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ORIGINAL ARTICLE

Process and optimization implementation of the α-ENSC value

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Abstract In this paper, we introduce a new value called α-ENSC value which is a convex combination of egalitarian non-separable contribution value (ENSC value) and the equal division value (ED value). The α-ENSC value reconciles two economic thoughts: egoism and altruism. We study an allocation process under the assumption that players are partially egocentric, and the final outcome happens to be the α-ENSC value. The α-ENSC value is also the optimal solution for corresponding optimization models under certain complaint criterion. Several new properties are proposed to characterize the α-ENSC value, including α-dual individual rationality, α-egocentric inessential game property and grand marginal contribution monotonicity.

Keywords Cooperative game · ENSC value · ED value · Allocation process · Axiomatization

1 Introduction

Cooperative game theory studies the mathematical models of cooperation between intelligent rational decision-makers (Myerson 1991). As a useful mathematical tool, it is widely used in economics and political science. Cooperative game theory provides

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methods to model the formation of grand value of the participants and to study the fair and reasonable allocation of this value.

In 1953, Shapley introduced a solution, called the Shapley value (Shapley 1953), which assigns to each player a payoff measuring his productivity within a cooperative game. The Shapley value can be interpreted as an average of marginal contributions, i.e., players enter the game one by one in the order $(\pi(1), \pi(2), \ldots, \pi(n))$, where π is a bijection on the player set, and each player obtains the marginal contribution he creates by joining the group of players already present, and the Shapley value averages all marginal vectors (Driessen 1988). Since every new entrant obtains his marginal contribution and does not share it with other players, in this sense, the Shapley value can be seen as an egocentric or self-interested allocation rule.

However, the marginal contribution may not be the ideal payoff of the players. Gillies (1953) stated that the grand marginal contribution is an upper bound of the core, which is the set of feasible allocations that cannot be improved upon by any coalition. When considering the allocation, the closer to the grand marginal contribution, the happier the player is. Under this circumstance, all the players would like to obtain their grand marginal contributions and do not have trend to share it with other players. In this paper, we call this sort of player *totally egocentric*. The ENSC value, which was proposed by Moulin (1985), is derived under this situation. In contrast, the player who takes out all his contributions and obtains nothing is called *totally altruistic* player. The ED value assigns to every player the same fraction of the total worth no matter how much their contributions are. In this sense, the ED value can be understood as an allocation under the principle of “altruism”.

In this paper, we consider the convex combination of the ED value and the the ENSC value. This combinatorial value reconciles two thoughts: egocentrism and altruism. We call this value the α-ENSC value. By introducing an individual preference parameter α, where $\alpha \in [0, 1]$, which reflects the players’ predilection to the grand marginal contribution. We study the dynamic preference situation which has the altruistic behavior and egocentric behavior as two extreme cases. When $\alpha = 0$, the player is totally altruistic, and the player is totally egocentric while $\alpha = 1$. The larger α is, the more selfish the player is. In order to reflect these two attitudes, several other combinatorial values have also been proposed. Notice that the α-ENSC value belongs to the class of the equal surplus sharing solutions, which were introduced by van den Brink and Funaki (2009). In this paper, we propose a process implementation of the value and study the value as the optimal solution of some optimization models. van den Brink and Funaki (2013) considered the convex combination of the Shapley value and the ED value. When compared with the ENSC value, the Shapley value seems less egocentric as an extreme point. Genjiu et al. (2015) introduced the α-CIS value as the combination of the CIS value and the ED value. But what they took into account is just the minimal right for players, that is, players’s individual worth. For that reason, the α-CIS value may fail to embody the egocentric part properly.

The excess of coalition with respect to a given allocation vector is a very important concept in cooperative game theory. It’s defined as the gap between the worth of coalition and what they can obtain from the proposed payoff, so excesses are also interpreted as the complaint of coalition. Many solution concepts for cooperative games in complaint form have been studied. The kernel, proposed by Davis and Maschler (1965), is a kind of multi-bilateral bargaining equilibrium based on excesses of coalitions without interpersonal utility comparisons. By minimizing the complaint function in the lexicographic order among the imputation set, Schmeidler (1969) in 1969 introduced the Nucleolus. In Maschler et al. (1979), clarified that the Nucleolus is the result of an arbitrator's desire to minimize the dissatisfaction of the most dissatisfied coalition. Instead of minimizing the maximal excess among coalitions in lexicographic order, Ruiz et al. (1996) proposed a Nucleolus-like solution concept called the least square nucleolus, which minimizes the variance of the excesses of all coalitions. All of these solutions are under the assumption that coalitions regard their own worth as ideal payoff, and the object is to optimize the gap between this ideal payoff and potential allocations under different criterions.

In this paper, we introduce a new kind of complaints for individual players instead of considering coalitions. In reality, players may be neither totally egocentric nor altruistic, and it is possible that the new entrant only obtains part of his grand marginal contribution. For different grand coalition forming orders, we assumed that all players are partially egocentric as we have announced, which means players could only get a fraction of his grand marginal contribution and the remaining is shared equally among the existing players. By considering players' predilection to the grand marginal contribution, it corresponds to the ideal payoff vector of individual players, yielding a gap between ideal payoff and proposed allocation, the difference of which is called the complaint. Our aim is to minimize the total variance of complaint for players among all formation orders under the least square criterion, which is similar to the method as Ruiz et al. have applied. Interestingly, the unique optimal solution for this problem coincides with the α-ENSC value. Moreover, the expected ideal payoff, which is the average ideal payoff deprived from considering all formation orders, also coincides with the α-ENSC value. One may notice that this process seems similar to the "procedural" value proposed by Malawski (2013). What players share during the process is the grand marginal contributions rather than marginal contributions to the coalitions formed by players joining in random order. Besides, the α-ENSC value does not belong to "procedural" value, since it fails to satisfy weakly monotonic, which is one of the axiomatizations to characterize "procedural" value.

van den Brink and Funaki (2009) characterized the ENSC value by means of dual individual rationality, which is under the assumption that all players regard their grand marginal contribution as ideal payoff. Notice that players in our model are partially egocentric, we propose a new property called α-dual individual rationality. Interestingly, together with efficiency, linearity and symmetry, α-dual individual rationality characterizes the α-ENSC value. Moreover, some other new properties are also proposed to characterize the α-ENSC value, including α-egocentric inessential game property and grand marginal contribution monotonicity. All of them reflect the dual characters of players: egocentrism and altruism.

