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ORIGINAL ARTICLE

Marginality and convexity in partition function form games

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Abstract

In this paper an order on the set of embedded coalitions is studied in detail. This allows us to define new notions of superaddivity and convexity of games in partition function form which are compared to other proposals in the literature. The main results are two characterizations of convexity. The first one uses non-decreasing contributions to coalitions of increasing size and can thus be considered parallel to the classic result for cooperative games without externalities. The second one is based on the standard convexity of associated games without externalities that we define using a partition of the player set. Using the later result, we can conclude that some of the generalizations of the Shapley value to games in partition function form lie within the cores of specific classic games when the original game is convex.

Keywords Game theory · Partition function · Partial order · Marginality · Convexity

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1 Introduction

Lately, the study of cooperative games with coalitional externalities has attracted the attention of some important researchers (see Maskin 2016). The basic ingredients of such games are coalitions of players embedded in a partition of the set of all players. Then, a game with coalitional externalities, or game in partition function form (Thrall and Lucas 1963), is a real valued function on the set of all such embedded coalitions with the convention that the value attached to the empty embedded coalition is zero. To date, most of the efforts have been devoted to the extensions of solution concepts like the core or the Shapley value from classic games, or games in characteristic function form, to games with externalities. In this paper we consider an order on the set of embedded coalitions implicitly used by Bolger (1990), Hu and Yang (2010), and Skibski et al. (2018) for different purposes. The study of the structure of this partially ordered set allows us to derive some interesting game theoretical results.

When dealing with embedded coalitions one has to consider two types of objects, namely subsets and partitions. Even if both objects have well known ordering relations that give rise to the Boolean algebra and the lattice of partitions, respectively, it is not clear how embedded coalitions should be ordered. Indeed, several approaches appeared in the literature. Myerson (1977) defined a value on partition function form games based on an order over the set of embedded coalitions. This order was studied in Grabisch (2010). Alonso-Meijide et al. (2019) proposed a value for partition function form games based on an order on the set of embedded coalitions that has been introduced and studied in Alonso-Meijide et al. (2017). In the present paper, we consider another partial order that was first used by Bolger (1990) and, more recently, by Hu and Yang (2010) and Skibski et al. (2018), but that has not been formally defined and analyzed yet. The three partial orders agree in considering that if one embedded coalition precedes another, then the coalition of the first should be contained in the coalition of the second. The difference lies in how they deal with the partition side. According to Grabisch (2010), the first partition should be finer than the second one while Alonso-Meijide et al. (2017) consider that it should be coarser.[1] In this paper, we consider that the second partition equals the first one after removing the agents in the second coalition. Then, this partial order can be considered a compromise between the other two. However, it turns out that these posets are quite different because the new one does not have a lattice structure. This order also provides an interpretation of how the grand coalition is formed. Initially, there is a partition of agents. Then, an agent leaves its group to form an active coalition[2]. Afterwards, another agent leaves its block to join the active coalition, and so on till all agents have joined in the active coalition. Our results are grounded in some properties of the structure of the poset of embedded coalitions. We characterize the maximal lower bounds and minimal upper bounds of two embedded coalitions, whenever they exist. We show that it is a graded poset and count the number of elements at a given level. Finally, we provide an isomorphism between the chains in our structure and the chains in the Boolean lattice. Based on this isomorphism, we count the total number of chains and describe its Möbius function.

[1] The precise definitions will soon follow.

[2] We call active coalition to the coalition whose worth is being evaluated.

The previous study leads us to introduce new notions of superadditivity and convexity. We first relate them to alternative definitions that can be found in the literature. Our superadditivity implies the one proposed by Maskin (2003) but, in contrast to the later, it is a sufficient condition for the efficiency of the grand coalition. Our notion of convexity is stronger than the ones introduced by Hafalir (2007) and Abe (2016), and thus implies them. Our main results are two characterizations of convex games in partition function form. The first is parallel to a well-known result for classic games. A game is convex if and only if the contributions of players to embedded coalitions of increasing size, with respect to our partial order, are non-decreasing. The second characterization uses the standard convexity of certain games in characteristic function form associated with the game in partition function form. For each partition of the set of agents we build a classic game assuming that an arbitrary coalition expects the complement to be organized according to this partition. This association is very similar to the one Bloch and van den Nouweland (2014) do using their exogenous expectation formation rules.

The second characterization result allows us to obtain additional results about some average Shapley values as defined in Macho-Stadler et al. (2007). More precisely, when the game with externalities is convex, the value defined in Pham Do and Norde (2007)[3] belongs to the core of the associated game without externalities obtained from the partition of singletons. Similar results are obtained for the values introduced by McQuillin (2009) and Hu and Yang (2010).

The rest of the paper is organized as follows. Section 2 is devoted to preliminaries and presents other partial orders on embedded coalitions introduced in the literature. In Sect. 3 we present and study the structural properties of the new poset. Section 4 presents the results on superadditivity and convexity. Section 5 features some implications of our results for some average Shapley values.

2 Preliminaries

Let $(\mathcal{A}, \leq)$ be a partially ordered finite set (in short, a poset). Let $A \subseteq \mathcal{A}$ and $x \in \mathcal{A}$. We say that x is a *lower bound* of A if and only if $x \leq y$, for every $y \in A$.[4] We say that x is an *upper bound* of A if and only if $y \leq x$, for every $y \in A$. We say that x is a *minimal* (*maximal*) element of A if there is no $y \in A\backslash\{x\}$ such that $y \leq x$ $(x \leq y)$. We say that x is the *supremum* of A, $sup(A)$, if x is an upper bound of A and $x \leq y$ for every upper bound y of A. We say that x is the *infimum* of A, $inf(A)$, if x is a lower bound of A and $y \leq x$ for every lower bound y of A. If there is an element $\hat{1} \in \mathcal{A}$ such that $y \leq \hat{1}$ for every $y \in \mathcal{A}$, we say that $\hat{1}$ is the *top* element of $\mathcal{A}$. Similarly, the *bottom* element $\hat{0}$ is an element of $\mathcal{A}$ such that $\hat{0} \leq y$ for every $y \in \mathcal{A}$. We say that x is *covered* by $y \in \mathcal{A}\backslash\{x\}$ or y *covers* x if $x \leq y$ and there is no $z \in \mathcal{A}\backslash\{x, y\}$ such that $x \leq z \leq y$. A *(irreducible) chain* $\mathcal{C}$ is a totally ordered subset of $\mathcal{A}$, $\mathcal{C} = \{x_0, x_1, \ldots, x_k\}$ such that x_{l+1} covers x_l, for every $l = 0, \ldots, k-1$. $(\mathcal{A}, \leq)$ satisfies the *Jordan-Dedekind condition* if all chains between two elements have the

[3] Named as externality-free value in de Clippel and Serrano (2008).

[4] We denote: $x = y$ if $x \leq y$ and $y \leq x$; $x < y$ if $x \leq y$, but $x \neq y$.

same length. A *rank function* is a function $\rho : \mathcal{A} \longrightarrow \mathbb{N}$ such that $\rho(y) = \rho(x) + 1$ for every $x, y \in \mathcal{A}$ where y covers x. If $x, y \in \mathcal{A}$ and $x \leq y$, we denote by $[x, y]_{\mathcal{A}}$ the set of elements $z \in \mathcal{A}$ such that $x \leq z \leq y$. If no confusion arises, we may simply write $[x, y]$.

Let $(\mathcal{A}_1, \leq_1), (\mathcal{A}_2, \leq_2)$ be two posets. An *isomorphism* ϕ is a bijective map $\phi : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ such that $\phi(x) \leq_2 \phi(y)$ if and only if $x \leq_1 y$, for every $x, y \in \mathcal{A}_1$.

A *finite lattice* is a poset $(\mathcal{A}, \leq)$ such that $sup(A) \in \mathcal{A}$ and $inf(A) \in \mathcal{A}$, for every $A \subseteq \mathcal{A}$. Apart from the Boolean lattice of a finite set, denoted by $(\mathcal{B}(N), \subseteq)$, we need to recall some notions related to the partition lattice. Let N be a finite set with cardinality $|N| = n$, and $\Pi(N)$ be the family of partitions of the set N. Let $S \subseteq N$ and $P \in \Pi(N)$. We denote by $|P|$ the number of elements in P, and by P_{-S} the partition of $N \backslash S$ given by $P_{-S} = \{T \backslash S \ : \ T \in P\}$. Sometimes we will refer to each element in P as a block. We denote by $\lceil S \rceil$ the partition of S given by $\{S\}$ and by $\lfloor S \rfloor$ the partition given by $\{\{i\} : i \in S\}$. Let $1 \leq k \leq n$. The total number of partitions of N with k elements is the *Stirling number of second kind*, that we denote by $S_{n,k}$. The *Bell number* of n is the total number of partitions of a finite set N with $|N| = n$, i.e., $B_n = \sum_{k=1}^{n} S_{n,k}$. A well-known partial order on $\Pi(N)$ is the following. Let $P, Q \in \Pi(N)$.

$$P \preceq Q \text{ if and only if for every } S \in P \text{ there is some } T \in Q \text{ such that } S \subseteq T.$$

We denote this poset by $(\Pi(N), \preceq)$. It is well-known that $(\Pi(N), \preceq)$ is a lattice. If $P, Q \in \Pi(N)$, we denote by $P \bigwedge Q$ the infimum of P and Q and by $P \bigvee Q$ the supremum of P and Q, according to the partial order $\preceq$.

