as well. Define an affine map $\alpha_1^{\Pi_{\text{conv}(W^{2d})}} : \Omega_1 \rightarrow \Omega_2 \times \Omega_3$ that lies on the supporting hyperplane $\Pi_{\text{conv}(W^{2d})}(u^{1d}, u^{2d}, u^{3d})$. Now, we need to check whether this map satisfies $\alpha_1^{\Pi_{\text{conv}(W^{2d})}}(\Omega_1) = \Omega_2 \times \Omega_3$. But this follows from the proof of Theorem 1 replacing W^{2d} by $\text{conv}(W^{2d})$. \square

Proposition 13 *Suppose that $\text{conv}(\bar{W}^{3d})$ is locally strictly convex at (u^{1d}, u^{2d}, u^{3d}) , $\bar{J}_3(u^2, u^3)$ is differentiable at (u^{2d}, u^{3d}) and $\Omega_2 = \mathbb{R}^{m_2}$, $\Omega_3 = \mathbb{R}^{m_3}$. Under the announced leader's optimal reverse Stackelberg strategy γ^{1*} , if (u^{1d}, u^{2d}, u^{3d}) is an exposed point of $\text{conv}(W^{2d})$ and $\nabla_{u^2} \bar{J}_3(u^{2d}, u^{3d}) \neq 0$, then, there exists the middle level player's optimal strategy $\gamma^{2*} : \Omega_3 \rightarrow \Omega_2$ that satisfies Eq. (3).*

Proof Since (u^{1d}, u^{2d}, u^{3d}) is assumed to be an exposed point of $\text{conv}(\bar{W}^{3d})$, the existence of a supporting hyperplane $\Pi_{\bar{W}^{3d}}(u^{1d}, u^{2d}, u^{3d})$ that supports $\text{conv}(\bar{W}^{3d})$ at (u^{1d}, u^{2d}, u^{3d}) follows from its definition. Furthermore, we have

$$\Pi_{\text{conv}(\bar{W}^{3d})}(u^{1d}, u^{2d}, u^{3d}) \cap \text{conv}(\bar{W}^{3d}) = (u^{1d}, u^{2d}, u^{3d}).$$

Noting that the desired equilibrium (u^{1d}, u^{2d}, u^{3d}) is in \bar{W}^{3d} and $\bar{W}^{3d} \subset \text{conv}(\bar{W}^{3d})$, the hyperplane $\Pi_{\text{conv}(\bar{W}^{3d})}(u^{1d}, u^{2d}, u^{3d})$ supports \bar{W}^{3d} at (u^{1d}, u^{2d}, u^{3d}) as well.

Define an affine map $\alpha_2^{\Pi_{\text{conv}(\bar{W}^{3d})}} : \Omega_2 \rightarrow \Omega_3$ that lie on the supporting hyperplane $\Pi_{\text{conv}(\bar{W}^{3d})}(u^{1d}, u^{2d}, u^{3d})$. Now, we check whether this map satisfies $\alpha_2^{\Pi_{\text{conv}(\bar{W}^{3d})}}(\Omega_2) = \Omega_3$.

Again the assertion that $\alpha_2^{\Pi_{conv}(\bar{W}^{3d})}(\Omega_2) = \Omega_3$ follows from Theorem 9 replacing \bar{W}^{3d} by $conv(\bar{W}^{3d})$. □

Remark 4 The idea described above can be extended to any finite n -level reverse Stackelberg problem that has similar structure as in problems 1 – 5. In deed, once the leaders optimal affine mapping γ^{1*} is constructed, since affine strategy of the leader (1st-level player) preserves convexity of sublevel sets of the followers, the structure of the objective functions of the remaining 2nd, . . . , $(n - 1)$ th, n th level players remain unaffected. Next, the 2nd-level player in the hierarchy constructs reverse Stackelberg strategy to preserve his/her team optimal solution induced by the announced strategy of the leader. This process continues up to the $(n - 1)$ th-level player in the vertical hierarchy which constructs a reverse Stackelberg strategy to obtain the team optimal solution induced by the strategy of the $(n - 2)$ th-level player in the hierarchy. This task can be achieved by repeating a procedure used for finding optimal reverse Stackelberg strategy of the middle level player in trilevel games $(n - 2)$ number of times sequentially

4 Characterization of optimal reverse Stackelberg strategies

In the first part of this Section, we provide illustrative examples to support the foregoing existence results. Reverse Stackelberg strategies for the leader are constructed based on Fréchet derivatives of J_2 and J_3 at the desired solution point (u^{1d}, u^{2d}, u^{3d}) . The solution method which is also discussed in (Mizukami et al. 1989) is not all inclusive. It results in a single leader’s affine reverse Stackelberg strategy, where in reality a number of strategies can be constructed. In the second part of the Section, we develop a more general method to construct multiple reverse Stackelberg strategies for the leader.