The paper is organized as follows: In Sect. 2, we give a brief introduction of the relevant game theoretic notions and solution concepts. In Sect. 3, we study an allocation process, of which the final outcome coincides with the α-ENSC value. In Sect. 4, two optimization models are discussed, and both of them leads to the α-ENSC value. In Sect. 5, we present the axiomatizations of the α-ENSC value, and the paper concludes with a brief summary and discussion of further research in Sect. 6.

2 Preliminaries

A cooperative game on a finite player set N is an ordered pair (N, v), where characteristic function $v : \mathcal{P}(N) \rightarrow \mathbb{R}$ is defined on $\mathcal{P}(N)$ satisfying $v(\emptyset) = 0$, and in short called a game v on N. Here $\mathcal{P}(N)$ denotes the power set of N, given by $\mathcal{P}(N) = \{S|S \subseteq N\}$. Denote Γ_n the class of all cooperative games with player set N. Although we do not require it, any cooperative game v is often assumed to be superadditive, i.e., $v(S \cup T) \geq v(S) + v(T)$, for all disjoint coalitions $S, T \subseteq N$. For notational convenience, throughout this paper, denote $v(\{i\})$ by $v(i)$, and $n = |N|$.

For any $T \subset N, T \neq \emptyset$, the *standard game* $b^T \in \Gamma_n$ is defined by

$$b^T(S) = \begin{cases} 1, & if\ S = T \\ 0, & otherwise \end{cases} \tag{1}$$

The set $\{b^T\}_{T \in \mathcal{P}(N)\backslash\emptyset}$ forms a basis of Γ_n. Given any $(N, v) \in \Gamma_n$, the dual game of v is defined by $v_*(S) = v(N) - v(N\backslash S)$ for all $S \subseteq N$.

Any $\mathbf{x} \in \mathbb{R}^n$ is called a payoff vector, and $\mathbf{x}(S) = \sum_{i \in S} \mathbf{x}_i$ denotes the total payoff for coalition S. A value or solution on Γ_n is a function $\phi : \Gamma_n \rightarrow \mathbb{R}^n$. The $i'th$ coordinate $\phi_i(v)$ of the vector $\phi(v)$ represents the payoff of player i in the game $v \in \Gamma_n$. We first list several well-known properties of value for cooperative games.

A value ϕ satisfies:

- **Efficiency**: if $\sum_{i \in N} \phi_i(v) = v(N)$ for all $(N, v) \in \Gamma_n$.
- **Linearity**: if for any $(N, v), (N, w) \in \Gamma_n, \alpha, \beta \in \mathbb{R}, \phi_i(\alpha v + \beta w) = \alpha\phi_i(v) + \beta\phi_i(w)$.
- **Symmetry**: for all $(N, v) \in \Gamma_n$, if $\phi_i(v) = \phi_j(v)$, where $i, j \in N$ are symmetric players in game (N, v) (that is $v(S \cup i) = v(S \cup j)$, for all $S \subseteq N\backslash\{i, j\}$).
- **Dual individual rationality**: if $\phi_i(v) \geq v_*(i)$, for all $i \in N$ and dual weakly essential game $(N, v) \in \Gamma_n$ (that is $\sum_{j \in N} v_*(j) \leq v(N)$).

The ENSC value and ED value are solution concepts that rely on egoism and altruism considerations respectively. The ENSC value is an egocentric distribution from the angle of player's attitude to grand marginal contribution, i.e., the players first obtain their grand marginal contribution, and the excessive worth is divided equally among other existing players. In contrast, the ED value reflects solidarity in the most radical manner, i.e., the overall worth generated is distributed evenly among the players. Based on these principles, we give specific definitions of two values and one set-valued solution for cooperative game as follows, the pre-imputation set $I^*(v)$, the ENSC value and the ED value.

Definition 1 The vectors $x \in \mathbb{R}^n$ which satisfy the efficiency principle are called pre-imputation (Driessen 1988), i.e.,

$$I^*(v) = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x}(N) = v(N)\}.$$

Definition 2 For any $(N, v) \in \Gamma_n$ and all players $i \in N$, the ED value is defined as

$$ED_i(v) = \frac{v(N)}{n}.$$

Definition 3 The ENSC value is introduced by Moulin (1985) as

$$ENSC_i(v) = b_i^v + \frac{1}{n}\left[v(N) - \sum_{j \in N} b_j^v\right].$$

where $b_j^v = v(N) - v(N \setminus j)$, which is the so-called grand marginal contribution.

The α-ENSC value, based on both the egoism and altruism, is a convex combination of the ENSC value and the ED value.

Definition 4 For any game $(N, v) \in \Gamma_n$ and $\alpha \in [0, 1]$, the α-ENSC value is defined as

$$\alpha\text{-}ENSC_i(v) = \alpha ENSC_i(v) + (1-\alpha)ED_i(v), \quad \text{for all} \quad i \in N. \tag{2}$$

We can rewrite the formula of the α-ENSC value as

$$\alpha\text{-}ENSC_i(v) = \alpha b_i^v + \frac{1}{n}\left[v(N) - \sum_{j \in N} \alpha b_j^v\right], \quad \text{for all} \quad i \in N. \tag{3}$$

In view of Eq. (3), the α-ENSC value can also be regarded as a generalization of the ENSC value by taking $0 \leq b_i^v \leq v(N) - v(N \setminus i)$ in Definition 3. The parameter α reflects the degree of aspiration for players towards their grand marginal contribution.

