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ORIGINAL ARTICLE

The stability and extended well-posedness of the solution sets for set optimization problems via the Painlevé–Kuratowski convergence

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Abstract

In this paper, we obtain the Painlevé–Kuratowski upper convergence and the Painlevé–Kuratowski lower convergence of the approximate solution sets for set optimization problems with the continuity and convexity of objective mappings. Moreover, we discuss the extended well-posedness and the weak extended well-posedness for set optimization problems under some mild conditions. We also give some examples to illustrate our main results.

Keywords Set optimization problem · Painlevé–Kuratowski convergence · Stability · Extended well-posedness

Mathematical Subject Classifications 49J40 · 49K40 · 90C31

1 Introduction

It is well known that the stability of the solution sets under certain perturbations (with respect to the feasible region and the objective function) has been of great interest in optimization theory with applications. Recently, some stability results have been derived for the vector optimization and vector equilibrium problems based on

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a sequence of sets converging. For instance, (Huang 2000a) obtained the stability results of the set of efficient solutions of vector-valued and set-valued optimization in the sense of Painlevé–Kuratowski; (Lucchetti and Miglierina 2004) studied the convergence of the solution sets under perturbations of both the objective function and the feasible region for convex vector optimization problem; (Crespi et al. 2009) obtained the stability properties of vector optimization problems under the assumption that the objective function is cone-quasiconvex; (Lalitha and Chatterjee 2012a) established the Painlevé–Kuratowski set-convergence of the sets of minimal, weak minimal and Henig proper minimal points of the perturbed problems to the corresponding minimal set of the original problem assuming the objective functions to be (strictly) properly quasi cone-convex; (Lalitha and Chatterjee 2012b) derived the Painlevé–Kuratowski convergence of the weak efficient solution sets, efficient solution sets and Henig proper efficient sets for the perturbed vector optimization problems by using generalized quasi convexities; (Fang and Li 2012) established the Painlevé–Kuratowski convergence of the efficient solution sets, the weak efficient solution sets and various proper efficient solution sets for the perturbed vector equilibrium problems under the C-strict monotonicity. Very recently, under new assumptions, which are weaker than the assumption of C-strict monotonicity, Peng and Yang (2014) obtained sufficient conditions for the Painlevé–Kuratowski convergence of the weak efficient solution sets and efficient solution sets for the perturbed vector equilibrium problems. Zhao et al. (2016) established Painlevé–Kuratowski upper convergence of weak efficient solutions for perturbed vector optimization problems with approximate equilibrium constraints. Anh et al. (2018) discussed Painlevé–Kuratowski upper convergence and Painlevé–Kuratowski lower convergence of solution sets for the perturbed vector quasi-equilibrium problems.

It is also well known that the stability analysis of the solution sets for set optimization problems has been investigated by many authors in the literature [see, for example, Khan et al. (2015) and the references therein]. Recently, Gutiérrez et al. (2016) establish external and internal stability of the solutions of a set optimization problem in the image space using set convergence notions. Xu and Li (2014) showed the lower and upper semicontinuity of the set of minimal and weak minimal solutions to a parametric set optimization problem by using converse u-property of objective mappings. Very recently, Han and Huang (2017) discussed the upper semicontinuity and the lower semicontinuity of solution mappings to parametric set optimization problems by using the level mappings. Han and Huang (2018) established the continuity and convexity of the nonlinear scalarizing function for sets, which was introduced by Hernández and Rodríguez-Marín Hernández and Rodríguez-Marín (2007); as applications, they derived the upper semicontinuity and the lower semicontinuity of strongly approximate solution mappings to the parametric set optimization problems. Khoshkhabar-amiranloo (2018) discussed the upper semicontinuity and lower semicontinuity and compactness of the minimal solutions of parametric set optimization problems. Karuna and Lalitha (2019) investigated external and internal stability in terms of the Hausdorff convergence and Painlevé–Kuratowski convergence of a sequence of solution sets of perturbed set optimization problems to the solution set of the original set optimization problem. However, to the best of our knowledge, the Painlevé–Kuratowski convergence of the approximate solution sets for set optimization problems has not been explored until now. Therefore, it would be quite natural and interesting to study the Painlevé–Kuratowski convergence of the approximate solution sets for set optimization problems under some mild conditions. The first aim of this paper is to make an attempt in this direction.

On the other hand, the well-posedness plays a significant role in the study of the stability theory of optimization problems. Recently, the well-posedness for set optimization problems has been studied under different conditions. Zhang et al. (2009) established the equivalent relations between the three kinds of well-posedness and the well-posedness of three kinds of scalar optimization problems by using a generalized Gerstewitz's function, respectively. Gutiérrez et al. (2012) obtained the well-posedness property in the setting of set optimization problems, which improves some results in Zhang et al. (2009) by relaxing the assumption of cone boundedness of the image of objective mappings. By using the generalized nonlinear scalarization function, Long et al. (2015) established the equivalence relations between the three kinds of pointwise well-posedness for set optimization problems and the well-posedness of three kinds of scalar optimization problems, respectively. Crespi et al. (2014) introduced a new notion of global well-posedness for set-optimization problems, which is a generalization of one of the global notion considered in Zhang et al. (2009). Very recently, Crespi et al. (2018) obtained some characterizations for pointwise and global well-posedness in set optimization.

We note that Zolezzi (1996) proposed the notion of extended well-posedness for optimization problems. In Huang (2000b, 2001), Huang generalized the notion of extended well-posedness to vector optimization problems. Crespi et al. (2009) discussed the extended well-posedness properties of vector minimization problems in which the objective function is C-quasiconvex. However, it seems that there are no authors to study the extended well-posedness for set optimization problems. Thus, it would be important and interesting to study the extended well-posedness of set optimization problems. The second aim of this paper is to give some characterizations for the extended well-posedness of set optimization problems under suitable conditions.

The rest of the paper is organized as follows. Section 2 presents some necessary notations and lemmas. In Sect. 3, we discuss the Painlevé–Kuratowski upper convergence and the Painlevé–Kuratowski lower convergence of the approximate solution sets for set optimization problems with the continuity and convexity of objective mappings. In Sect. 4, we introduce the notions of the extended well-posedness and the weak extended well-posedness for set optimization problems and derive the extended well-posedness and the weak extended well-posedness for set optimization problems under mild conditions.

2 Preliminaries

Throughout this paper, without special statements, let $X = \mathbb{R}^m$ and $Y = \mathbb{R}^l$. Assume that $C \subseteq Y$ is a nonempty, convex, closed and pointed cones with $\text{int}C \neq \emptyset$. We denote by $\text{int}A$, $\text{cl}A$, ∂A and A^c the topological interior, the topological closure, the topological boundary and the complementary set of A, respectively. Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}_+^0 = \{x \in \mathbb{R} : x > 0\}$. We denote by B_X and B_Y the closed unit balls in X and Y, respectively. Let A and B be two nonempty subsets of Y. The lower relation "$\leq^l$" and the weak lower relation "$\ll^l$" are defined, respectively, by

$$A \leq^l B \Leftrightarrow B \subseteq A + C$$

and

$$A \ll^l B \Leftrightarrow B \subseteq A + \mathrm{int}C.$$

Let e be a fixed point in $\mathrm{int}C$. For $\varepsilon \geq 0$, the ε-lower relation "$\leq^l_\varepsilon$" and the weak ε-lower relation "$\ll^l_\varepsilon$" are defined, respectively, by

$$A \leq^l_\varepsilon B \Leftrightarrow B \subseteq A + C + \varepsilon e$$

and

$$A \ll^l_\varepsilon B \Leftrightarrow B \subseteq A + \mathrm{int}C + \varepsilon e.$$

Let A be a nonempty subset of Y and $a \in A$. We say that a is a minimal point of A with respect to C, denoted by $a \in \mathrm{Min}\,(A)$, if $(A - a) \cap (-C) = \{0\}$.

Remark 2.1 It follows from Corollary 3.8 of Luc (1989) (page 48) that, if A is compact, then $\mathrm{Min}\,(A) \neq \emptyset$.

