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k-core covers and the core

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Abstract This paper extends the notion of individual minimal rights for a transferable utility game (TU-game) to coalitional minimal rights using minimal balanced families of a specific type, thus defining a corresponding minimal rights game. It is shown that the core of a TU-game coincides with the core of the corresponding minimal rights game. Moreover, the paper introduces the notion of the k-core cover as an extension of the core cover. The k-core cover of a TU-game consists of all efficient payoff vectors for which the total joint payoff for any coalition of size at most k is bounded from above by the value of this coalition in the corresponding dual game, and from below by the value of this coalition in the corresponding minimal rights game. It is shown that the core of a TU-game with player set N coincides with the largest integer below or equal to $\frac{|N|}{2}$-core cover. Furthermore, full characterizations of games for which a k-core cover is nonempty and for which a k-core cover coincides with the core are provided.

Keywords Core · Core cover · k-core cover · k-compromise admissibility · k-compromise stability · Assignment games

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1 Introduction

The core of a transferable utility game (TU-game), as introduced by Gillies (1953), consists of all efficient payoff vectors for the monetary value of the grand coalition from which no coalition has an incentive to deviate. Some well-known core catchers are the dominance core (cf. Gillies 1953, 1959), the Weber set (Weber 1988) and the core cover (Tijs and Lipperts 1982).

The literature shows that both convex games (Shapley 1971), for which the core equals the Weber set, and compromise stable games (Quant et al. 2005), which are balanced games for which the core equals the core cover, have several interesting and helpful properties. Restricting attention to compromise stability, we want to mention that the nucleolus (Schmeidler 1969) of any compromise stable game can be directly computed by using the Aumann Maschler rule (Aumann and Maschler 1985) of an associated bankruptcy game (cf. O'Neill 1982), thus unifying seemingly unrelated results on the nucleolus for, e.g., bankruptcy games and clan games (Potters et al. 1989).

It is well-known that the core of a game may be empty. Due to this, many researchers have defined set valued solution concepts that always contain the core, as the core cover. The core cover has a very simple structure since it is determined by the efficience hyperplane and bounded by the vectors of minimal rights and utopia values (and thus determined by $2|N| + 1$ hyperplanes). Its extreme points are, however, hard to determine due to the computational difficulty of the underlying minimal rights vector. Moreover, although the core is always contained in the core cover, the core cover may also be empty. In this paper, we exploit the simplicity of the core cover structure to better understand the structure of the core. With this purpose, we generalize the core cover by introducing k-core covers: an extension of the core cover based on coalitional considerations. Individual marginal contributions can readily be extended to coalitional marginal contributions using the corresponding dual game. This paper proposes to extend individual minimal rights to coalitional ones by using for each coalition S minimal balanced families on $N\backslash S$ with the size of its elements restricted to at most $|S|$. In this way, an associated minimal rights game is obtained. A first result shows that the core of a TU-game coincides with the core of its corresponding minimal rights game.

Using the dual game and the minimal rights game, we define the k-core cover, with $k \in \{1, \ldots, |N|\}$, of a TU-game with player set N as the set of all efficient payoff vectors for which the total joint payoff to any coalition S of size at most k is bounded from above by the value of S in the corresponding dual game, and from below by the value of S in the corresponding minimal rights game[1]. It is shown that the 1-core cover coincides with the core cover, that each k-core cover is a core catcher and, interestingly, that the core is a $\lfloor \frac{|N|}{2} \rfloor$-core cover[2].

[1] In a similar spirit, Grabisch and Miranda (2008) introduced the k-additive core, where coalitions of at most size k pay an important role. It turns out that the k-additive core in Grabisch and Miranda (2008) has no general relation with neither the k-core cover, nor the k-core introduced in this paper. Moreover, unlike the k-core cover and k-core here defined, k-additive cores need not be linear and coalitions of size at least $|N| - k$ do not have an influential role in their definition.