An *embedded coalition* of N is a pair $(S; P)$ with $\emptyset \neq S \subseteq N$ and P a partition of $N \backslash S$, i.e., $P \in \Pi(N \backslash S)$. If we have the embedded coalition $(T; Q)$ with $T = N$ then, $Q = \{\emptyset\}$ and we take $|Q| = 0$. For simplicity we denote by $(S; N \backslash S)$ the embedded coalition $(S; \{N \backslash S\})$, for every $S \subseteq N$. The set of embedded coalitions is denoted by EC^N. Several partial orders can be considered on the family of embedded coalitions of a finite set N, EC^N. One of them has been studied in Grabisch (2010) and it is defined as follows: for every $(S; P), (T; Q) \in EC^N$,

$$(S; P) \sqsubseteq_0 (T; Q) \text{ if and only if } S \subseteq T \text{ and } P \cup \lceil S \rceil \preceq Q \cup \lceil T \rceil.$$

Alonso-Meijide et al. (2017) study a different partial order on EC^N defined as

$$(S; P) \sqsubseteq_1 (T; Q) \text{ if and only if } S \subseteq T \text{ and } Q \preceq P_{-T} \tag{1}$$

for every $(S; P), (T; Q) \in EC^N$. Both partial orders consider a fictitious bottom element $\hat{0}$. Instead of that, here we consider empty embedded coalitions given by the family $\mathcal{F}_0(N) = \{(\emptyset; P) : P \in \Pi(N)\}$. We denote by $\mathcal{F}_N = EC^N \cup \mathcal{F}_0(N)$. Given $(S; P) \in \mathcal{F}_N$, we will sometimes name the coalition S as the active coalition.

Let N be a finite set. A *partition function form game* (in short, *a game*) with player set N is a function $v : \mathcal{F}_N \longrightarrow \mathbb{R}$ such that $v(\emptyset; P) = 0$, for every $(\emptyset; P) \in \mathcal{F}_0(N)$. The family of all partition function form games with player set N will be denoted by $\mathcal{G}_N$. A *cooperative game in characteristic function form* (in short, *a classic game*)

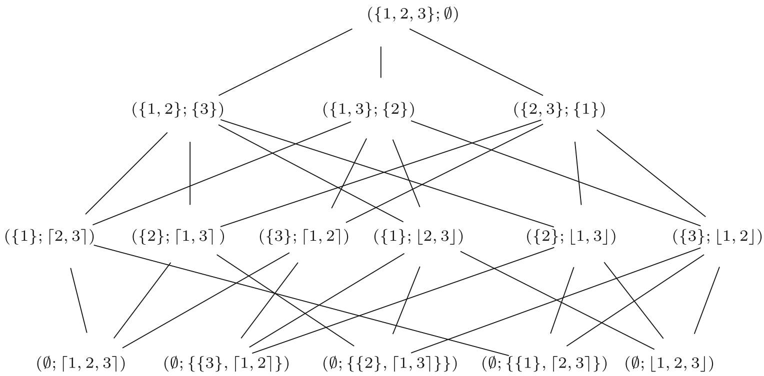

Fig. 1 The partial order $\sqsubseteq$ on $\mathcal{F}_N$ with $|N| = 3$

is a partition function form game $v \in \mathcal{G}_N$ such that $v(S; P) = v(S; Q)$ for every $(S; P), (S; Q) \in EC^N$. That is the worth of a coalition does not depend on how the remaining players are organized and we can simply write $v(S)$.

3 A new poset on $\mathcal{F}_N$

In this section we formulate and study the partial order that has been implicitly used by Bolger (1990), Hu and Yang (2010), and Skibski et al. (2018), among others, to describe intuitive properties of a value for games in partition function form. Let N be a finite set. First, we formalize this partial order defined on $\mathcal{F}_N$.

Definition 3.1 The *inclusion* in $\mathcal{F}_N$, denoted by $\sqsubseteq$, is defined as follows:

$$(S; P) \sqsubseteq (T; Q) \text{ if and only if } S \subseteq T \text{ and } Q = P_{-T} \tag{2}$$

for every $(S; P), (T; Q) \in \mathcal{F}_N$.[5]

Equation (2) implies a twofold relationship between the embedded coalitions $(S; P)$ and $(T; Q)$: on one hand all agents in S belong also to T and, on the other hand, the partitions $P \in \Pi(N \backslash S)$ and $Q \in \Pi(N \backslash T)$ must satisfy that agents outside T are organized in the same way in both P and Q.

This binary relation defines a partial order on $\mathcal{F}_N$. The next example illustrates the differences among the three partial orders defined above, $\sqsubseteq_0$, $\sqsubseteq_1$, and $\sqsubseteq$.

Example 3.1 Let us take $N = \{1, 2, 3\}$. Figure 1 depicts the Hasse diagram corresponding to $(\mathcal{F}_N, \sqsubseteq)$. Notice that $(\{1\}; \lceil 2, 3 \rceil)$ and $(\{1\}; \lfloor 2, 3 \rfloor)$ are not comparable according to $\sqsubseteq$. Nevertheless, $(\{1\}; \lceil 2, 3 \rceil) \sqsubseteq_1 (\{1\}; \lfloor 2, 3 \rfloor)$ and $(\{1\}; \lfloor 2, 3 \rfloor) \sqsubseteq_0 (\{1\}; \lceil 2, 3 \rceil)$.

Figure 1 also illustrates the fact that there is no bottom element in $(\mathcal{F}_N, \sqsubseteq)$, but there is a top element $(N; \emptyset)$.

[5] The *strict inclusion* in $\mathcal{F}._N$ is given by
$(S; P) \sqsubset (T; Q)$ if and only if $S \subset T$ and $Q = P_{-T}$, for every $(S; P), (T; Q) \in \mathcal{F}_N$.

In the rest of the section we study some properties of this poset that can be considered technical results that will be used in the definitions and results of the rest of the paper. We postpone their proofs to the “Appendix”.

We start studying (maximal) lower bounds of two embedded coalitions. Note that this will allow us to identify what the intersection of two embedded coalitions means. The role of these maximal lower bounds is parallel to the intersection in the Boolean lattice. They represent the closest embedded coalitions from which we can reach the initial ones. Example 3.2 shows that two elements in $(\mathcal{F}_N, \sqsubseteq)$ may not have a lower bound or that the lower bound may not be unique.

Example 3.2 Let $N = \{1, 2, 3\}$. Take $(S; P), (T; Q) \in \mathcal{F}_N$ defined as $(S; P) = (\{1\}; \lceil 2, 3\rceil)$, $(T; Q) = (\{1\}; \lfloor 2, 3\rfloor)$. It is easy to check using Fig. 1 that there is no $(L; M) \in \mathcal{F}_N$ such that $(L; M) \sqsubseteq (S; P)$ and $(L; M) \sqsubseteq (T; Q)$. Then, the set of lower bounds for $\{(S; P), (T; Q)\}$ is empty.

Take $(R; H), (L; Z) \in \mathcal{F}_N$ given by $(R; H) = (\{1\}; \lceil 2, 3\rceil)$, $(L; Z) = (\{2, 3\}; \{1\})$. Then, from Fig. 1 it is easy to see that the set of maximal lower bounds of $(R; H)$ and $(L; Z)$ is

$$\{(\emptyset; \lceil 1, 2, 3\rceil), (\emptyset; \{\{1\}, \lceil 2, 3\rceil\})\}.$$

Next, we characterize the set of maximal lower bounds of two elements of $\mathcal{F}_N$. This result will be critical to define what a convex game in partition function form means.

Proposition 3.1 *Let $(S; P), (T; Q) \in \mathcal{F}_N$ with $(S; P) \neq (T; Q)$.*

1. *If $Q_{-S} \neq P_{-T}$, a lower bound of $(S; P)$ and $(T; Q)$ does not exist.*
2. *If $P_{-T} = Q_{-S}$, then the maximal lower bounds of $(S; P)$ and $(T; Q)$ are embedded coalitions of the type $(S \cap T; H)$ where H is such that $R \in H$ if and only if*

$$R \in M \quad \text{or} \quad R = \lceil S' \cup T' \rceil$$

where $M = (P \cup \lfloor S\backslash T\rfloor) \bigvee (Q \cup \lfloor T\backslash S\rfloor)$ and $S', T' \in M$ such that $S' \subseteq S\backslash T$ and $T' \subseteq T\backslash S$.

This result shows that only those embedded coalitions that have the agents outside the active coalition organized in the same way can be reached from a common embedded coalition through the poset. Even when this holds, this embedded coalition might not be unique. The next example illustrates this fact.

Example 3.3 Let $N = \{1, 2, 3, 4\}$. Take $(S; P) = (\{1, 2\}; \lfloor 3, 4\rfloor)$ and $(T; Q) = (\{3, 4\}; \lfloor 1, 2\rfloor)$. It is clear that $S \cap T = \emptyset$ and $P_{-T} = Q_{-S}$. According to Proposition 3.1, $M = \lfloor 1, 2, 3, 4\rfloor$ and the set of lower bounds consists of

$$(\emptyset; \lfloor 1, 2, 3, 4\rfloor),\ (\emptyset; \lceil 1, 3\rceil, \lfloor 2, 4\rfloor),\ (\emptyset; \lceil 1, 4\rceil, \lfloor 2, 3\rfloor)$$
$$(\emptyset; \lceil 2, 3\rceil, \lfloor 1, 4\rfloor),\ (\emptyset; \lceil 2, 4\rceil, \lfloor 1, 3\rfloor),\ (\emptyset; \lceil 1, 3\rceil, \lceil 2, 4\rceil),\ (\emptyset; \lceil 2, 3\rceil, \lceil 1, 4\rceil).$$

Notice that all of them are maximal lower bounds.

In what follows, given a pair of embedded coalitions $\{(S;P),(T;Q)\} \subseteq \mathcal{F}_N$, we denote by $(S;P) \wedge (T;Q)$ its set of maximal lower bounds.

Next, we study (minimal) upper bounds of a set of two elements in $\mathcal{F}_N$. That is, the closest embedded coalitions that can be reached from these two elements through the poset. The example below illustrates the fact that the minimal upper bound might not be unique.