Let $F : X \to 2^Y$ be a set-valued mapping and $D \subseteq X$ with $D \neq \emptyset$. We consider the following set optimization problem:

$$\text{(SOP)} \quad \min F(x) \quad \text{subject to} \quad x \in D.$$

Definition 2.1 For $\varepsilon \geq 0$, an element $x_0 \in D$ is said to be

(i) l-minimal solution of (SOP) if, for $x \in D$, $F(x) \leq^l F(x_0)$ implies $F(x_0) \leq^l F(x)$.
(ii) weak l-minimal solution of (SOP) if, for $x \in D$, $F(x) \ll^l F(x_0)$ implies $F(x_0) \ll^l F(x)$.
(iii) l-minimal approximate solution of (SOP) if, for $x \in D$, $F(x) \leq^l_\varepsilon F(x_0)$ implies $F(x_0) \leq^l_\varepsilon F(x)$.
(iv) weak l-minimal approximate solution of (SOP) if, for $x \in D$, $F(x) \ll^l_\varepsilon F(x_0)$ implies $F(x_0) \ll^l_\varepsilon F(x)$.

Let $E_l(D)$, $W_l(D)$, $E_l(\varepsilon, D)$ and $W_l(\varepsilon, D)$ denote the l-minimal solution set of (SOP), the weak l-minimal solution set of (SOP), the l-minimal approximate solution set of (SOP) and the weak l-minimal approximate solution set of (SOP), respectively.

Remark 2.2 $E_l(\varepsilon, D)$ and $W_l(\varepsilon, D)$ depend on the choice of $e \in \mathrm{int}C$.

We give an example to illustrate Remark 2.2.

Example 2.1 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+ = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\right\}$. Define a set-valued mapping $F : X \to 2^Y$ as follows:

$$F(x) = \left((x-1)^2, (x-2)^2\right) + B_Y, \quad x \in X.$$

Let $D = [-5, 5]$. If we choose $e = (2.5, 0.5)$, then $0 \in E_l(1, D)$ and $0 \in W_l(1, D)$. However, if we choose $e = (0.5, 2.5)$, then $0 \notin E_l(1, D)$ and $0 \notin W_l(1, D)$.

Remark 2.3 For any $\varepsilon \geq 0$, we have $E_l(D) \subseteq E_l(\varepsilon, D)$ and $W_l(D) \subseteq W_l(\varepsilon, D)$. In fact, let $x_0 \in E_l(D)$. Suppose that there exists $y \in D$ such that $F(y) \leq^l_\varepsilon F(x_0)$. Then

$$F(x_0) \subseteq F(y) + C + \varepsilon e \subseteq F(y) + C$$

and so $F(y) \leq^l F(x_0)$. By $x_0 \in E_l(D)$, one has $F(x_0) \leq^l F(y)$. This shows that $F(y) \subseteq F(x_0) + C$. Thus,

$$F(y) \subseteq F(x_0) + C \subseteq F(y) + C + \varepsilon e \subseteq F(x_0) + C + \varepsilon e,$$

which means that $F(x_0) \leq^l_\varepsilon F(y)$. Therefore, $x_0 \in E_l(\varepsilon, D)$ and so $E_l(D) \subseteq E_l(\varepsilon, D)$. Similarly, we can prove that $W_l(D) \subseteq W_l(\varepsilon, D)$.

Remark 2.4 For any $\varepsilon \geq 0$, we have $E_l(\varepsilon, D) \subseteq W_l(\varepsilon, D)$. In fact, let $x_0 \in E_l(\varepsilon, D)$. Suppose that there exists $y \in D$ such that $F(y) \ll^l_\varepsilon F(x_0)$. Then

$$F(x_0) \subseteq F(y) + \text{int}C + \varepsilon e \subseteq F(y) + C + \varepsilon e$$

and so $F(y) \leq^l_\varepsilon F(x_0)$. By $x_0 \in E_l(\varepsilon, D)$, we have $F(x_0) \leq^l_\varepsilon F(y)$. This shows that

$$\begin{aligned} F(y) &\subseteq F(x_0) + C + \varepsilon e \\ &\subseteq F(y) + \text{int}C + C + 2\varepsilon e \\ &\subseteq F(x_0) + C + \text{int}C + C + 3\varepsilon e \\ &\subseteq F(x_0) + \text{int}C + \varepsilon e, \end{aligned}$$

which means that $F(x_0) \ll^l_\varepsilon F(y)$. Therefore, $x_0 \in W_l(\varepsilon, D)$.

Remark 2.5 For any $\varepsilon > 0$, we have $W_l(D) \subseteq E_l(\varepsilon, D)$. In fact, let $x_0 \in W_l(D)$. Suppose that there exists $y \in D$ such that $F(y) \leq^l_\varepsilon F(x_0)$. Then

$$F(x_0) \subseteq F(y) + C + \varepsilon e \subseteq F(y) + C + \text{int}C \subseteq F(y) + \text{int}C,$$

which implies $F(y) \ll^l F(x_0)$. It follows from $x_0 \in W_l(D)$ that $F(x_0) \ll^l F(y)$, and so $F(y) \subseteq F(x_0) + \mathrm{int}C$. Thus,

$$\begin{aligned} F(y) &\subseteq F(x_0) + \mathrm{int}C \\ &\subseteq F(y) + C + \varepsilon e + \mathrm{int}C \\ &\subseteq F(x_0) + \mathrm{int}C + C + \varepsilon e + \mathrm{int}C \\ &\subseteq F(x_0) + C + \varepsilon e. \end{aligned}$$

This shows that $F(x_0) \leq^l_\varepsilon F(y)$,and so $x_0 \in E_l(\varepsilon, D)$.

Now, let us recall the concept of the Painlevé–Kuratowski set-convergence [see, for example, Rockafellar and Wets (2004)]. Let $\{A_n\}$ be a sequence of nonempty subsets of $\mathbb{R}^m$. Set

$$\mathrm{Ls}A_n := \left\{x \in \mathbb{R}^m : x = \lim_{k\to+\infty} x_{n_k}, x_{n_k} \in D_{n_k}, \left\{x_{n_k}\right\} \text{is a subsequence of } \{x_n\}\right\},$$

$$\mathrm{Li}A_n := \left\{x \in \mathbb{R}^m : x = \lim_{n\to+\infty} x_n, x_n \in D_n \text{ for sufficiently large } n\right\}.$$

The set $\mathrm{Ls}A_n$ is called the upper limit of the sequence $\{A_n\}$, and the set $\mathrm{Li}A_n$ is called the lower limit of the sequence $\{A_n\}$. We say that the sequence $\{A_n\}$ converges in the sense of Painlevé–Kuratowski to the set A if

$$\mathrm{Ls}A_n \subseteq A \subseteq \mathrm{Li}A_n.$$

We denote the Painlevé–Kuratowski convergence by $A_n \xrightarrow{K} A$.

Definition 2.2 (Kuratowski 1968) Let (X, d) be a metric space, A and B be two nonempty subsets of X. The Hausdorff distance between A and B is defined by

$$H(A, B) := \max\{g(A, B), g(B, A)\},$$

where

$$g(A, B) := \sup_{a\in A} d(a, B)\, with d(a, B) = \inf_{b\in B} d(a, b).$$

Let $\{A_n\}$ be a sequence of nonempty subsets of $\mathbb{R}^m$. The sequence $\{A_n\}$ converges to $A \subseteq \mathbb{R}^m$ in the sense of Hausdorff iff $H(A_n, A) \to 0$, and we denote it by $A_n \xrightarrow{H} A$. Condition $g(A_n, A) \to 0$ is the upper part of Hausdorff convergence (denoted by $A_n \overset{H}{\rightharpoonup} A$), while condition $g(A, A_n) \to 0$ is the lower part of Hausdorff convergence (denoted by $A_n \overset{H}{\rightharpoondown} A$).

Definition 2.3 Let T and T_1 be two topological vector spaces. A set-valued mapping $\Phi : T \to 2^{T_1}$ is said to be

(i) upper semicontinuous (u.s.c.) at $u_0 \in T$ if, for any neighborhood V of $\Phi(u_0)$, there exists a neighborhood $U(u_0)$ of u_0 such that for every $u \in U(u_0)$, $\Phi(u) \subseteq V$;
(ii) lower semicontinuous (l.s.c.) at $u_0 \in T$ if, for any $x \in \Phi(u_0)$ and any neighborhood V of x, there exists a neighborhood $U(u_0)$ of u_0 such that for every $u \in U(u_0)$, $\Phi(u) \cap V \neq \emptyset$.

