

# Optimality conditions in optimization problems with convex feasible set using convexificators

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**Abstract** In this paper, we consider a nonsmooth optimization problem with a convex feasible set described by constraint functions which are neither convex nor differentiable nor locally Lipschitz necessarily. Utilizing upper regular convexificators, we characterize the normal cone of the feasible set and derive KKT type necessary and sufficient optimality conditions. Under some assumptions, we show that the set of KKT multipliers is bounded. We also characterize the set of optimal solutions and introduce a linear approximation corresponding to the original problem which is useful in checking optimality. The obtained outcomes extend various results existing in the literature to a more general setting.

**Keywords** Convex optimization · Nonsmooth optimization · Convexificator · KKT conditions · Linear approximation

## 1 Introduction

Our motivation in this article is recent works of [Lasserre \(2010\)](#) and [Dutta and Lalitha \(2013\)](#). We consider an optimization problem,

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0, \quad i = 1, \dots, m, \end{aligned} \tag{1}$$

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where  $f, g_i : \mathbb{R}^n \rightarrow \mathbb{R}$ , ( $i \in I := \{1, \dots, m\}$ ) are real-valued functions. The set of feasible solutions of (1) is

$$K := \{x \in \mathbb{R}^n : g_i(x) \leq 0, \quad i \in I\}.$$

The problem in the presence of convex objective function and convex constraint functions has been studied by various scholars; see e.g. [Bazaraa et al. \(2006\)](#), [Hiriart-Urruty and Lemarechal \(1993\)](#), [Rockafellar and Wets \(1998\)](#) and the references therein. In the current work, we focus on the case that constraint functions  $g_i$  are nonconvex, nondifferentiable and even discontinuous. These cases are important in both theory and application. In various practical problems, it occurs that some  $g_i$  function is not convex while  $K$  is convex. For example,  $K$  is a convex set if  $g_i$ 's are quasiconvex functions. It is simple to observe that  $K$  might be convex while  $g_i$  functions are neither convex nor quasiconvex. For example,

$$K = \{(x_1, x_2) \in \mathbb{R}^2 : 1 - x_1x_2 \leq 0, \quad x_1, x_2 \geq 0\},$$

is convex while the constraint function  $g_1(x_1, x_2) = 1 - x_1x_2$  is neither convex nor quasiconvex; see [Lasserre \(2010\)](#). As an interesting applied problem, consider an optimization model with probabilistic constraints (OMPC) as follows:

$$\begin{aligned} \min \quad & \mathbb{E}[h(x, Z)] \\ \text{s.t.} \quad & \mathbb{P}[r_i(x, Z) \leq 0, \quad i \in I] \geq p, \end{aligned}$$

where  $p \in (0, 1)$  is a fixed probability level.  $x \in \mathbb{R}^n$  and  $Z \in \mathbb{R}^p$  are respectively decision and random vectors. Here,  $\mathbb{E}[\cdot]$  and  $\mathbb{P}[\cdot]$  stand for expected value and probability function, respectively, and we assume that these functions are well-defined. The most important application of OMPC is appeared in portfolio selection (see [Dentcheva 2006](#), p. 50), where  $h$  is a return function and  $r_i$  functions are corresponding to a risk measure. One of the vital questions in this area is imposing suitable conditions on  $r_i(\cdot, \cdot)$  functions which make the feasible set of OMPC convex. If  $r_i(\cdot, \cdot)$  ( $i \in I$ ) are quasiconcave functions jointly in both arguments, and  $Z$  is a random variable with an  $\alpha$ -concave probability distribution (see [Dentcheva 2006](#), p. 53), then the feasible set  $\{x \in \mathbb{R}^n : \mathbb{P}[r_i(x, Z) \geq 0, i \in I] \geq p\}$  is convex; see [Dentcheva \(2006, Sect. 2.2.3\)](#). See also [Luedtke and Ahmed \(2008\)](#) and [Nemirovski and Shapiro \(2006\)](#).

As another practical example, [Marcille et al. \(2012\)](#) proposed a model for resource allocation for ad hoc networks which minimizes a convex function over a convex feasible set with nonconvex constraint functions (see [Marcille et al. 2012](#), Problem 1 and Lemma 1). They provided associated KKT conditions, invoking the Lasserre's (2010) results, leading to an optimal resource allocation algorithm. More examples of problems with nonconvex constrain functions and convex feasible set can be found in the literature, see e.g. problems with fractional constrains ([Schaible 1981](#)) and financial models under cash-subadditive risk measures ([Cerreia-Vioglio et al. 2011](#)) among others.

Lasserre (2010) considered the convexity of  $K$  without any convexity assumption on  $g_i$  functions and obtained KKT optimality condition under Slater constraint qualification and a non-degeneracy condition on  $g_i$ 's. He investigated only differentiable problems; and then Dutta and Lalitha (2013) extended Lasserre's work and derived KKT conditions when  $g_i$ 's are locally Lipschitz and regular in the sense of Clarke et al. (1998). Martínez-Legaz (2015) dealt with the problem with tangentially convex constraint functions and a pseudoconvex objective function. In these studies, objective and constraint functions are assumed to be continuous and at least directionally differentiable, while in the present paper we are going to work with discontinuous functions. Optimization models with appeared nonsmooth and even discontinuous functions may arise in various fields, including batch production (Imo and Leech 1984) and switching regression (Tishler and Zang 1979). See also Moreau and Aeyels (2000) and the references therein.

In the present article, we follow the problem without locally Lipschitz assumption; and we use convexificators in the presence of nonsmooth data (nondifferentiable/discontinuous appearing functions). The concept of convexificator has been introduced to apply for each function, leading to sharpen results in nonsmooth optimization. Convexificators enjoy nice properties including mean value theorem, calculus rules, optimality conditions, etc, under mild assumptions. However, in general, we do not have some calculus rules such as exact chain rule; see Jeyakumar and Luc (1999) for more details.

The convexificator notion has been studied in several recent studies, including Demyanov and Rubinov (1995), Dutta and Chandra (2004), Jeyakumar and Luc (1999), Luc (2002) and Luu (2014), for characterizing the optimality. In the first part of the work, a main result is obtained showing that the normal cone of  $K$  can be represented with respect to the upper regular convexificators of  $g_i$ 's under some conditions. Afterwards, KKT type optimality conditions are derived from this representation, leading to an extended version of some main results provided in Dutta and Lalitha (2013), Lasserre (2010) and Martínez-Legaz (2015). Furthermore, a result is proved to get KKT conditions from FJ conditions. We prove the boundedness of the set of KKT multipliers and derive a characterization for the solution set. Finally, we build a linear semi-infinite problem to check the optimality of a given feasible solution.

The rest of the paper unfolds as follows. After providing some preliminaries in Sect. 2, we briefly review convexificator notion and its relationship with other well-known generalized gradients in Sect. 3. This section addresses some constraint qualifications as well. The characterization of convexity of the feasible set and its normal cone are given in Sect. 4. Deriving KKT conditions from the characterization of normal cone, getting KKT conditions from FJ conditions, and establishing the boundedness of the set of KKT multipliers are presented in Sect. 5. Finally, in Sect. 6 we obtain some results to characterize the set of optimal solutions; and with a linear approximation, we relate main problem with a linear semi-infinite optimization problem to check the optimality of a given feasible solution.