3 Process implementation of the α-ENSC value

Given a formation order, player who claims all his contribution to the grand coalition when he joins the game is totally egocentric. But the general case is that players may also possess the altruistic quality and be partially egocentric. We will incarnate the degree of egocentrism by a parameter. The larger this parameter is, the more egocentric players are.

Definition 5 A player is α ***partially egocentric*** ($\alpha \in [0, 1]$), if for any coalition formation order in which he is not the first entrant, he would like to obtain α times his grand marginal contribution when he joins the game and what's left is distributed equally among the existing players.

To be special, when $\alpha = 0$, the new comer will obtain 0. If $\alpha = 1$, the new comer monopolizes his grand marginal contribution. We illustrate this thought by considering a three-person game.

Example 1 Given that all players are α partially egocentric, and the entrance order is $\pi = (3, 1, 2)$.

At the beginning, player 3 joins the game. Since he is the first entrant, the payoff is his individual worth $v(3)$, while player 1 and 2 get nothing. Payoff allocation for players at the moment is shown in Table 1.

Then, player 1 comes. As we have assumed that players are all α partially egocentric, player 1 gets α times his grand marginal contribution. The remaining worth $v(1, 3) -$

Table 1 Payoff allocation when only one player 3 joining in

Players	Player 1	Player 2	Player 3
Allocation	0	0	$v(3)$

Table 2 Payoff allocation when player 1 joins in

Players	Player 1	Player 2	Player 3
Allocation	αb_1^v	0	$v(1,3) - v(3) - \alpha b_1^v$

Table 3 Payoff allocation when player 2 joins in

Players	Player 1	Player 2	Player 3
Allocation	$\frac{v(N)-v(1,3)-\alpha b_2^v}{2}$	αb_2^v	$\frac{v(N)-v(1,3)-\alpha b_2^v}{2}$

Table 4 Payoff for α partially egocentric players under formation procedure $\pi = (3, 1, 2)$

N	Allocation in every step			Player's payoff
3 join in	$v(3)$	$v(1,3) - v(3) - \alpha b_1^v$	$\frac{v(N)-v(1,3)-\alpha b_2^v}{2}$	$-\alpha b_1^v + \frac{v(N)+v(1,3)-\alpha b_2^v}{2}$
1 join in	αb_1^v	$\frac{v(N)-v(1,3)-\alpha b_2^v}{2}$		$\alpha b_1^v + \frac{v(N)-v(1,3)-\alpha b_2^v}{2}$
2 join in	αb_2^v			αb_2^v

$v(3) - \alpha b_1^v$ is charged by player 3, while player 2 still obtains nothing. Table 2 shows the payoff allocation to players in this step.

Finally, the last player 2 joins and he obtains α times his grand marginal contribution and what's left is shared among players 1, 3 equally. Table 3 shows the outcome.

Summate the payoffs in different three stages, then we obtain the total payoff to players as is showed in Table 4.

Obviously, the final allocation scheme satisfies the efficiency principle. Now we give the general explicit expression of the total payoff to players in this procedure as follows:

Definition 6 Given that all players are α partially egocentric and for the coalition formation order π, player $i's$ **ideal payoff** is

$$\eta_i^{\alpha\pi} = \begin{cases} v(i) + \sum\limits_{k=\pi(i)+1}^{n} \frac{v\left(S_\pi^{\pi^{-1}(k)}\right) - v\left(S_\pi^{\pi^{-1}(k)} \setminus \pi^{-1}(k)\right) - \alpha b_{\pi^{-1}(k)}^v}{k-1}, & \pi(i) = 1, \\ \alpha b_i^v + \sum\limits_{k=\pi(i)+1}^{n} \frac{v\left(S_\pi^{\pi^{-1}(k)}\right) - v\left(S_\pi^{\pi^{-1}(k)} \setminus \pi^{-1}(k)\right) - \alpha b_{\pi^{-1}(k)}^v}{k-1}, & \pi(i) \neq 1, n, \\ \alpha b_i^v, & \pi(i) = n. \end{cases} \tag{4}$$

where $S_\pi^i = \{j \in N \mid \pi(j) \le \pi(i)\}$ is the set composed of player i and the players already present. $\pi(i)$ is the position of player i in order π and $\pi^{-1}(k)$ denotes the player who possesses position k in order π.

We call the payoff in Eq. (4) ideal payoff because under the assumption that all players are α partially egocentric, every player in any coalition formation order gets what he wants. The next theorem reveals the fact that the ultimate payoff is identical to the α-ENSC value when all formation order are taken into consideration.

Theorem 1 *For any $(N, v) \in \Gamma_n$, given that all the players are α partially egocentric, then player i's expected ideal payoff coincides with his α-ENSC value, i.e.,*

$$\sum_{\pi \in \Pi(N)} \frac{1}{n!}\eta_i^{\alpha\pi} = \alpha\text{-}ENSC_i(v) = \alpha ENSC_i(v) + (1-\alpha)ED_i(v), \tag{5}$$

where $\eta_i^{\alpha\pi}$ has the form in Eq. (4).

In order to prove the Theorem, the following two lemmas are taken into account. For notional convenience, $v(S_\pi^{\pi^{-1}(k)}) - v(S_\pi^{\pi^{-1}(k)} \backslash \pi^{-1}(k))$ is denoted by $m_{\pi(k)}$ throughout the paper.

