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ORIGINAL ARTICLE

An exact solution to a robust portfolio choice problem with multiple risk measures under ambiguous distribution

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Abstract This paper proposes a unified framework to solve distributionally robust mean-risk optimization problem that simultaneously uses variance, value-at-risk (VaR) and conditional value-at-risk (CVaR) as a triple-risk measure. It provides investors with more flexibility to find portfolios in the sense that it allows investors to optimize a return-risk profile in the presence of estimation error. We derive a closed-form expression for the optimal portfolio strategy to the robust mean-multiple risk portfolio selection model under distribution and mean return ambiguity (RMP). Specially, the robust mean-variance, robust maximum return, robust minimum VaR and robust minimum CVaR efficient portfolios are all special instances of RMP portfolios. We analytically and numerically show that the resulting portfolio weight converges to the minimum variance portfolio when the level of ambiguity aversion is in a high value. Using numerical experiment with simulated data, we demonstrate that our robust portfolios under ambiguity are more stable over time than the non-robust portfolios.

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1 Introduction

Since Markowitz (1952) developed the mean-variance portfolio model by incorporating the variance as a risk measure, the mean-risk framework is so intuitive and powerful that it has stimulated intensive research activities in finance and risk management. In mean-risk models, two scalers or criteria are attached to each random variable: the expected value and the value of a chosen risk measure. Rather than a single optimal solution, we derive a set of optimal portfolios through trade-off between risk and return. Thus, mean-risk models have a ready and proper interpretation of results and in most cases are very convenient from a computational point of view.

Variance is the first risk measure used in mean-risk models and, in spite of many subsequent proposals of risk measures, is still the most widely used measure of risk in the practice of portfolio selection. However, this measure fails to capture the downside risk. To circumvent this problem, researchers proposed value-at-risk (VaR) and conditional value-at-risk (CVaR), which have now been widely used as market risk measures. They measure the probable loss from a different perspective. In practice, however, no matter which type of risk measure is adopted, whether the risk of a portfolio can be well evaluated depends mainly on the reliability and the accuracy of the estimated parameters or the distributions of asset returns. Not only are we unable to obtain the exact distributions of risky assets, but also to get exact estimations of parameters. The common solution applied in practice is to replace the unknown parameters by their sample estimators. Since the data are often prone to errors, using estimates from limited historical data in the mean-risk models introduces estimation risk in portfolio selection. A popular approach to tackle this issue is to use robust portfolio optimization, which offers vehicles to incorporate estimation error into the decision making process in portfolio choice. There exist many studies on robust optimization methodology to deal with the effect of estimation errors on the estimates of expected returns; see, for instance, Goldfarb and Iyengar (2003) and Garlappi et al. (2007). To the best of our knowledge, with the exception of Garlappi et al. (2007), Pinar (2016), Tang and Ling (2014), Chen et al. (2011) and Pac and Pinar (2014), most of them concentrated on numerical solutions of robust portfolio optimization problems, especially conic programming (e.g. second-order cone programs or positive semidefinite programs) (Cornuejols and Tutuncu 2006).

Recently, researchers have recognized the usefulness of incorporating additional criteria beyond only one measure of risk into the portfolio selection model. The optimization approaches they proposed, however, are mostly based on scenario-based methods or a heuristic multi-objective genetic algorithms; see, e.g., Roman et al. (2007) and Baixauli-Soler et al. (2010). Despite the vast literature on robust portfolio optimization and many works on multiple risk measures in portfolio optimization, there are few works that concern both multiple risk measures and robust portfolios. The aim of this article is to propose a unified modeling framework and analysis of worst-case mean-risk models where optimal portfolios are selected based on the ambiguity of returns and three measures of risk (variance, VaR and CVaR). Motivated by recent development in robust portfolio optimization, we suppose that the decision-maker uses variance, VaR and CVaR to measure the portfolio risk and is ambiguous about both the distribution and parameters, simultaneously. In our worst-case optimization model of the portfolio selection, the expected return is maximized and the risk are minimized. The main contribution of this study is to formulate a robust mean-multiple risk optimization model that is elegant but simple enough to obtain a closed-form expression for the optimal robust portfolio selection rule. The robust portfolio has the nice property that it converges to the minimum variance portfolio under very high levels of ambiguity. In particular, the robust mean-variance portfolios (Garlappi et al. 2007), robust maximum return portfolios (Pinar 2016), robust minimum VaR portfolios (Chen et al. 2011; Pac and Pinar 2014) and robust minimum CVaR portfolios (Chen et al. 2011; Pac and Pinar 2014) are all special instances of our robust mean-multiple risk portfolios, see, Corollary 1, Remarks 4 and 5. To the best of our knowledge, this is the first analytical result of applying robust optimization approach to mean-multiple risk model optimization. Consequently, this study may offer a new insight into the multi-objective decision making process and robust portfolio optimization through finding the feasible analytical solution set.

The outline of the paper is organized as follows. In Sect. 2, we provide an overview of mean-risk optimization models and various kinds of risk measures. We derive an optimal portfolio rule for the robust mean-multiple risk optimization model under distribution ambiguity in Sects. 3. Section 4 analyzes several special cases of our robust optimization framework. In Sect. 5, we perform some numerical experiments using simulated data and real market data and report the results. The final section concludes the paper.

2 Risk measures and robustness

Mean-risk models are used to determine optimal portfolios. Consider a financial market consisting of n stocks.[1] The investment period starts at $t = 0$ and ends at time $t = 1$. Let $\xi \in \mathbb{R}^n$ be the random vector of the returns of n risky assets with mean vector μ and covariance matrix Σ. Let $x \in \mathbb{R}^n$ denote the portfolio percentage weights of an investor. The return of portfolio x is a random variable defined as $R_x = \xi_1 x_1 + \cdots +$

[1] As stated in Zymler et al. (2013), the derivative returns are uniquely determined by the underlying asset returns and modelled as convex piecewise linear or (possibly nonconvex) quadratic functions of the underlying asset returns, which are highly non-linear. Hence, there are difficulties in effectively implementing derivatives. In order to simplify our problem and get better results, the underlying financial instruments presented in this paper are constraint to stocks, as discussed in Garlappi et al. (2007), Chen et al. (2011) and many other papers.

$\xi_n x_n$ and the expected return of portfolio x is $\mathrm{E}(R_x)$. Then, the mean-risk (MR) model with the risk measure denoted by $\rho(R_x)$ is represented as follows:

$$\text{(MR)}: \min_{x \in \mathscr{X}} \quad -\mathrm{E}(R_x) + \tau \rho(R_x) \tag{1}$$

where $\tau \geq 0$ is the investor's risk aversion parameter that determines the trade-off between expected return and risk and $\mathscr{X} \subseteq \mathbb{R}^n$ denotes the admissible set of portfolios. In this Section, we will briefly review the concepts of three widely used risk measures and robustness. Readers can also refer to Fabozzi et al. (2010), Kim et al. (2014) and Qian et al. (2015) for a survey on this subject.

2.1 Risk measures

In mean-risk models, many different risk measures are used. Generally speaking, risk measures in finance can be divided into two main categories: moment based and quantile based. In this subsection, we will discuss three measures: variance, value-at-risk (VaR) and conditional value-at-risk (CVaR). Following Artzner et al. (1999), CVaR is a coherent risk measure, which satisfies four properties: monotonicity, translation invariance, homogeneity and subadditivity. However, it is known that VaR does not satisfies subadditivity and the optimization of VaR leads to a non-convex NP-hard problem, which is computationally intractable.

Variance of returns is a moment-based risk measure, which incorporates information about the first and second moments of the distribution of returns. As a traditional risk metric, variance measures the dispersion. Mathematically, it can be expressed as

$$\sigma^2(x) = \mathrm{E}[(R_x - \mathrm{E}(R_x))^2].$$

However, not only does the mean-variance method fail to control the risk of returns on the down side, but it might also restrain the possible gains on the up side.

In contrast to moment-based risk measures, quantile-based risk measures are concerned with the probability or magnitude of losses. The most popular quantile-based risk measures are VaR and CVaR. The VaR measures the worst portfolio loss over a given time under normal market conditions at a given confidence level. It sheds light on what the maximum loss is over a time interval. Suppose that ξ has a probability distribution $p(\xi)$ with mean vector μ and covariance matrix Σ (which is assumed to be positive definite throughout this paper) and the loss L associated with R_x is described by the random variable $-R_x$, i.e., $L = -R_x$. Then, given a confidence level β and a fixed x, the value-at-risk is defined as

$$\mathrm{VaR}_\beta(L) = \min \left\{ \alpha \in \mathbb{R} : \int_{\{\xi : L \leq \alpha\}} p(\xi) d\xi \geq \beta \right\}.$$

The corresponding CVaR risk measure is defined as the expected value of the loss L exceeding the VaR and can be expressed as

$$\mathrm{CVaR}_\beta(L) = \mathrm{E}[L \mid L \geq \mathrm{VaR}_\beta(L)] = \frac{1}{1-\beta}\int_{\{\xi:L\geq \mathrm{VaR}_\beta(L)\}} Lp(\xi)d\xi.$$

Rockafellar and Uryasev (2000) show that CVaR can be obtained by solving the following convex program:

$$\mathrm{CVaR}_\beta(L) = \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}E[(L-\alpha)^+],$$

where $[t]^+ = \max\{0, t\}$. This formula brings computational convenience as the objective function is explicit and convex in α.