## 2 Preliminaries

In this section we provide some preliminaries.

For a set  $\Omega \subseteq \mathbb{R}^n$ , we use the notations  $co \Omega$ ,  $int \Omega$ ,  $bd \Omega$  and  $cl \Omega$  to denote the convex hull, the interior, the boundary and the closure of  $\Omega$ , respectively. Throughout the paper, the considered norm  $\|\cdot\|$  is the Euclidean norm, i.e.,  $\|\cdot\| = \|\cdot\|_2$  and we use the convention  $\infty - \infty = \infty$ .

A nonempty set  $C \subseteq \mathbb{R}^n$  is called a cone if for each  $x \in C$  and each scalar  $\lambda \geq 0$ , we have  $\lambda x \in C$ . A cone  $C$  is said to be pointed whenever  $C \cap (-C) = \{0\}$ . For a convex set  $\Omega \subseteq \mathbb{R}^n$ , the cone of feasible directions, the tangent cone and the normal cone of  $\Omega$  at  $\bar{x} \in cl \Omega$ , denoted by  $D_\Omega(\bar{x})$ ,  $T_\Omega(\bar{x})$  and  $N_\Omega(\bar{x})$ , respectively, are defined as

$$\begin{aligned}
 D_\Omega(\bar{x}) &= \{d \in \mathbb{R}^n : \exists \delta > 0 \text{ s.t. } \bar{x} + \lambda d \in \Omega, \forall \lambda \in (0, \delta)\}, \\
 T_\Omega(\bar{x}) &= cl \left\{ \frac{u - \bar{x}}{t} : t > 0, u \in \Omega \right\}, \\
 N_\Omega(\bar{x}) &= \{\zeta \in \mathbb{R}^n : \langle \zeta, y - \bar{x} \rangle \leq 0, \forall y \in \Omega\}.
 \end{aligned}$$

The polar cone of a set  $\Omega \subseteq \mathbb{R}^n$  is defined by

$$\Omega^\circ = \{d \in \mathbb{R}^n : \langle d, x \rangle \leq 0, \forall x \in \Omega\}.$$

If  $\Omega$  is convex, then  $cl D_\Omega(\bar{x}) = T_\Omega(\bar{x})$  and  $T_\Omega^\circ(\bar{x}) = N_\Omega(\bar{x})$ . The convex cone generated by  $\Omega \subseteq \mathbb{R}^n$  is defined as follows:

$$cone(\Omega) := \left\{ y \in \mathbb{R}^n : \exists m \in \mathbb{N}; y = \sum_{i=1}^m \lambda_i y_i, \lambda_i \geq 0, y_i \in \Omega, i = 1, 2, \dots, m \right\}.$$

If  $\Omega_1, \dots, \Omega_l \subseteq \mathbb{R}^n$  are convex sets, then it can be shown that

$$cone\left(\bigcup_{i=1}^l \Omega_i\right) = \left\{ \sum_{i=1}^l \lambda_i d_i : d_i \in \Omega_i, \lambda_i \geq 0, i = 1, 2, \dots, l \right\}.$$

## 3 Convexicator and CQs

The upper Dini directional derivative, defined below, plays a central role in this work. Hereafter, we assume that  $h(\bar{x})$  is finite for a given  $\bar{x} \in \mathbb{R}^n$ .

**Definition 1** Let  $h : \mathbb{R}^n \rightarrow \mathbb{R}$ . The upper Dini directional derivative of  $h$  at  $\bar{x}$  in direction  $d \in \mathbb{R}^n$  is defined by

$$h^+(\bar{x}; d) := \limsup_{t \downarrow 0} \frac{h(\bar{x} + td) - h(\bar{x})}{t}.$$

If  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is locally Lipschitz, then upper Dini directional derivative exists finitely.

The directional derivative of  $h$  at  $\bar{x} \in \mathbb{R}^n$  in direction  $d \in \mathbb{R}^n$ , denoted by  $h'(\bar{x}; d)$ , is defined as

$$h'(\bar{x}; d) := \lim_{t \downarrow 0} \frac{h(\bar{x} + td) - h(\bar{x})}{t}.$$

**Definition 2 (Jeyakumar and Luc 1999)** The function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to have an upper regular convexificator (URC)  $\partial^*h(\bar{x})$  at  $\bar{x}$  if  $\partial^*h(\bar{x}) \subseteq \mathbb{R}^n$  is closed and, for each  $d \in \mathbb{R}^n$ ,

$$h^+(\bar{x}; d) = \sup_{\eta \in \partial^*h(\bar{x})} \langle \eta, d \rangle. \tag{2}$$

See [Jeyakumar and Luc \(1999\)](#) for properties and applications of URCs.

**Definition 3 (Clarke et al. 1998)** Let  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  be Lipschitz near  $\bar{x} \in \mathbb{R}^n$ . The Clarke generalized directional derivative of  $h$  at  $\bar{x}$  in the direction  $d \in \mathbb{R}^n$ , denoted by  $h^\circ(\bar{x}; d)$ , is defined as

$$h^\circ(\bar{x}; d) := \limsup_{\substack{y \rightarrow \bar{x} \\ t \downarrow 0}} \frac{h(y + td) - h(y)}{t}.$$

**Definition 4 (Clarke et al. 1998)** Let  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  be Lipschitz near  $\bar{x} \in \mathbb{R}^n$ . The Clarke generalized gradient of  $h$  at  $\bar{x}$ , denoted by  $\partial^{Cl}h(\bar{x})$ , is defined as

$$\partial^{Cl}h(\bar{x}) := \{ \zeta \in \mathbb{R}^n : h^\circ(\bar{x}; d) \geq \langle \zeta, d \rangle, \quad \forall d \in \mathbb{R}^n \}.$$

*Remark 1* If  $h$  is locally Lipschitz at  $\bar{x}$  with a URC  $\partial^*h(\bar{x})$  at this point, then due to the inequality  $h^+(\bar{x}; d) \leq h^\circ(\bar{x}; d)$ , one has

$$\sup_{\zeta \in co(\partial^*h(\bar{x}))} \langle \zeta, d \rangle = \sup_{\zeta \in \partial^*h(\bar{x})} \langle \zeta, d \rangle \leq \max_{\eta \in \partial^{Cl}h(\bar{x})} \langle \eta, d \rangle, \quad \forall d \in \mathbb{R}^n.$$

Now, from [Hiriart-Urruty and Lemarechal \(1993, Theorem V.3.3.1\)](#), we have

$$\partial^*h(\bar{x}) \subseteq co(\partial^*h(\bar{x})) \subseteq \partial^{Cl}h(\bar{x}).$$

This inclusion relation shows that one may obtain a URC smaller than Clarke generalized gradient. Notice that hunting smaller subdifferential sets is an important issue in optimization from a numerical standpoint.