4.1 A single solution case

In this Subsection an affine strategy $\gamma^1 : \Omega_2 \times \Omega_3 \rightarrow \Omega_1$ that yields the desired equilibrium point (u^{1d}, u^{2d}, u^{3d}) is characterized. Following the presentation in (Mizukami et al. 1989) we assume that the decision spaces $\Omega_1, \Omega_2, \Omega_3$ are unconstrained Hilbert spaces, J_2, J_3 are Fréchet differentiable and convex. The determination of leader’s function γ^1 in the finite dimensional case (with $\Omega_1 = \mathbb{R}^{m_1}, \Omega_2 = \mathbb{R}^{m_2}$ and $\Omega_3 = \mathbb{R}^{m_3}$), which is given by

$$u^1 := \gamma^1(u^2, u^3) = u^{1d} - Q_1(u^2 - u^{2d}) - Q_2(u^3 - u^{3d}), \tag{23}$$

satisfies Eq. (4) automatically, and is reduces to a computation of $m_1 \times m_2$ matrix Q_1 and $m_1 \times m_3$ matrix Q_2 . In the finite dimensional Hilbert space the linear operators Q_1 and Q_2 satisfy

$$Q_1^* \nabla_{u_1} J_2(u^{1d}, u^{2d}, u^{3d}) = \nabla_{u_2} J_2(u^{1d}, u^{2d}, u^{3d}), \tag{24}$$

$$Q_2^* \nabla_{u_1} J_2(u^{1d}, u^{2d}, u^{3d}) = \nabla_{u_3} J_2(u^{1d}, u^{2d}, u^{3d}), \tag{25}$$

where $Q_i^*, i = 1, 2$ denote the adjoint of $Q_i, i = 1, 2$. For finite dimensional spaces we use the fact that $Q_i^* = Q_i^T$.

In this case the leader’s affine reverse Stackelberg strategy is

$$\begin{aligned} \gamma^1(u^2, u^3) = & u^{1d} - \left(\frac{\nabla_{u_1} J_2(u^{1d}, u^{2d}, u^{3d})}{\langle \nabla_{u_1} J_2(u^{1d}, u^{2d}, u^{3d}), \nabla_{u_1} J_2(u^{1d}, u^{2d}, u^{3d}) \rangle} \right) \\ & \times (\langle \nabla_{u_2} J_2(u^{1d}, u^{2d}, u^{3d}), u^2 - u^{2d} \rangle) \\ & - \left(\frac{\nabla_{u_1} J_2(u^{1d}, u^{2d}, u^{3d})}{\langle \nabla_{u_1} J_2(u^{1d}, u^{2d}, u^{3d}), \nabla_{u_1} J_2(u^{1d}, u^{2d}, u^{3d}) \rangle} \right) \\ & \times (\langle \nabla_{u_3} J_2(u^{1d}, u^{2d}, u^{3d}), u^3 - u^{3d} \rangle). \end{aligned} \tag{26}$$

Substituting Eq. (23) into the supporting hyperplane in Eq. (13) and simplifying we get

$$\begin{aligned} & \langle \nabla_{u^2} J_3(u^{1d}, u^{2d}, u^{3d}) - Q_1^*(\nabla_{u^1} J_3(u^{1d}, u^{2d}, u^{3d})), u^2 - u^{2d} \rangle \\ & + \langle \nabla_{u^3} J_3(u^{1d}, u^{2d}, u^{3d}) - Q_2^*(\nabla_{u^1} J_3(u^{1d}, u^{2d}, u^{3d})), u^3 - u^{3d} \rangle = 0. \end{aligned} \tag{27}$$

The resulting Eq. (27) represents a hyperplane that is orthogonal to the product space $\Omega_2 \times \Omega_3$. As mentioned in (Mizukami et al. 1989), if

$$\nabla_{u^2} J_3(u^{1d}, u^{2d}, u^{3d}) - Q_1^*(\nabla_{u^1} J_3(u^{1d}, u^{2d}, u^{3d})) \neq 0,$$

then there exists a bounded linear operator Q_3^* that satisfies

$$\begin{aligned} & Q_3^*[\nabla_{u^2} J_3(u^{1d}, u^{2d}, u^{3d}) - Q_1^*(\nabla_{u^1} J_3(u^{1d}, u^{2d}, u^{3d}))] \\ & = [\nabla_{u^3} J_3(u^{1d}, u^{2d}, u^{3d}) - Q_2^*(\nabla_{u^1} J_3(u^{1d}, u^{2d}, u^{3d}))], \end{aligned} \tag{28}$$

such that an affine function

$$u^2 = \gamma^2(u^3) = u^{2d} - Q_3(u^3 - u^{3d}), \tag{29}$$

lies on the hyperplane determined by Eq. (27).