In addition, it is worth mentioning that when $L = -R_x$ and ξ has a normal distribution, both $\mathrm{VaR}_\beta(L)$ and $\mathrm{CVaR}_\beta(L)$ have closed-form expressions (Qian et al. 2015):

$$\mathrm{VaR}_\beta(L) = -\mu^T x + \kappa_v^n\sqrt{x^T\Sigma x}, \tag{2}$$

$$\mathrm{CVaR}_\beta(L) = -\mu^T x + \kappa_c^n\sqrt{x^T\Sigma x}, \tag{3}$$

where $\kappa_v^n = \Phi^{-1}(\beta)$, $\kappa_c^n = \frac{1}{\sqrt{2\pi}(1-\beta)}\exp(-(\Phi^{-1}(\beta))^2/2)$ and $\Phi^{-1}(\cdot)$ is the inverse of the cumulative distribution function of a standard normal distribution. The theoretical literature mostly assumes normally distributed returns to easily capture the complete dependence structure by the correlation coefficient (Alexander and Baptista 2002, 2004). Clearly, the computation of VaR (or CVaR) includes three components: the mean, the variance, and the distribution of portfolio returns. It implies that an investor who regards VaR (or CVaR) as the risk measure can adjust κ_v^n (or κ_c^n) and $\mu^T x$ to reduce the portfolio risk. In this sense, the VaR or CVaR risk measure is more flexible than the variance.

As we mentioned before, different measures of risk focus on different properties of returns. The variance measures the dispersion of returns and the VaR or CVaR measure the probable loss. VaR focuses on the maximum likely loss of a portfolio for a given confidence level. It does not take into account the shape of the tail, which is actually non-negligible in many practical problems. In case of non-normal (leptokurtic, asymmetric) return distributions, it may be useful to consider tail risk separately. Unlike VaR risk measure, CVaR as an alternative measure of risk captures these losses, and measures the expected loss if the loss is above a specified quantile. Therefore, CVaR can be used in conjunction with VaR when estimating the risk with non-symmetric return distribution. The two values coincide only if the tail is cut off. In summary, each measure captures only one particular aspect of the uncertainty and each has its own advantages and disadvantages.

2.2 Robust portfolios

Since the introduction of the mean-variance model for portfolio selection, many criticisms have been raised on its practical relevance, especially in regard to the sensitivity of the optimal portfolios with respect to the statistical errors in the parameters. However, the estimating errors of the mean value and the variance can not be avoided if the sample estimate of ξ is used in practice. To overcome these difficulties, in the last decade portfolio models based on the robust optimization technique have become a focal point of many researchers. The idea of robust optimization that address uncertainties is to look for a solution which is feasible in every possible scenario and optimal in the worst case. We briefly review two cases of robustness, which are commonly considered in the literature.

2.2.1 Parameter uncertainty

Several authors have pointed out that the Markowitz optimal portfolio is extremely sensitive to distributional input parameters, and amplifies the estimation errors (Broadie 1993; DeMiguel and Nogales 2009). Therefore, it is meaningful to handle uncertain parameters by requiring the user to specify a uncertainty set based on some limited information about their values. The key idea of parameter robust optimization is to find an optimal solution to the problem that remain feasible for any realization of the uncertain coefficients within the pre-specified (deterministic) uncertainty set. Lobo and Boyd (2000) were the first to apply the worst-case analysis in portfolio selection, and they presented several different methods for modelling the uncertainty sets for the expected return and covariance matrices, such as box or ellipsoidal sets. In addition, Tutuncu and Koenig (2004) considered a portfolio selection problem with box uncertainty sets for the return mean and covariance and solved the resulting worst-case Markowitz problem via a saddle-point algorithm. Instead of specifying the return by an uncertainty set directly, Goldfarb and Iyengar (2003) defined asset returns by robust factor models in which uncertainty was modeled by ellipsoidal sets and showed that their portfolio choice problems can be reformulated as second-order cone program.

Since the effect of estimation error is known to be greater for the expected returns of assets than the covariance of asset returns (Jagannathan and Ma 2003), to facilitate subsequent analysis, we only focus on the uncertainty for the expected returns and leave interested readers to review Lobo and Boyd (2000) and Fabozzi et al. (2007). The robust counterpart analogous to the mean-risk optimization problem (MR), where the optimal decision is based on the worst scenario of the underlying uncertainty, becomes

$$(\text{PRP}): \min_{x\in\mathscr{X}} \max_{\mu\in\mathscr{U}} \{-\mathrm{E}(R_x) + \tau\rho(R_x)\} \tag{4}$$

where $\mathscr{U}$ is a deterministic uncertainty set of the mean return. Typically, the uncertainty set is convex, and its size is related to some kind of guarantee on the probability that the constraint involving the uncertain data will not be violated. The most often used structure for tractability in robust optimization is

$$\mathscr{U} = \{\mu \in \mathbb{R}^n \mid (\mu - \hat{\mu})^T \Sigma^{-1} (\mu - \hat{\mu}) \leq \gamma\}, \tag{5}$$

which is an n-dimensional ellipsoid centered at the sample mean $\hat{\mu}$ with radius γ (this parameter is called "the level of ambiguity aversion" in Fabretti et al. 2014).

The uncertainty set of expected returns describes a geometric structure around the estimate $\hat{\mu}$ and measures the combined deviation of all assets. Scherer (2007) points out that there is no way to consistently determine the value of the parameter γ. Generally speaking, however, the larger the number of samples S, the more reliable the estimated mean. Therefore, for simplicity, it may be straightforward to assume that the level of uncertainty γ is a function of data sample size S, i.e., $\gamma := d(S)$, which is nonincreasing as S increases. Furthermore, γ tends to zero as S tends to infinity, especially in the case where the return distribution follows a stationery process.

Decision making under uncertainty has been an active research area for several domains; see e.g., Deng et al. (2005), Pinar (2014) and Fabretti et al. (2014). The returns of risk assets in these models are all assumed to follow a known distribution with uncertain parameters. However, such assumption can not reflect the real and complex financial market, and in most cases we know little about the distributional form of asset returns.

2.2.2 Distribution ambiguity

In uncertain environments, one mostly has only partial information on the underlying probability distributional information which can be available. Therefore, it will be more realistic for decision-maker to hedge the worst case over a pre-defined set of probability measure $\mathbb{D}$, which is defined by the limited available information. Then, a mean-risk portfolio optimization model under ambiguous distribution, which explicitly trades risk against return in the objective function, is

$$\text{(DRP)}: \min_{x \in \mathscr{X}} \max_{\text{P} \in \mathbb{D}} \{-\text{E}(R_x) + \tau \rho(R_x)\} \tag{6}$$

Several different choices of the ambiguity set $\mathbb{D}$ based on the historical data sampled from true probability distribution have been proposed; see, e.g., Qian et al. (2015). For example, since the information on empirical estimates of the mean and covariance matrix of random vector ξ can be relatively easily obtained from historical data, decision makers in the finance industry may describe the uncertainty in returns of assets by the mean and covariance. Hence, the moment-based ambiguity set can be naturally constructed by letting $\text{E}(\xi)$ and $\text{Cov}(\xi)$ equal to their estimates respectively, i.e.,

$$\mathbb{D}_0 = \left\{ \text{P} \in \mathscr{M}_+ : \text{E}(\xi) = \hat{\mu}, \text{Cov}(\xi) = \hat{\Sigma} \right\},$$

where $\mathscr{M}_+$ represents the set of all probability distributions, and $\hat{\mu}$ and $\hat{\Sigma}$ are statistically inferred by the historical data.

There has been much research on this topic. Ghaoui et al. (2003) proposed to maximize the VaR of a given portfolio over all asset return distributions when only the first and second order moment information is available. Chen et al. (2011) considered the worst-case lower partial moments and the worst-case conditional value-at-risk with respect to the only reliable data consisting of fixed first and second order moments, extending the work of Zhu and Fukushima (2009), and derived tight bounds for these two risk measures. However, this type of ambiguity set might be insufficient to describe the uncertainty in real markets and thus not a very reasonable assumption, since in practice the inferred sample mean $\hat{\mu}$ and covariance matrix $\hat{\Sigma}$ can not completely represent the actual values of mean and covariance. A criticism leveled against the aforementioned ambiguity set is the sensitivity to uncertainties or estimation error in the mean return data. To address this issue and further control the uncertainty of moments, the ambiguity set $\mathbb{D}$ of distributions who take into account the knowledge of the distribution's support and of a confidence region for its mean and second-moment matrix is presented. Natarajan et al. (2010) derived exact and approximate optimal strategies for a worst-case expected utility model of portfolio selection under distribution ambiguity using a piecewise linear concave utility function. In addition, Delage and Ye (2010) took into account the model ambiguity in terms of the uncertainty in the first and second moments. They demonstrated that the portfolio optimization problem has an equivalent semi-definite programming reformulation. More recently, Pac and Pinar (2014) considered the problem of optimal portfolio choice using lower partial moments and CVaR measure, respectively, when the mean return is subject to ellipsoidal uncertainty set in addition to distribution ambiguity.

In summary, each of these risk measures (variance, VaR and CVaR) captures a different aspect of risk. Therefore, it could be worthy to introduce them jointly in portfolio selection. Additionally, in most cases we only have a series of data samples which can be collected from the true distribution. It may be unreasonable for decision-maker to assume that the distribution of losses is precisely known, from a practical point of view. As a result, a remedy for both these difficulties is to adopt a distributionally robust approach, which will be stated in the next section.