As mentioned in the preceding section, related results in various papers have been obtained for directionally differentiable functions. Moreover, in some works, the functions in question are assumed to be regular (i.e.  $h'(\bar{x}; d)$  exists and equals to  $h^\circ(\bar{x}; d)$ )

for any  $d \in \mathbb{R}^n$ ). However, even Lipschitz functions are not always directionally differentiable. For example, look at  $h : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$h(x) = \begin{cases} 0, & \text{if } x \in (-\infty, 0] \cup [1, \infty) \\ -2x + \frac{2}{3^n}, & \text{if } x \in \left[\frac{2}{3^{n+1}}, \frac{1}{3^n}\right), n \in \mathbb{N} \cup \{0\} \\ 2x - \frac{2}{3^{n+1}}, & \text{if } x \in \left[\frac{1}{3^{n+1}}, \frac{2}{3^{n+1}}\right), n \in \mathbb{N} \cup \{0\} \end{cases}$$

This function is Lipschitz on  $\mathbb{R}$  with a Lipschitz constant  $k = 2$ , while it is not directionally differentiable at  $\bar{x} = 0$  (see [Jeyakumar and Luc 2008](#), p. 7). However,

$$h^+(\bar{x}; d) = \begin{cases} d, & d > 0, \\ 0, & d \leq 0, \end{cases}$$

which implies that  $\partial^*h(\bar{x}) = \{0, 1\}$  is a URC of  $h$  at  $\bar{x}$ . It is seen that  $\partial^*h(\bar{x}) \not\subseteq \partial^{Cl}h(\bar{x}) = [-2, 2]$ .

According to the discussion and example above, even for locally Lipschitz functions, the URC notion may provide sharper results than Clarke generalized gradient.

The class of functions that admit a URC is rich. Gâteaux differentiable functions and regular functions in the sense of [Clarke et al. \(1998\)](#) are important members of this class. Tangential subdifferential (of a tangentially convex function) is also a URC (see [Martínez-Legaz 2015](#), Definition 5). Moreover, if  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is locally Lipschitz, then the Clarke’s generalized gradient ([Clarke et al. 1998](#)) and the Michel and Penot ([1992](#)) subdifferential are upper semi regular convexificators ([Dutta and Chandra 2002](#)). In the present study we use URC as a strong tool for working with discontinuous and nondifferentiable functions.

Now we recall some constraint qualification conditions from the literature.

Let

$$K = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i \in I\}, \tag{3}$$

where  $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$  for each  $i \in I$  is not necessarily continuous or differentiable.

**Definition 5** We say that the Slater Constraint Qualification (SCQ) holds for (1) if there exists  $x \in K$  such that  $g_i(x) < 0$  for all  $i \in I$ .

[Lasserre \(2010\)](#), [Dutta and Lalitha \(2013\)](#) and [Martínez-Legaz \(2015\)](#) use SCQ. Since in their work, the constraint functions are continuous, SCQ implies  $int K \neq \emptyset$ . However, we use  $int K \neq \emptyset$  for our results because we do not assume continuity.

For a given  $\bar{x} \in K$ , set

$$I(\bar{x}) := \{i \in I : g_i(\bar{x}) = 0\}.$$

**Assumption 1** Hereafter, given  $\bar{x} \in K$  we assume  $I(\bar{x}) \neq \emptyset$ , and  $g_i, i \in I(\bar{x})$  has a URC  $\partial^*g_i(\bar{x})$  at  $\bar{x}$ .

**Definition 6** We say that the Linear Independent Constraint Qualification (LICQ) holds at  $\bar{x} \in K$  if

$$0 = \sum_{i \in I(\bar{x})} \lambda_i \zeta_i \ \& \ \zeta_i \in co(\partial^* g_i(\bar{x})) \Rightarrow \lambda_i = 0, \ i \in I(\bar{x}).$$

**Definition 7** We say that the Constraint Qualification (CQ) holds at  $\bar{x} \in K$  if

$$0 \notin co \left\{ \bigcup_{i \in I(\bar{x})} co(\partial^* g_i(\bar{x})) \right\}.$$

**Definition 8** We say that the generalized Lasserre Constraint Qualification (GLCQ) holds at  $\bar{x} \in K$  if  $0 \notin \partial^* g_i(\bar{x})$  for each  $i \in I(\bar{x})$ .

It is not difficult to show that  $LICQ \Rightarrow CQ \Rightarrow GLCQ$ , but the opposite statements are not true generally.

### 4 Convexity and characterization of normal cone

In this section, we characterize the convexity of the feasible set  $K$  and its normal cone at a given point  $\bar{x} \in K$ , which  $K$  is as represented in (3). It is shown that the normal cone is the closure of convex cone generated by  $\cup_{i \in I(\bar{x})} \partial^* g_i(\bar{x})$ .

The following theorem characterizes the convexity of  $K$  using URCs. This theorem extends [Lasserre \(2010, Lemma 2.2\)](#), [Dutta and Lalitha \(2013, Proposition 2.2\)](#) and [Martínez-Legaz \(2015, Proposition 6\)](#). We assume

$$\{x \in K : g_i(x) < 0, \ \forall i \in I\} \cap bd K = \emptyset. \tag{4}$$

If constraint functions are continuous, then this condition automatically holds.

**Theorem 1** *Let  $int K \neq \emptyset$  and GLCQ hold. Furthermore, assume that  $K$  is closed and (4) holds. Then  $K$  is convex if and only if for every  $i \in I$  and every  $x \in K$  with  $g_i(x) = 0$ , there exists  $\zeta \in \partial^* g_i(x)$  such that  $\langle \zeta, y - x \rangle \leq 0$  for each  $y \in K$ .*

*Proof* Assume that  $K$  is convex. By indirect proof, assume that there exist  $i \in I$  and  $x \in K$  with  $g_i(x) = 0$  such that for each  $\zeta \in \partial^* g_i(x)$ ,  $\langle \zeta, y - x \rangle > 0$  for some  $y \in K$ . Hence,  $g_i^+(x; y - x) > 0$  which leads to  $g_i(x + t(y - x)) > 0$  for some sufficiently small  $t > 0$ . This contradicts the convexity of  $K$ .

Converse: Let  $\bar{x}$  be any boundary point of  $K$ . Due to (4), there exists some  $j \in I$  such that  $g_j(\bar{x}) = 0$ . Hence, by assumption, there exists  $\zeta \in \partial^* g_j(\bar{x})$  such that  $\langle \zeta, y - \bar{x} \rangle \leq 0$  for each  $y \in K$ . Since GLCQ holds, we have  $\zeta \neq 0$ . Thus,  $\zeta$  is the normal vector of a non-trivial supporting hyperplane to the set  $K$  at  $\bar{x}$ . Now, since  $int K \neq \emptyset$ , by Theorem 1.3.3 in [Schneider \(1994\)](#),  $K$  is convex. □

Theorem 2 is devoted to characterization of the normal cone of  $K$  at a given point  $\bar{x} \in K$ .

**Assumption 2** In the rest of this section, we consider  $K$  as a convex set as defined in (3) where  $g_i$ 's are not necessary locally Lipschitz, continuous or convex.

**Theorem 2** Let  $\bar{x} \in K$ . If  $g_i$  for  $i \notin I(\bar{x})$  is upper semicontinuous at  $\bar{x}$ ,  $\text{int } K \neq \emptyset$  and GLCQ holds at  $\bar{x}$ , then

$$N_K(\bar{x}) = \text{cl cone} \left( \cup_{i \in I(\bar{x})} \partial^* g_i(\bar{x}) \right).$$

*Proof* As  $N_K(\bar{x})$  is a closed convex cone, to justify the inclusion

$$\text{cl cone} \left( \cup_{i \in I(\bar{x})} \partial^* g_i(\bar{x}) \right) \subseteq N_K(\bar{x}),$$

it is sufficient to show that  $\partial^* g_i(\bar{x}) \subseteq N_K(\bar{x})$  for each  $i \in I(\bar{x})$ .