Lemma 1 *For any cooperative game $(N, v) \in \Gamma_n$, the ENSC value has the form as follows:*

$$ENSC_i(v) = \frac{1}{n}b_i^v + \sum_{\pi:\pi(i)=1} \frac{1}{n!}\left[v(i) + \sum_{k=\pi(i)+1}^{n} \frac{m_{\pi(k)} - b^v_{\pi^{-1}(k)}}{k-1}\right] + \sum_{\pi:\pi(i)\neq 1,n} \frac{1}{n!}\left[b_i^v + \sum_{k=\pi(i)+1}^{n} \frac{m_{\pi(k)} - b^v_{\pi^{-1}(k)}}{k-1}\right]$$

The proof of Lemma 1 is lengthy, therefore we put it in Appendix 1. By considering the particular case of Lemma 1, $b_i^v = 0$ for all $i \in N$, we can obtain the following conclusion.

Lemma 2 *For any cooperative game $(N, v) \in \Gamma_n$, the ED value has the form as follows:*

$$ED_i(v) = \sum_{\pi:\pi(i)=1} \frac{1}{n!}\left[v(i) + \sum_{k=\pi(i)+1}^{n} \frac{m_{\pi(k)}}{k-1}\right] + \sum_{\pi:\pi(i)\neq 1,n} \frac{1}{n!} \sum_{k=\pi(i)+1}^{n} \frac{m_{\pi(k)}}{k-1}$$

Proof of Theorem 1 In view of Eq. (4), we have

$$
\begin{aligned}
\sum_{\pi\in\Pi(N)}\frac{1}{n!}\eta_i^{\alpha\pi} &= \sum_{\pi:\pi(i)=1}\frac{1}{n!}\eta_i^{\alpha\pi} + \sum_{\pi:\pi(i)\neq 1,n}\frac{1}{n!}\eta_i^{\alpha\pi} + \sum_{\pi:\pi(i)=n}\frac{1}{n!}\eta_i^{\alpha\pi}\\
&= \sum_{\pi:\pi(i)=1}\frac{1}{n!}\left[v(i) + \sum_{k=\pi(i)+1}^{n}\frac{m_{\pi(k)}-\alpha b^v_{\pi^{-1}(k)}}{k-1}\right]\\
&\quad + \sum_{\pi:\pi(i)\neq 1,n}\frac{1}{n!}\left[\alpha b_i^v + \sum_{k=\pi(i)+1}^{n}\frac{m_{\pi(k)}-\alpha b^v_{\pi^{-1}(k)}}{k-1}\right] + \frac{(n-1)!}{n!}\alpha b_i^v\\
&= \sum_{\pi:\pi(i)=1}\frac{1}{n!}\left[\alpha v(i) + (1-\alpha)v(i) + \sum_{k=\pi(i)+1}^{n}\frac{\alpha(m_{\pi(k)}-b^v_{\pi^{-1}(k)}) + (1-\alpha)m_{\pi(k)}}{k-1}\right]\\
&\quad + \sum_{\pi:\pi(i)\neq 1,n}\frac{1}{n!}\left[\alpha b_i^v + \sum_{k=\pi(i)+1}^{n}\left(\frac{\alpha m_{\pi(k)}-\alpha b^v_{\pi^{-1}(k)}}{k-1} + \frac{(1-\alpha)m_{\pi(k)}}{k-1}\right)\right] + \frac{1}{n}\alpha b_i^v\\
&= \frac{1}{n}\alpha b_i^v + \sum_{\pi:\pi(i)=1}\frac{1}{n!}\left[\alpha v(i) + \sum_{k=\pi(i)+1}^{n}\frac{\alpha m_{\pi(k)}-\alpha b^v_{\pi^{-1}(k)}}{k-1}\right]\\
&\quad + \sum_{\pi:\pi(i)\neq 1,n}\frac{1}{n!}\left[\alpha b_i^v + \sum_{k=\pi(i)+1}^{n}\frac{\alpha m_{\pi(k)}-\alpha b^v_{\pi^{-1}(k)}}{k-1}\right]\\
&\quad + \sum_{\pi:\pi(i)=1}\frac{1}{n!}\left[(1-\alpha)v(i) + \sum_{k=\pi(i)+1}^{n}\frac{(1-\alpha)m_{\pi(k)}}{k-1}\right]\\
&\quad + \sum_{\pi:\pi(i)\neq 1,n}\frac{1}{n!}\sum_{k=\pi(i)+1}^{n}\frac{(1-\alpha)m_{\pi(k)}}{k-1}\\
&= \alpha ENSC_i(v) + (1-\alpha)ED_i(v)\\
&= \alpha\text{-}ENSC_i(v)
\end{aligned}
$$

The last but one equation is derived from Lemmas 1 and 2. □

Malawski (2013) introduced the concept of "procedural" values, the family of which includes several classical values for cooperative games, such as the Shapley value, the ED value and solidarity value. Although the α-ENSC value also has the procedural form, it does not belong to the family since it fails to obey the weakly monotonic property (Weber 1988), which is one of the axiomatizations to characterize the "procedural" values.

Remark 1 Suppose that players have different degrees of altruism and egoism in the model, we could get a more general result. Let α_i denote the degree of egoism for player i. Similar to the procedure in Example 1, players join the game one by one, but every new entrant i obtains α_i times his grand marginal contribution, then the corresponding ideal payoff with respect to coalition formation order π is as follows.