3 Robust mean-multiple risk model under ambiguity

While people all agree to take expected return as one common measure of portfolio performance, there exists no consensus on which risk measure can best capture investors' risk attitudes. The question of which risk measure is most appropriate for all portfolio selection problems is still the subject of much debate. In this section, a robust optimization model with three risk measures (variance, VaR, CVaR), which uses more information of the underlying distribution of portfolio returns, will be analyzed. The proposed model allows aforementioned one moment and two percentile measures of risk to manage the risk of portfolio loss.

Since the real mean μ and covariance matrix Σ are not observable, their estimates $\hat{\mu}$ and $\hat{\Sigma}$ based on the available sample information have to be used. It is well-known that variance-covariance estimates are relatively stable over time and can be predicted more reliably than expected return; see Merton (1980) and Jorion (1985). Additionally, Jagannathan and Ma (2003) have further argued that estimation error in sample means has larger effect on portfolio optimization than that in sample covariance matrices. In the remainder, we will ignore the estimation risk in the sample covariance matrix $\hat{\Sigma}$ and suppose that the estimate is reasonably accurate, i.e., $\Sigma = \hat{\Sigma}$.

Moreover, in practice, one only has limited information about the probability distributions of asset returns. This implies that we do not assume the exact return distributions and the exact moment information are known. However, investors can rely on empirical estimates of the mean $\hat{\mu}$ and covariance matrix $\hat{\Sigma}$ of random returns to construct an informational set $\mathbb{D}$ of the underlying distributions. In other words, there may be an ambiguity set of many different distributions that are consistent with the available information, and the decision maker has almost no possibility to single out the true distribution among all conceivable distributions. Instead of pretending to have full knowledge of the true distribution, therefore, we will assume that the true distribution is ambiguous and only known to be an element of a given set, which can be viewed as a confidence region in the space of return distributions.

We apply a simple multiple-objective weighted method by Fliege and Werner (2014) to deal with our multiple risk measures. This leads to our robust mathematical programming (RMP) model for the robust mean-multiple risk portfolio choice problem with distribution ambiguity as below:

$$\text{(RMP)}: \quad \min_{x\in\mathscr{X}} \max_{\mathrm{P}\in\mathbb{D}_\gamma} \left\{-\mathrm{E}(\xi^T x) + \omega \mathrm{VaR}_\beta(-\xi^T x) + \theta \mathrm{CVaR}_\beta(-\xi^T x) + \lambda x^T \Sigma x\right\}, \tag{7}$$

where

$$\mathbb{D}_\gamma = \left\{\mathrm{P} \in \mathscr{M}_+ : \begin{array}{l} (\mathrm{E}(\xi) - \hat{\mu})^T \Sigma^{-1} (\mathrm{E}(\xi) - \hat{\mu}) \le \gamma \\ \mathrm{Cov}(\xi) = \Sigma \succ 0 \end{array}\right\}$$

is the associated distributional set and $\mathscr{M}_+$ is the set of all probability measures on the measure space $(\mathbb{R}^n, \mathscr{B})$, $\mathscr{B}$ is the Boreal σ-algebra on $\mathbb{R}^n$, ω, θ and λ are the non-negative weights balancing the importance of the three risk measures and can be regarded as the degrees of VaR aversion, CVaR aversion and variance aversion, and the parameter γ reflects the investor's level of ambiguity aversion. It should be mentioned that our model (RMP) has four objectives: the expected return is maximized, while the variance, the VaR and the CVaR that measure risk from different perspectives are minimized. Comparing with the mean-variance, the mean-VaR and the mean-CVaR models, we use triple-risk measures instead of one single risk measure, and our model includes robust mean-variance, the robust mean-VaR and the robust mean-CVaR models as special cases. On the other hand, we assume not only knowledge of the mean and covariance matrix of return distributions, but also uncertainty of its first moment information.

Remark 1 In our robust model, although it is more flexible for investors to choose different weighting coefficients according to their investment preferences and risk, there is still no formal rule to guide an investor regarding an appropriate choice of ω, θ and λ (Roman et al. 2007).

We start our analysis of the "max" function by assuming that the vector of portfolio x is given. For the sake of our discussion, we first define the auxiliary function $\Upsilon(x)$: $\mathbb{R}^n \mapsto \mathbb{R}$ by

$$\Upsilon(x) = \max_{\mathrm{P}\in\mathbb{D}_\gamma} \left\{-\mathrm{E}(\xi^T x) + \omega \mathrm{VaR}_\beta(-\xi^T x) + \theta \mathrm{CVaR}_\beta(-\xi^T x)\right\}. \tag{8}$$

Then, (RMP) can be written as

$$\min_{x \in \mathscr{X}} \ \Upsilon(x) + \lambda x^T \Sigma x. \tag{9}$$

Lemma 1 *Assume that* $\beta \in (0.5, 1)$ *and random vector* $\xi \in \mathbb{R}^n$ *with mean value* μ *and covariance* Σ *follows a family of distributions* $\mathscr{F}$, *which is defined by* $\mathscr{F} = \{\mathrm{P} \in \mathscr{M}_+ |\ \mathrm{E}(\xi) = \mu, \mathrm{Cov}(\xi) = \Sigma\}$. *Then we have*

$$\max_{\mathrm{P} \in \mathscr{F}} \left\{ VaR_\beta(-\xi^T x) + CVaR_\beta(-\xi^T x) \right\} = \max_{\mathrm{P} \in \mathscr{F}} VaR_\beta(-\xi^T x) + \max_{\mathrm{P} \in \mathscr{F}} CVaR_\beta(-\xi^T x). \tag{10}$$

Proof Obviously,

$$\max_{\mathrm{P} \in \mathscr{F}} \left\{ \mathrm{VaR}_\beta(-\xi^T x) + \mathrm{CVaR}_\beta(-\xi^T x) \right\} \leq \max_{\mathrm{P} \in \mathscr{F}} \mathrm{VaR}_\beta(-\xi^T x) + \max_{\mathrm{P} \in \mathscr{F}} \mathrm{CVaR}_\beta(-\xi^T x).$$

Since the function $F_\beta(x, \alpha) = \alpha + \frac{1}{1-\beta}\mathrm{E}[(-\xi^T x - \alpha)^+]$ is convex in α for every ξ and $\mathscr{F}$ is a convex set, by Theorem 2.4 of Shapiro (2011), we have

$$\begin{aligned}
\max_{\mathrm{P} \in \mathscr{F}} \mathrm{CVaR}_\beta(-\xi^T x) &= \max_{\mathrm{P} \in \mathscr{F}} \min_{\alpha \in R} F_\beta(x, \alpha) \\
&= \min_{\alpha \in R} \max_{\mathrm{P} \in \mathscr{F}} F_\beta(x, \alpha) \\
&= \min_{\alpha \in R} \left\{ \alpha + \frac{1}{1-\beta} \max_{\mathrm{P} \in \mathscr{F}} \mathrm{E}\left[(-\xi^T x - \alpha)^+ \right] \right\}.
\end{aligned}$$

Similarly, the worst-case VaR can be obtained from

$$\begin{aligned}
\max_{\mathrm{P} \in \mathscr{F}} \mathrm{VaR}_\beta(-\xi^T x) &= \max_{\mathrm{P} \in \mathscr{F}} \arg\min_{\alpha \in R} F_\beta(x, \alpha) \\
&= \arg\min_{\alpha \in R} \max_{\mathrm{P} \in \mathscr{F}} F_\beta(x, \alpha) \\
&= \arg\min_{\alpha \in R} \left\{ \alpha + \frac{1}{1-\beta} \max_{\mathrm{P} \in \mathscr{F}} \mathrm{E}\left[(-\xi^T x - \alpha)^+ \right] \right\}.
\end{aligned}$$

That is to say, to obtain the maximum values of $\mathrm{CVaR}_\beta(-\xi^T x)$ and $\mathrm{VaR}_\beta(-\xi^T x)$, the first thing we should do is to derive the upper bound of first-order lower partial moment $\mathrm{E}[(-\xi^T x - \alpha)^+]$, which has been provided in Lemmas 2.2 and 2.4 of Chen et al. (2011). Hence, there exists a member of $\mathscr{F}$ that allows the worst-case CVaR and the worst-case VaR of losses to get the maximum value at the same time. This leads to the desired conclusion. □

Remark 2 The left and right-hand sides of (10) can be considered as the different robust counterparts to CVaR and VaR combined optimization for distribution ambiguity. Generally, the robust counterpart of the left-hand side is less conservative, which requires the distribution used in the CVaR and VaR of losses to be equal. However, by assuming $F_\beta(x, \alpha)$ has a unique minimum, we can observe that they are identical.

We know from Appendix 1 in Chen et al. (2011) that the explicit expression for the worst-case CVaR and the worst-case VaR over the family $\mathscr{F}$ which is composed of all distributions with given mean and covariance can be obtained.

Define $\kappa_c = \sqrt{\frac{\beta}{1-\beta}}$ and $\kappa_v = \frac{2\beta-1}{2\sqrt{\beta(1-\beta)}}$.