Let  $i \in I(\bar{x})$  be arbitrary. Since  $K$  is convex, for each  $x \in K$  we have  $g_i^+(\bar{x}; x - \bar{x}) \leq 0$ , leading to

$$\langle \zeta, x - \bar{x} \rangle \leq 0, \quad \forall x \in K, \forall \zeta \in \partial^* g_i(\bar{x}).$$

Hence, according to the definition of normal cone of a convex set, we have  $\partial^* g_i(\bar{x}) \subseteq N_K(\bar{x})$ . Therefore,

$$\text{cl cone} \left( \cup_{i \in I(\bar{x})} \partial^* g_i(\bar{x}) \right) \subseteq N_K(\bar{x}). \tag{5}$$

Set

$$\Gamma(\bar{x}) := \cup_{i \in I(\bar{x})} \partial^* g_i(\bar{x}).$$

To show the converse inclusion, we claim that  $\Gamma^\circ(\bar{x}) \subseteq T_K(\bar{x})$ . Let  $y \in \text{int } K$  be arbitrary. There exists some  $\epsilon > 0$  such that for each  $d \in \mathbb{R}^n$  with  $\|d\| \leq 1$ ,  $y + \epsilon d \in K$ . Hence, from the convexity of  $K$ , for each  $i \in I(\bar{x})$  and  $d \in \mathbb{R}^n$  with  $\|d\| \leq 1$ ,

$$g_i^+(\bar{x}; \epsilon d + y - \bar{x}) \leq 0.$$

Therefore,

$$\langle \zeta, y - \bar{x} \rangle \leq -\epsilon \langle \zeta, d \rangle, \quad \forall d \in \mathbb{R}^n; \|d\| \leq 1, \forall i \in I(\bar{x}), \forall \zeta \in \partial^* g_i(\bar{x}).$$

By setting  $d = \frac{\zeta}{\|\zeta\|}$ , we get

$$\langle \zeta, y - \bar{x} \rangle \leq -\epsilon \|\zeta\|, \quad \forall i \in I(\bar{x}), \forall \zeta \in \partial^* g_i(\bar{x}).$$

Hence, because of  $0 \notin \partial^* g_i(\bar{x})$  and the closedness of  $\partial^* g_i(\bar{x})$ , we have

$$g_i^+(\bar{x}; y - \bar{x}) = \sup_{\zeta \in \partial^* g_i(\bar{x})} \langle \zeta, y - \bar{x} \rangle \leq -\epsilon \inf_{\zeta \in \partial^* g_i(\bar{x})} \|\zeta\| < 0.$$

Therefore,

$$g_i^+(\bar{x}; y - \bar{x}) < 0, \quad \forall y \in \text{int } K, \forall i \in I(\bar{x}). \tag{6}$$



Let  $d \in \Gamma^\circ(\bar{x})$  and  $y \in \text{int } K$ . From (6) and the definition of URC, we have

$$g_i^+(\bar{x}; d + t(y - \bar{x})) \leq g_i^+(\bar{x}; d) + t g_i^+(\bar{x}; y - \bar{x}) < 0, \quad \forall t > 0, \forall i \in I(\bar{x}).$$

Therefore, for sufficiently small  $\lambda$  values,  $g_i(\bar{x} + \lambda(d + t(y - \bar{x}))) \leq 0$  for each  $t > 0$  and  $i \in I(\bar{x})$ . On the other hand, considering  $t > 0$ , for each  $i \notin I(\bar{x})$ , due to the upper semicontinuity assumption,  $g_i(\bar{x} + \lambda(d + t(y - \bar{x}))) \leq 0$  for sufficiently small  $\lambda$  values. Hence  $d + t(y - \bar{x}) \in D_K(\bar{x})$  for each  $t > 0$ . This implies  $d \in \text{cl } D_K(\bar{x}) = T_K(\bar{x})$ . Therefore,

$$\Gamma^\circ(\bar{x}) \subseteq T_K(\bar{x}).$$

Now, we have

$$N_K(\bar{x}) = T_K^\circ(\bar{x}) \subseteq \Gamma^{\circ\circ}(\bar{x}) = \text{cl cone}(\Gamma(\bar{x})).$$

Therefore, according to (5),  $N_K(\bar{x}) = \text{cl cone}(\cup_{i \in I(\bar{x})} \partial^* g_i(\bar{x}))$  and the proof is completed. □

*Remark 2* Under the assumptions of Theorem 2, if  $\text{cone}(\Gamma(\bar{x}))$  is closed, then by mentioned theorem and because of

$$\text{cone}(\cup_{i \in I(\bar{x})} \partial^* g_i(\bar{x})) = \text{cone}(\cup_{i \in I(\bar{x})} \text{co}(\partial^* g_i(\bar{x}))),$$

for each  $d \in N_K(\bar{x})$ ,

$$d \in \sum_{i \in I(\bar{x})} \lambda_i \text{co}(\partial^* g_i(\bar{x})),$$

for some  $\lambda_i \geq 0, (i \in I(\bar{x}))$ . Furthermore, due to (6),  $0 \notin \text{co}(\partial^* g_i(\bar{x}))$  for any  $i \in I(\bar{x})$ . So,  $\text{cone}(\Gamma(\bar{x}))$  is closed provided that  $\partial^* g_i(\bar{x})$  is bounded for each  $i \in I(\bar{x})$ .

For sake of convenience and getting sharper results, in some theorems we use a base of the cone generated by a convexificator instead of the convexificator. Given a cone  $C$ , a set  $B \subseteq C$  is called a base of  $C$  if  $0 \notin B$  and for each  $c \in C \setminus \{0\}$  there are unique  $b \in B$  and  $t > 0$  such that  $c = tb$ .

Let assumptions of Theorem 2 hold and  $\text{cone}(\partial^* g_i(\bar{x}))$  be closed for each  $i \in I(\bar{x})$ . From (6), we have

$$\langle \zeta, y - \bar{x} \rangle < 0, \quad \forall y \in \text{int } K, \quad \forall \zeta \in \text{cone}(\partial^* g_i(\bar{x})) \setminus \{0\}, \quad \forall i \in I(\bar{x}). \quad (7)$$

Therefore,  $\text{cone}(\partial^* g_i(\bar{x}))$  is a closed pointed cone, and it has a closed convex bounded base  $B_i$  (see Luc 1989, Remark 1.6), leading to

$$\text{cone}(\cup_{i \in I(\bar{x})} \partial^* g_i(\bar{x})) = \text{cone}(\cup_{i \in I(\bar{x})} B_i).$$

**Corollary 1** *Under assumptions of Theorem 2, if  $\text{cone}(\partial^* g_i(\bar{x}))$  is closed for each  $i \in I(\bar{x})$ , then*

$$N_K(\bar{x}) = \text{cone} \left( \bigcup_{i \in I(\bar{x})} B_i \right).$$

There are known ways for obtaining the base  $B_i$ ; see e.g. [Luc \(1989, Proposition 1.10\)](#). The representation given in [Corollary 1](#) will help us to prove the boundedness of KKT multipliers in the next section.

## 5 KKT condition

### 5.1 KKT via FJ

We start this section by recalling FJ and KKT point notions.