We say that Φ is u.s.c. and l.s.c. on T if it is u.s.c. and l.s.c. at each point $u \in T$, respectively. We call that Φ is continuous on T if it is both u.s.c. and l.s.c. on T.

Definition 2.4 (Han and Huang 2018) Let D be a nonempty convex subset of X. A set-valued mapping $\Phi : X \to 2^Y$ is said to be

(i) natural quasi C-convex on D if, for any $x_1, x_2 \in D$ and for any $t \in [0, 1]$, there exists $\lambda \in [0, 1]$ such that

$$\lambda\Phi(x_1) + (1-\lambda)\Phi(x_2) \subseteq \Phi(tx_1 + (1-t)x_2) + C.$$

(iii) strictly natural quasi C-convex on D if, for any $x_1, x_2 \in D$ with $x_1 \neq x_2$ and for any $t \in (0, 1)$, there exists $\lambda \in [0, 1]$ such that

$$\lambda\Phi(x_1) + (1-\lambda)\Phi(x_2) \subseteq \Phi(tx_1 + (1-t)x_2) + \text{int}C.$$

Definition 2.5 Let $F : X \to 2^Y$ be a set-valued mapping and D be a nonempty subset of X. The ε-level set $Q_l(\varepsilon, x, D)$ is defined as follows:

$$Q_l(\varepsilon, x, D) = \left\{u \in D : F(u) \leq_\varepsilon^l F(x)\right\} \bigcup \{x\}.$$

Remark 2.6 Crespi et al. (2017) defined general level sets for any map g from a domain D to a range R with a binary relation $\precsim$ on R, as

$$\text{Lev}\left(g, \precsim, r\right) = \left\{d \in D : g(d) \precsim r\right\},$$

for any $r \in R$. However, in Definition 2.5, since the ε-lower relation "$\leq_\varepsilon^l$" is not reflexive for $\varepsilon > 0$, $x \in \left\{u \in D : F(u) \leq_\varepsilon^l F(x)\right\}$ may not be true. Thus, we define the ε-level set $Q_l(\varepsilon, x, D)$ by

$$Q_l(\varepsilon, x, D) = \left\{u \in D : F(u) \leq_\varepsilon^l F(x)\right\} \bigcup \{x\}.$$

In the following two lemmas, let T and T_1 be two normed vector spaces.

Lemma 2.1 (Aubin and Ekeland 1984) *A set-valued mapping $G : T \to 2^{T_1}$ is l.s.c. at $u_0 \in T$ if and only if for any sequence $\{u_n\} \subseteq T$ with $u_n \to u_0$ and for any $x_0 \in G(u_0)$, there exists $x_n \in G(u_n)$ such that $x_n \to x_0$.*

Lemma 2.2 (Göpfert et al. 2003) *Let $G : T \to 2^{T_1}$ be a set-valued mapping. For any given $u_0 \in T$, if $G(u_0)$ is compact, then G is u.s.c. at $u_0 \in T$ if and only if for any sequence $\{u_n\} \subseteq T$ with $u_n \to u_0$ and for any $x_n \in G(u_n)$, there exist $x_0 \in G(u_0)$ and a subsequence $\left\{x_{n_k}\right\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$.*

Lemma 2.3 (Rockafellar and Wets 2004) *Let* $A_n \subseteq \mathbb{R}^m$ *with* $n = 1, 2, \ldots$ *and* $A \subseteq \mathbb{R}^m$. *Then* $A \subseteq \mathrm{Li}A_n$ *if and only if for any open set* W *with* $W \cap A \neq \emptyset$, *there exists* $n_0 \in \mathbb{N}$ *such that* $W \cap A_n \neq \emptyset$ *for any* $n \geq n_0$.

Lemma 2.4 (Karuna and Lalitha 2019) *Let* A *be a nonempty subset of* X *and* A_n *be a sequence of nonempty subsets of* X. *Then the following assertions hold:*

(i) *If* $A_n \overset{H}{\rightharpoonup} A$ *and* A *is closed, then* $\mathrm{Ls}A_n \subseteq A$.
(ii) $A_n \overset{H}{\rightharpoonup} A$ *if and only if for any* $\varepsilon > 0$, *there exists* $n_\varepsilon \in \mathbb{N}$ *such that* $A_n \subseteq A + \varepsilon B_X$ *for all* $n \geq n_\varepsilon$.

Lemma 2.5 (Han and Huang 2017; Alonso and Rodríguez-Marín 2005) *Assume that* D *is nonempty compact and* F *is u.s.c. on* D. *Then* $E_l(D) \neq \emptyset$.

Lemma 2.6 (Han et al. 2019) *Assume that* D *is convex and* F *is strictly natural quasi* C*-convex on* D *with nonempty compact values. Then* $E_l(D) = W_l(D)$.

Lemma 2.7 *Assume that* D *is closed and* F *is u.s.c. on* D *with nonempty compact values. Then* $Q_l(\varepsilon, x, D)$ *is closed.*

Proof It suffices to prove that $\left\{u \in D : F(u) \leq_\varepsilon^l F(x)\right\}$ is closed. Assume that

$$\{u_n\} \subseteq \left\{u \in D : F(u) \leq_\varepsilon^l F(x)\right\}$$

with $u_n \to u_0$. Then $u_0 \in D$ and $F(u_n) \leq_\varepsilon^l F(x)$, which means that $F(x) \subseteq F(u_n) + C + \varepsilon e$. For any $z \in F(x)$, there exists $v_n \in F(u_n)$ such that

$$z - v_n \in C + \varepsilon e. \tag{1}$$

Since F is u.s.c. at u_0, it follows from Lemma 2.2 that there exist $v_0 \in F(u_0)$ and a subsequence $\left\{v_{n_k}\right\}$ of $\{v_n\}$ such that $v_{n_k} \to v_0$. This together with (1) implies that $z - v_0 \in C + \varepsilon e$ and so $F(x) \subseteq F(u_0) + C + \varepsilon e$. Therefore, $u_0 \in \left\{u \in D : F(u) \leq_\varepsilon^l F(x)\right\}$. □

Lemma 2.8 *Assume that* $x_0 \in D$ *and* $F(x_0)$ *is compact.*

(i) *If* $\varepsilon > 0$, *then* $x_0 \in E_l(\varepsilon, D)$ *if and only if there does not exist* $y \in D$ *satisfying* $F(y) \leq_\varepsilon^l F(x_0)$*;*
(ii) *Then* $x_0 \in W_l(\varepsilon, D)$ *if and only if there does not exist* $y \in D$ *satisfying* $F(x) \leq_\varepsilon^u F(x_0)$.

Proof (i). It suffices to prove the necessity. Suppose that there exists $y_0 \in D$ such that $F(y_0) \leq_\varepsilon^l F(x_0)$, and so

$$F(x_0) \subseteq F(y_0) + C + \varepsilon e. \tag{2}$$

It follows from $x_0 \in E_l(\varepsilon, D)$ that $F(x_0) \leq_\varepsilon^l F(y_0)$. Then,

$$F(y_0) \subseteq F(x_0) + C + \varepsilon e. \tag{3}$$

Due to (2) and (3), we have

$$F(x_0) \subseteq F(y_0) + C + \varepsilon e \subseteq F(x_0) + C + 2\varepsilon e \subseteq F(x_0) + C \backslash \{0\}. \tag{4}$$

In view of Remark 2.1, we have $\operatorname{Min}(F(x_0)) \neq \emptyset$. Let $z_0 \in \operatorname{Min}(F(x_0))$. Consequently,

$$(F(x_0) - z_0) \cap (-C) = \{0\}. \tag{5}$$

It follows from (4) that there exists $u_0 \in F(x_0)$ and $c_0 \in C \backslash \{0\}$ such that $z_0 = u_0 + c_0$. Thus,

$$0 \neq -c_0 = u_0 - z_0 \in (F(x_0) - z_0) \cap (-C),$$

which contradicts (5).