Lemma 2 (Chen et al. 2011) *Assume that* $\beta \in (0.5, 1)$ *and random vector* $\xi \in \mathbb{R}^n$ *with mean value* μ *and covariance* Σ *follows a family of distributions* $\mathscr{F}$*. Then for any* $x \in \mathbb{R}^n$*,*

$$\max_{P \in \mathscr{F}} CVaR_\beta(-\xi^T x) = -\mu^T x + \kappa_c \sqrt{x^T \Sigma x}, \tag{11}$$

$$\max_{P \in \mathscr{F}} VaR_\beta(-\xi^T x) = -\mu^T x + \kappa_v \sqrt{x^T \Sigma x}. \tag{12}$$

Since $\kappa_c > \kappa_v$, Lemma 2 implies that the worst-case VaR is always less than the worst-case CVaR. As a consequence of Lemmas 1 and 2, the auxiliary function $\Upsilon(x)$ can be expressed explicitly.

Lemma 3 *The following relation holds:*

$$\Upsilon(x) = -(1+\omega+\theta)\hat{\mu}^T x + \left[(1+\omega+\theta)\sqrt{\gamma} + \omega\kappa_v + \theta\kappa_c\right]\sqrt{x^T \Sigma x}. \tag{13}$$

Proof Define

$$\mathscr{U}_{\hat{\mu}} = \left\{\mu \in \mathbb{R}^n |\ (\mu - \hat{\mu})^T \Sigma^{-1}(\mu - \hat{\mu}) \leq \gamma\right\}.$$

From Lemmas 1 and 2, we have

$$\begin{aligned}
\Upsilon(x) &= \max_{\mu \in \mathscr{U}_{\hat{\mu}}} \max_{P \in \mathscr{F}} \left\{-\mathrm{E}(\xi^T x) + \omega \mathrm{VaR}_\beta(-\xi^T x) + \theta \mathrm{CVaR}_\beta(-\xi^T x)\right\} \\
&= \max_{\mu \in \mathscr{U}_{\hat{\mu}}} \left\{-\mu^T x + \max_{P \in \mathscr{F}} \omega \mathrm{VaR}_\beta(-\xi^T x) + \max_{P \in \mathscr{F}} \theta \mathrm{CVaR}_\beta(-\xi^T x)\right\} \\
&= \max_{\mu \in \mathscr{U}_{\hat{\mu}}} \left\{-\mu^T x + \omega(-\mu^T x + \kappa_v \sqrt{x^T \Sigma x}) + \theta(-\mu^T x + \kappa_c \sqrt{x^T \Sigma x})\right\} \\
&= -(1+\omega+\theta) \min_{\mu \in \mathscr{U}_{\hat{\mu}}} x^T \mu + \omega\kappa_v \sqrt{x^T \Sigma x} + \theta k_c \sqrt{x^T \Sigma x}.
\end{aligned}$$

Note that x is fixed in problem $\min_{\mu \in \mathscr{U}_{\hat{\mu}}} x^T \mu$, we are optimizing over variable μ. The optimal solution to this problem can be shown to be

$$\mu^* = \hat{\mu} - \frac{\sqrt{\gamma}\Sigma x}{\sqrt{x^T \Sigma x}}. \tag{14}$$

Substituting this into the representation $\Upsilon(x)$ gives our desired result. □

The inclusion $x \in \mathscr{X}$ usually denotes the budget constraints that forces the sum of the weights to be one, i.e., $e^T x = 1$, where e denotes the vector of 1s. In the following, we set $\mathscr{X} = \{x \in \mathbb{R}^n | e^T x = 1\}$ unless otherwise stated, and define

$$A = e^T \Sigma^{-1} e, B = \hat{\mu}^T \Sigma^{-1} e, C = \hat{\mu}^T \Sigma^{-1} \hat{\mu}, \Delta = AC - B^2,$$
$$\sqrt{\gamma'} = (1 + \omega + \theta)\sqrt{\gamma} + \omega\kappa_v + \theta\kappa_c.$$

By Cauchy–Schwarz inequality, we can check that $\Delta > 0$. The following theorem gives the main results of this paper.

Theorem 1 *Suppose that $\beta \in (0.5, 1)$. Then the robust mean-multiple risk model (RMP) under distribution and mean return ambiguity has the optimal portfolio*

$$x^*_{\mathbf{RMP}} = \left(\frac{\sigma_p^*}{\sqrt{\gamma'} + 2\lambda\sigma_p^*}\right)\Sigma^{-1} \times \left[(1 + \omega + \theta)\hat{\mu} - \frac{1}{A}\left((1 + \omega + \theta)B - \frac{\sqrt{\gamma'} + 2\lambda\sigma_p^*}{\sigma_p^*}\right)e\right], \quad (15)$$

where σ_p^* is the variance of the optimal portfolio and can be obtained from solving the fourth degree polynomial equation

$$A\lambda'^2\sigma_p^4 + 2A\lambda'\sqrt{\gamma'}\sigma_p^3 + (A\gamma' - (1+\omega+\theta)^2\Delta - \lambda'^2)\sigma_p^2 - 2\lambda'\sqrt{\gamma'}\sigma_p - \gamma' = 0. \quad (16)$$

See Appendix A for the proof of the theorem.

The (RMP) model is useful for an individual investor who decides to use variance, VaR, and CVaR as one or more measures of risk but dose not impose a assumption on distributions of asset returns. The constraint set $\mathscr{X}$ of (RMP) can be more general, for example, the one with no-short selling. In this case, however, the explicit solution may no longer be obtained.

The robust portfolio strategy $x^*_{\mathbf{RMP}}$ evidently depends on the ambiguity aversion parameter γ. Its limiting behaviour is given by the following proposition.

Proposition 1 *For (RMP), the robust portfolio strategy $x^*_{\mathbf{RMP}}(\gamma)$ converges to the minimum variance portfolio (MVP) $x^*_{\mathbf{min}} = \frac{\Sigma^{-1}e}{A}$ when the level of ambiguity aversion γ goes to infinite, that is,*

$$\lim_{\gamma\to\infty} x^*_{\mathbf{RMP}}(\gamma) = \frac{\Sigma^{-1}e}{A}. \quad (17)$$

The proof is straightforward from (44) in Appendix A. It suggests that an investor with very high uncertainty in the reference noisy sample mean invests in the minimum variance portfolio, which is free of ambiguity. If the risky assets are not correlated, that is, Σ is a diagonal matrix only containing variance, then we find that the weight of each asset i is inversely proportional to its variance σ_i^2 in the limit of levels of ambiguity aversion, i.e., $x_i^* = \frac{(\sigma_i^2)^{-1}}{\sum_{i=1}^n (\sigma_i^2)^{-1}}$. Further, when $\Sigma = c\mathrm{I}_n$ ($c > 0$), where I_n is the identity matrix, the robust portfolio strategy converges towards the "$\frac{1}{n}$-naive rule". This strategy has been served as a well-known benchmark by academic research and the investment management industry (DeMiguel et al. 2009).

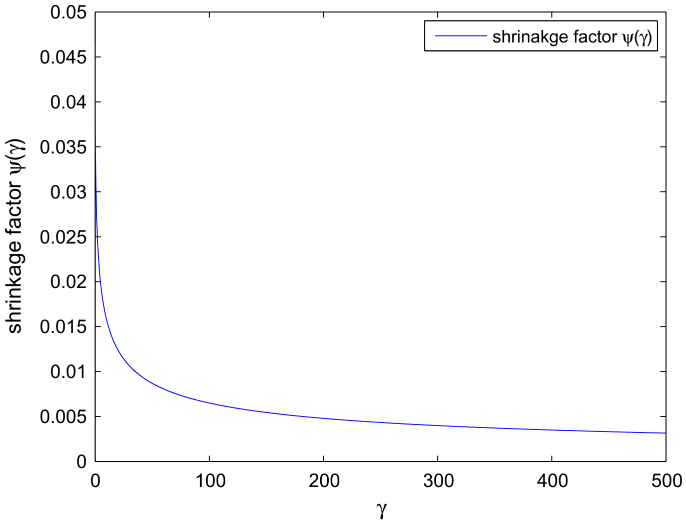

Fig. 1 Effect of increasing γ for the RMP strategy with $n = 3$, $\lambda = \omega = \theta = 0.5$. The data $\hat{\mu}$, Σ used by Rockafellar and Uryasev (2000)

To provide a better understanding of the robust mechanisms of the optimal strategy for (RMP), we numerically show that $x^*_{\mathbf{RMP}}(\gamma)$ converges to the MVP as γ goes to ∞. According to (44), we define a shrinkage factor

$$\psi(\gamma) = \left(1 - \frac{\sqrt{\gamma'}}{\sqrt{\gamma'} + 2\lambda\sigma_p^*}\right)(1 + \omega + \theta). \tag{18}$$

As Scherer (2007) claimed, the Markowitz mean-variance optimal portfolio $x^*_{\mathbf{Mark}}$ can be represented as the combination of the minimum variance portfolio $x^*_{\mathbf{min}}$ and a speculative demand $x^*_{\mathbf{spec}} = \frac{1}{2\lambda}\Sigma^{-1}(\hat{\mu} - \frac{B}{A}e)$, i.e.,

$$x^*_{\mathbf{Mark}} = x^*_{\mathbf{spec}} + x^*_{\mathbf{min}}. \tag{19}$$

Hence, we can rewrite the optimal portfolio for (RMP) as

$$x^*_{\mathbf{RMP}} = \psi(\gamma)x^*_{\mathbf{spec}} + x^*_{\mathbf{min}}. \tag{20}$$

Since σ_p^* is contained in $\psi(\gamma)$ and relates to γ, the curve of the shrinkage factor as a function of ambiguity aversion level γ needs to be simulated numerically. A simulated result is shown in Fig. 1. The figure demonstrates that $\psi(\gamma) \geq 0$ for $\gamma > 0$ and that when $\gamma \to \infty$, $\psi(\gamma)$ is decreasing and slowly close to 0 (maybe at very high values of γ). This together with (20) implies that, with $\gamma > 0$ increasing, our robust optimal portfolio $x^*_{\mathbf{RMP}}$ invests less and less in $x^*_{\mathbf{spec}}$ and hence more and more in $x^*_{\mathbf{min}}$. Therefore, due to that estimation errors only affect $x^*_{\mathbf{spec}}$ but not

$x^*_{\mathbf{min}}$, we can significantly reduce the effect of estimation errors on the optimal portfolio by incorporating robustness (embodied by γ) into the portfolio construction process. That is to say, robustness indeed can markedly improve the stability of optimal portfolios.