Example 3.4 Take $N = \{1,2,3\}$, $(S;P) = (\{1\};\lfloor 2,3 \rfloor)$ and $(T;Q) = (\{1\};\lceil 2,3 \rceil)$. From Fig. 1, it is derived that both $(\{1,3\};\{2\})$ and $(\{1,2\};\{3\})$ are minimal upper bounds of $(S;P)$ and $(T;Q)$.

The result below specifies the (minimal) upper bounds of a set of two embedded coalitions. It is of paramount importance for the definitions of superadditivity and convexity in Sect. 4 since it generalizes the union of coalitions in the Boolean lattice to embedded coalitions. These minimal upper bounds are the closest ones that can be reached from those embedded coalitions.

Proposition 3.2 *Let $(S;P),(T;Q) \in \mathcal{F}_N$ with $(S;P) \neq (T;Q)$.*

1. *$(R;M')$ is an upper bound of $\{(S;P),(T;Q)\}$ if and only if $R = S \cup T \cup L$ with $L \subseteq N \backslash (S \cup T)$ and $M' = P_{-(T \cup L)} = Q_{-(S \cup L)}$.*
2. *$(R;M')$ is a minimal upper bound of $\{(S;P),(T;Q)\}$ if and only if $R = S \cup T \cup L$ with $L \subseteq N \backslash (S \cup T)$, $M' = P_{-(T \cup L)} = Q_{-(S \cup L)} = M_{-L}$ with $M = P_{-T} \bigwedge Q_{-S}$, and for every $L' \subseteq N \backslash (S \cup T)$ with $P_{-(T \cup L')} = Q_{-(S \cup L')} = M_{-L'}$ it holds $L \subseteq L'$ or $L \cap L' = \emptyset$.*

Proposition 3.2 characterizes the embedded coalitions that can be reached from two arbitrary embedded coalitions. The active part of such embedded coalitions must guarantee that the structure of the remaining agents is the same according to P and Q. There are cases where this cannot be achieved by just taking the union of S and T, as Example 3.5 illustrates.

Example 3.5 Let $N = \{1,2,3,4,5\}$. Take $(S;P) = (\{1\};\{\lceil 2,3,4 \rceil,\{5\}\})$ and $(T;Q) = (\{3\};\{\lceil 1,2 \rceil,\lceil 4,5 \rceil\})$. Then, $S \cup T = \{1,3\}$, $M = P_{-T} \bigwedge Q_{-S} = \lfloor 2,4,5 \rfloor$ and

$$\{L \subseteq N \backslash (S \cup T) : P_{-(T \cup L)} = Q_{-(S \cup L)} = M_{-L}\} = \{\{4\},\{4,5\},\{2,4\},\{2,5\},\{2,4,5\}\}.$$

As a consequence of that, the upper bounds of $\{(S;P),(T;Q)\}$ are

$$(\{1,3,4\};\lfloor 2,5 \rfloor), (\{1,3,4,5\};\{2\}), (\{1,2,3,4\};\{5\}), (\{1,2,3,5\};\{4\}), (N;\emptyset).$$

Thus, the set of minimal upper bounds is given by

$$\{(\{1,3,4\};\lfloor 2,5 \rfloor), (\{1,2,3,5\};\{4\})\}.$$

Nevertheless, we can identify pairs of embedded coalitions that have a unique minimal upper bound.

Corollary 3.1 *Let* $(S; P), (T; Q) \in \mathcal{F}_N$ *such that* $P_{-T} = Q_{-S}$. *Then,* $\{(S; P), (T; Q)\}$ *has a unique minimal upper bound, the supremum of* $\{(S; P), (T; Q)\}$*, given by* $sup\{(S; P), (T; Q)\} = (S \cup T; P_{-T})$.

In what follows, given a pair of embedded coalitions $\{(S; P), (T; Q)\} \subseteq \mathcal{F}_N$, we denote by $(S; P) \vee (T; Q)$ its set of minimal upper bounds.

The next result describes how many embedded coalitions are covered by a particular element in $\mathcal{F}_N$. Note that this may be important to define Shapley like values identifying the links in the poset as contributions.

Proposition 3.3 *Let* $(S; P) \in \mathcal{F}_N$.

1. *If* $(S; P) \neq (N; \emptyset)$*, then the number of elements in* $\mathcal{F}_N$ *that cover* $(S; P)$ *is* $|N \backslash S|$.
2. *If* $S \neq \emptyset$*, then the number of elements in* $\mathcal{F}_N$ *covered by* $(S; P)$ *is* $|S|(|P| + 1)$.

The result below identifies an isomorphism between the chains in our poset and in the Boolean lattice that will allow us to determine the coefficients of an arbitrary game with externalities in the basis of unanimity games with respect to the inclusion relationship presented in Definition 3.1.

Proposition 3.4 *Consider the partially ordered set* $(\mathcal{F}_N, \sqsubseteq)$.

1. *For every* $(S; P),\ (T; Q) \in \mathcal{F}_N$ *such that* $(S; P) \sqsubseteq (T; Q)$*, then* $[(S; P), (T; Q)]$ *is isomorphic to* $[S, T]_{\mathcal{B}(N)}$.
2. $(\mathcal{F}_N, \sqsubseteq)$ *is graded.*
3. *Let* $0 \leq k \leq n$. *Then, there are* $\binom{n}{k} B_{n-k}$ *elements of* $\mathcal{F}_N$ *of rank k with* B_{n-k} *the Bell number of* $n - k$.

From the above result, $(\mathcal{F}_N, \sqsubseteq)$ has the Jordan-Dedekind property and the length of any maximal chain is n. Next we obtain some results related to the number of chains that again could play a role to introduce new Shapley like solutions.

Proposition 3.5 *Let* $P \in \Pi(N)$*,* $(\emptyset; P), (T; Q) \in \mathcal{F}_N$ *with* $(\emptyset; P) \sqsubseteq (T; Q)$.

1. *The number of chains in* $[(\{i\}; P_{-\{i\}}), (T; Q)]$ *is* $(|T| - 1)!$*, for every* $i \in T$.
2. *The number of chains in* $[(\emptyset; P), (T; Q)]$ *is* $|T|!$.
3. *The total number of chains in* $(\mathcal{F}_N, \sqsubseteq)$ *is* $|N|!B_n$*, being* B_n *the Bell number of* n.

To conclude the Section we will use the isomorphism presented in Proposition 3.4 to characterize the Möbius function of $(\mathcal{F}_N, \sqsubseteq)$. Next, we recall the definition of the Möbius function of a finite poset. Let $(\mathcal{A}, \leq)$ be a finite poset. The *Möbius function* of $(\mathcal{A}, \leq)$, μ, is given by

$$\mu(x, y) = \begin{cases} 1 & \text{if } x = y \\ -\sum_{x \leq z < y} \mu(x, z) = -\sum_{x < z \leq y} \mu(z, y) & \text{if } x < y \end{cases}$$

for every $x, y \in \mathcal{A}$ with $x \leq y$. The Möbius function of $(\mathcal{B}(N), \subseteq)$ is given by $\hat{\mu}(S, T) = (-1)^{|T|-|S|}$, for every $S \subseteq T \subseteq N$. Using Proposition 3.4 we obtain

$$\mu((S; P), (T; Q)) = \begin{cases} (-1)^{|T|-|S|} & \text{if } (S; P) \sqsubseteq (T; Q) \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Additionally, we can express any partition function form game v as a linear combination of certain unanimity games as follows. Let $(S; P) \in EC^N$. The *unanimity game* of the embedded coalition $(S; P)$ associated with the ordering $\sqsubseteq$ is defined by

$$u_{(S;P)}(T; Q) = \begin{cases} 1 & \text{if } (S; P) \sqsubseteq (T; Q) \\ 0 & \text{otherwise} \end{cases}$$

for every $(T; Q) \in \mathcal{F}_N$. The family of unanimity games $\mathcal{U} = \left\{u_{(S;P)} : (S; P) \in EC^N\right\}$ is a basis of the vector space $\mathcal{G}^N$. Then,

$$v = \sum_{(S;P)\in EC^N} \delta_{(S;P)} u_{(S;P)}$$

for every partition function form game v. Using the Möbius function characterized in Eq. (3), we can obtain an explicit expression of the coefficients of a game in the basis $\mathcal{U}$.

Proposition 3.6 *Let v be a partition function form game and $(S; P) \in EC^N$. Then,*

$$\delta_{(S;P)} = \sum_{(T;Q)\sqsubseteq(S;P)} (-1)^{|S|-|T|} v(T; Q).$$

4 Convex games with externalities

Superadditivity and convexity are two well-known and interesting properties for games in characteristic function. Nevertheless, their generalization to games in partition function form is not straightforward and one can find different proposals in the literature. Thanks to the poset structure of $(\mathcal{F}_N, \sqsubseteq)$ studied in the previous section it is quite natural to introduce what it means for a game in partition function form to be superadditive or convex. For instance, to define superadditivity we can just replace the union of bare coalitions by the supremum of embedded coalitions whenever it exists.

Definition 4.1 Let $v \in \mathcal{G}_N$. We say that v is *superadditive* if and only if

$$v(S \cup T; P_{-T}) \geq v(S; P) + v(T; Q) \tag{4}$$

for every $(S; P), (T; Q) \in \mathcal{F}_N$ such that $S \cap T = \emptyset$ and $P_{-T} = Q_{-S}$.

Recall that for every $(S; P), (T; Q) \in \mathcal{F}_N$ such that $P_{-T} = Q_{-S}$ there are maximal lower bound. Moreover, if $S \cap T = \emptyset$, then all of them are of the type $(\emptyset; H)$ for some $H \in \Pi(N)$ (see Proposition 3.1). Besides, there is $sup\{(S; P), (T; Q)\} = (S \cup T; P_{-T})$ (see Corollary 3.1). Even though we compare the worths of embedded coalitions with different partitions, we require the organization of agents outside $S \cup T$ to be the same. Next example shows a partition function form game that is superadditive but not a classic game.