Denote expressions in Eq. (28) as

$$\begin{aligned} \bar{u}_{321} &= \nabla_{u^2} J_3(u^{1d}, u^{2d}, u^{3d}) - Q_1^*(\nabla_{u^1} J_3(u^{1d}, u^{2d}, u^{3d})), \\ \bar{u}_{331} &= \nabla_{u^3} J_3(u^{1d}, u^{2d}, u^{3d}) - Q_2^*(\nabla_{u^1} J_3(u^{1d}, u^{2d}, u^{3d})). \end{aligned}$$

Then Eq. (29) can be re-written as

$$\gamma^2(u^3) = u^{2d} - \frac{\bar{u}_{321}}{\langle \bar{u}_{321}, \bar{u}_{321} \rangle} \langle \bar{u}_{331}, u^3 - u^{3d} \rangle. \tag{30}$$

That means, given a trilevel game with differentiable cost functions of followers, the top level player can construct optimal reverse Stackelberg strategy γ^{1*} using the

expression given in Eq. (26). The strategy declared by the leader induces the middle level player to construct γ^{2*} using the expressions given in Eq. (30). These strategies are set up in such a way that the leader can achieve his/her desired equilibrium.

Example 2 For the problem in Example 1, we obtain the team optimal solution $(u^{1d}, u^{2d}, u^{3d}) = (2, 1, 3)$ by minimizing J_1 over \mathbb{R}^3 .

So, the leader’s optimal reverse Stackelberg strategy γ^{1*} that modifies follower’s objective functions so as to reach his/her desired equilibrium must satisfy

$$\begin{aligned} \operatorname{argmin}_{u^2, u^3} J_2(\gamma^{1*}(u^2, u^3), u^2, u^3) &= (1, 3), \\ \gamma^{1*}(1, 3) &= 2. \end{aligned}$$

Since J_2 is differentiable, $\nabla J_2(u^1, u^2, u^3) = (2(u^1 - 1), 2u^2, 2u^3)^\top$ and at the desired equilibrium point $(2, 1, 3)$, $\nabla J_2(2, 1, 3) = (2, 2, 6)^\top$. From Eqs. (24) and (25) at the team optimal solution $Q_1 = 1, Q_2 = 3$. In this case, the optimal reverse Stackelberg strategy of the leader is

$$\gamma^{1*} = -u^2 - 3u^3 + 12.$$

When the leader announces his/her optimal reverse Stackelberg strategy γ^{1*} we verify that Eqs. (4) and (5) are satisfied.

$$\begin{aligned} \operatorname{argmin}_{u^2, u^3} [(-u^2 - 3u^3 + 12) - 1]^2 + (u^2)^2 + (u^3)^2 &= (1, 3), \\ \gamma^{1*}(1, 3) &= 2. \end{aligned}$$

In reaction to announcement of γ^{1*} by the leader, the middle level player constructs γ^{2*} that satisfies Eqs. (2) and (3) by solving:

$$\begin{aligned} \operatorname{argmin}_{u^3} J_3(\gamma^{1*}(u^2, u^3), \gamma^{2*}(u^3), u^3) &= 3, \\ \gamma^{2*}(3) &= 1. \end{aligned}$$

Since $\nabla J_3(u^1, u^2, u^3) = (2u^1, 2(u^2 - 2), 2u^3)^\top$, the Fréchet derivative at the desired equilibrium is $\nabla J_3(2, 1, 3) = (4, -2, 6)^\top$. It follows from Eq. (28) that $Q_3 = 1$ and from Eq. (26) that optimal strategy of the middle level player is

$$\gamma^{2*} = -u^3 + 4.$$

Example 3 Consider the general quadratic trilevel dynamic game with the objective functions for each $i = 1, 2, 3$ is given by

$$J_i(u^1, u^2, u^3) = \sum_{j,k=1,2,3} \langle u^j, A^i_{jk} u^k \rangle + \sum_{k=1,2,3} \langle u^k, l^i_k \rangle, \tag{31}$$

where $k \geq j$ and A_{ik}^j are linear bounded operators, A_{ii}^j are strongly positive, $A_{ki}^i = 0$ for $i \neq k$, $l_k^i \in \Omega_k$ are known and the conditions on A_{jk}^i makes the Hessian matrix non negative definite for all players.

For $i = 1, 2, 3$, the cost function of each of the players becomes

$$J_i = \langle u^1, A_{11}^i u^1 \rangle + \langle u^1, A_{12}^i u^2 \rangle + \langle u^2, A_{22}^i u^2 \rangle + \langle u^1, A_{13}^i u^3 \rangle + \langle u^2, A_{23}^i u^3 \rangle + \langle u^3, A_{33}^i u^3 \rangle + \langle u^1, l_1^i \rangle + \langle u^2, l_2^i \rangle + \langle u^3, l_3^i \rangle.$$

The Fréchet derivatives of J_i with respect to u^i , $i = 1, 2, 3$ are

$$\nabla_{u^1} J_i(u^1, u^2, u^3) = 2A_{11}^i u^1 + A_{12}^i u^2 + A_{13}^i u^3 + l_1^i, \tag{32}$$

$$\nabla_{u^2} J_i(u^1, u^2, u^3) = A_{12}^i u^1 + 2A_{22}^i u^2 + A_{23}^i u^3 + l_2^i, \tag{33}$$

$$\nabla_{u^3} J_i(u^1, u^2, u^3) = A_{13}^i u^1 + A_{23}^i u^2 + 2A_{33}^i u^3 + l_3^i. \tag{34}$$

The desired equilibrium for the leader (u^{1d}, u^{2d}, u^{3d}) is found as a solution of the system

$$\begin{pmatrix} 2A_{11}^i & A_{12}^i & A_{13}^i \\ A_{12}^i & 2A_{22}^i & A_{23}^i \\ A_{13}^i & A_{23}^i & 2A_{33}^i \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} = \begin{pmatrix} -l_1^i \\ -l_2^i \\ -l_3^i \end{pmatrix}, \tag{35}$$

for each $i = 1, 2, 3$.