Table 5 Payoff for players under formation procedure $\pi = (3, 1, 2)$

N	Allocation in every step			Player's payoff
3 join in	$v(3)$	$v(1,3) - v(3) - \alpha_1 b_1^v$	$\frac{v(N)-v(1,3)-\alpha_2 b_2^v}{2}$	$-\alpha_1 b_1^v + \frac{v(N)+v(1,3)-\alpha_2 b_2^v}{2}$
1 join in	$\alpha_1 b_1^v$	$\frac{v(N)-v(1,3)-\alpha_2 b_2^v}{2}$		$\alpha_1 b_1^v + \frac{v(N)-v(1,3)-\alpha b_2^v}{2}$
2 join in	$\alpha_2 b_2^v$			$\alpha_2 b_2^v$

$$\eta_i^{\vec{\alpha}\pi} = \begin{cases} v(i) + \sum\limits_{k=\pi(i)+1}^{n} \frac{v\left(S_\pi^{\pi^{-1}(k)}\right) - v\left(S_\pi^{\pi^{-1}(k)} \backslash \pi^{-1}(k)\right) - \alpha_{\pi^{-1}(k)} b^v_{\pi^{-1}(k)}}{k-1}, & \pi(i) = 1, \\ \alpha_i b_i^v + \sum\limits_{k=\pi(i)+1}^{n} \frac{v\left(S_\pi^{\pi^{-1}(k)}\right) - v\left(S_\pi^{\pi^{-1}(k)} \backslash \pi^{-1}(k)\right) - \alpha_{\pi^{-1}(k)} b^v_{\pi^{-1}(k)}}{k-1}, & \pi(i) \neq 1, n, \\ \alpha_i b_i^v, & \pi(i) = n. \end{cases} \tag{6}$$

Without loss of generality, we write $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. For any $i \in N$, when considering all the formation orders of the grand coalition, the average final payoff is given by

$$\sum_{\pi \in \Pi(N)} \frac{1}{n!} \eta_i^{\vec{\alpha}\pi} = \alpha_1 ENSC_i(v) + \sum_{k=2}^{n} (\alpha_k - \alpha_{k\text{-}1}) ENSC_i(v^k) + (1 - \alpha_n) ED_i(v), \tag{7}$$

where v^k, $k \in \{2, 3, \ldots, n\}$, is defined as

$$v^k(S) = \begin{cases} v(N), & if\ S = N \backslash \{l\}, l < k, \\ v(S), & otherwise. \end{cases} \tag{8}$$

The proof of Eq. (7) is similar with that of Theorem 1, while the difference is that it entails the decomposition of the payoff in every step for players. Notice that the degree of egoism α_i for player i is decomposed into different levels, $\alpha_1, \alpha_2 - \alpha_1, \ldots, \alpha_i - \alpha_{i-1}$. For games v^k with $k \leq i$, player i will obtain a portion of the ENSC value as his egoistic action. Once $k > i$, the grand marginal contribution for i in game v^k equals zero and what he gets in these games is just originated from other players' altruistic action. Moreover, the α-ENSC value is one of the special cases when all the degrees are the same.

We conclude this section by reconsidering Example 1 to illustrate the validity of Eq. (7).

Example 2 Given that player i is α_i partially egocentric, and the entrance order is $\pi = (3, 1, 2)$. Here we omit the allocation for players in every step and only show the final payoff under this entrance order in Table 5.

By considering all the permutations of the grand coalition, the average payoff of players in this model is given as

$$\sum_{\pi\in\Pi(N)} \frac{1}{n!}\eta_i^{\vec{\alpha}\pi} = \alpha_i b_i^v + \frac{v(N) - \sum_{j\in N}\alpha_j b_j^v}{3}, \quad \forall i \in N. \tag{9}$$

In view of Eq. (8), we have

$$v^2(S) = \begin{cases} v(N), & if\ S = \{2, 3\}, \\ v(S), & otherwise. \end{cases} \tag{10}$$

and

$$v^3(S) = \begin{cases} v(N), & if\ S = \{2, 3\}, \{1, 3\}, \\ v(S), & otherwise. \end{cases} \tag{11}$$

According to the formula in Eq. (7), we can obtain the average payoff for players as follows

$$\sum_{\pi\in\Pi(N)} \frac{1}{n!}\eta_i^{\vec{\alpha}\pi} = \alpha_i b_i^v + \frac{v(N) - \sum_{j\in N}\alpha_j b_j^v}{3}, \quad \forall i \in N, \tag{12}$$

which is the same as in Eq. (9).

4 Optimization models for the α-ENSC value

Nucleolus and the least square nucleolus both are the optimal solutions for certain optimization problems of which the aims are to minimize the complaint of coalitions under different criterion. In this section, we discuss complaint from the perspective of individual player and explore the optimal solutions for the same optimization models as in the determination of Nucleolus and the least square nucleolus based on the new complaint we define.

Definition 7 Given that players are all α partially egocentric and the ideal payoff $\eta_i^{\alpha\pi}$ is in the form of (4), then denote $e(i, \mathbf{x}, \eta^{\alpha\pi})$ the complaint of player i at the payoff vector $\mathbf{x}$ with respect to the ideal payoff $\eta^{\alpha\pi}$ of the game v, i.e.,

$$e(i, \mathbf{x}, \eta^{\alpha\pi}) = \eta_i^{\alpha\pi} - \mathbf{x}_i \quad i \in N, \mathbf{x} \in I^*(v). \tag{13}$$

Recall that the least square nucleolus is the imputation payoff for which the complaint vector is closest to vector zero. One may wonder that what will the optimal solution be when considering the new complaint of individual players with the similar optimization model. Formally, the following problem is taken into account so as to minimize the variance of complaint for players.

Problem 1 *Minimize* $\sum_{i\in N}\sum_{\pi}(\eta_i^{\alpha\pi} - \boldsymbol{x}_i)^2$ *s.t.* $\sum_{i\in N}\boldsymbol{x}_i = v(N), \boldsymbol{x} \in \mathbb{R}^n$.

Theorem 2 *For any $(N, v) \in \Gamma_n$, there exists a unique optimal solution $\boldsymbol{x}^* \in \mathbb{R}^n$ of Problem 1, which coincides with the α-ENSC value, i.e.,*

$$\boldsymbol{x}^* = \alpha\text{-}ENSC(v) = \alpha ENSC(v) + (1-\alpha)ED(v) \tag{14}$$

According to Theorem 1, since the α-ENSC value is an average ideal payoff over all the orders, it is clear that it minimizes the variance in Problem 1. Thus, Theorem 2 is straightforward.