**Definition 9** (i) We say that  $\bar{x} \in K$  is a Fritz John (FJ) point for (1) if there exist  $\lambda_0, \lambda_1, \dots, \lambda_m \geq 0$  not all zero, such that

$$0 \in \lambda_0 \text{co}(\partial^* f(\bar{x})) + \sum_{i=1}^m \lambda_i \text{co}(\partial^* g_i(\bar{x})), \tag{8}$$

$$\lambda_i g_i(\bar{x}) = 0, \quad \forall i \in I. \tag{9}$$

(ii) We say that  $\bar{x} \in K$  is a KKT point for (1) if it is a FJ point with  $\lambda_0 = 1$ .

[Dutta and Chandra \(2002, Theorem 3.3\)](#) derived a necessary FJ optimality condition for (1) in terms of convexificators. [Theorem 3](#) shows that KKT conditions are derived if one adds  $\text{int } K \neq \emptyset$  and GLCQ to the assumptions imposed by [Dutta and Chandra \(2002, Theorem 3.3\)](#).

**Theorem 3** *Let  $\bar{x} \in K$  be a FJ point. If  $\text{int } K \neq \emptyset$  and GLCQ holds, then  $\bar{x}$  is a KKT point.*

*Proof* Assume  $\lambda_0 = 0$  in (8). Then  $0 = \sum_{i=1}^m \lambda_i \zeta_i$ , for some  $\zeta_i \in \text{co}(\partial^* g_i(\bar{x}))$ ,  $i = 1, 2, \dots, m$ . Thus

$$0 = \sum_{i \in J} \lambda_i \langle \zeta_i, y - \bar{x} \rangle, \quad \forall y \in K,$$

where  $J := \{i \in I : \lambda_i > 0\} \subseteq I(\bar{x})$ . For each  $i \in I(\bar{x})$  and  $y \in K$ , as  $K$  is convex,  $g_i^+(\bar{x}; y - \bar{x}) \leq 0$  which implies  $\langle \zeta_i, y - \bar{x} \rangle \leq 0$ . Thus

$$\langle \zeta_i, y - \bar{x} \rangle = 0, \quad \forall y \in K, \quad \forall i \in J.$$

Since there exists some  $\hat{x} \in \text{int } K$ , for any  $v \in \mathbb{R}^n$  and  $t > 0$  sufficiently small we have  $\hat{x} + tv \in K$ ; and hence

$$\langle \zeta_i, \hat{x} + tv - \bar{x} \rangle = 0, \quad \forall i \in J.$$

Therefore,  $\langle \zeta_i, \hat{x} - \bar{x} \rangle + \langle \zeta_i, tv \rangle = 0$  which leads to

$$\langle \zeta_i, v \rangle = 0, \quad \forall i \in J.$$

Hence for each  $v \in \mathbb{R}^n$  and for each  $i \in J$ ,  $\langle \zeta_i, v \rangle = 0$ . This implies  $\zeta_i = 0$  for each  $i \in J$ . Therefore,  $0 \in co(\partial^* g_i(\bar{x}))$  for all  $i \in J$ . On the other hand, invoking GLCQ and  $int K \neq \emptyset$ , by a manner similar to the proof of Theorem 2, Eq. (6) is derived, leading to  $0 \notin co(\partial^* g_i(\bar{x}))$  for any  $i \in J$ . This contradiction shows  $\lambda_0 > 0$  and without loss of generality one may take  $\lambda_0 = 1$ . □

### 5.2 KKT without FJ

In Dutta and Lalitha (2013), Lasserre (2010) and Martínez-Legaz (2015), the authors investigated KKT conditions at optimal solutions of (1). To this end, they consider FJ conditions and then they impose some constraint qualifications to get KKT conditions. Their manner can be summarized in the following diagram:

$$\text{Minimizer} \Rightarrow \text{FJ} \Rightarrow \text{KKT}$$

Now we are going to use the characterization of  $N_K(\bar{x})$ , proved in Theorem 2, to get KKT conditions without using FJ conditions. The diagram below clarifies our manner:

$$\text{Minimizer} + (N_K \text{ characterization}) \Rightarrow \text{KKT}$$

**Theorem 4** *Assume that  $f$  has a URC  $\partial^* f(\bar{x})$  at  $\bar{x} \in K$ . If  $\bar{x}$  is an optimal solution of (1), then*

$$0 \in cl \left( co(\partial^* f(\bar{x})) + N_K(\bar{x}) \right).$$

*Proof* We claim

$$\sup_{\eta \in \partial^* f(\bar{x})} \langle \eta, d \rangle \geq 0, \quad \forall d \in D_K(\bar{x}). \tag{10}$$

By indirect proof suppose  $f^+(\bar{x}; d) = \sup_{\eta \in \partial^* f(\bar{x})} \langle \eta, d \rangle < 0$  for some  $d \in D_K(\bar{x})$ . This implies the existence of some  $t > 0$  satisfying  $\bar{x} + td \in K$  and  $f(\bar{x} + td) < f(\bar{x})$ , which contradict the optimality of  $\bar{x}$ . From (10), we have

$$\sup_{\eta \in \partial^* f(\bar{x})} \langle \eta, d \rangle \geq 0, \quad \forall d \in cl D_K(\bar{x}) = T_K(\bar{x}).$$

Therefore,

$$\sup_{\eta \in co(\partial^* f(\bar{x}))} \langle \eta, d \rangle + I_{T_K(\bar{x})}(d) \geq 0, \quad \forall d \in \mathbb{R}^n,$$

where  $I_{T_K(\bar{x})}(\cdot)$  stands for the indicator function ( $I_{T_K(\bar{x})}(d)$  equals to zero if  $d \in T_K(\bar{x})$  and  $\infty$  otherwise). By [Hiriart-Urruty and Lemarechal \(1993, Example V.2.3.1\)](#),  $I_{T_K(\bar{x})}(d) = \sup_{\eta \in N_K(\bar{x})} \langle \eta, d \rangle$  for each  $d$ . Thus

$$\sup_{\eta \in \text{co}(\partial^* f(\bar{x})) + N_K(\bar{x})} \langle \eta, d \rangle \geq 0, \quad \forall d \in \mathbb{R}^n.$$

Now, invoking [Hiriart-Urruty and Lemarechal \(1993, Theorem V.3.3.1\)](#), we have

$$0 \in \text{cl} \left( \text{co}(\partial^* f(\bar{x})) + N_K(\bar{x}) \right),$$

and the proof is completed. □

**Corollary 2** *Assume that  $\bar{x} \in K$ ,  $\text{int } K \neq \emptyset$ , GLCQ holds, and  $f$  has a bounded URC  $\partial^* f(\bar{x})$  at  $\bar{x}$ . Furthermore, assume that  $g_i$  for  $i \notin I(\bar{x})$  is upper semicontinuous at  $\bar{x}$ . If  $\bar{x}$  is an optimal solution of (1) and  $\text{cone}(\Gamma(\bar{x}))$  is closed, then there exist  $\lambda_i \geq 0$ ,  $i \in I(\bar{x})$  such that*

$$0 \in \text{co}(\partial^* f(\bar{x})) + \sum_{i \in I(\bar{x})} \lambda_i \text{co}(\partial^* g_i(\bar{x})).$$

*Proof* It results from [Theorem 4](#) and [Remark 2](#). □

[Theorem 4](#) and [Corollary 2](#) present necessary optimality conditions. In [Theorem 5](#), we obtain a sufficient optimality condition in the presence of an asymptotic pseudoconvex objective function. Along the lines of [Yang \(2005\)](#) we use the following generalized definition of pseudoconvexity w.r.t convexificators.