Since J_1 is convex, minimization of J_1 with respect to (u^1, u^2, u^3) provides the desired team optimal solution, say (u^{1d}, u^{2d}, u^{3d}) . The sets W^{2d} and W^{3d} , given by

$$W^{2d} = \{(u^1, u^2, u^3) \in \Omega_1 \times \Omega_2 \times \Omega_3 : J_2(u^1, u^2, u^3) \leq J_2(u^{1d}, u^{2d}, u^{3d})\}$$

$$W^{3d} = \{(u^1, u^2, u^3) \in \Omega_1 \times \Omega_2 \times \Omega_3 : J_3(u^1, u^2, u^3) \leq J_3(u^{1d}, u^{2d}, u^{3d})\}$$

are convex (as each is a sublevel set of a convex function).

Supporting hyperplanes of W^{2d} and W^{3d} at (u^{1d}, u^{2d}, u^{3d}) are given by

$$\langle \nabla_{u^1} J_i(u^{1d}, u^{2d}, u^{3d}), u^1 - u^{1d} \rangle + \langle \nabla_{u^2} J_i(u^{1d}, u^{2d}, u^{3d}), u^2 - u^{2d} \rangle + \langle \nabla_{u^3} J_i(u^{1d}, u^{2d}, u^{3d}), u^3 - u^{3d} \rangle = 0,$$

where $i = 2, 3$. To construct optimal reverse Stackelberg strategy of the leader, gradients of J_2 and J_3 at the team optimal solution (u^{1d}, u^{2d}, u^{3d}) are

$$\nabla_{u^1} J_i(u^{1d}, u^{2d}, u^{3d}) = 2A_{1d}^i u^{1d} + A_{12}^i u^{2d} + A_{13}^i u^{3d} + l_1^i,$$

$$\nabla_{u^2} J_i(u^{1d}, u^{2d}, u^{3d}) = A_{12}^i u^{1d} + 2A_{22}^i u^{2d} + A_{23}^i u^{3d} + l_2^i,$$

$$\nabla_{u^3} J_i(u^{1d}, u^{2d}, u^{3d}) = A_{13}^i u^{1d} + A_{23}^i u^{2d} + 2A_{33}^i u^{3d} + l_3^i.$$

Then according to Eqs. (23)–(26), the optimal reverse Stackelberg strategy of the leader is

$$\gamma^{1*}(u^2, u^3) = u^{1d} - Q_1(u^2 - u^{2d}) - Q_3(u^3 - u^{3d}), \tag{36}$$

where

$$Q_1 = \frac{\langle 2A_{11}^1 u^{1d} + A_{12}^1 u^{2d} + A_{13}^1 u^{3d} + l_1^1, A_{12}^1 u^{1d} + 2A_{22}^1 u^{2d} + A_{23}^1 u^{3d} + l_1^1 \rangle}{\langle 2A_{11}^1 u^{1d} + A_{12}^1 u^{2d} + A_{13}^1 u^{3d} + l_1^1, 2A_{11}^1 u^{1d} + A_{12}^1 u^{2d} + A_{13}^1 u^{3d} + l_1^1 \rangle},$$

$$Q_2 = \frac{\langle 2A_{11}^1 u^{1d} + A_{12}^1 u^{2d} + A_{13}^1 u^{3d} + l_1^1, A_{13}^1 u^{1d} + A_{23}^1 u^{2d} + 2A_{33}^1 u^{3d} + l_1^1 \rangle}{\langle 2A_{11}^1 u^{1d} + A_{12}^1 u^{2d} + A_{13}^1 u^{3d} + l_1^1, 2A_{11}^1 u^{1d} + A_{12}^1 u^{2d} + A_{13}^1 u^{3d} + l_1^1 \rangle}.$$

Substituting the fixed leader’s optimal affine reverse Stackelberg strategy $u^1 = \gamma^{1*}(u^2, u^3)$ in $J_3(u^1, u^2, u^3)$ the supporting hyperplane for

$$\bar{W}^{3d} = \{(u^2, u^3) \in \Omega_2 \times \Omega_3 \mid \bar{J}_3(u^2, u^3) \leq \bar{J}_3(u^{2d}, u^{3d})\},$$

is given by

$$\langle \nabla_{u^2} \bar{J}_3(u^{2d}, u^{3d}), u^2 - u^{2d} \rangle + \langle \nabla_{u^3} \bar{J}_3(u^{2d}, u^{3d}), u^3 - u^{3d} \rangle = 0.$$