Definition 8 Given that players are all α partially egocentric, denote $e^*(i, \mathbf{x})$ the expected complaint of i at the payoff vector $\mathbf{x}$ by considering $n!$ different coalition formation orders in complaint criterion (13), i.e., for all $i \in N$, $\mathbf{x} \in I^*(v)$

$$e^*(i, \mathbf{x}) = \sum_{\pi} \frac{1}{n!}[\eta_i^{\alpha\pi} - \mathbf{x}_i]. \tag{15}$$

One may notice that all players have the same expected complaint with respect to the allocation under α-ENSC value. Given expected complaint criterion in form (15), a value ϕ satisfies

- **Equal expected complaint property**: if $e^*(i, \phi) = e^*(j, \phi)$, for any $i, j \in N$.

Obviously, the equal expected complaint property and efficiency define the α-ENSC value.

The Nucleolus proposed by Schmeidler (1969) is an optimal value which minimizes the complaint function in the lexicographic order over the imputation set. Given the expected payoff complaint criterion in the form of (15), we define the so-called ***expected complaint vector*** $\theta^*(\mathbf{x}) \in \mathbb{R}^n$, for which components are the expected complaints $e^*(i, \mathbf{x})$ arranged in non-increasing order. The next theorem states that the α-ENSC value is the optimal solution that minimize the expected complaint vector in lexicographic order among pre-imputation set.

Theorem 3 *For any $v \in \Gamma_n$, the α-ENSC value is the unique pre-imputation that satisfies*

$$\theta^*(\alpha\text{-}ENSC(v)) \leq_L \theta^*(\boldsymbol{x}) \quad \textit{for all } \boldsymbol{x} \in I^*(v)$$

Where $\leq_L$ is the lexicographic order.

Proof Given any $\mathbf{x} \in I^*(v)$, the expected complaint of player i is

$$e^*(i, \mathbf{x}) = \sum_{\pi} \frac{1}{n!}\left[\eta_i^{\alpha\pi} - \mathbf{x}_i\right] = \alpha\text{-}ENSC_i(v) - \mathbf{x}_i.$$

The last equation is derived from Theorem 1. Moreover, $\sum_{i\in N} \alpha\text{-}ENSC_i(v) = \mathbf{x}(N) = v(N)$, thus for any $\mathbf{x} \in I^*(v)$, it holds that $\sum_{i\in N} \theta_i^*(\mathbf{x}) = 0$. Together with $\theta^*(\alpha\text{-}ENSC(v)) = (0, 0, \ldots, 0)$, we can obtain that the α-ENSC value is the unique pre-imputation in the above conditions satisfying the lexicographic order. □

5 Axiomatizations of the α-ENSC value

van den Brink and Funaki (2009) applied dual individual rationality to characterize the ENSC value. Dual individual rationality indicates that players will get at least his marginal contribution to the grand coalition for dual weakly essential games, which states that the total grand marginal contribution for players does not exceed the worth of the grand coalition. But as we have assumed in Sect. 3 that players are all α partially egocentric, which implies that marginal contribution to the grand coalition may no longer be the objective for players, so the following property is put forward.

A value ϕ satisfies

- α**-dual individual rationality**: if $\forall i \in N$, $\phi_i(v) \geq \alpha[v(N) - v(N\backslash i)]$, for any α-dual weakly essential game $(N, v) \in \Gamma_n$.

Game v is α-dual weakly essential if $\alpha \sum_{i\in N}[v(N) - v(N\backslash i)] \leq v(N), \alpha \in [0, 1]$.

The α-dual individual rationality states that players could only guarantee a fraction α of his grand marginal contribution. However, there always exists gap between reality and ideality. This can only be realized when the worth of the grand coalition is adequate, that is, for α-dual weakly essential game.

It is not difficult to verity that the α-ENSC value satisfies this property. Moreover, we could also apply this property together with other properties to characterize the α-ENSC value. In order to do that, we first give a new class game of Γ_n. For any $T \subsetneqq N$,

$$Z^T(S) = \begin{cases} \alpha - 1, & if\ T = N\backslash S \\ 0, & if\ T \subsetneqq N\backslash S \\ \alpha, & otherwise \end{cases} \tag{16}$$

As to $T = N$, we define $Z^N(S) = 1$, for all $S \subset N$. Besides, we consider the dual game of the standard game b^T in form of (1) denoted by $b^T_*(S) = b^T(N) - b^T(N\backslash S)$, for all $S, T \subset N$, and $T \neq \emptyset$.

Theorem 4 *The α-ENSC value is the unique value that satisfies efficiency, linearity, symmetry and α-dual individual rationality.*

Proof One could easily verify that the α-ENSC value satisfies the four properties. The remaining is to prove the uniqueness. Suppose that solution ϕ obeys above mentioned properties.

Given a coalition T, consider cooperative game (N, Z^T) in (16). When $|T| = 1$, $\alpha \sum_{i\in N}[Z^T(N) - Z^T(N\backslash i)] = \alpha[n\alpha - (\alpha - 1) - (n - 1)\alpha] = \alpha = Z^T(N)$, that is, game Z^T is α-dual weakly essential. Thus the α-dual individual rationality implies that

$$\phi_i(Z^T) \geq \alpha[Z^T(N) - Z^T(N\backslash i)] = \begin{cases} \alpha, & if\ i \in T \\ 0, & if\ i \notin T \end{cases} \tag{17}$$

Together with efficiency principle $\sum_{i\in N} \phi_i(Z^T) = Z^T(N) = \alpha$, we have

$$\phi_i(Z^T) = \begin{cases} \alpha, & if\ i \in T \\ 0, & if\ i \notin T \end{cases} \tag{18}$$

When $2 \leq |T| \leq n-1$, observe that $\alpha \sum_{i \in N}[b_*^T(N) - b_*^T(N\backslash i)] = 0 = b_*^T(N)$, the α-dual individual rationality implies that $\phi_i(b_*^T) \geq \alpha[b_*^T(N) - b_*^T(N\backslash i)] = 0, \forall i \in N$. Besides, efficiency principle states that $\sum_{i \in N} \phi_i(b_*^T) = b_*^T(N) = 0$, thus $\phi_i(b_*^T(N)) = 0$, for all $i \in N$.