Example 4.1 Let us take $N = \{1, 2, 3\}$. We consider $v \in \mathcal{G}_N$ as follows:

$$\begin{aligned} &v(N; \emptyset) = 7,\ v(\{1\}; \lceil 2, 3 \rceil) = 1,\ v(\{1\}; \lfloor 2, 3 \rfloor) = 0, \\ &v(\{i\}; P_{-\{i\}}) = 2,\ \text{for every } i \in N \backslash \{1\},\ P \in \Pi(N),\ \text{and} \\ &v(\{j, k\}; \{i\}) = 4,\ \text{for every } j, k \in N \backslash \{i\},\ j \neq k, i \in N. \end{aligned}$$

Maskin (2003) (see also Hafalir 2007) consider the following definition of superadditivity, that we call *M-superadditivity*. A game $v \in \mathcal{G}_N$ is *M-superadditive*[6] if and only if

$$v(S \cup T; P) \geq v(S; P \cup \lceil T \rceil) + v(T; P \cup \lceil S \rceil) \tag{5}$$

for every $S, T \subseteq N$ such that $S \cap T = \emptyset$ and $P \in \Pi(N \backslash (S \cup T))$.

Notice that $(S; P \cup \lceil T \rceil)$ and $(T; P \cup \lceil S \rceil)$ in Eq. (5) satisfy the conditions of Definition 4.1. In case of an M-superadditive game, only comparisons between embedded coalitions with the same partition are taken into account. Then, if a partition function form is superadditive in our sense, it is also M-superadditive. But both notions are not equivalent. We see this by revisiting Example 1 in Hafalir (2007).

Example 4.2 (Hafalir 2007) Let $N = \{1, 2, 3\}$ and $v \in \mathcal{G}_N$ defined by

$$\begin{aligned} &v(N; \emptyset) = 11, \\ &v(\{i\}; \lfloor j, k \rfloor) = 4,\ \text{for every } j, k \in N \backslash \{i\},\ j \neq k, i \in N, \\ &v(\{j, k\}; \{i\}) = 9,\ \text{for every } j, k \in N \backslash \{i\},\ j \neq k, i \in N,\ \text{and} \\ &v(\{i\}; \lceil j, k \rceil) = 1,\ \text{for every } j, k \in N \backslash \{i\},\ j \neq k, i \in N. \end{aligned}$$

This game is M-superadditive, but it is not superadditive according to Definition 4.1. Take, for instance, the embedded coalitions $(S; P) = (\{1\}; \lfloor 2, 3 \rfloor)$ and $(T; Q) = (\{2, 3\}; \{1\})$. Clearly, $\{1\} \cap \{2, 3\} = \emptyset$ and $P_{-T} = Q_{-S}$. Besides,

$$v(S \cup T; \emptyset) = v(N; \emptyset) = 11 < 4 + 9 = v(S; P) + v(T; Q)$$

An interesting property for games in partition function form is *efficiency*. A game $v \in \mathcal{G}_N$ is efficient if for every $P \in \Pi(N)$,

$$\sum_{S \in P} v(S; P_{-S}) \leq v(N; \emptyset).$$

In general, an M-superadditive game is not efficient (see Example 4.2). Our notion of superadditivity guarantees efficiency.

Proposition 4.1 *Let $v \in \mathcal{G}_N$ be a superadditive game. Then, v is efficient.*

Proof Let $v \in \mathcal{G}_N$ be a superadditive game. Let $P = \{S_1, \ldots, S_p\} \in \Pi(N)$. Notice that $\cup_{k=1}^{p} S_k = N$ and $S_k \cap S_l = \emptyset$ for every $k, l \in \{1, \ldots, p\}$, $k \neq l$. If $p = 1$, the

[6] Or *superadditive* in Maskin's sense.

result immediately follows. Let us assume that $p > 1$. Using that v is superadditive, we have

$$\sum_{k=1}^{p} v(S_k; P_{-S_k}) \leq v(S_1 \cup S_2; P_{-(S_1 \cup S_2)}) + \sum_{k=3}^{p} v(S_k; P_{-S_k}) \leq \dots$$
$$\leq v(\cup_{k=1}^{p} S_k; \emptyset) = v(N; \emptyset).$$

□

The convexity or supermodularity property of a function on a poset is related to the value of the function on the infimum and supremum of two elements in the poset. In the literature there are several extensions of the classic convexity property to games in partition function form. For instance, Hafalir (2007) provides the following definition of convexity, that we named *H-convexity*. A game $v \in \mathcal{G}_N$ is *H-convex* if and only if

$$v(S \cup T; P) + v(S \cap T; P \cup \lceil T \backslash S \rceil \cup \lceil S \backslash T \rceil)$$
$$\geq v(S; P \cup \lceil T \backslash S \rceil) + v(T; P \cup \lceil S \backslash T \rceil)$$

for every $S, T \subseteq N$ and $P \in \Pi(N \backslash (S \cup T))$. Notice that for every $S, T \subseteq N$ and $P \in \Pi(N \backslash (S \cup T))$, we have $(P \cup \lceil T \backslash S \rceil)_{-T} = P = (P \cup \lceil S \backslash T \rceil)_{-S}$. Moreover,

$$(P \cup \lceil T \backslash S \rceil \cup \lfloor S \backslash T \rfloor) \bigvee (P \cup \lceil S \backslash T \rceil \cup \lfloor T \backslash S \rfloor) = P \cup \lceil S \backslash T \rceil \cup \lceil T \backslash S \rceil.$$

Then, the set of maximal lower bounds of $(S; P \cup \lceil T \backslash S \rceil)$ and $(T; P \cup \lceil S \backslash T \rceil)$, using Proposition 3.1, is given by

$$\{(S \cap T; P \cup \lceil S \backslash T \rceil \cup \lceil T \backslash S \rceil), (S \cap T; P \cup \lceil (S \cup T) \backslash (S \cap T) \rceil)\}.$$

In the definition of an H-convex game, only one of the maximal lower bounds is used. Convexity can be interpreted as a relation between the sum of values of two elements with the sum of values of the closest common starting point (maximal lower bound) and the closest common end point (minimal upper bound). With this idea in mind, we extend the notion of convexity to the framework of partition function form games using the poset $(\mathcal{F}_N, \sqsubseteq)$.

Definition 4.2 Let $v \in \mathcal{G}_N$. We say that v is *convex* if and only if

$$v(S \cup T; P_{-T}) + v(S \cap T; M') \geq v(S; P) + v(T; Q) \tag{6}$$

for every $(S; P), (T; Q) \in \mathcal{F}_N$ with $P_{-T} = Q_{-S}$ and $(S \cap T; M')$ a maximal lower bound of $\{(S; P),\ (T; Q)\}$.

If v is convex according to Definition 4.2, then, for every $(S; P), (T; Q) \in \mathcal{F}_N$ with $P_{-T} = Q_{-S}$

$$v(S \cup T; P_{-T}) + \frac{1}{|(S; P) \wedge (T; Q)|} \sum_{(S \cap T; M') \in (S; P) \wedge (T; Q)} v(S \cap T; M') \geq v(S; P) + v(T; Q)$$

where $|(S; P) \wedge (T; Q)|$ denotes the number of maximal lower bounds of $(S; P)$ and $(T; Q)$. Finally, it is enough to check Inequality (6) for an embedded coalition in

$$\arg\min_{(S \cap T; M') \in (S; P) \wedge (T; Q)} v\left(S \cap T; M'\right).$$

It is clear that if $v \in \mathcal{G}_N$ is convex, then it is also superadditive. Besides, if a game is convex according to Definition 4.2, it is also H-convex, while the reverse implication does not hold in general as Example 4.3 shows.

Example 4.3 Let $N = \{1, 2, 3\}$ and $v \in \mathcal{G}_N$ defined as follows:

$$v(\{i\}; \lceil j, k \rceil) = 1, \ v(\{i\}; \lfloor j, k \rfloor) = 2, \ v(\{i, j\}; \{k\}) = 4, \ v(N; \emptyset) = 6.$$

Clearly, this game is superadditive according to Definition 4.1 and H-convex. Nevertheless, it is not convex according to Definition 4.2. For instance, take $(S; P) = (\{1, 3\}; \{2\})$ and $(T; Q) = (\{1, 2\}; \{3\})$. The set of maximal lower bounds is $\{(\{1\}; \lfloor 2, 3 \rfloor), (\{1\}; \lceil 2, 3 \rceil)\}$. The notion of H-convexity only checks Eq. (6) for the embedded coalition $(\{1\}; \lfloor 2, 3 \rfloor)$, while in our definition we use both lower bounds. Taking $(\{1\}; \lceil 2, 3 \rceil)$ we have

$$v(N; \emptyset) + v(\{1\}; \lceil 2, 3 \rceil) = 6 + 1 < 4 + 4 = v(\{1, 3\}; \{2\}) + v(\{1, 2\}; \{3\}).$$

Thus, this game is not convex according to Definition 4.2.

If we consider $w \in \mathcal{G}_N$ defined as $w(S; P) = v(S; P)$ if $(S; P) \neq (N; \emptyset)$ and $w(N; \emptyset) = 10$, we obtain a convex game according to Definition 4.2.

Next, we present a characterization of convexity that is parallel to a well known result for classic games. Roughly speaking, the agent's contribution to an active coalition grows as we move from it along a chain to the top element of the poset.