4 Special cases

In this section, we examine some variations or special cases of problem (RMP) with different values of risk aversion parameters λ, ω and θ. It can be shown that the robust mean-variance, robust maximum return, robust minimum VaR (CVaR) optimal portfolios are all special cases of our robust optimal portfolios.

4.1 Robust mean-variance portfolios

We consider the special case where the risk aversion parameter ω and θ are zero. In this case, only variance measure is considered and thus less emphasis is placed on the tail risk, and the robust mean-multiple risk model under distribution and mean return ambiguity is just the classical robust mean-variance model proposed by Garlappi et al. (2007).

Corollary 1 *When* $\omega = \theta = 0$, (RMP) reduces to the classical robust mean-variance model

$$(\text{RMv}): \min_{x \in \mathscr{X}} \max_{\mathrm{P} \in \mathbb{D}_\gamma} \left\{ -\mathrm{E}(\xi^T x) + \lambda x^T \Sigma x \right\} \tag{21}$$

and its optimal portfolio becomes

$$\begin{aligned} x^*_{\mathbf{RMv}} &= \left(1 - \frac{\sqrt{\gamma}}{\sqrt{\gamma} + 2\lambda\sigma_p^*}\right) \frac{1}{2\lambda} \Sigma^{-1} \left(\hat{\mu} - \frac{B}{A} e\right) + \frac{\Sigma^{-1} e}{A} \\ &= \left(\frac{2\lambda\sigma_p^*}{\sqrt{\gamma} + 2\lambda\sigma_p^*}\right) x^*_{\mathbf{Mark}} + \left(1 - \frac{2\lambda\sigma_p^*}{\sqrt{\gamma} + 2\lambda\sigma_p^*}\right) x^*_{\mathbf{min}}, \end{aligned} \tag{22}$$

where σ_p^*, the variance of the optimal portfolio, is a positive solution of the fourth degree polynomial equation

$$4A\lambda^2\sigma_p^4 + 4A\lambda\sqrt{\gamma}\sigma_p^3 + (A\gamma - \Delta - 4\lambda^2)\sigma_p^2 - 4\lambda\sqrt{\gamma}\sigma_p - \gamma = 0. \tag{23}$$

The result (22) is consistent with the one of Garlappi et al. (2007), and demonstrates that the (RMv) portfolio is equivalent to a convex combination of two benchmark portfolios: the mean-variance portfolio and the minimum variance portfolio, where the weights reflect the investor's degrees of ambiguity aversion and variance aversion. Furthermore, the portfolio selection model (21) can accommodate a wide class of portfolio selection models, including mean-variance model, minimum variance model and so on.

Case A $\omega = \theta = 0$, $\gamma \to 0$ (mean-variance portfolio).

For a low level of ambiguity aversion ($\gamma \to 0$), the investor is nearly ambiguity neutral and does not doubt that the estimated mean $\hat{\mu}$ is equal to the true mean, and the (RMP) reduces to the sample mean-variance model

$$\text{(Mv)}: \quad \min_{x \in \mathscr{X}} \quad -\hat{\mu}^T x + \lambda x^T \Sigma x \tag{24}$$

with the optimal portfolio $x^*_{\mathbf{Mv}} = x^*_{\mathbf{spec}} + x^*_{\mathbf{min}}$.

Case B $\omega = \theta = 0$, $\gamma \to 0$ and $\lambda \to \infty$ (minimum variance portfolio).

Low ambiguity aversion ($\gamma \to 0$) and high variance aversion ($\lambda \to \infty$) will lead to a portfolio close to the minimum variance portfolio $x^*_{\mathbf{min}}$, which totally prevents the effect of sampling errors in the mean and often leads to a better out-of-sample performance than a mean-variance portfolios (Jagannathan and Ma 2003).

Case C $\omega = \theta = 0$, $\gamma \to \infty$ (minimum variance portfolio).

As pointed out by Proposition 1, an investor with high ambiguity aversion ($\gamma \to \infty$) will chose the minimum variance portfolio, because she considers the worst-case occurrence of the true mean within a bigger set and hence takes a more conservative choice.

Case D $\omega = \theta = 0$, $\lambda = 0$ (robust maximum return portfolio).

In this case, the model (RMP) reduces to

$$\text{(RR)}: \quad \min_{x \in \mathscr{X}} \quad \max_{\mu \in \mathscr{U}_{\hat{\mu}}} -\mu^T x, \tag{25}$$

which actually maximizes the worst-case expected return. By virtue of (23), if $\gamma A > \Delta$, then the problem (25) has the optimal portfolio

$$x^*_{\mathbf{RR}} = \frac{1}{\sqrt{A\gamma - \Delta}} \Sigma^{-1} \left(\hat{\mu} - \frac{B}{A} e \right) + \frac{\Sigma^{-1} e}{A}. \tag{26}$$

The (RR) model is exactly identical to the one presented by Pinar (2016) (see, Propositions 5 and 6 there for detail).

Case E $\omega = \theta = 0$, $\lambda = 0$ and $\gamma \to 0$ (maximum return portfolio).

In this case, the model (RMP) reduces to the one that maximizes the expected return:

$$\min_{x \in \mathscr{X}} \quad -\hat{\mu}^T x. \tag{27}$$

If short-selling is not allowed, this problem achieves the highest expected return by allocating all money in the asset with highest return. Otherwise, this problem has no optimal solution.

4.2 Robust mean-CVaR and robust mean-VaR portfolios

When the investor only considers the risk measured by CVaR or VaR, that is, when $\lambda = \omega = 0$ or $\lambda = \theta = 0$, the model (RMP) becomes the robust mean-CVaR model

$$(\text{RMC}): \min_{x\in\mathscr{X}} \max_{\text{P}\in\mathbb{D}_\gamma} -\text{E}(\xi^T x) + \theta\text{CVaR}_\beta(-\xi^T x) \tag{28}$$

or the robust mean-VaR model

$$(\text{RMV}): \min_{x\in\mathscr{X}} \max_{\text{P}\in\mathbb{D}_\gamma} -\text{E}(\xi^T x) + \omega\text{VaR}_\beta(-\xi^T x). \tag{29}$$

Define

$$\eta = \sqrt{A((1+\theta)\sqrt{\gamma} + \theta\kappa_c)^2 - (1+\theta)^2\Delta},$$
$$\vartheta = \sqrt{A((1+\omega)\sqrt{\gamma} + \omega\kappa_v)^2 - (1+\omega)^2\Delta}.$$

Corollary 2 *Let* $\beta \in (0.5, 1]$. *The model* (RMC) has the optimal portfolio

$$x^*_{\mathbf{RMC}} = \frac{1}{\eta}(1+\theta)\Sigma^{-1}\left(\hat{\mu} - \frac{B}{A}e\right) + \frac{\Sigma^{-1}e}{A} \tag{30}$$

if $\kappa_c > \frac{1+\theta}{\theta}(\sqrt{\frac{\Delta}{A}} - \sqrt{\gamma})$, and is unbounded otherwise. The model (RMV) has the optimal portfolio

$$x^*_{\mathbf{RMV}} = \frac{1}{\vartheta}(1+\omega)\Sigma^{-1}\left(\hat{\mu} - \frac{B}{A}e\right) + \frac{\Sigma^{-1}e}{A} \tag{31}$$

if $\kappa_v > \frac{1+\omega}{\omega}(\sqrt{\frac{\Delta}{A}} - \sqrt{\gamma})$, and is unbounded otherwise.

This corollary implies that the investor must be careful in choosing the confidence level β so that the problem (RMC) or (RMV) has a solution. We observe that the functions κ_c and κ_v are increasing in β on $(0.5, 1]$. Thus we can find a minimal level $\beta_0 \in (0.5, 1]$ to guarantee that the condition $\kappa_c > \frac{1+\theta}{\theta}(\sqrt{\frac{\Delta}{A}} - \sqrt{\gamma})$ or $\kappa_v > \frac{1+\omega}{\omega}(\sqrt{\frac{\Delta}{A}} - \sqrt{\gamma})$ holds for $\beta \in (\beta_0, 1]$.

Remark 3 By comparing Eqs. (30) and (31) with (19), we find that $x^*_{\mathbf{RMC}}$ and $x^*_{\mathbf{RMV}}$ can be obtained from $x^*_{\mathbf{Mark}}$ by letting $\lambda = \frac{\eta}{2(1+\theta)}$ and $\lambda = \frac{\vartheta}{2(1+\omega)}$ respectively. That is to say, the optimal portfolios of models (RMC) and (RMV) are mean-variance efficient. This will be further illustrated by the following example.