When $T = N$, since all players are symmetric in game b_*^N, symmetry and efficiency principle imply that $\phi_i(b_*^N) = \frac{b_*^N(N)}{n} = \frac{1}{n}$, for all $i \in N$.

From above discussion, we can see that solution ϕ is unique for game Z^T with $|T| = 1$ and b_*^T with $|T| \geq 2$. The last part we have to prove is that the set $\{Z^T\}_{T \subset N, |T|=1} \cup \{b_*^T\}_{T \subset N, |T| \geq 2}$ forms a basis of Γ_n.

Denote all non-empty subsets of N by $S_1, S_2, \ldots, S_{2^n-1}$ which are arranged in lexicographic order as discussed by Genjiu et al. (2008). Let $\mathbf{C} = [c_{ij}]$ be a $2^n - 1$ square matrix defined by

$$c_{ij} = \begin{cases} Z^{S_i}(S_j), & if\ |S_i| = 1 \\ b_*^{S_i}(S_j), & if\ |S_i| \geq 2 \end{cases} \tag{19}$$

Observe that when $n < i < 2^n - 1$, there is only one nonzero element(which equals -1) in row i and all elements of the last row are equal to 1. By expanding the matrix in these rows step by step, we have $|det\mathbf{C}| = |det\mathbf{C}^{'}|$, where $\mathbf{C}^{'} = [c_{ij'}]_{(n+1)\times(n+1)}$ is the final cofactor of matrix $\mathbf{C}$. The specific form of $\mathbf{C}^{'}$ is

$$c^{'}_{ij} = \begin{cases} \alpha - 1, & if\ i + j = n + 1 \\ 1, & if\ i = n + 1 \\ \alpha, & otherwise \end{cases} \tag{20}$$

By simple calculation, we find that $|det\mathbf{C}| = |det\mathbf{C}^{'}| = 1 \neq 0$, and the proof is left to the reader. So we conclude that $\{Z^T\}_{T \subset N, |T|=1} \cup \{b_*^T\}_{T \subset N, |T| \geq 2}$ forms a basis of Γ_n. Applying the linearity of ϕ, we get the conclusion that solution ϕ is unique for any game $v \in \Gamma_n$, and it is just the α-ENSC value. □

We now introduce other two properties to characterize the α-ENSC value called α-inessential game property and grand marginal contribution monotonicity respectively.

A value ϕ satisfies

- **α-egocentric inessential game property**: if $\forall i \in N$, $\phi_i(v) = \alpha v(i) + (1-\alpha)\frac{v(N)}{n}$, $\alpha \in [0, 1]$, for any inessential game.

A game $(N, v) \in \Gamma_n$ is inessential if $v(S) = \sum_{i \in S} v(i), \forall S \subset N$. When a game is inessential, the general consideration is that players have no association with others and act alone, so a proper value allocates total worth according to their own power, that is, $v(i)$ for all $i \in N$. Such idea bases on the assumption that players are all totally egocentric. But as discussed earlier, player may be not that egocentric, they could also behave altruistically. α-egocentric inessential game property, in some sense, reflects the partial egoism.

A value ϕ satisfies

– **Grand marginal contribution monotonicity**: for any given $(N, v) \in \Gamma_n$, $i, j \in N$, if $v(N) - v(N\backslash i) \geq v(N) - v(N\backslash j)$, then $\phi_i(v) \geq \phi_j(v)$.

Since every player views his grand marginal contribution as his ideal payoff, the more his contribution is, the more he should get in the final allocation. These two properties together with efficiency and linearity could characterize the α-ENSC value.

Theorem 5 *The α-ENSC value is the unique value that satisfies efficiency, linearity, α-egocentric inessential game property and grand marginal contribution monotonicity.*

Proof It is trivial that the α-ENSC value satisfies the four properties. It remains to prove the uniqueness part.

For any $(N, v) \in \Gamma_n$, suppose that ϕ is a value with the four mentioned properties. We now construct a new game $w \in \Gamma_n$ defined as $w(S) = v(S) - \sum_{j \in S}[v(N) - v(N\backslash j)]$, $\forall S \subset N$. Denote $v^0 = v - w$, then one could verify that game v^0 is an inessential game. Applying α-egocentric inessential game property, we have $\phi_i(v^0) = \alpha v^0(i) + (1-\alpha)\frac{v^0(N)}{n}$, for all $i \in N$. Further, $w(N) - w(N\backslash i) = w(N) - w(N\backslash j)$, $\forall i, j \in N$. According to grand marginal contribution monotonicity, we have $\phi_i(w) = \phi_j(w)$, for all $i, j \in N$. Since ϕ also obeys efficiency principle, then $\phi_i(w) = \frac{w(N)}{n}$, $\forall i \in N$. Eventually, linearity implies that $\phi(v) = \phi(w) + \phi(v^0)$, which is equal to the α-ENSC value. □

6 Conclusion

In this paper, we propose a combinatorial value called the α-ENSC value, which reflects both egoism and altruism thoughts in allocation concepts. The procedural form of the α-ENSC value are discussed. Moreover, under certain complaint criterion, we reveal that the α-ENSC value is also the optimal solution for several optimization models which are inspired by the raise of the least square nucleolus and the Nucleolus. Some new properties are proposed to characterize the α-ENSC value. The above method obviously can be applied to other values, such as the CIS value, solidarity value, but the most important step is how to define the complaint criterion for players.