The proof of (ii) is similar to the proof of (i) and so we omit it here. □

Lemma 2.9 *Assume that $\varepsilon > 0$, $x_0 \in D$ and $F(x_0)$ is compact. Then $x_0 \in E_l(\varepsilon, D)$ if and only if $Q_l(\varepsilon, x_0, D) = \{x_0\}$.*

Proof In view of Lemma 2.8, it is easy to see that the necessity is true. Next, we prove the sufficiency. We claim that

$$\left\{u \in D : F(u) \leq_\varepsilon^l F(x_0)\right\} = \emptyset. \tag{6}$$

In fact, if not, due to $Q_l(\varepsilon, x_0, D) = \{x_0\}$, we have $F(x_0) \leq_\varepsilon^l F(x_0)$ and so

$$F(x_0) \subseteq F(x_0) + C + \varepsilon e \subseteq F(x_0) + C \backslash \{0\}. \tag{7}$$

In view of Remark 2.1, similar to the proof of Lemma 2.8, we can see that (7) is not true. This shows that (6) holds. Thus, it follows from (6) and Lemma 2.8 that $x_0 \in E_l(\varepsilon, D)$. □

Lemma 2.10 *Assume that $x \in D$ and $F(y)$ is compact for any $y \in D$. Then*

$$E_l(\varepsilon, Q_l(\varepsilon, x, D)) \subseteq E_l(\varepsilon, D).$$

Proof Suppose to the contrary that there exists

$$v_0 \in E_l(\varepsilon, Q_l(\varepsilon, x, D)) \tag{8}$$

such that $v_0 \notin E_l(\varepsilon, D)$. It follows from $v_0 \notin E_l(\varepsilon, D)$ and Lemma 2.8 that there exists $z_0 \in D$ such that $F(z_0) \leq_\varepsilon^l F(v_0)$ and so

$$F(v_0) \subseteq F(z_0) + C + \varepsilon e. \tag{9}$$

We claim that $z_0 \in Q_l(\varepsilon, x, D)$. In fact, in view of $v_0 \in Q_l(\varepsilon, x, D)$, there are two cases to be considered.

Case 1. $v_0 = x$. It is clear that $F(z_0) \leq^l_\varepsilon F(x)$ and so $z_0 \in Q_l(\varepsilon, x, D)$.
Case 2. $F(v_0) \leq^l_\varepsilon F(x)$. Then

$$F(x) \subseteq F(v_0) + C + \varepsilon e. \tag{10}$$

Due to (9) and (10), we have

$$F(x) \subseteq F(v_0) + C + \varepsilon e \subseteq F(z_0) + C + C + 2\varepsilon e \subseteq F(z_0) + C + \varepsilon e,$$

which means that $F(z_0) \leq^l_\varepsilon F(x)$. Thus, $z_0 \in Q_l(\varepsilon, x, D)$.

In view of Lemma 2.8, $z_0 \in Q_l(\varepsilon, x, D)$ and $F(z_0) \leq^l_\varepsilon F(v_0)$ show that $v_0 \notin E_l(\varepsilon, Q_l(\varepsilon, x, D))$, which contradicts (8). □

3 Painlevé–Kuratowski convergence

In this section, we discuss the Painlevé–Kuratowski upper convergence and the Painlevé–Kuratowski lower convergence of the approximate solution sets for set optimization problems.

Lemma 3.1 *Let $\{D_n\}$ be a sequence of subsets of X, D be a bounded subset of X, $x_n \in D_n$ with $x_n \to x \in D$ and $\{\varepsilon_n\} \subseteq \mathbb{R}_+$ with $\varepsilon_n \to \varepsilon_0$. Assume that F is continuous on D with nonempty compact values and any of the following conditions is satisfied:*

(a) $\mathrm{Ls} D_n \subseteq D$ *and there exist* $\delta > 0$ *and* $n_0 \in \mathbb{N}$ *such that* $D_n \subseteq D + \delta B_X$ *for any* $n \geq n_0$*;*
(b) $D_n \stackrel{H}{\rightharpoonup} D$ *and D is closed;*
(c) $\varepsilon_0 = 0$, $\mathrm{Ls} D_n \subseteq D$*, D_n is convex and and F is naturally quasi C-convex on D_n.*

Then, for any $\alpha > 0$, there exists $\bar{n} \in \mathbb{N}$ such that

$$Q_l(\varepsilon_n, x_n, D_n) \subseteq Q_l(\varepsilon_0, x, D) + \alpha B_X, \quad \forall n \geq \bar{n}.$$

Proof Suppose to the contrary that there exists $\alpha_0 > 0$ such that, for any $n \in \mathbb{N}$, there exists $m_n \geq n$ satisfying

$$Q_l\left(\varepsilon_{m_n}, x_{m_n}, D_{m_n}\right) \not\subseteq Q_l(\varepsilon_0, x, D) + \alpha_0 B_X.$$

Without loss of generality, we assume that

$$Q_l(\varepsilon_n, x_n, D_n) \not\subseteq Q_l(\varepsilon_0, x, D) + \alpha_0 B_X, \quad \forall n \in \mathbb{N}.$$

Then there exists

$$v_n \in Q_l(\varepsilon_n, x_n, D_n) \tag{11}$$

such that

$$v_n \notin Q_l(\varepsilon_0, x, D) + \alpha_0 B_X. \tag{12}$$

It is clear that $v_n \in D_n$. It follows from $x_n \to x$ and $x \in Q_l(\varepsilon_0, x, D)$ that

$$x_n \in Q_l(\varepsilon_0, x, D) + \alpha_0 B_X \tag{13}$$

for n large enough. This together with (12) implies that $x_n \neq v_n$ for n large enough. Thus, by (11), we have $F(v_n) \preceq^l_{\varepsilon_n} F(x_n)$ and so

$$F(x_n) \subseteq F(v_n) + C + \varepsilon_n e. \tag{14}$$

(a) By virtue of condition (a), we can see that $\{v_n\} \subseteq X$ is bounded. Without loss of generality, we assume that $v_n \to v_0 \in X$. In view of $\mathrm{Ls}D_n \subseteq D$, we have $v_0 \in D$. We claim that

$$F(x) \subseteq F(v_0) + C + \varepsilon_0 e. \tag{15}$$

In fact, if not, then there exists $z_0 \in F(x)$ such that

$$z_0 \notin F(v_0) + C + \varepsilon_0 e. \tag{16}$$

Since F is l.s.c. at x, by Lemma 2.1, there exists $z_n \in F(x_n)$ such that $z_n \to z_0$. Due to (14), there exists $u_n \in F(v_n)$ such that

$$z_n - u_n \in C + \varepsilon_n e. \tag{17}$$

Since F is u.s.c. at v_0, it follows from Lemma 2.2 that there exist $u_0 \in F(v_0)$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to u_0$. In view of (17), we have $z_0 - u_0 \in C + \varepsilon_0 e$. which contradicts (16). Therefore, (15) holds, which means that $v_0 \in Q_l(\varepsilon_0, x, D)$. Noting that

$$v_n \to v_0 \in Q_l(\varepsilon_0, x, D) + \alpha_0 B_X,$$

we can see that $v_n \in Q_l(\varepsilon_0, x, D) + \alpha_0 B_X$ for n large enough, which contradicts (12).

(b) In view of Lemma 2.4, it is easy to see that condition (b) implies condition (a).