**Definition 10** *Assume that  $h : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  admits a URC at  $\bar{x} \in \Omega$ . The function  $h$  is said to be asymptotic pseudoconvex (a-pseudoconvex in brief) at  $\bar{x}$  if for every  $y \in \Omega$ ,*

$$\left( \{\zeta_k\} \subseteq \text{co}(\partial^* h(\bar{x})), \quad \lim_{k \rightarrow \infty} \langle \zeta_k, y - \bar{x} \rangle \geq 0 \right) \Rightarrow h(y) \geq h(\bar{x}).$$

This function is called a-pseudoconvex on  $\Omega$  if it is a-pseudoconvex at any point of  $\Omega$ .

**Theorem 5** *Assume that  $f$  is a-pseudoconvex with a URC  $\partial^* f(\bar{x})$  at  $\bar{x} \in K$ . If there exist  $\lambda_i \geq 0$ ,  $i \in I(\bar{x})$  such that*

$$0 \in \text{co}(\partial^* f(\bar{x})) + \sum_{i \in I(\bar{x})} \lambda_i \text{co}(\partial^* g_i(\bar{x})), \tag{11}$$

*then  $\bar{x}$  is an optimal solution of (1).*

*Proof* According to the first part of the proof of [Theorem 2](#), [Eq. \(11\)](#) implies  $0 \in \text{co}(\partial^* f(\bar{x})) + N_K(\bar{x})$ . Hence,  $-\eta \in N_K(\bar{x})$  for some  $\eta \in \text{co}(\partial^* f(\bar{x}))$ , which implies

$$\langle \eta, y - \bar{x} \rangle \geq 0, \quad \forall y \in K.$$

Now, the proof is completed due to the  $\alpha$ -pseudoconvexity of  $f$  at  $\bar{x}$ . □

**Corollary 3** *Let  $\bar{x} \in K$ . Assume  $g_i$  for  $i \notin I(\bar{x})$  is upper semicontinuous at  $\bar{x}$ ,  $\text{int } K \neq \emptyset$  and GLCQ holds at  $\bar{x}$ . Furthermore, assume that  $f$  has a bounded URC  $\partial^* f(\bar{x})$ . If  $\bar{x}$  is an optimal solution of (1) and for each  $i \in I(\bar{x})$ ,  $\text{cone}(\partial^* g_i(\bar{x}))$  is closed, then there exist  $\lambda_i \geq 0$ ,  $i \in I(\bar{x})$  such that*

$$0 \in \text{co}(\partial^* f(\bar{x})) + \sum_{i \in I(\bar{x})} \lambda_i B_i,$$

where  $B_i$  is a convex compact base of  $\text{cone}(\partial^* g_i(\bar{x}))$ . If  $f$  is  $\alpha$ -pseudoconvex at  $\bar{x}$ , this condition is sufficient for optimality.

*Proof* Apply Corollary 1, Theorem 4, and Theorem 5. □

As mentioned before, in the current subsection, KKT conditions are gotten from optimality directly, in contrast to Dutta and Lalitha (2013, Theorem 2.4), Lasserre (2010, Theorem 2.3), and Martínez-Legaz (2015, Theorem 9). Furthermore, here we have used upper regular convexificator instead of usual gradient [used in Lasserre (2010)], the Clarke generalized gradient [used in Dutta and Lalitha (2013)], and the tangential subdifferential [used in Martínez-Legaz (2015)]. Moreover, here  $g_i$  functions are not assumed to be locally Lipschitz or directionally differentiable. It is worth mentioning that Dutta and Lalitha (2013, Theorem 2.4), Lasserre (2010, Theorem 2.3), and Martínez-Legaz (2015, Theorem 9) result from Corollary 2 and Theorem 5.

We conclude this subsection with an example. In the following example, the constraint function  $g$  is not locally Lipschitz while the feasible set is closed and convex.

*Example 1* Consider

$$\min f(x_1, x_2) \quad \text{s.t.} \quad g(x_1, x_2) \leq 0,$$

where  $f(x_1, x_2) = -x_1 - x_2$  and

$$g(x_1, x_2) = \begin{cases} x_1 + x_2, & x_1, x_2 \leq 0, \\ \sqrt{|x_1|} + \sqrt{|x_2|}, & \text{otherwise} \end{cases}$$

Set  $K = \{(x_1, x_2) : g(x_1, x_2) \leq 0\}$ . It can be seen that

$$g^+((0, 0); (v_1, v_2)) = \begin{cases} v_1 + v_2, & v_1, v_2 \leq 0, \\ +\infty, & \text{otherwise} \end{cases}$$

Two sets  $\partial^* f(0, 0) = \{(-1, -1)\}$  and  $\partial^* g(0, 0) = \{(1 + t, 1 + \frac{1}{t}) : t > 0\} \cup \{(1, 1)\}$  are URCs for  $f$  and  $g$  at  $\bar{x} = (0, 0)$ , respectively. Furthermore,  $(0, 0) \notin \partial^* g(0, 0)$ . On

the other hand,  $(1, 1) \in \partial^* g(0, 0)$ . Hence,  $(0, 0) \in \partial^* f(0, 0) + \partial^* g(0, 0)$ . Therefore,  $(0, 0)$  is a KKT point and hence it is an optimal solution.

### 5.3 Boundedness of KKT multipliers

This subsection is devoted to a result about boundedness of the set of KKT multipliers. The boundedness of KKT multipliers is very useful in sketching numerical algorithms and duality results (Li and Zhang 2010; Nguyen et al. 1980). Without loss of generality assume that  $I(\bar{x}) = \{1, \dots, s\} \subseteq I$ . Set

$$\mathcal{M}(\bar{x}) := \left\{ (\lambda_1, \dots, \lambda_s) \geq 0 : 0 \in \text{co}(\partial^* f(\bar{x})) + \sum_{i=1}^s \lambda_i B_i \right\},$$

where  $B_i$  is a convex compact base of  $\text{cone}(\partial^* g_i(\bar{x}))$ .