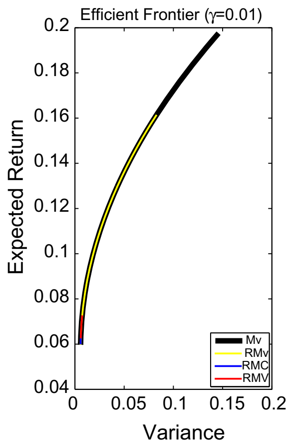

Fig. 2 Mean-variance efficient frontier and the trajectories of variance and return of RMv, RMC, RMV portfolios

Example 1 We take the sample mean return and covariance matrix from Gao et al. (2016) as follows:

$$\hat{\mu} = \begin{pmatrix} 0.1213 \\ 0.0522 \\ 0.1645 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0.0390 & 0.0028 & 0.0504 \\ 0.0028 & 0.006 & 0.0023 \\ 0.0504 & 0.0023 & 0.0917 \end{pmatrix}.$$

With λ ranging from 0.5 to 30, Fig. 2 plots the trajectory of points (return, variance) corresponding to optimal portfolios for a model. The black thick curve is the mean-variance efficient frontier and the yellow, red, and blue thin curves correspond to trajectories for (RMv), (RMC), (RMV) respectively. We observe that the other three trajectories are simply parts of the mean-variance efficient frontier, which verifies our conclusion that the (RMv), (RMC) and (RMV)'s portfolios are nothing but mean-variance portfolios with different values of λ.

Remark 4 When the level of CVaR aversion θ or the level of VaR aversion ω tends to infinity, the model (RMC) or (RMV) becomes the robust minimum CVaR model

$$(\text{RC}): \quad \min_{x\in\mathscr{X}} \max_{\text{P}\in\mathbb{D}_\gamma} \text{CVaR}_\beta(-\xi^T x),$$

or the robust minimum VaR model

$$(\text{RV}): \quad \min_{x\in\mathscr{X}} \max_{\text{P}\in\mathbb{D}_\gamma} \text{VaR}_\beta(-\xi^T x),$$

and our Corollary 2 is exactly Theorem 5 of Pac and Pinar (2014).

In particular, if we choose the ambiguity aversion level γ equal to zero, then the models (RMC) and (RMV) respectively reduce to

$$(\mathrm{RMC}_0): \min_{x\in\mathscr{X}} \max_{\mathrm{P}\in\mathbb{D}_0} -\mathrm{E}(\xi^T x) + \theta \mathrm{CVaR}_\beta(-\xi^T x), \quad (32)$$

$$(\mathrm{RMV}_0): \min_{x\in\mathscr{X}} \max_{\mathrm{P}\in\mathbb{D}_0} -\mathrm{E}(\xi^T x) + \omega \mathrm{VaR}_\beta(-\xi^T x). \quad (33)$$

Corollary 3 *Let* $\beta \in (0.5, 1]$. *The model* (RMC_0) has the optimal portfolio

$$x^*_{\mathbf{RMC_0}} = \frac{1}{\sqrt{A\theta^2\kappa_c^2 - (1+\theta)^2\Delta}}(1+\theta)\Sigma^{-1}\left(\hat{\mu} - \frac{B}{A}e\right) + \frac{\Sigma^{-1}e}{A} \quad (34)$$

if $\kappa_c > \frac{1+\theta}{\theta}\sqrt{\frac{\Delta}{A}}$, and is unbounded otherwise. The model (RMV_0) has the optimal portfolio

$$x^*_{\mathbf{RMV_0}} = \frac{1}{\sqrt{A\omega^2\kappa_v^2 - (1+\omega)^2\Delta}}(1+\omega)\Sigma^{-1}\left(\hat{\mu} - \frac{B}{A}e\right) + \frac{\Sigma^{-1}e}{A} \quad (35)$$

if $\kappa_v > \frac{1+\omega}{\omega}\sqrt{\frac{\Delta}{A}}$, and is unbounded otherwise.

Remark 5 When the level of CVaR aversion θ or the level of VaR aversion ω tends to infinity, the model (RMC_0) or (RMV_0) becomes the robust minimum CVaR model

$$(\mathrm{RC}_0): \min_{x\in\mathscr{X}} \max_{\mathrm{P}\in\mathbb{D}_0} \mathrm{CVaR}_\beta(-\xi^T x),$$

or the robust minimum VaR model

$$(\mathrm{RV}_0): \min_{x\in\mathscr{X}} \max_{\mathrm{P}\in\mathbb{D}_0} \mathrm{VaR}_\beta(-\xi^T x),$$

and our Corollary 3 is identical to Theorem 2.9 of Chen et al. 2011.

It's well-known that the mean-variance model (Mv) has the famous two-fund separation theorem, which plays a key role in a wide range of modern portfolio theory, such as in the classical capital asset pricing model of Sharpe (1964). Here, we can easily show that the two-fund separation theorem remains true for the models (RMC), (RMv) and (RMV). Because the statement is similar, we only give the one for (RMC) to conserve space.

Let $x(\theta)$ be the optimal solution of problem (RMC) associated with the CVaR aversion level θ. Let $S(x)$ be the set of all optimal solutions $x(\theta)$. By Corollary 2,

$$S(x) = \{x(\theta) : \theta > \max\{\tilde{\theta}, 0\}\},$$

where $\tilde{\theta} = \frac{\delta}{1-\delta}$ and $\delta = \frac{\sqrt{\frac{\Delta}{A}} - \sqrt{\gamma}}{\kappa_c}$.

Proposition 2 *Assume that* $x(\theta_1), x(\theta_2) \in S(x), \theta_1 \neq \theta_2$. *Then for any* $\theta > \max\{\tilde{\theta}, 0\}$ *and the corresponding solution* $x(\theta)$, *there exists a real number* ϖ, *such that*

$$x(\theta) = \varpi x(\theta_1) + (1 - \varpi)x(\theta_2), \tag{36}$$

that is, the two-fund separation theorem holds.

Proof It is directly from the solution expression (30). □

It states that two efficient "mutual funds" (portfolios) can be established, so that any efficient portfolio can be represented as a combination of these two. In other words, an investor only needs to invest in two different funds. What she needs to do is to choose the investment proportions ϖ and $1 - \varpi$ in these two funds.

4.3 Robust mean-variance-CVaR and robust mean-variance-VaR portfolios

When $\omega = 0$ or $\theta = 0$, the model (RMP) reduces to the robust mean-variance-CVaR model

$$\text{(RMvC)}: \min_{x \in \mathscr{X}} \max_{\mathrm{P} \in \mathbb{D}_\gamma} - \mathrm{E}(\xi^T x) + \theta \mathrm{CVaR}_\beta \left(-\xi^T x\right) + \lambda x^T \Sigma x, \tag{37}$$

or the robust mean-variance-VaR model

$$\text{(RMvV)}: \min_{x \in \mathscr{X}} \max_{\mathrm{P} \in \mathbb{D}_\gamma} - \mathrm{E}(\xi^T x) + \omega \mathrm{VaR}_\beta \left(-\xi^T x\right) + \lambda x^T \Sigma x. \tag{38}$$

Corollary 4 *Let* $\beta \in (0.5, 1]$. *The model* (RMvC) has the optimal portfolio

$$x^*_{\mathbf{RMvC}} = \left(1 - \frac{\sqrt{\gamma'}}{\sqrt{\gamma'} + 2\lambda\sigma_p^*}\right)(1+\theta)\frac{1}{2\lambda}\Sigma^{-1}\left(\hat{\mu} - \frac{B}{A}e\right) + \frac{\Sigma^{-1}e}{A}, \tag{39}$$

where $\sqrt{\gamma'} = (1+\theta)\sqrt{\gamma} + \theta\kappa_c$ and σ_p^*, the variance of the optimal portfolio, is a positive solution of the fourth degree polynomial equation

$$4A\lambda^2\sigma_p^4 + 4A\lambda\sqrt{\gamma'}\sigma_p^3 + (A\gamma' - (1+\theta)^2\Delta - 4\lambda^2)\sigma_p^2 - 4\lambda\sqrt{\gamma'}\sigma_p - \gamma' = 0.$$

The model (RMvV) has the optimal portfolio

$$x^*_{\mathbf{RMvV}} = \left(1 - \frac{\sqrt{\gamma'}}{\sqrt{\gamma'} + 2\lambda\sigma_p^*}\right)(1+\omega)\frac{1}{2\lambda}\Sigma^{-1}\left(\hat{\mu} - \frac{B}{A}e\right) + \frac{\Sigma^{-1}e}{A}, \tag{40}$$

where $\sqrt{\gamma'} = (1+\omega)\sqrt{\gamma} + \omega\kappa_v$ and σ_p^*, the variance of the optimal portfolio, is a positive solution of the fourth degree polynomial equation

$$4A\lambda^2\sigma_p^4 + 4A\lambda\sqrt{\gamma'}\sigma_p^3 + (A\gamma' - (1+\omega)^2\Delta - 4\lambda^2)\sigma_p^2 - 4\lambda\sqrt{\gamma'}\sigma_p - \gamma' = 0.$$

4.4 Robust mean-VaR-CVaR portfolios

When $\lambda = 0$, that is, the investor uses VaR and CVaR to evaluate risk, the model (RMP) becomes the robust mean-VaR-CVaR model

$$\text{(RMVC)}: \min_{x\in\mathscr{X}} \max_{\mathrm{P}\in\mathbb{D}_\gamma} -\mathrm{E}(\xi^T x) + \omega\mathrm{VaR}_\beta(-\xi^T x) + \theta\mathrm{CVaR}_\beta(-\xi^T x).$$

Define

$$\zeta = \sqrt{A((1+\omega+\theta)\sqrt{\gamma} + \omega\kappa_v + \theta\kappa_c)^2 - (1+\omega+\theta)^2\Delta}.$$

Corollary 5 *Let* $\beta \in (0.5, 1]$. *The model* (RMVC) has the optimal portfolio

$$x^*_{\mathbf{RMVC}} = \frac{1}{\zeta}(1+\omega+\theta)\Sigma^{-1}\left(\hat{\mu} - \frac{B}{A}e\right) + \frac{\Sigma^{-1}e}{A} \tag{41}$$

if $\omega\kappa_v + \theta\kappa_c > (1+\omega+\theta)(\sqrt{\frac{\Delta}{A}} - \sqrt{\gamma})$, and is unbounded otherwise.