(c) Since $F(x)$ is compact and F is l.s.c. at x. we can see that F is H-l.s.c. at x. Noting that $C - \frac{1}{n}e$ is a neighborhood of $0 \in X$, then there exists $m_n \geq n$ such that $F(x) \subseteq F(x_{m_n}) + C - \frac{1}{n}e$. Without loss of generality, we assume that

$$F(x) \subseteq F(x_n) + C - \frac{1}{n}e, \quad \forall n \in \mathbb{N}. \tag{18}$$

Due to (14) and (18), we have

$$F(x) \subseteq F(v_n) + C + \left(\varepsilon_n - \frac{1}{n}\right)e. \tag{19}$$

Let $x_n(t) = tv_n + (1-t)x_n$ for $t \in [0,1]$. By the convexity of D_n, we have $x_n(t) \in D_n$. Since F is naturally quasi C-convex on D_n, for the above $t \in [0,1]$, there exists $\lambda \in [0,1]$ such that

$$\lambda F(v_n) + (1-\lambda) F(x_n) \subseteq F(x_n(t)) + C. \tag{20}$$

Applying (18)–(20), we have

$$\begin{aligned} F(x) &\subseteq \lambda F(x) + (1-\lambda) F(x) \\ &\subseteq \lambda F(v_n) + \lambda C + \lambda \left(\varepsilon_n - \frac{1}{n}\right) e \\ &\quad + (1-\lambda) F(x_n) + (1-\lambda) C - (1-\lambda) \frac{1}{n} e \\ &\subseteq F(x_n(t)) + \lambda C - \frac{1}{n} e + (1-\lambda) C + C + \lambda \varepsilon_n e \\ &\subseteq F(x_n(t)) + C - \frac{1}{n} e. \end{aligned} \tag{21}$$

It follows from (12) and (13) that there exists $t_n \in [0,1]$ such that $x_n(t_n) \in \partial [Q_l(\varepsilon_0, x, D) + \alpha_0 B_X]$. Since $Q_l(\varepsilon_0, x, D) + \alpha_0 B_X$ is bounded and $X = \mathbb{R}^m$, it is easy to see that $\partial [Q_l(\varepsilon_0, x, D) + \alpha_0 B_X]$ is compact. Without loss of generality, we assume that

$$x_n(t_n) \to w \in \partial [Q_l(\varepsilon_0, x, D) + \alpha_0 B_X] \tag{22}$$

Due to (22), $x_n(t_n) \in D_n$ and $\mathrm{Ls} D_n \subseteq D$, we have $w \in D$. It follows from (21) that

$$F(x) \subseteq F(x_n(t_n)) + C - \frac{1}{n} e. \tag{23}$$

We claim that

$$F(x) \subseteq F(w) + C = F(w) + C + \varepsilon_0 e. \tag{24}$$

In fact, for any $v \in F(x)$, it follows from (23) that there exists $z_n \in F(x_n(t_n))$ such that

$$v - z_n + \frac{1}{n} e \in C. \tag{25}$$

Since F is u.s.c. at $w \in D$, by Lemma 2.2, there exist $z_0 \in F(w)$ and a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \to z_0$. Due to (25) and the closedness of C, we have $v - z_0 \in C$, and so $v \in z_0 + C \subseteq F(w) + C$. Thus, $F(x) \subseteq F(w) + C = F(w) + C + \varepsilon_0 e$. This together with $w \in D$ implies that $w \in Q_l(\varepsilon_0, x, D)$, which contradicts (22). This completes the proof.

□

Theorem 3.1 *Let $\{D_n\}$ be a sequence of subsets of X and $\{\varepsilon_n\} \subseteq \mathbb{R}_+$ with $\varepsilon_n \to \varepsilon_0$. Assume that F is continuous on D with nonempty compact values and $D_n \stackrel{K}{\to} D$. Then* $\mathrm{Ls} W_l(\varepsilon_n, D_n) \subseteq W_l(\varepsilon_0, D)$.

Proof Let $x_0 \in \text{Ls}W_l(\varepsilon_n, D_n)$. Then there exist a subsequence $\{n_k\}$ of the integers and $x_{n_k} \in W_l(\varepsilon_{n_k}, D_{n_k})$ such that $x_{n_k} \to x_0$. Due to $D_n \overset{K}{\to} D$, we have $x_0 \in D$.

We now show that $x_0 \in W_l(\varepsilon_0, D)$. Suppose to the contrary that $x_0 \notin W_l(\varepsilon_0, D)$. Then, in view of Lemma 2.8, there exists $y_0 \in D$ such that $F(y_0) \ll^l_{\varepsilon_0} F(x_0)$ and so

$$F(x_0) \subseteq F(y_0) + \text{int}C + \varepsilon_0 e. \tag{26}$$

Due to $y_0 \in D$ and $D_n \overset{K}{\to} D$, there exists $y_n \in D$ such that $y_n \to y_0$. We claim that there exists $k_0 \in \mathbb{N}$ such that

$$F(x_{n_k}) \subseteq F(y_{n_k}) + \text{int}C + \varepsilon_{n_k} e, \quad \forall k \geq k_0. \tag{27}$$

In fact, if not, then there exist a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ and a subsequence $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ such that

$$F\left(x_{n_{k_j}}\right) \not\subseteq F\left(y_{n_{k_j}}\right) + \text{int}C + \varepsilon_{n_{k_j}} e.$$

Without loss of generality, we assume that

$$F(x_{n_k}) \not\subseteq F(y_{n_k}) + \text{int}C + \varepsilon_{n_k} e, \quad \forall k \in \mathbb{N}.$$

Then there exists $v_{n_k} \in F(x_{n_k})$ such that

$$v_{n_k} \notin F(y_{n_k}) + \text{int}C + \varepsilon_{n_k} e. \tag{28}$$

Since F is u.s.c. at x_0, by Lemma 2.2, without loss of generality, we assume that $v_{n_k} \to v_0 \in F(x_0)$. It follows from (26) that there exists $u_0 \in F(y_0)$ such that

$$v_0 - u_0 \in \text{int}C + \varepsilon_0 e. \tag{29}$$

Noting that F is l.s.c. at y_0, in view of Lemma 2.1, there exists $u_{n_k} \in F(y_{n_k})$ such that $u_{n_k} \to u_0$. Due to (29), we have $v_{n_k} - u_{n_k} \in \text{int}C + \varepsilon_{n_k} e$ for k large enough, which contradicts (28). This shows that (27) holds and so $F(y_{n_k}) \ll^l_{\varepsilon_{n_k}} F(x_{n_k})$. It follows from Lemma 2.8 that $x_{n_k} \notin W_l(\varepsilon_{n_k}, D_{n_k})$, which contradicts $x_{n_k} \in W_l(\varepsilon_{n_k}, D_{n_k})$. This shows that $x_0 \in W_l(\varepsilon_0, D)$. □

Theorem 3.2 *Let $\{D_n\}$ be a sequence of subsets of X, D be a bounded subset of X and $\{\varepsilon_n\} \subseteq \mathbb{R}_+$ with $\varepsilon_n \to \varepsilon_0 \in \mathbb{R}_+$. Assume that F is continuous on D with nonempty compact values, $D \subseteq \text{Li}D_n$, D_n is closed and F is u.s.c. on D_n, and suppose that any of the following conditions is satisfied:*

(a) *$\varepsilon_0 > 0$, $\text{Ls}D_n \subseteq D$ and there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $D_n \subseteq D + \delta B_X$ for any $n \geq n_0$;*

(b) *$\varepsilon_0 > 0$, $D_n \overset{H}{\rightharpoonup} D$ and D is closed;*

(c) $\varepsilon_0 = 0$, $\mathrm{Ls}D_n \subseteq D$, D_n *and* D *are convex,* F *is naturally quasi* C*-convex on* D_n *and* F *is strictly naturally quasi* C*-convex on* D.

Then, $E_l(\varepsilon_0, D) \subseteq \mathrm{Li}E_l(\varepsilon_n, D_n)$. *Moreover, if* D *is convex and* F *is strictly naturally quasi* C*-convex on* D*, then* $W_l(\varepsilon_0, D) \subseteq \mathrm{Li}W_l(\varepsilon_n, D_n)$.