The middle level player reacts to the leader’s reverse Stackelberg strategy Eq. (36) by constructing γ^{2*} based on Eq. (30) as

$$\gamma^{2*}(u^3) = u^{2d} - \frac{\bar{u}_{321}}{\langle \bar{u}_{321}, \bar{u}_{321} \rangle} (\bar{u}_{331}, u^3 - u^{3d}),$$

provided that $\bar{u}_{332} \neq 0$ and

$$\bar{u}_{321} = A_{12}^3 u^{1d} + 2A_{22}^3 u^{2d} + A_{23}^3 u^{3d} + l_2^3 - Q_1(2A_{1d}^3 u^{1d} + A_{12}^3 u^{2d} + A_{13}^3 u^{3d} + l_1^3),$$

$$\bar{u}_{331} = A_{13}^3 u^{1d} + A_{23}^3 u^{2d} + 2A_{33}^3 u^{3d} + l_3^3 - Q_2(A_{13}^3 u^{1d} + A_{23}^3 u^{2d} + 2A_{33}^3 u^{3d} + l_3^3).$$

4.2 General characterization

In Sect. 4.1, we briefly discussed construction of affine reverse Stackelberg strategies for trilevel games based on the Fréchet derivatives of objective functions of followers. However, as illustrated in the following trilevel game, it is possible that the leader can have an unlimited number of strategies to achieve his/her team optimal solution. So, in this section we present a method to construct multiple optimal affine reverse Stackelberg strategies of the leader.

Example 4 Consider a three person trilevel game with decision spaces $u^1 \in \mathbb{R}^2, u^2 \in \mathbb{R}$ and $u^3 \in \mathbb{R}$ characterized by

$$\begin{aligned} J_1(u^1, u^2, u^3) &= (u_1^1)^2 + (u_2^1)^2 + (u^2)^2 + (u^3)^2 + 5u_1^1 + 3u_2^1 + u^2 + u^3, \\ J_2(u^1, u^2, u^3) &= (u_1^1)^2 + (u_2^1)^2 + (u^2)^2 + (u^3)^2 + 3u^2, \\ J_3(u^1, u^2, u^3) &= (u_1^1)^2 + (u_2^1)^2 + (u^2)^2 + (u^3)^2, \end{aligned} \tag{37}$$

where $u^1 = \begin{pmatrix} u_1^1 \\ u_2^1 \end{pmatrix}$.

In this example the leader’s desired equilibrium is $(\frac{-5}{2}, \frac{-3}{2}, \frac{-1}{2}, \frac{-1}{2})$ and the reverse Stackelberg strategy of the leader is

$$\gamma^1 = \begin{pmatrix} \frac{-5}{2} \\ \frac{-3}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{5}(2 - 3t_1) \\ t_1 \end{pmatrix} (u^2 + \frac{1}{2}) + \begin{pmatrix} -\frac{1}{5}(1 + 3t_2) \\ t_2 \end{pmatrix} (u^3 + \frac{1}{2}), \tag{38}$$

where t_1 and t_2 are free parameters. Therefore, there are infinitely many reverse Stackelberg strategies for this problem. Note that, the characterization of strategies that is used in Sect. 4.1 only yields a single solution if we apply it also to this example.

Here, results in (Groot et al. 2016) for bi-level games were extended to derive general characterizations of leader’s affine reverse Stackelberg strategies for trilevel games. We propose the following leader’s function.

$$\gamma^1 = u^{1d} - R_1 s_1 - R'_1 s_2 \tag{39}$$

where R_1, R'_1, s_1 and s_2 are matrices of appropriate dimensions satisfying

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} u^{1d} \\ u^{2d} \end{pmatrix} + \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} s_1, \tag{40}$$

$$\begin{pmatrix} u^1 \\ u^3 \end{pmatrix} = \begin{pmatrix} u^{1d} \\ u^{3d} \end{pmatrix} + \begin{pmatrix} R'_1 \\ R_3 \end{pmatrix} s_2, \tag{41}$$

provided that R_2 and R_3 are nonsingular. Solving for s_1 and s_2 from Eqs. (40) and (41), Eq. (39) can be written as

$$u^1 := \gamma^1(u^2, u^3) = u^{1d} - R_1 R_2^{-1}(u^2 - u^{2d}) - R'_1 R_3^{-1}(u^3 - u^{3d}). \tag{42}$$

In order to explore the general characterization of γ^1 in Eqs. (39)–(42), set $R^1 = [R_1^\top \ R_2^\top], R^1 \in \mathbb{R}^{(m_1+m_2) \times m_2}$ and $R^2 = [R'_1{}^\top \ R_3^\top], R^2 \in \mathbb{R}^{(m_1+m_3) \times m_3}$.

Lemma 4 *If the leader function γ^1 given in Eq. (42) is optimal for $R^1 = [R_1^\top \ R_2^\top]$ and $R^2 = [R'_1{}^\top \ R_3^\top]$, the following conditions hold.*

1. *The matrices R^1 and R^2 satisfy*

$$[\nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}) \ \nabla_{u^2} J_2(u^{1d}, u^{2d}, u^{3d})]^\top R^1 = 0, \tag{43}$$

$$[\nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}) \ \nabla_{u^3} J_2(u^{1d}, u^{2d}, u^{3d})]^\top R^2 = 0. \tag{44}$$

2. The columns of R_2 and R_3 should form bases for Ω_2 and Ω_3 respectively, i.e., R_2 and R_3 are of full rank matrices of size m_2 and m_3 respectively.