We have proven that all robust mean-risk models under mean return and distribution ambiguity can be solved explicitly under certain conditions. Table 1 shows that our robust mean-multiple risk portfolio optimization framework is general enough to capture a set of well-studied portfolios. The resulting robust optimal portfolios are all mean-variance efficient. In next section, we briefly test the robustness and study the cumulative wealth of different optimal portfolios.

Table 1 A unified formula $x^*_{\text{Port}} = \phi_{\text{Port}}\Sigma^{-1}(\hat{\mu} - \frac{B}{A}e) + \frac{\Sigma^{-1}e}{A}$ for optimal portfolios of different models, where Port = {RMP, Mv, minimum variance, RC, RV, RMv, RMC, RMV, RMvC, RMvV, RMVC} and $\sqrt{\gamma'} = (1+\omega+\theta)\sqrt{\gamma} + \omega k_v + \theta k_c$

Portfolio strategy	ϕ_{Port}		
RMP	$\frac{\sigma_p^*}{\sqrt{\gamma'}+2\lambda\sigma_p^*}(1+\omega+\theta)$		
Portfolio strategy	ϕ_{Port}	Portfolio strategy	ϕ_{Port}
Mv	$\frac{1}{2\lambda}$	Minimum variance	0
RC	$\frac{1}{\sqrt{A(\sqrt{\gamma}+k_c)^2-\Delta}}$	RV	$\frac{1}{\sqrt{A(\sqrt{\gamma}+k_v)^2-\Delta}}$
RMv	$\frac{\sigma_p^*}{\sqrt{\gamma}+2\lambda\sigma_p^*}$	RMvC	$\frac{\sigma_p^*}{(1+\theta)\sqrt{\gamma}+\theta\kappa_c+2\lambda\sigma_p^*}(1+\theta)$
RMC	$\frac{1+\theta}{\eta}$	RMvV	$\frac{\sigma_p^*}{(1+\omega)\sqrt{\gamma}+\omega\kappa_v+2\lambda\sigma_p^*}(1+\omega)$
RMV	$\frac{1+\omega}{\vartheta}$	RMVC	$\frac{1+\omega+\theta}{\zeta}$

The variance σ_p^* is derived from the fourth degree polynomial equation corresponding to the robust model

5 Numerical experiments

In this section, we briefly present some numerical results based on the simulated data and real market data to show the robustness and performance of the models discussed in the previous section. The rolling horizon procedure similar to that in DeMiguel and Nogales (2009) will be used to test our models. All computations are performed on a PC with Intel Celeron(R) 1.70GHz processor and 2GB RAM and carried out in MATLAB 2008a using CPLEX 12.1 (2009).

5.1 On the stability of RMC portfolios under ambiguity

This subsection presents a simulation that highlights the impact of estimation error on optimal asset allocation. We evaluate and compare the (RMC) model under distribution ambiguity and the non-robust mean-CVaR model (MC). Our experiment is conducted as follows. We first randomly generate a time-series of 200 asset returns by sampling from the true asset return distribution. Then, we carry out a rolling-horizon experiment (DeMiguel and Nogales 2009) based on the time series data. We use the first 150 returns in the time series to estimate the sample mean of asset returns, and then repeat this procedure by "rolling" the estimation window forward one period at a time and simultaneously drop the data for the earliest period until the end of the time series is reached. The investor is assumed to have a CVaR aversion level $\theta = 0.5$ and a confidence level $\beta = 0.95$.

We consider the case where returns of $n = 2$ assets follow multivariate Student's t distribution with 5 degrees of freedom and use monte carlo method to generate return series (see Appendix B). That is,

$$\xi \sim t_\nu(\mu, \Sigma),$$

where $\mu = 0.01e$, $\Sigma = 0.05\mathrm{I}_n$ and ν is the degrees of freedom. The reason we use t-distribution to generate the asset returns is that the multivariate normal distribution assumption may not be satisfied in real-world situation. Some asset returns may capture a heavy-tailed distribution.

In practical calculations, CVaR optimization problem is usually solved by scenario method (Rockafellar and Uryasev 2000). Therefore, given $\xi_{[1]}, \xi_{[2]}, \ldots, \xi_{[S]}$, where each $\xi_{[i]}$ is an independent sample of the mean return vector from its assumed distribution and S is the number of chosen samples or scenarios, the (MC) model takes the form by introducing new variables z_i,

$$\begin{aligned}
(\text{MC}):\quad & \min_{(x,\alpha,z)\in\mathbb{R}^{n+1+S}} \left\{ -\frac{1}{S}\sum_{i=1}^{S} \xi_{[i]}^T x + \theta\left(\alpha + \frac{1}{(1-\beta)S}\sum_{i=1}^{S} z_i\right)\right\} \\
& \quad \text{s.t.} \qquad z_i \geq -\xi_{[i]}^T x - \alpha,\ i = 1, \ldots, S, \\
& \qquad\qquad\quad z_i \geq 0,\ i = 1, \ldots, S, \\
& \qquad\qquad\quad x \in \mathscr{X}.
\end{aligned}$$

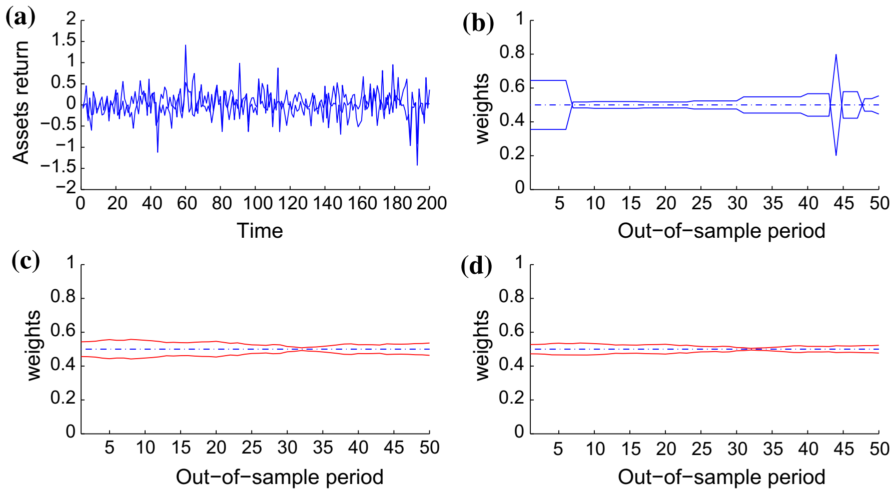

Fig. 3 Time-varying portfolio weights for (MC) and (RMC) with $n = 2$

Figure 3 depicts the times series of 200 returns for the two assets and the time-varying portfolio weights for each portfolio strategy. We find that (RMC)'s portfolios provide an improved stability in the portfolio composition over (MC)'s portfolios and that the robust optimization approach yields more stable strategies over time, and the greater the level of ambiguity aversion γ, the better the robustness of the resulting portfolio performs. And from Panels (c) and (d), as γ is increasing, (RMC)'s portfolios do not distinguish the assets and allocates the wealth equally among them.

5.2 Out-of-sample evaluation using real market data

In this part, we briefly compare the (RMvC) model with the (RMv) and (RMC) models to study the cumulative wealth of different portfolios. Our experiment setting follows closely the one in Rockafellar and Uryasev (2000), where the risky assets are the Standard & Poor index (S&P 500), the long-term US Government Bond (Bond) and the US Small capital index (SmallCap). Historical return data is obtained from the Center for Research in Security Prices (CRSP) database, which is one of the most complete sources of the U.S. Equity Indexes available. For the three assets, monthly returns span January 1998 to December 2016, for a total of 228 observations. We now evaluate the portfolio strategies under the same rolling-horizon procedure described in the previous section but with real market data. We use an estimation window of 84 months and rebalance on a monthly basis. The level of ambiguity aversion is set to 0.1.