Proof Let $x_0 \in E_l(\varepsilon_0, D)$. Due to $x_0 \in D$ and $D \subseteq \mathrm{Li}D_n$, there exists $x_n \in D_n$ such that $x_n \to x_0$. If $\varepsilon_0 > 0$, in view of Lemma 2.9 and $x_0 \in E_l(\varepsilon_0, D)$, we have $Q_l(\varepsilon_0, x_0, D) = \{x_0\}$. If $\varepsilon_0 = 0$, it follows from Lemma 3.1 of Han et al. (2019) that $Q_l(\varepsilon_0, x_0, D) = \{x_0\}$. By Lemma 3.1 , we can see that for any $\alpha > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$Q_l(\varepsilon_n, x_n, D_n) \subseteq Q_l(\varepsilon_0, x_0, D) + \alpha B_X = \{x_0\} + \alpha B_X, \quad \forall n \geq n_0. \tag{30}$$

It follows from Lemma 2.7 that $Q_l(\varepsilon_n, x_n, D_n)$ is closed. Due to (30), we know that $Q_l(\varepsilon_n, x_n, D_n)$ is bounded, and consequently, $Q_l(\varepsilon_n, x_n, D_n)$ is compact. In view of Remark 2.3 and Lemma 2.5, we have

$$E_l(\varepsilon_n, Q_l(\varepsilon_n, x_n, D_n)) \neq \emptyset.$$

Let $v_n \in E_l(\varepsilon_n, Q_l(\varepsilon_n, x_n, D_n))$. From Lemma 2.10, one has

$$v_n \in E_l(\varepsilon_n, Q_l(\varepsilon_n, x_n, D_n)) \subseteq E_l(\varepsilon_n, D_n).$$

Due to $v_n \in Q_l(\varepsilon_n, x_n, D_n)$ and (30), we have $v_n \to x_0$. Therefore, $x_0 \in \mathrm{Li}E_l(\varepsilon_n, D_n)$ and so $E_l(\varepsilon_0, D) \subseteq \mathrm{Li}E_l(\varepsilon_n, D_n)$.

We next show that $W_l(\varepsilon_0, D) \subseteq \mathrm{Li}W_l(\varepsilon_n, D_n)$. For any open set V with $V \cap W_l(\varepsilon_0, D) \neq \emptyset$, we claim that

$$V \cap E_l(\varepsilon_0, D) \neq \emptyset. \tag{31}$$

In fact, let $z_0 \in V \cap W_l(\varepsilon_0, D)$. Suppose that $V \cap E_l(\varepsilon_0, D) = \emptyset$. Then $z_0 \notin E_l(\varepsilon_0, D)$. It follows from Lemma 2.8 that there exists $y_0 \in D$ such that $F(y_0) \preceq^l_{\varepsilon_0} F(z_0)$ and so

$$F(z_0) \subseteq F(y_0) + C + \varepsilon_0 e \subseteq F(y_0) + C \setminus \{0\}. \tag{32}$$

Noting Remark 2.1 and (4), from the proof of Lemma 2.8, we can see that $z_0 \neq y_0$. Since F is strictly naturally quasi C-convex on D, for any $t \in (0, 1)$, there exists $\lambda \in [0, 1]$ such that

$$\lambda F(z_0) + (1 - \lambda) F(y_0) \subseteq F(tz_0 + (1 - t) y_0) + \mathrm{int}C. \tag{33}$$

By (32) and (33), we have

$$F(z_0) \subseteq F(tz_0 + (1 - t) y_0) + \mathrm{int}C, \quad \forall t \in (0, 1). \tag{34}$$

In fact, if (34) does not hold, then there exist $t_0 \in (0, 1)$ and $u_0 \in F(z_0)$ such that

$$u_0 \notin F(t_0 z_0 + (1 - t_0) y_0) + \mathrm{int} C. \tag{35}$$

It follows from (32) that there exist $s_0 \in F(y_0)$ and $c_0 \in C$ such that $u_0 = s_0 + c_0$. Due to (33), one has

$$\lambda u_0 + (1 - \lambda) s_0 \in F(t_0 z_0 + (1 - t_0) y_0) + \mathrm{int} C.$$

This together with $u_0 = s_0 + c_0$ implies that $u_0 \in F(t_0 z_0 + (1 - t_0) y_0) + \mathrm{int} C$, which contradicts (35). Therefore, (34) is true. Let $z(t) = t z_0 + (1 - t) y_0$ for $t \in (0, 1)$. It is clear that there exists $\hat{t} \in (0, 1)$ such that $z(\hat{t}) \in V$. This together with $V \cap E_l(\varepsilon_0, D) = \emptyset$ implies that $z(\hat{t}) \notin E_l(\varepsilon_0, D)$. In view of Lemma 2.8, there exists $w_0 \in D$ such that $F(w_0) \preceq^l_{\varepsilon_0} F(z(\hat{t}))$ and so

$$F(z(\hat{t})) \subseteq F(w_0) + C + \varepsilon_0 e. \tag{36}$$

Due to (34) and (36), we have

$$F(z_0) \subseteq F(z(\hat{t})) + \mathrm{int} C \subseteq F(w_0) + \mathrm{int} C + C + \varepsilon_0 e \subseteq F(w_0) + \mathrm{int} C + \varepsilon_0 e,$$

which means that $F(w_0) \ll^l_{\varepsilon_0} F(z_0)$. From Lemma 2.8, we have $z_0 \notin W_l(\varepsilon_0, D)$, which contradicts $z_0 \in W_l(\varepsilon_0, D)$. Hence, (31) holds. Applying (31), $E_l(\varepsilon_0, D) \subseteq \mathrm{Li} E_l(\varepsilon_n, D_n)$ and Lemma 2.3, we know that there exists $\bar{n} \in \mathbb{N}$ such that

$$V \cap E_l(\varepsilon_n, D_n) \neq \emptyset, \quad \forall n \geq \bar{n}. \tag{37}$$

It follows from Remark 2.4 that $E_l(\varepsilon_n, D_n) \subseteq W_l(\varepsilon_n, D_n)$. This together with (37) implies that

$$V \cap W_l(\varepsilon_n, D_n) \neq \emptyset, \quad \forall n \geq \bar{n}.$$

Thus, from Lemma 2.3, we have $W_l(\varepsilon_0, D) \subseteq \mathrm{Li} W_l(\varepsilon_n, D_n)$. □

Now, we give an example to illustrate Theorems 3.1 and 3.2.

Example 3.1 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$. Define a set-valued mapping $F : X \to 2^Y$ as follows:

$$F(x) = \left(-5 \cos \frac{1}{6} x, x^2 - x + 3\right) + B_Y, \quad x \in X.$$

Let $D = [-9, 9]$ and $D_n = \left[-9 + \sin \frac{1}{n}, 8 + \cos \frac{1}{n}\right]$. Then it is easy to see that $D_n \xrightarrow{K} D$ and F is strictly naturally quasi C-convex on D. Let $\varepsilon_0 = 1$, $\varepsilon_n = \frac{n+1}{n}$ and $e = (1, 1)$. Then,

$$0 \in E_l(\varepsilon_0, D) \subseteq W_l(\varepsilon_0, D), \quad 0 \in E_l(\varepsilon_n, D_n) \subseteq W_l(\varepsilon_n, D_n).$$

It is easy to check that all conditions of Theorems 3.1 and 3.2 are satisfied. Thus, Theorem 3.1 shows that $\mathrm{Ls}W(\varepsilon_n, D_n) \subseteq W(\varepsilon_0, D)$ and Theorem 3.2 implies that $E_l(\varepsilon_0, D) \subseteq \mathrm{Li}E_l(\varepsilon_n, D_n)$ and $W_l(\varepsilon_0, D) \subseteq \mathrm{Li}W_l(\varepsilon_n, D_n)$.

Theorem 3.3 *Let $\{D_n\}$ be a sequence of subsets of X, D be a convex subset of X and $\{\varepsilon_n\} \subseteq \mathbb{R}_+$ with $\varepsilon_n \to \varepsilon_0 = 0$. Assume that*

(i) F is continuous on D with nonempty compact values and $D_n \xrightarrow{K} D$.
(ii) F is strictly naturally quasi C-convex on D.

Then, $\mathrm{Ls}E_l(\varepsilon_n, D_n) \subseteq E_l(\varepsilon_0, D)$.