Appendix 1: The Proof of Lemma 1

Proof

$$\begin{aligned}
&\frac{1}{n}b_i^v + \sum_{\pi:\pi(i)=1} \frac{1}{n!}\left[v(i) + \sum_{k=\pi(i)+1}^{n} \frac{m_{\pi(k)} - b^v_{\pi^{-1}(k)}}{k-1}\right] \\
&\quad + \sum_{\pi:\pi(i)\neq 1,n} \frac{1}{n!}\left[b_i^v + \sum_{k=\pi(i)+1}^{n} \frac{m_{\pi(k)} - b^v_{\pi^{-1}(k)}}{k-1}\right] \\
&= \frac{1}{n}b_i^v + \frac{(n-1)!}{n!}v(i) + \sum_{\pi:\pi(i)=1} \frac{1}{n!} \sum_{k=\pi(i)+1}^{n} \frac{m_{\pi(k)} - b^v_{\pi^{-1}(k)}}{k-1}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(n-2)(n-1)!}{n!}b_i^v + \sum_{\pi:\pi(i)\neq 1,n}\frac{1}{n!}\sum_{k=\pi(i)+1}^{n}\frac{m_{\pi(k)}-b^v_{\pi^{-1}(k)}}{k-1}\\
&=\frac{1}{n}v(i)+\frac{n-1}{n}b_i^v+\sum_{\pi(i)\neq n}\frac{1}{n!}\sum_{k=\pi(i)+1}^{n}\frac{m_{\pi(k)}-b^v_{\pi^{-1}(k)}}{k-1}\\
&=\frac{1}{n}v(i)+\frac{n-1}{n}b_i^v+\sum_{l\in N\backslash\{i\}}\sum_{\pi:S_\pi^l\ni i}\frac{1}{n!}\frac{\left(v(S_\pi^l)-v(S_\pi^l\backslash l)\right)-b_l^v}{|S_\pi^l|-1}\\
&=\frac{1}{n}v(i)+\frac{n-1}{n}b_i^v+\sum_{l\in N\backslash\{i\}}\sum_{S\ni i,l}\frac{(v(S)-v(S\backslash l))-b_l^v}{s-1}\cdot\frac{(s-1)!(n-s)!}{n!}\\
&=\frac{1}{n}v(i)+\frac{n-1}{n}b_i^v+\sum_{l\in N\backslash\{i\}}\sum_{S\ni i,l}\left[(v(S)-v(S\backslash l))-b_l^v\right]\cdot\frac{(s-2)!(n-s)!}{n!}\\
&=\frac{1}{n}v(i)+\frac{n-1}{n}b_i^v+\sum_{l\in N\backslash\{i\}}\sum_{S\ni i,l}v(S)\cdot\frac{(s-2)!(n-s)!}{n!}\\
&\quad-\sum_{l\in N\backslash\{i\}}\sum_{S\ni i,l}v(S\backslash l)\cdot\frac{(s-2)!(n-s)!}{n!}-\sum_{l\in N\backslash\{i\}}\sum_{S\ni i,l}b_l\cdot\frac{(s-2)!(n-s)!}{n!}\\
&=\frac{1}{n}v(i)+\frac{n-1}{n}b_i^v+\sum_{S\ni i,|S|\geq 2}\sum_{l\in S\backslash\{i\}}v(S)\cdot\frac{(s-2)!(n-s)!}{n!}\\
&\quad-\sum_{l\in N\backslash\{i\}}\sum_{T\subseteq N\backslash l,T\ni i}v(T)\cdot\frac{(t+1-2)!(n-(t+1))!}{n!}\\
&\quad-\sum_{l\in N\backslash\{i\}}\sum_{S\ni i,l}b_l^v\cdot\frac{(s-2)!(n-s)!}{n!}\\
&=\frac{1}{n}v(i)+\frac{n-1}{n}b_i^v+\sum_{S\ni i,|S|\geq 2}(s-1)v(S)\cdot\frac{(s-2)!(n-s)!}{n!}\\
&\quad-\sum_{T\ni i,|T|\leq n-1}\sum_{l\notin T}v(T)\cdot\frac{(t-1)!(n-t-1)!}{n!}\\
&\quad-\sum_{l\in N\backslash\{i\}}\sum_{s=2}^{n}b_l^v\cdot\frac{(s-2)!(n-s)!}{n!}\binom{n-2}{s-2}\\
&=\frac{1}{n}v(i)+\frac{n-1}{n}b_i^v+\sum_{S\ni i,|S|\geq 2}v(S)\cdot\frac{(s-1)!(n-s)!}{n!}\\
&\quad-\sum_{T\ni i,|T|\leq n-1}(n-t)v(T)\cdot\frac{(t-1)!(n-t-1)!}{n!}-\sum_{l\in N\backslash\{i\}}\sum_{s=2}^{n}b_l^v\cdot\frac{1}{n(n-1)}\\
&=\frac{1}{n}v(i)+\frac{n-1}{n}b_i^v+\sum_{S\ni i,|S|\geq 2}v(S)\cdot\frac{(s-1)!(n-s)!}{n!}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{T \ni i, |T| \leq n-1} v(T) \cdot \frac{(t-1)!(n-t)!}{n!} - \sum_{l \in N \setminus \{i\}} \frac{b_l^v}{n} \\
&= \frac{1}{n} v(i) + \frac{n-1}{n} b_i^v - \frac{1}{n} v(i) + \frac{v(N)}{n} - \sum_{l \in N \setminus \{i\}} \frac{b_l^v}{n} \\
&= b_i^v + \frac{v(N) - \sum_{l \in N} b_l^v}{n} \\
&= ENSC_i(v)
\end{aligned}
$$

□

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