Theorem 4.1 *Let $v \in \mathcal{G}_N$. The game v is* convex *if and only if for every $i \in N$ and $(S; P) \sqsubseteq (T; Q) \sqsubseteq (N \backslash \{i\}; \{i\})$ we have*

$$v\left(T \cup \{i\}; Q_{-\{i\}}\right) - v(T; Q) \geq v\left(S \cup \{i\}; P_{-\{i\}}\right) - v(S; P). \tag{7}$$

Proof Let $v \in \mathcal{G}_N$ be a convex game. Take $i \in N$ and $(S; P) \sqsubseteq (T; Q) \sqsubseteq (N \backslash \{i\}; \{i\})$. The elements $\left(S \cup \{i\}; P_{-\{i\}}\right)$ and $(T; Q)$ satisfy the conditions in Definition 4.2 because $P_{-(T \cup \{i\})} = Q_{-\{i\}}$. This holds because $(S; P) \sqsubseteq (T; Q)$ implies $S \subseteq T$

and $P_{-T} = Q$. Using Corollary 3.1, the supremum of $\left(S \cup \{i\}; P_{-\{i\}}\right)$ and $(T; Q)$ exists and it is given by $\left(T \cup \{i\}; Q_{-\{i\}}\right) = \left(T \cup \{i\}; P_{-(T\cup\{i\})}\right)$. Notice that $(S; P)$ is a maximal lower bound of $\left\{\left(S \cup \{i\}; P_{-\{i\}}\right), (T; Q)\right\}$. Applying Inequality (6) to $\left(S \cup \{i\}; P_{-\{i\}}\right)$, $(T; Q)$, and $(S; P)$, we get

$$v\left(T \cup \{i\}; Q_{-\{i\}}\right) + v(S; P) \geq v\left(S \cup \{i\}; P_{-\{i\}}\right) + v(T; Q),$$

and Inequality (7) holds.

Now let us assume that v satisfies Inequality (7) for every $i \in N$ and $(S; P) \sqsubseteq (T; Q) \sqsubseteq (N\backslash\{i\}; \{i\})$. Let us take $(S; P), (T; Q) \in \mathcal{F}_N$ such that $P_{-T} = Q_{-S}$, $M = (P \cup \lfloor S\backslash T \rfloor) \bigvee (Q \cup \lfloor T\backslash S \rfloor)$, and $(S \cap T; M')$ be a maximal lower bound of $\{(S; P), (T; Q)\}$. We proceed by induction on $|S\backslash T|$. If $|S\backslash T| = 0$ then, $S \subseteq T$. Since $P_{-T} = Q$, we have $(S; P) \sqsubseteq (T; Q)$. Then, $\left(S \cap T; M'\right) = (S; P)$, $(S \cup T; P_{-T}) = (T; Q)$, and Inequality (6) holds. Let us assume that $|S\backslash T| = 1$ and $S\backslash T = \{i\}$. Then, $S\cap T = S\backslash\{i\}$. A maximal lower bound of $(S; P)$ and $(T; Q)$ is $\left(S\backslash\{i\}; M'\right)$ satisfying $R \in M'$ if and only if

$$R \in M \quad \text{or} \quad R = \lceil S' \cup T' \rceil \tag{8}$$

with $S', T' \in M$, $S' \subseteq S\backslash T$, and $T' \subseteq T\backslash S$. Let $\left(S\backslash\{i\}; M'\right)$ be a maximal lower bound of $(S; P)$ and $(T; Q)$. Then, $\left(S\backslash\{i\}; M'\right) \sqsubseteq (S; P)$ and $P = M'_{-S} = M'_{-\{i\}}$. Besides, $\left(S\backslash\{i\}; M'\right) \sqsubseteq (T; Q) \sqsubseteq (N\backslash\{i\}; \{i\})$ which implies $M'_{-T} = Q$. Since $P_{-T} = Q_{-S}$ and the choice of i, we get $P_{-(T\cup\{i\})} = P_{-T} = Q_{-S} = Q_{-\{i\}}$. Applying Inequality (7) to $\left(S\backslash\{i\}; M'\right)$ and $(T; Q)$, we obtain

$$v(T \cup i; P_{-(T\cup\{i\})}) - v(T; Q) \geq v(S; P) - v(S\backslash\{i\}; M'), \text{ or}$$
$$v(S \cup T; P_{-T}) + v(S \cap T; M') \geq v(S; P) + v(T; Q),$$

and Inequality (6) holds.

Let us assume that the result holds for every $(S; P), (T; Q) \in \mathcal{F}_N$ with $|S\backslash T| \leq k$, $P_{-T} = Q_{-S}$, and $(S \cap T; M')$ a maximal lower bound of $(S; P)$ and $(T; Q)$. Now, take $(S; P), (T; Q) \in \mathcal{F}_N$ such that $|S\backslash T| = k + 1$, $P_{-T} = Q_{-S}$, and $(S \cap T; M')$ a maximal lower bound of $(S; P)$ and $(T; Q)$. Let us take $i \in S\backslash T$ and $\left(T \cup (S\backslash\{i\}); Q_{-(S\backslash\{i\})}\right)$. We can assume that $\left(S \cap T; M'\right)$ is not covered by $(S; P)$. Otherwise, $|S\backslash T| = 1$ and we have just proved the result for this situation. Take $(S\backslash\{i\}; \hat{M})$ such that $(S\backslash\{i\}; \hat{M})$ covers $(S \cap T; M')$ and $(S \cap T; M') \sqsubset (S\backslash\{i\}; \hat{M}) \sqsubset (S; P)$. Since $i \in S\backslash T$ and $(S \cap T; M')$ is a maximal lower bound of $\{(S; P), (T; Q)\}$, $(S\backslash\{i\}; \hat{M})$ does not precede $(T; Q)$. Nevertheless, $\left(S \cap T; M'\right) \sqsubseteq \left(T \cup (S\backslash\{i\}); Q_{-(S\backslash\{i\})}\right)$ because $S \cap T \subset T \cup (S\backslash\{i\})$, $M'_{-T} = Q$, and $M'_{-(T\cup(S\backslash\{i\}))} = Q_{-(S\backslash\{i\})}$. Besides, $\left(S\backslash\{i\}; \hat{M}\right) \sqsubseteq \left(T \cup (S\backslash\{i\}); Q_{-(S\backslash\{i\})}\right)$ because $S\backslash\{i\} \subseteq T \cup (S\backslash\{i\})$ and $\hat{M}_{-T} = M'_{-(T\cup(S\backslash\{i\}))} = Q_{-(S\backslash\{i\})}$ as a consequence of $M'_{-(S\backslash\{i\})} = \hat{M}$ and $M'_{-(T\cup(S\backslash\{i\}))} = Q_{-(S\backslash\{i\})}$. Applying Inequality (7) to

i, $\left(S\backslash\{i\};\, \hat{M}\right)$, and $\left(T\cup(S\backslash\{i\});\, Q_{-(S\backslash\{i\})}\right)$, we obtain

$$v(T\cup S;\, Q_{-S}) - v\left(T\cup(S\backslash\{i\});\, Q_{-(S\backslash\{i\})}\right) \geq v(S;\, P) - v\left(S\backslash\{i\};\, \hat{M}\right). \quad (9)$$

Notice that $|S\backslash(T\cup\{i\})| = k$. We apply the induction hypothesis to $\left(S\backslash\{i\};\, \hat{M}\right)$, $(T;\, Q)$ and $(S\cap T;\, M')$ because $\hat{M}_{-T} = Q_{-(S\backslash\{i\})}$ and $(S\cap T;\, M')$ is also a maximal lower bound of $\{(S\backslash\{i\};\, \hat{M}),\ (T;\, Q)\}$. Thus,

$$v\left(T\cup(S\backslash\{i\});\, Q_{-(S\backslash\{i\})}\right) + v(S\cap T;\, M') \geq v(T;\, Q) + v\left(S\backslash\{i\};\, \hat{M}\right). \quad (10)$$

Finally, adding up Inequalities (9)-(10), we obtain Inequality (6), concluding the proof. □

As our notion of convexity is different from H-convexity, so is Inequality (7) with respect to the concept of *weak convexity* defined in Abe (2016). A game $v \in \mathcal{G}_N$ is *weakly convex* if for every $S, T \subseteq N$ with $|S\backslash T| = |T\backslash S| \leq 1$ and $P \in \Pi(N\backslash(S\cup T))$, it holds

$$\begin{aligned} &v(S\cup T;\, P) + v(S\cap T;\, P\cup\lceil S\backslash T\rceil\cup\lceil T\backslash S\rceil) \\ &\geq v(S;\, P\cup\lceil T\backslash S\rceil) + v(T;\, P\cup\lceil S\backslash T\rceil). \end{aligned}$$

Notice that this inequality is meaningful if $|S\backslash T| = |T\backslash S| = 1$, otherwise the inequality always holds. Then, if $|S\backslash T| = |T\backslash S| = 1$, there are $i, j \in N$ such that $i \neq j$, $S = (T\backslash\{j\})\cup\{i\}$ and $T = (S\backslash\{i\})\cup\{j\}$. Taking $(S\backslash\{i\};\, \lfloor i, j\rfloor\cup P)$ and $(T;\, \{i\}\cup P)$, we have $(S\backslash\{i\};\, \lfloor i, j\rfloor\cup P) \sqsubseteq (T;\, \{i\}\cup P)$. Thus, if the game is convex we have

$$v(S\cup T;\, P) - v(T;\, \{i\}\cup P) \geq v(S;\, \{j\}\cup P) - v(S\backslash\{i\};\, \lfloor i, j\rfloor\cup P),$$

which means that the game is weakly convex. In the next example we revisit Example 5 introduced in Hafalir (2007) and used also in Abe (2016) (as Example 2.6) to show that the reverse implication does not hold.