Proof Noting that $\varepsilon_0 = 0$ and Lemma 2.6, we have $E_l(\varepsilon_0, D) = E_l(D) = W_l(D) = W_l(\varepsilon_0, D)$. From Remark 2.4, we have $E_l(\varepsilon_n, D_n) \subseteq W_l(\varepsilon_n, D_n)$ and so $\mathrm{Ls}E_l(\varepsilon_n, D_n) \subseteq \mathrm{Ls}W_l(\varepsilon_n, D_n)$. It follows from Theorem 3.1 that $\mathrm{Ls}W(\varepsilon_n, D_n) \subseteq W(\varepsilon_0, D)$. Consequently,

$$\mathrm{Ls}E_l(\varepsilon_n, D_n) \subseteq \mathrm{Ls}W_l(\varepsilon_n, D_n) \subseteq W_l(\varepsilon_0, D) = E_l(\varepsilon_0, D).$$

This completes the proof. □

If $\varepsilon_0 > 0$ and F is strictly naturally quasi C-convex on D, then $E_l(\varepsilon_0, D) = W_l(\varepsilon_0, D)$ may not be true. We give the following counterexample to illustrate it.

Example 3.2 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+ = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\right\}$. Define a set-valued mapping $F : X \to 2^Y$ as follows:

$$F(x) = \left(-2\cos\frac{1}{3}x, x^2\right) + B_Y, \quad x \in X.$$

Let $D = [-4, 4]$, $\varepsilon_0 = 1$ and $e = (1, 1)$. Then it is easy to see that F is strictly naturally quasi C-convex on D. Moreover, from Lemma 2.8, we can see that $E_l(\varepsilon_0, D) = (-\pi, \pi)$ and $W_l(\varepsilon_0, D) = [-\pi, \pi]$. Thus, $E_l(\varepsilon_0, D) \neq W_l(\varepsilon_0, D)$.

Inspired by Example 3.2, we give the following proposition.

Proposition 3.1 *Let $\varepsilon_0 > 0$, D be a nonempty and compact subset of X and F be continuous on D with nonempty and compact values. Then $E_l(\varepsilon_0, D)$ is open in D and $W_l(\varepsilon_0, D)$ is closed in D.*

Proof We show that $E_l(\varepsilon_0, D)$ is open in D. For any $\{x_n\} \subseteq D\backslash E_l(\varepsilon_0, D)$ with $x_n \to x_0 \in D$, it follows from Lemma 2.8 that there exists $y_n \in D$ such that $F(y_n) \preceq^l_{\varepsilon_0} F(x_n)$ and so

$$F(x_n) \subseteq F(y_n) + C + \varepsilon_0 e. \tag{38}$$

Since D is compact, without loss of generality, we assume that $y_n \to y_0 \in D$. Similar to the proof of (15), by (38), it is easy to prove that $F(x_0) \subseteq F(y_0) + C + \varepsilon_0 e$. This together with Lemma 2.8 implies that $x_0 \notin E_l(\varepsilon_0, D)$. Therefore, $E_l(\varepsilon_0, D)$ is open in D.

Next, we prove that $W_l(\varepsilon_0, D)$ is closed in D. For any $\{x_n\} \subseteq W_l(\varepsilon_0, D)$ with $x_n \to x_0 \in D$, it suffices to show that $x_0 \in W_l(\varepsilon_0, D)$. Suppose that $x_0 \notin W_l(\varepsilon_0, D)$. Then it follows from Lemma 2.8 that there exists $v_0 \in D$ such that $F(v_0) \ll^l_{\varepsilon_0} F(x_0)$, i.e.,

$$F(x_0) \subseteq F(v_0) + \text{int}C + \varepsilon_0 e.$$

It is clear that $F(v_0) + \text{int}C + \varepsilon_0 e$ is a neighborhood of $F(x_0)$. Since F is u.s.c. at x_0, there exists $n_0 \in \mathbb{N}$ such that

$$F(x_n) \subset F(v_0) + \text{int}C + \varepsilon_0 e, \quad \forall n \geq n_0,$$

which means that $F(v_0) \ll^l_{\varepsilon} F(x_n)$. This together with Lemma 2.8 implies that $x_n \notin W_l(\varepsilon_0, D)$, which contradicts $\{x_n\} \subseteq W_l(\varepsilon_0, D)$. This shows that $x_0 \in W_l(\varepsilon_0, D)$ and so $W_l(\varepsilon_0, D)$ is closed in D. ◻

Remark 3.1 In Theorem 3.3, we show that $\text{Ls}E_l(\varepsilon_n, D_n) \subseteq E_l(\varepsilon_0, D)$ under the assumption that $\varepsilon_0 = 0$. If $\varepsilon_0 > 0$, then it follows from Proposition 3.1 that $E_l(\varepsilon_0, D) \neq W_l(\varepsilon_0, D)$ in the general case. Therefore, it is interesting to obtain some suitable conditions to ensure $\text{Ls}E_l(\varepsilon_n, D_n) \subseteq E_l(\varepsilon_0, D)$ holds when $\varepsilon_0 > 0$.

4 Extended well-posedness

The concept of the extended well-posedness for vector optimization problems is due to Huang (2000b, 2001). In this section, we introduce the notions of the extended well-posedness and the weak extended well-posedness for set optimization problems. We denote by $d(x, A) = \inf\{\|y - a\|, a \in A\}$ the distance of a point $x \in X$ to a set $A \subseteq X$.

Definition 4.1 Let $\{D_n\}$ be a sequence of subsets of X and D be a subset of X. We say that (SOP) is extended well-posed (resp., weak extended well-posed) with respect to the perturbation defined by the sequence $\{D_n\}$ if $E_l(D) \neq \emptyset$ (resp., $W_l(D) \neq \emptyset$) and for every sequence $\{x_n\} \subset D_n$ with $x_n \in E_l(\varepsilon_n, D_n)$ (resp., $x_n \in W_l(\varepsilon_n, D_n)$) for some sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0^+$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k}, E_l(D)) \to 0$ (resp., $d(x_{n_k}, W_l(D)) \to 0$) as $k \to +\infty$.

Theorem 4.1 *Let $\{D_n\}$ be a sequence of subsets of X and D be a compact subset of X. Assume that*

(i) *F is continuous on D with nonempty compact values and for any $x \in D_n$, $F(x)$ is compact;*

(ii) *$D_n \xrightarrow{K} D$ and there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $D_n \subseteq D + \delta B_X$ for any $n \geq n_0$.*

Then (SOP) is weak extended well-posed. Moreover, if D is convex and F is strictly naturally quasi C-convex on D, then (SOP) is extended well-posed.

Proof In view of Proposition 2.7 of Hernández and Rodríguez-Marín (2007), we have $E_l(D) \subseteq W_l(D)$. This together with Lemma 2.5 implies that $W_l(D) \neq \emptyset$. Suppose that (SOP) is not weak extended well-posed. Then we can find a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0^+$ and $x_n \in W_l(\varepsilon_n, D_n)$ such that $d(x_n, W_l(D)) \nrightarrow 0$ as $n \to +\infty$. Thus, there exists $\delta > 0$ such that, for any $n \in \mathbb{N}$, there exists $m_n \geq n$ satisfying $x_{m_n} \notin W_l(D) + \delta B_X$. Without loss of generality, we assume that

$$x_n \notin W_l(D) + \delta B_X, \quad \forall n \in \mathbb{N}. \tag{39}$$

Then it is easy to see that $x_n \in D_n$. By condition (ii), we can see that $\{x_n\} \subseteq X$ is bounded. Without loss of generality, we assume that $x_n \to x_0 \in X$. It follows from $D_n \overset{K}{\to} D$ that $x_0 \in D$. Suppose that $x_0 \notin W_l(D)$. In view of Lemma 2.8, there exists $y_0 \in D$ such that $F(y_0) \ll^l F(x_0)$ and so

$$F(x_0) \subseteq F(y_0) + \mathrm{int}C. \tag{40}$$

Due to $D \subseteq \mathrm{Li}D_n$, there exists $y_n \in D_n$ such that $y_n \to y_0$.