**Theorem 6** *Let  $\bar{x} \in K$ . Assume that  $\text{int } K \neq \emptyset$ , GLCQ holds at  $\bar{x}$ , objective function  $f$  has a bounded URC  $\partial^* f(\bar{x})$ . Then  $\mathcal{M}(\bar{x})$  is a bounded set.*

*Proof* By the indirect proof, without loss of generality, assume that there is a sequence  $\{(\lambda_1^k, \dots, \lambda_s^k)\}_{k \in \mathbb{N}} \subseteq \mathcal{M}(\bar{x})$  such that  $\lambda_1^k \rightarrow \infty$  as  $k \rightarrow \infty$ . For each  $k \in \mathbb{N}$ , there exist  $\eta^k \in \text{co}(\partial^* f(\bar{x}))$  and  $\zeta_i^k \in B_i$  such that

$$0 = \langle \eta^k, y - \bar{x} \rangle + \sum_{i=1}^s \lambda_i^k \langle \zeta_i^k, y - \bar{x} \rangle, \quad \forall y \in \text{int } K.$$

By a manner similar to the second part of the proof of Theorem 2, we have  $\langle \eta_i, y - \bar{x} \rangle < 0$  for each  $i \in I(\bar{x})$  and each  $\eta_i \in \partial^* g_i(\bar{x})$ . Thus

$$0 \leq \langle \eta^k, y - \bar{x} \rangle + \lambda_1^k \langle \zeta_1^k, y - \bar{x} \rangle.$$

Since  $\text{co}(\partial^* f(\bar{x}))$  and  $B_1$  are compact, by working with subsequences if necessary one may assume  $\eta^k \rightarrow \eta \in \text{co}(\partial^* f(\bar{x}))$  and  $\zeta_1^k \rightarrow \zeta_1 \in B_1$  as  $k \rightarrow \infty$ . Hence, by  $k \rightarrow \infty$ ,

$$0 \leq \frac{\langle \eta^k, y - \bar{x} \rangle}{\lambda_1^k} + \langle \zeta_1^k, y - \bar{x} \rangle \rightarrow 0 + \langle \zeta_1, y - \bar{x} \rangle < 0.$$

This contradiction proves the theorem. □

## 6 Characterizations of the solution set

### 6.1 Convexifiers at optimality

In this subsection, we characterize the solution set of optimization problem (1) in terms of convexifiers invoking KKT type conditions obtained in Sect. 5. Mangasarian

(1988) first introduced several characterizations of the solution set of a convex optimization problem. After that, many attempts were performed on this subject, especially for making the convexity requirements on objective function weaker and for generalizing the results to nonsmooth case; see e.g. [Burke and Ferris \(1991\)](#), [Dinh et al. \(2006\)](#), [Zhao and Yang \(2013\)](#), and the references therein.

Assume that the solution set of (1), denoted by

$$S := \{x \in K : f(x) \leq f(y), \quad \forall y \in K\},$$

is nonempty. First we obtain some inclusion characterizations.

**Proposition 1** *Let  $\bar{x} \in S$  be given. Assume that  $S$  is convex. Then we have*

$$S \subseteq S_1 := \{x \in K : co(\partial^* f(\bar{x})) \cap -N_K(\bar{x}) \subseteq -N_K(x)\},$$

$$S \subseteq S_2 := \{x \in K : co(\partial^* f(x)) \cap -N_K(x) \subseteq -N_K(\bar{x})\}.$$

Furthermore, for any convex set  $A$  satisfying  $S \subseteq A \subseteq K$ , we have

$$S \subseteq A_1 := \{x \in A : co(\partial^* f(\bar{x})) \cap -N_A(\bar{x}) \subseteq -N_A(x)\},$$

$$S \subseteq A_2 := \{x \in A : co(\partial^* f(x)) \cap -N_A(x) \subseteq -N_A(\bar{x})\}.$$

*Proof* We prove only  $S \subseteq S_1$ . The proof for other inclusions is similar. Let  $x \in S$  be arbitrary. By convexity of  $S$ , for sufficiently small  $t > 0$ ,  $\bar{x} + t(x - \bar{x}) \in S$ . This implies  $f^+(\bar{x}; x - \bar{x}) = 0$ , leading to  $\langle \eta, x - \bar{x} \rangle \leq 0$  for each  $\eta \in \partial^* f(\bar{x})$ . Therefore,  $\langle \eta, x - \bar{x} \rangle = 0$  for each  $\eta \in co(\partial^* f(\bar{x})) \cap -N_K(\bar{x})$ . Let  $y \in K$  be arbitrary. For each  $\bar{\eta} \in co(\partial^* f(\bar{x})) \cap -N_K(\bar{x})$ , we have

$$\langle \bar{\eta}, y - x \rangle = \langle \bar{\eta}, y - \bar{x} \rangle + \langle \bar{\eta}, \bar{x} - x \rangle \geq 0.$$

Hence,  $\bar{\eta} \in -N_K(x)$  and the proof is completed. □

Convexity of  $S$  assumed in this subsection holds if  $f$  is quasiconvex. If  $f$  is continuous and a-pseudoconvex, then this function is quasiconvex ([Yang 2005](#), Definitions 3.1 and 3.2 and Theorem 3.1) and hence  $S$  is convex.

Proposition 2 provides a full characterization of the solution set  $S$ .

**Proposition 2** *Let  $\bar{x} \in S$  be given. Assume that  $S$  is convex,  $co(\partial^* f(\bar{x})) + N_K(\bar{x})$  is closed, and  $f$  is a-pseudoconvex with URC on  $K$ . Then  $S = S_3 = S_4 = S_5$ , where*

$$S_3 := \{x \in K : \exists \zeta \in co(\partial^* f(x)), \quad \langle \zeta, \bar{x} - x \rangle = 0\},$$

$$S_4 := \{x \in K : \exists \zeta \in co(\partial^* f(x)), \quad \langle \zeta, \bar{x} - x \rangle \geq 0\},$$

$$S_5 := \{x \in K : \exists \zeta \in co(\partial^* f(x)) \text{ s.t. } \forall \eta \in \partial^* f(\bar{x}), \quad \langle \eta, x - \bar{x} \rangle \leq \langle \zeta, \bar{x} - x \rangle\}.$$

*Proof* It is clear that  $S_3 \subseteq S_4$ . Furthermore, from a-pseudoconvexity, we get  $S_4 \subseteq S$ . We establish  $S \subseteq S_3$ ,  $S \subseteq S_5$ , and  $S_5 \subseteq S_4$ .

$S \subseteq S_3$ : Let  $x \in S$ . By Theorem 4 and closedness of  $co(\partial^* f(\bar{x})) + N_K(\bar{x})$ , there exists some  $\zeta \in co(\partial^* f(x)) \cap -N_K(x)$ , and similar to the proof of Proposition 1 we get  $\langle \zeta, \bar{x} - x \rangle = 0$ .

$S \subseteq S_5$ : Let  $x \in S$ . Similar to the preceding part there exists some  $\zeta \in co(\partial^* f(x))$  satisfying  $\langle \zeta, \bar{x} - x \rangle = 0$ . As  $S$  is convex and  $\bar{x} \in S$ , we have  $f^+(\bar{x}, x - \bar{x}) = 0$ . Thus for each  $\eta \in \partial^* f(\bar{x})$ ,  $\langle \eta, x - \bar{x} \rangle \leq 0 = \langle \zeta, \bar{x} - x \rangle$ .

$S_5 \subseteq S_4$ : Let  $x \in S_5$ . Since  $\bar{x} \in S$  and  $K$  is convex,  $f^+(\bar{x}; x - \bar{x}) \geq 0$ . Hence, due to  $x \in S_5$ , there exists some  $\zeta \in co(\partial^* f(x))$  such that

$$0 \leq f^+(\bar{x}; x - \bar{x}) = \sup_{\eta \in \partial^* f(\bar{x})} \langle \eta, x - \bar{x} \rangle \leq \langle \zeta, \bar{x} - x \rangle.$$

□

### 6.2 Linear approximation

In this subsection, we obtain a linear semi-infinite problem (Goberna and López 1998) to check the optimality of a given feasible solution of (1). To the best of our knowledge, Soleimani-damaneh (2008, 2010) is the first scholar who dealt with this problem under locally Lipschitz data.