Example 4.4 Let $N = \{1, 2, 3, 4, 5\}$ and the symmetric game v given by

$v(N;\emptyset) = 25,\quad v(S;\, \lfloor N\backslash S\rfloor) = 18,$ for every S with $|S| = 4$,
$v(\{i\};\, \lceil N\backslash\{i\}\rceil) = 3,$ for every $i \in N$, $\quad v(S;\, \lceil N\backslash S\rceil) = 17,$ for every S with $|S| = 3$,
$v(S;\, \lceil N\backslash S\rceil) = 6,$ for every S with $|S| = 2$, $\quad v(S;\, \lfloor N\backslash S\rfloor) = 12,$
for every S with $|S| = 3$,
$v(\{i\};\, \lceil N\backslash\{i, j\}\rceil\cup\{j\}) = 3,$ for every $i, j \in N, i \neq j$,
$v(S;\, \lceil N\backslash(S\cup\{i\})\rceil\cup\{i\}) = 9,$ for every S with $|S| = 2, i \in N\backslash S$,
$v(\{i\};\, \lceil T\rceil\cup\lceil N\backslash(T\cup\{i\})\rceil) = 8,$ for every T with $|T| = 2, T\subset N\backslash\{i\},\ i \in N$,
$v(S;\, \lfloor N\backslash S\rfloor) = 7,$ for every S with $|S| = 2$,
$v(\{i\};\, \lceil T\rceil\cup\lfloor N\backslash(T\cup\{i\})\rfloor) = 3,$ for every T with $|T| = 2, T\subset N\backslash\{i\},\ i \in N$,
$v(\{i\};\, \lfloor N\backslash\{i\}\rfloor) = 3,$ for every $i \in N$.

Abe (2016) shows that this game is weakly convex. Notice that it is not convex because

$$1 = 18 - 17 = v(\{1,2,3,4\}; \{5\}) - v(\{1,2,3\}; \lceil 4,5 \rceil)$$
$$< v(\{1,2,4\}; \lfloor 3,5 \rfloor) - v(\{1,2\}; \{3\} \cup \lceil 4,5 \rceil) = 12 - 9 = 3.$$

In the remainder of this section, we present our second main result that relates our notion of convexity for games in partition function form with the standard convexity of certain classic games. Recall that a classic game w is convex if and only if for every $i \in N$ and $S \subseteq T \subseteq N \backslash \{i\}$,

$$w(T \cup \{i\}) - w(T) \geq w(S \cup \{i\}) - w(S).$$

Given a game with externalities, $v \in \mathcal{G}_N$, and a partition of the set of agents, $P \in \Pi(N)$, the associated classic game, v^P, is defined for every $S \subseteq N$ by

$$v^P(S) = v\left(S; P_{-S}\right).$$

That is, externalities are removed since every coalition expects the agents in the complementary coalition to be organized according to the projection of P. This is very similar to the characteristic function that Bloch and van den Nouweland (2014) associate using a so-called P-exogenous rule. The difference is that they consider the superadditive cover of v^P.

The convexity of a game with externalities is characterized by the convexity of the classic games associated with it for any possible partition of the set of agents.

Theorem 4.2 *Let $v \in \mathcal{G}_N$. The game v is convex if and only if, for every $P \in \Pi(N)$, the classic game v^P is convex.*

Proof Let $v \in \mathcal{G}_N$. First, let us assume that v is convex. Let $P \in \Pi(N)$ and consider the classic game v^P defined as follows: $v^P(S) = v(S; P_{-S})$, for every $S \subseteq N$. Let $i \in N$ and $S \subseteq T \subseteq N \backslash \{i\}$. Take $(S; P_{-S}), (T; P_{-T}) \in \mathcal{F}_N$. It is clear that $(S; P_{-S}) \sqsubseteq (T; P_{-T}) \sqsubseteq (N \backslash \{i\}; \{i\})$. Since v is convex, using Inequality (7), we have

$$v(T \cup \{i\}; P_{-(T \cup \{i\})}) - v(T; P_{-T}) \geq v(S \cup \{i\}; P_{-(S \cup \{i\})}) - v(S; P_{-S}). \quad (11)$$

Rewriting both sides of Inequality (11), we have

$$v^P(T \cup \{i\}) - v^P(T) \geq v^P(S \cup \{i\}) - v^P(S),$$

concluding that the classic game v^P is convex.

Second, let us assume that for every $P \in \Pi(N)$, the classic game v^P is convex. We check Inequality (7). Let $i \in N$ and $(S; P), (T; Q) \in \mathcal{F}_N$ such that $(S; P) \sqsubseteq (T; Q) \sqsubseteq (N \backslash \{i\}; \{i\})$. Let $H \in \Pi(N)$ such that $H_{-S} = P$[7]. Then, $H_{-T} = P_{-T} = Q$

[7] Notice that we can take $H = P \cup \lceil S \rceil$.

because $(S; P) \sqsubseteq (T; Q)$ and the choice of H. Take the classic game v^H. Since v^H is a convex game, we get

$$v^H(T \cup \{i\}) - v^H(T) \geq v^H(S \cup \{i\}) - v^H(S).$$

Taking into account the definition of v^H, and the fact that $H_{-T} = P_{-T} = Q$ and $H_{-S} = P$, we have

$$v\left(T \cup \{i\}; Q_{-\{i\}}\right) - v(T; Q) \geq v\left(S \cup \{i\}; P_{-\{i\}}\right) - v(S; P),$$

concluding that v is convex. □

5 Convexity and average values

The order analyzed in Sect. 3 induces a notion of marginality for games with externalities. Let $i \in N$ and $(S; P) \in \mathcal{F}_N$ with $i \notin S$, the *marginal contribution of agent i* to $(S; P)$ is given by

$$v\left(S \cup \{i\}; P_{-\{i\}}\right) - v(S; P),$$

that is, the change of v on two endpoints of a link in $(\mathcal{F}_N, \sqsubseteq)$ that represents the incorporation of agent i to the active coalition. This notion has already been introduced in Bolger (1989) in order to characterize the value proposed therein. The marginal contribution measures the surplus generated by an agent who moves from a non-active block to the active block. When the non-active block is a singleton, the marginal contribution is called *intrinsic marginal contribution* by de Clippel and Serrano (2008). Bolger (1990) proposed a family of power indices for multicandidate voting games using marginal contributions. Sánchez-Pérez (2016) also presented a family of values for games in partition function form based on marginal contributions. This family includes, among others, the values proposed in Pham Do and Norde (2007), which coincides with the externality free value (de Clippel and Serrano 2008), and the average value proposed in Macho-Stadler et al. (2007), but it does not include either the value proposed in Myerson (1977) or the value given in Albizuri et al. (2005). Moreover, marginal contributions are also used in the definition of what a null player is in a game with externalities (see, for instance Sánchez-Pérez 2016).

In Theorem 4.1 we characterize convex games in terms of non-decreasing marginal contributions with respect to the order defined in Sect. 3. Moreover, in Theorem 4.2 we also characterize convex games through the convexity of some classic games. Many extensions of the Shapley value to games with externalities in the literature are built using an associated classic game. This is the case of the *average values* defined in Macho-Stadler et al. (2007). Let α be a real-valued function defined on the family $\mathcal{F}_N$ such that

$$\sum_{P \in \Pi(N \setminus S)} \alpha(S; P) = 1, \quad \text{for every } S \subseteq N.$$

When the real number of $\alpha(S; P)$ is non-negative it can be interpreted as the probability that coalition S assigns to the rest of agents being organized according to P. Let $v \in \mathcal{G}_N$ be a game. The classic game associated with v with respect to α is defined by:

$$v^{\alpha}(S) = \sum_{P \in \Pi(N \setminus S)} \alpha(S; P) v(S; P).$$

An *average value* Φ^{α} is given as $\Phi^{\alpha}(v) = Sh(v^{\alpha})$, with Sh the Shapley value of a classic game. Each function α provides an average value. Some examples of values in this class are the ones proposed in de Clippel and Serrano (2008), McQuillin (2009), and Hu and Yang (2010). In the context of classic games, the convexity property of the game implies that the Shapley value belongs to the core. In this section we study if the implication carries on to these average games using our definition of convexity for partition function form games.

First, we analyze the *externality free* value (Pham Do and Norde 2007, de Clippel and Serrano (2008)). It is defined as

$$\Phi^{CS}(v) = Sh(v^{\lfloor N \rfloor}),$$

for every $v \in \mathcal{G}_N$. Using Theorem 4.2, if v is convex, then the classic game $v^{\lfloor N \rfloor}$ is also convex and $\Phi^{CS}(v)$ belongs to the core of $v^{\lfloor N \rfloor}$.

A counterpart of the externality free value is the McQuillin value (McQuillin (2009)), Φ^{MQ}, defined as follows:

$$\Phi^{MQ}(v) = Sh(v^{\lceil N \rceil}),$$

for every $v \in \mathcal{G}_N$. Using Theorem 4.2, if v is convex, then the classic game $v^{\lceil N \rceil}$ is also convex and $\Phi^{MQ}(v)$ belongs to the core of $v^{\lceil N \rceil}$.

Finally, we consider the value defined and characterized in Hu and Yang (2010). It is an extension of the Shapley value to games in partition function form. Let $v \in \mathcal{G}_N$ and $i \in N$. The *Hu-Yang* value is given by

$$\Phi_i^{HY}(N, v) = \sum_{P \in \Pi(N)} \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!|\Pi(N)|} \left(v(S \cup \{i\}; P_{-(S \cup \{i\})}) - v(S; P_{-S}) \right).$$

This value has several interpretations using the ordering studied in this paper. First, this value weights agent i's marginal contribution from $(S; P_{-S})$ to $\left(S \cup \{i\}; P_{-(S \cup \{i\})}\right)$ by the proportion of chains that join $(\emptyset; P)$ and $(N; \emptyset)$ having the link from $(S; P_{-S})$ to $(S \cup \{i\}; P_{-(S \cup \{i\})})$. Second, Φ^{HY} can be seen as the average of the Shapley value for classic games v^P, for every $P \in \Pi(N)$ or, equivalently, Φ^{HY} is also the Shapley value of the classic game defined by

$$v^{HY}(S) = \frac{1}{|\Pi(N)|} \sum_{P \in \Pi(N)} v^P(S).$$

If v is convex, then v^{HY} is also convex and Φ^{HY} provides an element in the core of v^{HY}.