We claim that there exists $n_0 \in \mathbb{N}$ such that

$$F(x_n) \subseteq F(y_n) + \mathrm{int}C + \varepsilon_n e, \quad \forall n \geq n_0. \tag{41}$$

In fact, if not, then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$F\left(x_{n_k}\right) \not\subseteq F\left(y_{n_k}\right) + \mathrm{int}C + \varepsilon_{n_k} e.$$

Without loss of generality, we assume that

$$F(x_n) \not\subseteq F(y_n) + \mathrm{int}C + \varepsilon_n e, \quad \forall n \in \mathbb{N}.$$

Then there exists $v_n \in F(x_n)$ such that

$$v_n \notin F(y_n) + \mathrm{int}C + \varepsilon_n e. \tag{42}$$

Since F is u.s.c. at x_0, by Lemma 2.2, there exist $v_0 \in F(x_0)$ and a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $v_{n_k} \to v_0$. It follows from (40) that there exists $u_0 \in F(y_0)$ such that

$$v_0 - u_0 \in \mathrm{int}C. \tag{43}$$

Noting that F is l.s.c. at y_0, in view of Lemma 2.1, there exists $u_n \in F(y_n)$ such that $u_n \to u_0$. By (43), we have $v_{n_k} - u_{n_k} \in \mathrm{int}C + \varepsilon_{n_k} e$ for k large enough, which contradicts (42). This shows that (41) holds and so $F(y_n) \ll^l_{\varepsilon_n} F(x_n)$. It follows from Lemma 2.8 that $x_n \notin W_l(\varepsilon_n, D_n)$, which contradicts $x_n \in W_l(\varepsilon_n, D_n)$.

Therefore, $x_0 \in W_l(D)$. Noting that $x_n \to x_0 \in W_l(D) + \delta B_X$, we can see that $x_n \in W_l(D) + \delta B_X$ for n large enough, which contradicts (39). This shows that (SOP) is weak extended well-posed.

Next, we show that (SOP) is extended well-posed. Since F is strictly naturally quasi C-convex on D, it follows from Lemma 2.6 that $E_l(D) = W_l(D)$. For every sequence $\{x_n\} \subset D_n$ with $x_n \in E_l(\varepsilon_n, D_n)$ for some sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0^+$, it follows from Remark 2.4 that $x_n \in E_l(\varepsilon_n, D_n) \subseteq W_l(\varepsilon_n, D_n)$. Noting that (SOP) is weak extended well-posed, then there exists a subsequence $\left\{x_{n_k}\right\}$ of $\{x_n\}$ such that $d\left(x_{n_k}, W_l(D)\right) \to 0$ as $k \to +\infty$. This together with $E_l(D) = W_l(D)$ implies that $d\left(x_{n_k}, E_l(D)\right) \to 0$ as $k \to +\infty$. This show that (SOP) is extended well-posed. □

From Lemma 2.4 and Theorem 4.1, we can get the following corollary.

Corollary 4.1 *Let $\{D_n\}$ be a sequence of subsets of X and D be a compact subset of X. Assume that*

(i) *F is continuous on D with nonempty compact values and for any $x \in D_n$, $F(x)$ is compact;*
(ii) *$D_n \overset{H}{\rightharpoonup} D$ and $D \subseteq \mathrm{Li} D_n$.*

Then (SOP) is weak extended well-posed. Moreover, if D is convex and F is strictly naturally quasi C-convex on D, then (SOP) is extended well-posed.

Now, we give an example to illustrate Theorem 4.1.

Example 4.1 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+ = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\right\}$. Define a set-valued mapping $F : X \to 2^Y$ as follows:

$$F(x) = \left(-2\sin\frac{1}{3}x, (x-4)^2 + 1\right) + B_Y, \quad x \in X.$$

Let $D = [0, 3\pi]$ and $D_n = \left[\sin\frac{1}{n}, 3\pi - 1 + \cos\frac{1}{n}\right]$. Then it is easy to see that $D_n \overset{K}{\to} D$ and F is strictly naturally quasi C-convex on D. Let $\varepsilon_0 = 1$, $\varepsilon_n = \frac{n+1}{n}$ and $e = (1, 1)$. Then we can see that $4 \in E_l(\varepsilon_0, D) \subseteq W_l(\varepsilon_0, D)$. Moreover, we can check that all conditions of Theorem 4.1 are satisfied. Thus, Theorem 4.1 shows that (SOP) is weak extended well-posed and extended well-posed.

Theorem 4.2 *Let $\{D_n\}$ be a sequence of convex subsets of X and D be a convex and compact subset of X. Assume that*

(i) *F is continuous and strictly naturally quasi C-convex on D with nonempty compact values;*
(ii) *$D_n \overset{K}{\to} D$ and F is naturally quasi C-convex on D_n with nonempty compact values;*
(iii) *for any $\varepsilon > 0$ and for any $n \in \mathbb{N}$, $W_l(\varepsilon, D_n)$ is connected.*

Then (SOP) is weak extended well-posed. Moreover, (SOP) is extended well-posed.

Proof Suppose that (SOP) is not weak extended well-posed. Then we can find a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0^+$ and $x_n \in W_l(\varepsilon_n, D_n)$ such that $d(x_n, W_l(D)) \nrightarrow 0$

as $n \to +\infty$. Thus, there is a constant $\delta > 0$ such that, for any $n \in \mathbb{N}$, there exists $m_n \geq n$ satisfying $x_{m_n} \notin W_l(D) + \delta B_X$. Without loss of generality, we assume that

$$x_n \notin W_l(D) + \delta B_X, \quad \forall n \in \mathbb{N}. \tag{44}$$

Now we claim that there exists

$$z_n \in \partial [W_l(D) + \delta B_X] \cap W_l(\varepsilon_n, D_n). \tag{45}$$

In fact, if $\partial [W_l(D) + \delta B_X] \cap W_l(\varepsilon_n, D_n) = \emptyset$, then

$$W_l(\varepsilon_n, D_n) \subseteq \operatorname{int}(W_l(D) + \delta B_X) \cup (W_l(D) + \delta B_X)^c. \tag{46}$$

It follows from (44) that

$$W_l(\varepsilon_n, D_n) \cap (W_l(D) + \delta B_X)^c \neq \emptyset. \tag{47}$$

Now we show that

$$W_l(\varepsilon_n, D_n) \cap \operatorname{int}(W_l(D) + \delta B_X) \neq \emptyset. \tag{48}$$

Let $v_0 \in W_l(D)$. In view of Theorem 3.2, we have $W_l(D) = W_l(0, D) \subseteq \mathrm{Li}W_l(\varepsilon_n, D_n)$ and so there exists $v_n \in W_l(\varepsilon_n, D_n)$ such that $v_n \to v_0 \in \operatorname{int}(W_l(D) + \delta B_X)$. This means that $v_n \in \operatorname{int}(W_l(D) + \delta B_X)$ for n large enough, which implies that (48) holds. Due to Proposition 2.3 of Han and Huang (2017), we can see that $W_l(D)$ is closed. Since B_X is compact, we obtain that $W_l(D) + \delta B_X$ is closed. Thus, it follows from (46)–(48) that $W_l(\varepsilon_n, D_n)$ is not connected, which contradicts condition (iii). Therefore, we know that (45) holds.

Noting that $\partial [W_l(D) + \delta B_X]$ is compact, without loss of generality, we assume that

$$z_n \to z_0 \in \partial [W_l(D) + \delta B_X]. \tag{49}$$

Due to $\mathrm{Ls}D_n \subseteq D$ and $z_n \in D_n$, we have $z_0 \in D$.

Next we claim that $z_0 \in W_l(D)$. In fact, if not, it follows from Lemma 2.8 that there exists $y_0 \in D$ such that $F(y_0) \ll^l F(z_0)$ and so $F(z_0) \subseteq F(y_0) + \mathrm{int}C$. Due to $D \subseteq \mathrm{Li}D_n$, there exists $y_n \in D_n$ such that $y_n \to y_0$. Similar to the proof of (41), we can prove that there exists $n_0 \in \mathbb{N}$ such that

$$F(z_n) \subseteq F(y_n) + \mathrm{int}C + \varepsilon_n e, \quad \forall n \geq n_0. \tag{50}$$

(50) yields $F(y_n) \ll^l_{\varepsilon_n} F(z_n)$. This together with 2.8 implies that $z_n \notin W_l(\varepsilon_n, D_n)$, which contradicts (45). Therefore, $z_0 \in W_l(D)$, which contradicts (49).

Similar to the proof of Theorem 4.1, we can show that (SOP) is extended well-posed. This completes the proof. □

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