Proof 1. Substituting the proposed strategy Eq. (42) for the leader in Eq. (10) we have

$$\begin{aligned} &\langle \nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}), R_1 R_2^{-1}(u^2 - u^{2d}) + R_1' R_3^{-1}(u^3 - u^{3d}) \rangle \\ &\quad + \langle \nabla_{u^2} J_2(u^{1d}, u^{2d}, u^{3d}), u^2 - u^{2d} \rangle \\ &\quad + \langle \nabla_{u^3} J_2(u^{1d}, u^{2d}, u^{3d}), u^3 - u^{3d} \rangle = 0. \end{aligned}$$

Then it follows that

$$\begin{aligned} &\langle \nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}), R_1 R_2^{-1}(u^2 - u^{2d}) \rangle \\ &\quad + \langle \nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}), R_1' R_3^{-1}(u^3 - u^{3d}) \rangle \\ &\quad + \langle \nabla_{u^2} J_2(u^{1d}, u^{2d}, u^{3d}), u^2 - u^{2d} \rangle \\ &\quad + \langle \nabla_{u^3} J_2(u^{1d}, u^{2d}, u^{3d}), u^3 - u^{3d} \rangle = 0. \end{aligned}$$

Thus

$$\begin{aligned} &\langle (R_1 R_2^{-1})^\top \nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}), (u^2 - u^{2d}) \rangle \\ &\quad + \langle (R_1' R_3^{-1})^\top \nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}), (u^3 - u^{3d}) \rangle \\ &\quad + \langle \nabla_{u^2} J_2(u^{1d}, u^{2d}, u^{3d}), u^2 - u^{2d} \rangle \\ &\quad + \langle \nabla_{u^3} J_2(u^{1d}, u^{2d}, u^{3d}), u^3 - u^{3d} \rangle \\ &= \langle (R_1 R_2^{-1})^\top \nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}) \\ &\quad + \nabla_{u^2} J_2(u^{1d}, u^{2d}, u^{3d}), (u^2 - u^{2d}) \rangle \\ &\quad + \langle (R_1' R_3^{-1})^\top \nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}) \\ &\quad + \nabla_{u^3} J_2(u^{1d}, u^{2d}, u^{3d}), (u^3 - u^{3d}) \rangle = 0 \end{aligned}$$

from which the first statement of the lemma follows.

2. According to Theorem 4, for an affine leader’s function γ^1 of the form Eq. (41) to be admissible, R_2 and R_3 must be basis of Ω_2 and Ω_3 respectively. \square

Lemma 5 *If there exists an optimal affine mapping γ^1 characterized by Eq. (42), one can select $R_2 = I_{m_2}$ and $R_3 = I_{m_3}$, without any loss of generality.*

Proof The selection $R_2 = I_{m_2}$ and $R_3 = I_{m_3}$ satisfies full dimensionality requirement mentioned in Lemma 4. Now, we prove under a selection $R_2 = I_{m_2}$ and $R_3 = I_{m_3}$ whether the affine strategy given in Eq. (42) lies on the supporting hyperplane determined by Eq. (10). This simply follows from the proof of Lemma 4 by substituting I_{m_2} and I_{m_3} for R_2 and R_3 respectively. \square

When $R_2 = I_{m_2}$ and $R_3 = I_{m_3}$, the following equations can be derived from Eqs. (43) and (44).

$$\nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d})R_1 = -\nabla_{u^2} J_2(u^{1d}, u^{2d}, u^{3d})^\top I_{m_2}, \tag{45}$$

$$\nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d})R'_1 = -\nabla_{u^3} J_2(u^{1d}, u^{2d}, u^{3d})^\top I_{m_3}. \tag{46}$$

We conclude the general characterization of leader’s reverse Stackelberg strategy by the following theorem that follows from Theorem 4, Lemmas 3 and 4. Moreover, equations that help to find matrices R_1 and R'_1 satisfying Eqs. (45) and (46) under fixed basis matrices $R_2 = I_{m_2}$ and $R_3 = I_{m_3}$ are given.

Theorem 14 *Let $\Omega_1 = \mathbb{R}^{m_1}, \Omega_2 = \mathbb{R}^{m_2}, \Omega_3 = \mathbb{R}^{m_3}$ and conditions of Theorem 4 are satisfied. Then an affine leader’s function of the form Eq. (42) exists where the matrices R_1 and R'_1 belong to affine spaces of the form*

$$\begin{aligned} \mathcal{R}_1 &= \{R_1 = R_{11}^0 + B_N t_1\}, \\ \mathcal{R}'_1 &= \{R'_1 = R_{12}^0 + B_N t_2\}, \end{aligned} \tag{47}$$

where R_{11}^0 and R_{12}^0 are particular solutions of (45) and (46) respectively and $B_N \in \text{Null}\{\nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d})\}$.