Let  $\bar{x}$  be a feasible solution of (1). Assume that  $f$  has a URC at  $\bar{x}$  as  $\partial^* f(\bar{x})$ . Let  $\bar{\eta} \in co(\partial^* f(\bar{x}))$ . Consider the following linear semi-infinite programming problem:

$$\inf_{x \in K_1} f(\bar{x}) + \langle \bar{\eta}, x - \bar{x} \rangle, \tag{12}$$

where

$$K_1 = \{x \in \mathbb{R}^n : \langle \zeta_i, x - \bar{x} \rangle \leq 0, \quad \forall i \in I(\bar{x}), \forall \zeta_i \in \partial^* g_i(\bar{x})\}.$$

Theorems 7 and 8 study the relationships between Problems (1) and (12).

**Theorem 7** *Assume that  $\partial^* f(\bar{x})$  is bounded,  $int K \neq \emptyset$  and GLCQ holds at  $\bar{x}$ . Furthermore, assume that  $g_i$  for  $i \notin I(\bar{x})$  is upper semicontinuous at  $\bar{x}$ . If  $\bar{x}$  is an optimal solution of (1) and  $cone(\Gamma(\bar{x}))$  is closed, then  $\bar{x}$  is an optimal solution of (12) for some  $\bar{\eta} \in co(\partial^* f(\bar{x}))$ .*

*Proof* From Corollary 2, there exist  $\lambda_i \geq 0, i \in I(\bar{x})$  such that

$$0 \in co(\partial^* f(\bar{x})) + \sum_{i \in I(\bar{x})} \lambda_i co(\partial^* g_i(\bar{x})).$$

Therefore,

$$0 = \bar{\eta} + \sum_{i \in I(\bar{x})} \lambda_i \bar{\zeta}_i, \tag{13}$$



for some  $\bar{\eta} \in co(\partial^* f(\bar{x}))$  and  $\bar{\zeta}_i \in co(\partial^* g_i(\bar{x}))$ . Let  $x'$  be a feasible solution to (12) corresponding to  $\bar{\eta}$ . By (13),

$$f(\bar{x}) + \langle \bar{\eta}, x' - \bar{x} \rangle = f(\bar{x}) - \sum_{i \in I(\bar{x})} \lambda_i \langle \bar{\zeta}_i, x' - \bar{x} \rangle \geq f(\bar{x}).$$

Therefore,  $\bar{x}$  is an optimal solution for (12) corresponding to  $\bar{\eta}$ . □

Now, we are going to prove the converse of Theorem 7.

**Theorem 8** *Assume that  $f$  is  $\alpha$ -pseudoconvex with a URC  $\partial^* f(\bar{x})$  at  $\bar{x}$ . If  $\bar{x}$  is a optimal solution to (12) for some  $\bar{\eta} \in co(\partial^* f(\bar{x}))$ , then  $\bar{x}$  is a optimal solution to (1).*

*Proof* Let  $x \in K$  be arbitrary. According to the convexity of  $K$ , for each  $i \in I(\bar{x})$ ,  $g_i^+(\bar{x}; x - \bar{x}) \leq 0$ . Hence,

$$\langle \zeta, x - \bar{x} \rangle \leq 0, \quad \forall i \in I(\bar{x}), \forall \zeta \in \partial^* g_i(\bar{x}),$$

which leads to  $x \in K_1$ . Therefore,  $K \subseteq K_1$  which implies  $N_{K_1}(\bar{x}) \subseteq N_K(\bar{x})$ .

Since  $\bar{x}$  solves (12) for some  $\bar{\eta} \in co(\partial^* f(\bar{x}))$ , the objective function  $f(\bar{x}) + \langle \bar{\eta}, x - \bar{x} \rangle$  is linear (w.r.t.  $x$ ), and the feasible set  $K_1$  is convex, we have  $0 \in \bar{\eta} + N_{K_1}(\bar{x})$ . Hence,

$$\langle \eta, y - \bar{x} \rangle \geq 0, \quad \forall y \in K.$$

Therefore,  $\bar{x}$  solves (1), because of  $\alpha$ -pseudoconvexity assumption on  $f$ . □

### 6.3 An example

To illustrate our results, we look at a specific model arising in stochastic programming with probabilistic constraints. Assume that  $A$  is a  $s \times n$  matrix and  $C \subseteq \mathbb{R}^n$  is a convex set. Consider the following optimization problem:

$$\min\{f(x) : x \in C, F(Ax) \geq p\},$$

where  $p \in (0, 1)$  is given and  $F(\cdot)$  is corresponding to a probability distribution function; see [Henrion and Römisch \(1999, pp. 55–56\)](#) for more detail about applications of this model.

*Example 2* Consider

$$\min \left\{ f(x) : F(x) \leq \frac{1}{2} \right\},$$

where

$$f(x) = \begin{cases} -x + 3, & x < 1 \\ 1, & x = 1, \\ -x + 2, & x > 1, \end{cases}$$

and the probability distribution function  $F$  is given by

$$F(x) = \begin{cases} 0, & x \leq 0 \\ \frac{1}{2}x, & 0 \leq x \leq 1, \\ 1, & x > 1 \end{cases}$$

The function  $F$  is not convex or concave while

$$K = \{x : F(x) \leq \frac{1}{2}\} = (-\infty, 1]$$

is a convex set. Both  $F$  and  $f$  are discontinuous at  $\bar{x} = 1$  while two sets  $\partial^* F(\bar{x}) = [\frac{1}{2}, \infty)$  and  $\partial^* f(\bar{x}) = (-\infty, -1]$  are URCs of these functions at  $\bar{x}$ , respectively.

Let  $y \in K$ . Let  $\{\eta_k\}_{k \in \mathbb{N}} \subseteq \partial^* f(\bar{x})$  be such that  $\lim_{k \rightarrow \infty} \eta_k(y - \bar{x}) \geq 0$ . Due to  $\eta_k \leq -1$ , this implies  $y \leq \bar{x}$  and then  $f(y) \geq f(\bar{x})$ . Therefore,  $f$  is a-pseudoconvex at  $\bar{x}$ . Furthermore,  $\text{int } K \neq \emptyset$ . Moreover,  $F(\bar{x}) = \frac{1}{2}$  and  $0 \notin \partial^* F(\bar{x})$ , leading to GLCQ at  $\bar{x}$ . Thus, according to Theorem 2,

$$N_K(\bar{x}) = \text{cl cone}(\partial^* F(\bar{x})) = [0, \infty)$$

and one may consider  $B = \{1\}$  as a convex compact base for this closed pointed convex cone. Since  $0 = \eta + \lambda\zeta$  with  $\eta = -1 \in \partial^* f(\bar{x})$ ,  $\lambda = 1$ , and  $\zeta = 1 \in B$ , by Theorem 5 (sufficient part in Corollary 3),  $\bar{x}$  is an optimal solution.

Notice that here the objective function is quasiconvex and hence the solution set  $S$  is convex. Now consider  $\hat{x} \in K$ ,  $\hat{x} \neq \bar{x}$ . We have  $\hat{x} < 1$  and  $f$  is differentiable at  $x$  with  $f'(x) = -1$ . Hence,  $f'(\hat{x})(\hat{x} - \bar{x}) \neq 0$  which implies  $\hat{x} \notin S$  because of Proposition 2. Therefore,  $S = \{\bar{x}\}$ .

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