In Fig. 4 we plot the cumulated wealth of the (Mv)'s portfolio with $\lambda = 0.5$, the (RMv)'s portfolio with $\lambda = 0.5$, the (RMC)'s portfolio with $\theta = 0.5$, and the (RMvC)'s portfolio with $\lambda = 0.5$ and $\theta = 0.5$ over the time from January 2005 to December 2016. We find that the optimal portfolios based on the robust approach can help preserve the accumulated wealth when the market is volatile (for instance, during the financial crisis), whereas the optimal portfolio obtained from the non-robust (Mv)

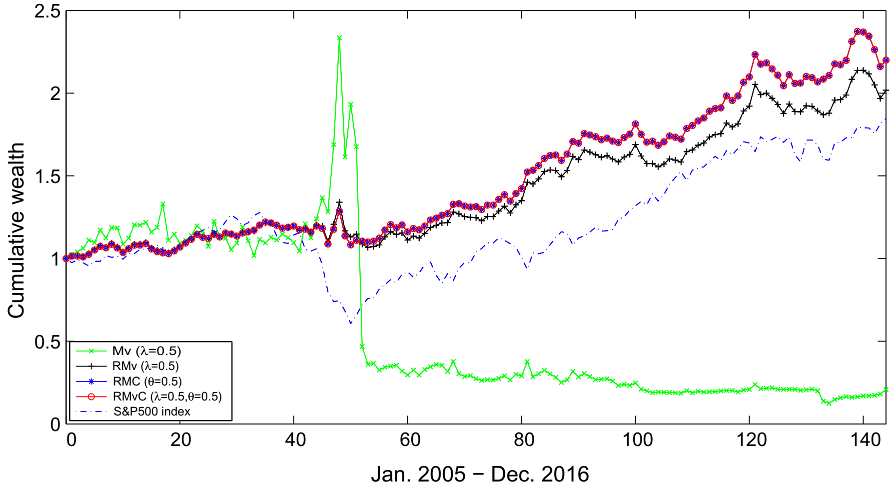

Fig. 4 The cumulative wealth of the optimal portfolios using monthly rebalancing between January 2005 and December 2016. The evolution of S&P500 index is also provided for reference purposes

approach is very erratic. That is, for this particular data set, the optimal portfolios of (RMv), (RMC) and (RMvC) systematically outperform the the one of (Mv). This is because the (Mv)'s portfolios may have extreme negative weights in a number of assets. Moreover, we observe that the portfolios of (RMC) and (RMvC) perform better than the ones of (Mv) and (RMv), and (RMvC) is very close to (RMC) over time.

6 Conclusions

In this paper, we combine the variance, value-at-risk and conditional value-at-risk as a risk measure in a mean-risk optimization model under mean return and distribution ambiguity. The main advantage of our model is that it makes no assumption on probability distribution and leads to a simple closed-form expression for the portfolio strategies. In this manner, the robust mean-variance, robust maximum return, robust minimum VaR and robust minimum CVaR models are all special cases of ours. The numerical experiments using simulated data indicate that our robust model under ambiguity is able to deliver more stability in the portfolio weights in comparison to non-robust approaches. The analysis conducted so far is not intended to provide specific advice or recommendations for any investor. The main purpose of this paper is to provide an analytical framework for investors with more flexibility to find portfolios in that it allows investors to optimize a return-risk profile in the presence of estimation error and meanwhile make a better understanding of the mechanics of robustness. More comprehensive performance analysis of different investment strategies are outside the scope of this article and, therefore, left for future research.

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Appendix A: Proof for Theorem 1

Proof Let $\lambda^{'} = 2\lambda$. By Lemma 3, we can rewrite the model (9) as

$$\min_{x\in\mathscr{X}} \quad -(1+\omega+\theta)\hat{\mu}^T x + \sqrt{\gamma'}\sqrt{x^T \Sigma x} + \frac{\lambda^{'}}{2} x^T \Sigma x. \tag{42}$$

Similar to the proof of Proposition 2 in Garlappi et al. (2007), we first define $\Lambda(x) \equiv \Sigma(\lambda^{'} + \frac{2\sqrt{\gamma'}}{\sqrt{x^T \Sigma x}})$ for $x \in \mathscr{X}$. Then the above model can be further reformulated as

$$\min_{x\in\mathscr{X}} \quad -(1+\omega+\theta)\hat{\mu}^T x + \frac{1}{2} x^T \Lambda(x) x. \tag{43}$$

The Lagrangian function of the above optimization problem is

$$L(x,\nu) = -(1+\omega+\theta)\hat{\mu}^T x + \frac{1}{2} x^T \Lambda(x) x - \nu\left(1 - e^T x\right),$$

where ν is the Lagrangian multiplier for the budget constraint. Clearly, the first order conditions with respect to x is given by

$$\frac{\partial L}{\partial x} = -(1+\omega+\theta)\hat{\mu} + \left(\frac{\sqrt{\gamma'} + \lambda^{'}\sqrt{x^T \Sigma x}}{\sqrt{x^T \Sigma x}}\right) \Sigma x + \nu e = 0.$$

Let $\sigma_p = \sqrt{x^T \Sigma x}$. We have

$$-(1+\omega+\theta)\hat{\mu} + \left(\frac{\sqrt{\gamma'} + \lambda^{'}\sigma_p}{\sigma_p}\right) \Sigma x + \nu e = 0.$$

Thus, we get the candidate solution

$$x = \left(\frac{\sigma_p}{\sqrt{\gamma'} + \lambda^{'}\sigma_p}\right) \Sigma^{-1}\left[(1+\omega+\theta)\hat{\mu} - \nu e\right].$$

Substituting it into the budget constraint $e^T x = 1$, we get

$$\left(\frac{\sigma_p}{\sqrt{\gamma'} + \lambda^{'}\sigma_p}\right) \left[(1+\omega+\theta)\hat{\mu} - \nu e\right]^T \Sigma^{-1} e = 1.$$

That is,

$$\left(\frac{\sigma_p}{\sqrt{\gamma'} + \lambda^{'}\sigma_p}\right) \left[(1+\omega+\theta)B - \nu A\right],$$

where $A = e^T \Sigma^{-1} e$, $B = \hat{\mu}^T \Sigma^{-1} e$. This means that

$$\nu = \frac{1}{A}\left[(1+\omega+\theta)B - \frac{\sqrt{\gamma'} + \lambda' \sigma_p}{\sigma_p}\right].$$

Substituting ν into the candidate solution x yields

$$x = \left(\frac{\sigma_p}{\sqrt{\gamma'} + \lambda' \sigma_p}\right) \Sigma^{-1}\left[(1+\omega+\theta)\hat{\mu} - \frac{1}{A}\left((1+\omega+\theta)B - \frac{\sqrt{\gamma'} + \lambda' \sigma_p}{\sigma_p}\right)e\right].$$

Substituting it into $\sigma_p = \sqrt{x^T \Sigma x}$, we obtain the following polynomial equation about σ_p^*:

$$A\lambda'^2 \sigma_p^4 + 2A\lambda' \sqrt{\gamma'} \sigma_p^3 + \left(A\gamma' - (1+\omega+\theta)^2 \Delta - \lambda'^2\right)\sigma_p^2 - 2\lambda' \sqrt{\gamma'} \sigma_p - \gamma' = 0,$$

where $C = \hat{\mu}^T \Sigma^{-1} \hat{\mu}$, $\Delta = AC - B^2$. Define the function

$$\begin{aligned} g(\sigma_p) = {} & A\lambda'^2 \sigma_p^4 + 2A\lambda' \sqrt{\gamma'} \sigma_p^3 + \left(A\gamma' - (1+\omega+\theta)^2 \Delta - \lambda'^2\right)\sigma_p^2 \\ & - 2\lambda' \sqrt{\gamma'} \sigma_p - \gamma'. \end{aligned}$$

for $\sigma_p \in [0, +\infty)$. Because $g(0) = -\gamma' < 0$ and $\lim_{\sigma_p \to +\infty} g(\sigma_p) = +\infty$, the Eq. (16) has at least one positive real root σ_p^*. Consequently, the optimal portfolio is given by

$$\begin{aligned} x_{RMP}^* &= \left(\frac{\sigma_p^*}{\sqrt{\gamma'} + 2\lambda \sigma_p^*}\right) \Sigma^{-1} \\ &\quad \times \left[(1+\omega+\theta)\hat{\mu} - \frac{1}{A}\left((1+\omega+\theta)B - \frac{\sqrt{\gamma'} + 2\lambda \sigma_p^*}{\sigma_p^*}\right)e\right] \\ &= \left(1 - \frac{\sqrt{\gamma'}}{\sqrt{\gamma'} + 2\lambda \sigma_p^*}\right)(1+\omega+\theta)\frac{1}{2\lambda}\Sigma^{-1}\left(\hat{\mu} - \frac{B}{A}e\right) + \frac{\Sigma^{-1} e}{A}. \end{aligned} \tag{44}$$

This completes the proof. □

Appendix B: Generation of random samples

Procedure: Generating multivariate t-distribution return series of $t_\nu(\mu, \Sigma)$ with degree of freedom ν, expected returns μ and covariance matrix Σ.

(1) Decompose Σ via Cholesky decomposition to obtain a lower triangular matrix G such that $\Sigma = GG^T$.

(2) Generate an $n \times 1$ vector z with $z \sim \mathcal{N}(0, I_n)$, where I_n is an $n \times n$ identity matrix. Setting $y = Gz$, then $y \sim \mathcal{N}(0, \Sigma)$.
(3) Generate a random variable d with chi-squared distribution χ^2_ν.
(4) Setting $h = y\sqrt{\frac{\nu}{d}} + \mu$, then $h \sim t_\nu(\mu, \Sigma)$.

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