Proof Since $\nabla_{u^1} J_2(u^{1d}, u^{2d}, u^{3d}) \neq 0$, Eqs. (45) and (46) can be solved as a system of equalities. Moreover, for each zero element $\nabla_{u^{1,j}} J_2(u^{1d}, u^{2d}, u^{3d}), j = 1, \dots, m_2$ the corresponding entry R_{1j}^0 is free. Therefore, Eq. (47) is a possible solution of Eqs. (45) and (46). \square

Now, given the leader’s optimal reverse Stackelberg strategy γ^{1*} , the objective functions $J_2(\gamma^{1*}, u^2, u^3)$ and $J_3(\gamma^{1*}, u^2, u^3)$ become a function of u^2 and u^3 only, and hence we denote the later by $\tilde{J}_3(u^2, u^3)$ for which a corresponding result for two level problem works.

Example 5 In Example 4 the leader’s reverse Stackelberg strategy depends on the free parameters t_1 and t_2 . For example, fixing $t_1 = t_2 = 0$ in Eq. (38), the leader’s reverse Stackelberg strategy becomes

$$\begin{pmatrix} u^1_1 \\ u^1_2 \end{pmatrix} = \begin{pmatrix} \gamma^1_1 \\ \gamma^1_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{5}(2u^2 - u^3 - 12) \\ \frac{-3}{2} \end{pmatrix}. \tag{48}$$

Substituting Eq. (48) for J_2 and J_3 in Eq. (37) we get

$$\begin{aligned} J_2(\gamma^{1*}, u^2, u^3) &= \left(\frac{1}{5}(2u^2 - u^3 - 12)\right)^2 + \left(\frac{-3}{2}\right)^2 + (u^2)^2 + (u^3)^2 + 3u^2, \\ J_3(\gamma^{1*}, u^2, u^3) &= \left(\frac{1}{5}(2u^2 - u^3 - 12)\right)^2 + \left(\frac{-3}{2}\right)^2 + (u^2)^2 + (u^3)^2. \end{aligned} \tag{49}$$

Now Eq. (49) is a two level problem for which the middle level player's reverse Stackelberg strategy is

$$\gamma^{2*} = -\frac{1}{2}.$$

Remark 5 The choice of $t_1 = t_2 = 0$ for the free parameters, in Example 5 above, is random. However, the leader can choose the values of t_1 and t_2 by considering secondary optimization problem like optimizing the loss incurred, to the leader and punishment to the followers, in case followers deviate from the desired solution.

5 Constrained decision space

In this section, we present a constrained trilevel reverse Stackelberg games. Suppose that $\Omega_1 \times \Omega_2 \times \Omega_3 \subsetneq \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3}$ and $\Omega_1, \Omega_2, \Omega_3$ are compact. In the presence of constraints, the leader solves constrained nonlinear optimization problem to determine his/her desired equilibrium (u^{1d}, u^{2d}, u^{3d}) . To utilize the geometric existence theorems presented in Sects. 3 and 4 for a constrained decision space it should be verified that supporting hyperplanes $\Pi_{W^{2d}}(u^{1d}, u^{2d}, u^{3d})$ and $\Pi_{W^{3d}}(u^{1d}, u^{2d}, u^{3d})$ containing affine strategies γ^1 and γ^2 belonged to the constrained decision space $\Omega_1 \times \Omega_2 \times \Omega_3$. Therefore, existence conditions of optimal affine reverse Stackelberg strategies presented in Sects. 3 and 4 for unconstrained version of the trilevel game provides only necessary conditions in case of a constrained game.

In (Groot et al. 2016), it is reported that proving conditions for existence of leader's reverse Stackelberg strategy for bilevel games in constrained decision space is a challenging task. In the same article, the authors were able to filter feasible optimal strategies of a constrained game among infinitely many reverse Stackelberg strategies in unconstrained case of the bilevel problem. We shall extend this concept to the trilevel (and hence to the general n -level) case

Consider a trilevel problem with linear constraints:

$$\begin{aligned} & \min_{u^1 \in \Omega_1} J_1(u^1, u^2, u^3), \\ & \min_{u^2 \in \Omega_2} J_2(u^1, u^2, u^3), \\ & \min_{u^3 \in \Omega_3} J_3(u^1, u^2, u^3), \end{aligned} \tag{50}$$

subject to:

$$A^1 u^1 + A^2 u^2 + A^3 u^3 \leq b^1, A^i \in \mathbb{R}^{k \times m_i}, b^1 \in \mathbb{R}^k, \text{ for } i = 1, 2, 3.$$

Here, since the constraints are linear the feasible set is convex.

The first task towards obtaining reverse Stackelberg strategy of the leader is the determination of a desired equilibrium. The leader searches for a team solution of a

constrained nonlinear optimization problem

$$\begin{aligned} & \min_{(u^1, u^2, u^3) \in \Omega_1 \times \Omega_2 \times \Omega_3} J_1(u^1, u^2, u^3), \\ & \text{subject to:} \\ & A^1 u^1 + A^2 u^2 + A^3 u^3 \leq b^1, A^i \in \mathbb{R}^{k \times m_i}, b^1 \in \mathbb{R}^k, \text{ for } i = 1, 2, 3., \end{aligned} \tag{51}$$

which gives the desired equilibrium (u^{1d}, u^{2d}, u^{3d}) under appropriate conditions.