An open question is under which conditions the classic game v^{α} is convex. This fact will imply that the corresponding average value belongs to its core. It would also be interesting to study what values in the family of Sánchez-Pérez (2016) can be obtained applying the Shapley value to a classic game and studying its convex nature. There are other values for partition function form games that do not belong to this class such as Myerson (1977), Albizuri et al. (2005), Ju (2007), and Borm et al. (2015). It would be worth studying the implications of convexity on them.

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Appendix

Proof of Proposition 3.1. Let $(S; P), (T; Q) \in \mathcal{F}_N$ with $(S; P) \neq (T; Q)$ and define $M = (P \cup \lfloor S \backslash T \rfloor) \bigvee (Q \cup \lfloor T \backslash S \rfloor)$.

1. $Q_{-S} \neq P_{-T}$. Suppose that $(L; H) \in \mathcal{F}_N$ is a lower bound of $\{(S; P), (T; Q)\}$. Then, $(L; H) \sqsubseteq (S; P)$ and $(L; H) \sqsubseteq (T; Q)$. This implies that $L \subseteq S \cap T$, $H_{-S} = P$, and $H_{-T} = Q$. Notice that

$$H_{-(S \backslash L)} = H_{-S} = P = P_{-L} \text{ and } H_{-(T \backslash L)} = H_{-T} = Q = Q_{-L}.$$

Thus,

$$P_{-T} = P_{-(T \backslash L)} = H_{-((S \cup T) \backslash L)} = Q_{-(S \backslash L)} = Q_{-S},$$

which is a contradiction.

2. $P_{-T} = Q_{-S}$. Take $(L; H)$ with $L = S \cap T$ and H defined by $R \in H$ if and only if

$$R \in M \quad \text{or} \quad R = \lceil S' \cup T' \rceil \tag{12}$$

for some $S', T' \in M$ such that $S' \subseteq S \backslash T$ and $T' \subseteq T \backslash S$.

□

We shall prove that $(L; H) \sqsubseteq (S; P)$, and $(L; H) \sqsubseteq (T; Q)$. Clearly $L \subseteq S$ and $L \subseteq T$. It remains to prove that $H_{-S} = P$ and $H_{-T} = Q$. The following Claim will be useful.

Claim A. *Let N be a finite set, P, $Q \in \Pi(N)$, $S \subseteq N$ such that $\lfloor S \rfloor \in Q$. Then, $(P \bigvee Q)_{-S} = P_{-S} \bigvee Q_{-S}$.*

Proof If $S = N$, the result immediately follows. Let us assume that $S \subset N$ and take $L \in P \bigvee Q$. If $L \cap S = \emptyset$, then $L \in P_{-S} \bigvee Q_{-S}$. Let us assume that $L \cap S \neq \emptyset$. By the choice of L, there are $L_1, \ldots, L_k \in P$ such that $L = \cup_{j=1}^{k} L_j$ with $L_j \cap S \neq \emptyset$ for some $j \in \{1, \ldots, k\}$. Besides, there are $L'_1, \ldots L'_r \in Q$ with $L'_j \cap S = \emptyset$, for every $j = 1, \ldots, r$, such that $L = (\cup_{j=1}^{r} L'_j) \cup (\lfloor L \cap S \rfloor)$. Then,

$$L \backslash S = \cup_{j=1}^{r} L'_j = \cup_{j=1}^{k} (L_j \backslash S)$$

and $L \backslash S \in P_{-S} \bigvee Q_{-S}$.

Now, take $L \in P_{-S} \bigvee Q_{-S}$. There are $L_1, \ldots, L_k \in P_{-S}$ and $L'_1, \ldots L'_r \in Q_{-S}$ such that $L = \cup_{j=1}^{k} L_j = \cup_{j=1}^{r} L'_j$. Take $R_1, \ldots, R_k \in P$ such that $L_j \subseteq R_j$, for every $j = 1, \ldots, k$ and define $R = \cup_{j=1}^{k} R_j$. Thus,

$$R = \cup_{j=1}^{k} R_j = L \cup (R \cap S) = (\cup_{j=1}^{r} L'_j) \cup (R \cap S),$$

$R \backslash S = L$, and $R \in P \bigvee Q$. □

We distinguish two cases. First, let us assume that $H = M = (P \cup \lfloor S \backslash T \rfloor) \bigvee (Q \cup \lfloor T \backslash S \rfloor)$. Using Claim A and the fact that $P_{-T} = Q_{-S}$,

$$\begin{aligned} M_{-S} = M_{-(S \backslash T)} &= [(P \cup \lfloor S \backslash T \rfloor) \bigvee (Q \cup \lfloor T \backslash S \rfloor)]_{-(S \backslash T)} \\ &= P \bigvee (Q \cup \lfloor T \backslash S \rfloor)_{-(S \backslash T)} = P \bigvee (Q_{-(S \backslash T)} \cup \lfloor T \backslash S \rfloor) \\ &= P \bigvee (Q_{-S} \cup \lfloor T \backslash S \rfloor) = P \bigvee (P_{-T} \cup \lfloor T \backslash S \rfloor) = P. \end{aligned}$$

In a similar way, we can prove that $M_{-T} = Q$. Thus, we find out that $(L; M)$ is a lower bound of $(S; P)$ and $(T; Q)$.

Second, let us assume that $H \neq M$. Then, there is $R \in H$ such that $R \notin M$ given by $R = \lceil S' \cup T' \rceil$ for some $S', T' \in M$ such that $S' \subseteq S \backslash T$ and $T' \subseteq T \backslash S$. Take any $R \in H$. If $R \in M$, then as before $R \backslash S \in P$ and $\backslash T \in Q$. Otherwise $R = \lceil S' \cup T' \rceil$ for some $S', T' \in M$ such that $S' \subseteq S\backslash T$ and $T' \subseteq T\backslash S$. Then, $R\backslash S = T' \in M_{-S} = P$ and $R\backslash T = S' \in M_{-T} = Q$. Thus, we have proved that $(L; H)$ is a lower bound of $(S; P)$ and $(T; Q)$.

Next, we prove that these lower bounds are maximal. Take $(S \cap T; H)$ satisfying the conditions of Eq. (12). If there is $(U; W) \in \mathcal{F}_N$ such that $(S\cap T; H) \sqsubset (U; W) \sqsubset (S; P)$ and $(S\cap T; H) \sqsubset (U; W) \sqsubset (T; Q)$, then $S\cap T \subset U \subset S$ and $S\cap T \subset U \subset T$ which is a contradiction. Then, $(S \cap T; H)$ is a maximal lower bound.

It remains to check that these embedded coalitions are the unique maximal lower bounds. Let us assume that $(L; K)$ is a maximal lower bound of $(S; P)$ and $(T; Q)$. It is clear that $L = S \cap T$ and $K_{-S} = P$ and $K_{-T} = Q$. Take $R \in K$. We have $R\backslash S \in K_{-S} = P = M_{-S}$ and $R\backslash T \in K_{-T} = Q = M_{-T}$. We distinguish several cases.

- $R \cap S = R \cap T = \emptyset$. Then, $R = R\backslash S = R\backslash T \in K_{-S} \cap K_{-T}$. Since $K_{-S} = P$, $K_{-T} = Q$ and the definition of M, we have $R \in M$.
- $R \cap S = \emptyset$ but $R \cap T \neq \emptyset$. Then, $R \in K_{-S} = P = M_{-S}$, $R\backslash T \in Q = M_{-T}$, and $R \in M$. We can reason in a similar way if $R \cap S \neq \emptyset$ and $R \cap T = \emptyset$.
- $R\cap S \neq \emptyset$ and $R\cap T \neq \emptyset$. Then, $R = (R\backslash(S\cup T))\cup(R\cap(S\backslash T))\cup(R\cap(T\backslash S))$ and we have $R\backslash S = (R\backslash(S \cup T)) \cup (R \cap (T\backslash S)) \in P = M_{-S}$, $R\backslash T = (R\backslash(S \cup T)) \cup (R \cap (S\backslash T)) \in Q = M_{-T}$. If $R\backslash(S \cup T) = \emptyset$, using the definition of M, we have $R\backslash S, R\backslash T \in M$ and $R = (R\backslash S) \cup (R\backslash T)$. If $R\backslash(S \cup T) \neq \emptyset$, by the definition of M, we have $R \in M$.

Thus, any $R \in K$ is of the type described in Eq. (12), concluding the proof. □

Proof of Proposition 3.2. Let $(S; P), (T; Q) \in \mathcal{F}_N$ and $M = P_{-T} \bigwedge Q_{-S}$.

1. Let $(R; M') \in \mathcal{F}_N$ be an upper bound of $\{(S; P), (T; Q)\}$. Then, $S \cup T \subseteq R$ and $P_{-R} = M' = Q_{-R}$. In other words, there is some $L \subseteq N\backslash(S \cup T)$ such that $R = S \cup T \cup L$ and $P_{-R} = P_{-(T\cup L)} = M' = Q_{-R} = Q_{-(S\cup L)}$.
   On the other hand, if $R = S \cup T \cup L$ with $L \subseteq N\backslash(S \cup T)$ and $M' = P_{-(T\cup L')} = Q_{-(S\cup L')}$, clearly $(S; P), (T; Q) \sqsubseteq (R; M')$, and $(R; M')$ is an upper bound.