The next task will be to construct feasible affine reverse Stackelberg strategies that satisfy Eqs.(2)–(5). The strategy γ^1 of the leader is feasible in a constrained decision space if its domain is $\Omega_2 \times \Omega_3$ and $\gamma^1(\Omega_2 \times \Omega_3)$ is a subset of Ω_1 . Similarly, the corresponding strategy γ^2 is feasible in a constrained decision space if its domain is Ω_3 and $\gamma^2(\Omega_3)$ is a subset of Ω_2 . The optimal affine strategies constructed for unconstrained version of the problem fulfill all these conditions except the condition for feasibility. Since the desired equilibrium Eq. (51) is in the constrained decision space, strategies that are optimal for unconstrained game are also optimal for a constrained game provided that they are feasible. Therefore, feasible optimal solutions for a constrained game can be obtained from set of affine optimal solutions of the unconstrained version of the problem.

Given a set of optimal affine solutions for unconstrained trilevel reverse Stackelberg game, we verify solvability of a constrained version of the game by determining feasible strategies that are admissible. Thus, leader’s optimal strategy γ^1 in the unconstrained game is also admissible in a constrained game if it satisfies the condition that for all $(u^2, u^3) \in \Omega_2 \times \Omega_3$ there exists $u^1 \in \Omega_1$ such that $u^1 = \gamma^1(u^2, u^3)$. Therefore, optimal affine functions discussed in Sect. 4 for unconstrained game satisfying $\gamma^1(\Omega_2 \times \Omega_3) \subseteq \Omega_1$ are admissible for a constrained version of the game provided that γ^2 exists. Furthermore, among set of optimal affine reverse Stackelberg strategies γ^2 for unconstrained game, those that satisfy $\gamma^2(\Omega_3) \subseteq \Omega_2$ are also optimal in a constrained game.

In what follows, we present some cases for which the existence of strategies can be verified based on our characterization of multiple optimal strategies in Sect. 4.

- If the leader’s decision space is constrained by the linear constraints

$$A^1 u^1 \leq b^1, A^1 \in \mathbb{R}^{k \times m_1}, b^1 \in \mathbb{R}^k, \tag{52}$$

the set of optimal solutions of unconstrained version of the problem can be reduced by adding a constraint

$$A^1 [u^{1d} - R_1(u^2 - u^{2d}) - R'_1(u^3 - u^{3d})] \leq b^1, A^1 \in \mathbb{R}^{k \times m_1}, b^1 \in \mathbb{R}^k,$$

to the expression in Eq. (47).

- Suppose $\Omega_3 \subset \mathbb{R}^{m_3}$ is constrained but $\Omega_1 = \mathbb{R}^{m_1}, \Omega_2 = \mathbb{R}^{m_2}$ are unconstrained. In this case if $\gamma^1(\Omega_2 \times \Omega_3) \subseteq \Omega_1$ and $\gamma^2(\Omega_3) \subseteq \Omega_2$, all affine optimal strategies characterized in Sect. 4 for unconstrained game are also feasible for constrained version of the game.

- If decision spaces of all the players are constrained by linear constraints, the feasible region is a convex polyhedron in which affine mappings γ^1 and γ^2 preserve convexity. However, one must verify the satisfaction of the conditions that $\gamma^1(\Omega_2 \times \Omega_3) \subseteq \Omega_1$ and $\gamma^2(\Omega_3) \subseteq \Omega_2$.

6 Conclusion

In a static multilevel Stackelberg game, the decision process is sequential from the top 1st-level player to 2nd, . . . , $(n - 1)^{\text{th}}$, n^{th} -level player (the bottom follower). Players at each level optimize their own objective functions which is affected by actions of decision makers at other levels. Reverse Stackelberg strategy of the leader is a mapping from the followers' decision space to the leader's decision space enabling him/her to achieve a desired equilibrium. The results on the existence, and construction of such strategies in the current literature applies only to problems where objective functions of followers are strictly convex and provide only a single strategy to the leader.

This article formulates existence conditions of affine reverse Stackelberg strategy in multilevel static game where the sublevel sets of objective functions of followers' at the desired equilibrium are required to be connected. Moreover, the construction of leader's multiple optimal reverse Stackelberg strategies is developed. The attainment of more than one optimal strategy provides an opportunity to consider secondary optimization criteria as well as to solve a constrained game.

However, more research is still needed to address similar type problems with multiple decision entities that are acting according to Nash game at each level of hierarchy. The development of numerical solution techniques is also another area to be worked on. One can also consider development of nonlinear reverse Stackelberg strategies which are more stable as compared to the affine strategies.

Another important direction for future work is existence and construction of reverse Stackelberg strategy for multilevel differential game, in which case the state evolves according to a differential equation and the performance criteria is integral. In continuous time setting, the strategies of the leader and the middle level player should satisfy causality constraint to be admissible as opposed to the static case.

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