□

The following Claim will be useful to prove that these upper bounds are minimal.
Claim B. *Let* $P, Q \in \Pi(N)$, *and* $S \subseteq N$. Then, $P_{-S} \bigwedge Q_{-S} = (P \bigwedge Q)_{-S}$.

Proof Let $P, Q \in \Pi(N)$ and $S \subseteq N$. Let $R \in P$ and $\tilde{R} \in Q$. Then, $(R\backslash S)\cap(\tilde{R}\backslash S) \in P_{-S} \bigwedge Q_{-S}$ and $(R\backslash S) \cap (\tilde{R}\backslash S) = (R \cap \tilde{R})\backslash S \in (P \bigwedge Q)_{-S}$. Let $R \in (P \bigwedge Q)_{-S}$. Then, there are $R' \in P$ and $\tilde{R} \in Q$ such that $R = (R' \cap \tilde{R})\backslash S = (R'\backslash S) \cap (\tilde{R}\backslash S) \in P_{-S} \bigwedge Q_{-S}$. □

2. Let $(S\cup T\cup L; M_{-L})$ with $L \subseteq N\backslash(S\cup T)$ such that $P_{-(T\cup L)} = Q_{-(S\cup L)} = M_{-L}$ and for every $L' \subseteq N\backslash(S\cup T)$ with $P_{-(T\cup L')} = Q_{-(S\cup L')} = M_{-L'}$ it holds $L \subseteq L'$ or $L \cap L' = \emptyset$. From Item 1 and Claim B, we have that $(S \cup T \cup L; M_{-L})$ is an upper bound of $\{(S; P), (T; Q)\}$. It remains to check that there is no $(R; H) \in \mathcal{F}_N$ such that $(S; P) \sqsubset (R; H) \sqsubset (S \cup T \cup L; M_{-L})$ and $(T; Q) \sqsubset (R; H) \sqsubset (S \cup T \cup L; M_{-L})$. If $L = \emptyset$, it is clear that $(S \cup T; M)$ is a minimal upper

bound. Let us assume that $L \neq \emptyset$ and that $(S \cup T \cup L; M_{-L})$ is not a minimal upper bound of $\{(S, P), (T; Q)\}$. Then, $R = S \cup T \cup L' \subset S \cup T \cup L$, for some $\emptyset \neq L' \subseteq N\backslash(S \cup T)$ with $P_{-(T\cup L')} = Q_{-(S\cup L')} = M_{-L'}$. Thus, $L' \subset L$, which contradicts the choice of L.

It remains to check that there are no other minimal upper bounds. Take $L \subseteq N\backslash(S\cup T)$ such that $(S \cup T \cup L; P_{-(T\cup L)})$ is a minimal upper bound of $\{(S, P), (T; Q)\}$. Then, $(S; P), (T; Q) \sqsubseteq (S\cup T\cup L; P_{-(T\cup L)})$ and we have $P_{-(T\cup L)} = Q_{-(S\cup L)} = M_{-L}$ by Claim B. Let $\emptyset \neq L' \subseteq N\backslash(S \cup T)$ with $P_{-(T\cup L')} = Q_{-(S\cup L')} = M_{-L'}$. We consider two cases. If $\big(S \cup T \cup L; P_{-(T\cup L)}\big) \sqsubseteq \big(S \cup T \cup L'; P_{-(T\cup L')}\big)$, then $L \subseteq L'$. It remains to study the case in which $\big(S \cup T \cup L; P_{-(T\cup L)}\big)$ and $\big(S \cup T \cup L'; P_{-(T\cup L')}\big)$ are not comparable. Since both are upper bounds of $\{(S; P), (T; Q)\}$, we have $P_{-(T\cup L')} = Q_{-(S\cup L')}$, $S \cup T \subseteq S \cup T \cup L$, and $S \cup T \subseteq S \cup T \cup L'$. Then, L and L' are not comparable. If $L \cap L' = \emptyset$, we are done. Let us assume that $R = L \cap L' \neq \emptyset$ and $L\backslash L' \neq \emptyset$. By Claim B, $P_{-(T\cup R)} \bigwedge Q_{-(S\cup R)} = M_{-R}$, then $(S \cup T \cup R; M_{-R})$ is also an upper bound of $\{(S; P), (T; Q)\}$. But $(S \cup T \cup R; M_{-R}) \sqsubset \big(S \cup T \cup L; P_{-(T\cup L)}\big)$ and then $(S\cup T\cup L; P_{-(T\cup L)})$ would not be minimal. Summarizing, $L \subseteq L'$ or $L\cap L' = \emptyset$. □

Proof of Corollary 3.1. Let $(S; P), (T; Q) \in \mathcal{F}_N$ such that $P_{-T} = Q_{-S}$. Then, $P_{-T} \bigwedge Q_{-S} = P_{-T} = Q_{-S}$. It is clear that $(S \cup T; P_{-T})$ is an upper bound of $\{(S; P), (T; Q)\}$. Using Proposition 3.2, any upper bound is given by $(S \cup T \cup L; P_{-(T\cup L)})$ with $L \subseteq N\backslash(S \cup T)$, and $P_{-(T\cup L)} = Q_{-(S\cup L)}$. If $L \neq \emptyset$, we have $(S\cup T; P_{-T}) \sqsubset (S\cup T\cup L; P_{-(T\cup L)})$ and $(S\cup T\cup L; P_{-(T\cup L)})$ is not a minimal upper bound. Thus, $(S\cup T; P_{-T})$ is the unique minimal upper bound of $\{(S; P), (T; Q)\}$. □

Proof of Proposition 3.3. Let $(S; P) \in \mathcal{F}_N$.

1. If $(S; P) \neq (N; \emptyset)$. Then, for every $i \in N\backslash S$, $(S; P) \sqsubseteq \big(S \cup \{i\}; P_{-\{i\}}\big)$ and there is no $(L; H) \in \mathcal{F}_N$ such that $(S; P) \sqsubset (L; H) \sqsubset \big(S \cup \{i\}; P_{-\{i\}}\big)$.
2. Suppose that $S \neq \emptyset$. For every $i \in S$ and $R \in P$, we have $(S\backslash\{i\}; P_{-R}\cup\{R\cup\{i\}\}) \sqsubseteq (S; P)$ and there is no $(T; Q) \in \mathcal{F}_N$ such that $(S\backslash\{i\}; P_{-R}\cup\lceil R\cup\{i\}\rceil) \sqsubset (T; Q) \sqsubset (S; P)$. Additionally, for every $i \in S$, $(S\backslash\{i\}; P \cup \{i\}) \sqsubseteq (S; P)$ and there is no $(T; Q) \in \mathcal{F}_N$ such that $(S\backslash\{i\}; P \cup \{i\}) \sqsubset (T; Q) \sqsubset (S; P)$. Notice that if $(S; P) = (N; \emptyset)$, then $|P| = 0$ and $(N; \emptyset)$ covers $|N| = n$ embedded coalitions.

□

Proof of Proposition 3.4. 1. Let $(S; P), (T; Q) \in \mathcal{F}_N$ such that $(S; P) \sqsubseteq (T; Q)$. Notice that $S \subseteq T$ and $Q = P_{-T}$. We define the mapping ϕ from $[(S; P), (T; Q)]$ to $[S, T]_{\mathcal{B}(N)}$ as follows: $\phi(L; P_{-L}) = L$ for every $(L; P_{-L}) \in [(S; P), (T; Q)]$. It is clear that if $(L; P_{-L}), (L'; P_{-L'}) \in [(S; P), (T; Q)]$ with $(L; P_{-L}) \sqsubseteq (L'; P_{-L'})$ we have, in particular, $L \subseteq L'$ and then, $\phi(L; P_{-L}) \subseteq \phi(L'; P_{-L'})$. Take the mapping ϕ^{-1} from $[S, T]_{\mathcal{B}(N)}$ to $[(S; P), (T; Q)]$ defined by $\phi^{-1}(L) = (L; P_{-L})$ for every $L \in [S, T]_{\mathcal{B}(N)}$. ϕ and ϕ^{-1} are inverse maps and if $L \subseteq L'$ we have $\phi^{-1}(L) \sqsubseteq \phi^{-1}(L')$.
2. This follows immediately from the isomorphism above. Besides, the unique rank function is given by $\rho(S; P) = |S|$, for every $(S; P) \in \mathcal{F}_N$.

3. It follows directly from Item 1, Item 2, and the structure of each embedded coalition. □

Proof of Proposition 3.5. Items 1 and 2 follow directly from the isomorphism presented in Proposition 3.4.

Let us prove Item 3. Taking into account the first item, the total number of chains in $(\mathcal{F}_N, \sqsubseteq)$ from $(\{i\}; H)$ to $(N; \emptyset)$ is $(|N|-1)!$, for every $i \in N$ and $H \in \Pi(N\backslash\{i\})$. Additionally, there are $|H|+1$ elements of rank 1 linked to $(\{i\}; H)$. Thus, the total number of chains is

$$\sum_{i\in N}\sum_{H\in\Pi(N\backslash\{i\})}(|N|-1)!(|H|+1) = |N|!\sum_{H\in\Pi(N\backslash\{i\})}(|H|+1)$$
$$= |N|!\sum_{r=1}^{n-1}(r+1)S_{n-1,r} = |N|!B_n,$$

using the generalized recurrence expression provided in Spivey (2008) applied to $n-1$ and 1. □

Proof of Proposition 3.6. Let $(S; P) \in EC^N$. We obtain the coefficient $\delta_{(S;P)}$ through the Möbius inversion formula as follows

$$\delta_{(S;P)} = \sum_{(T;Q)\sqsubseteq(S;P)} \mu((T;Q),(S;P))v(T;Q) = \sum_{(T;Q)\sqsubseteq(S;P)} (-1)^{|S|-|T|}v(T;Q).$$

□

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