[2] $\lfloor \frac{|N|}{2} \rfloor$ is the integer part of $\frac{|N|}{2}$

Defining a game to be k-compromise admissible if the k-core cover is nonempty, and k-compromise stable if it is balanced and the k-core cover and core coincide, this paper characterizes k-compromise admissible games and k-compromise stable games by means of conditions on specific minimal balanced families in both the dual game and the minimal rights game. Finally, we show that assignment games (Shapley and Shubik 1972) are a specific case of 2-compromise stable games.

The paper is structured as follows. Section 2 presents basic definitions and notations regarding TU-games and balanced families. Section 3 introduces and analyzes minimal right games and the k-core cover, while Sect. 4 characterizes k-compromise admissible games and k-compromise stable games. Section 5 shows that assignment games are 2-compromise stable.

2 Preliminaries

A transferable utility game (TU-game) is an ordered pair (N, v) where N is a finite set of players and $v : 2^N \to \mathbb{R}$ satisfies $v(\emptyset) = 0$. In general, $v(S)$ represents the value of coalition S, that is, the joint payoff that can be obtained by this coalition when its members decide to cooperate. Let G^N be the set of all TU-games with player set N. Given $S \subseteq N$, let $|S|$ be the number of players in S.

The main focus within a cooperative setting is on how to share the total joint payoff obtained when all players decide to cooperate. Given a TU-game $v \in G^N$, the *core* of v, $\mathcal{C}ore(v)$, is defined as the set of efficient allocations (for which exactly $v(N)$ is allocated) that are stable, in the sense that no coalition has an incentive to deviate. Formally,

$$\mathcal{C}ore(v) = \left\{ x \in \mathbb{R}^N \ : \ \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N \right\}.$$

It is well known that the core of a game may be empty. In Tijs and Lipperts (1982), the core cover is introduced as a core catcher. The core cover is the set of efficient allocations in which every player gets an amount no lower than his minimal right and no higher than his utopia value. Given a game $v \in G^N$ and a player $i \in N$, the *utopia value* of player i, $M_i(v)$, is defined by

$$M_i(v) = v(N) - v(N \backslash \{i\})$$

and the *minimal right* of player i, $m_i(v)$, is defined by

$$m_i(v) = \max_{S \subseteq N \, : \, S \ni i} \left\{ v(S) - \sum_{j \in S \backslash \{i\}} M_j(v) \right\}.$$

The *core cover* of $v \in G^N$, $\mathcal{CC}(v)$, is defined by

$$\mathcal{CC}(v) = \left\{ x \in \mathbb{R}^N \ : \ \sum_{i \in N} x_i = v(N) \text{ and } m(v) \leq x \leq M(v) \right\}$$

where $m(v) = (m_i(v))_{i\in N}$ and $M(v) = (M_i(v))_{i\in N}$. A game $v \in G^N$ is *compromise admissible* if $\mathcal{CC}(v)$ is nonempty. Formally, if

$$m(v) \leq M(v) \quad \text{and} \quad \sum_{i\in N} m_i(v) \leq v(N) \leq \sum_{i\in N} M_i(v). \tag{1}$$

A compromise admissible game is *compromise stable* if the core cover coincides with the core.

Theorem 2.1 (Tijs and Lipperts (1982)) *Let $v \in G^N$ be compromise admissible. Then,*

(i) $\mathcal{C}ore(v) \subseteq \mathcal{CC}(v)$.
(ii) If $|N| = 3$, then, $\mathcal{C}ore(v) = \mathcal{CC}(v)$.

Quant et al. (2005) characterize the class of compromise stable games.

Theorem 2.2 (Quant et al. (2005)) *Let $v \in G^N$ be compromise admissible. Then, $\mathcal{C}ore(v) = \mathcal{CC}(v)$ if, and only if, for every $S \subseteq N$,*

$$v(S) \leq \max\left\{\sum_{i\in S} m_i(v), v(N) - \sum_{i\in N\setminus S} M_i(v)\right\}.$$

Let $\emptyset \neq S \subseteq N$. A *family* $\mathcal{B}$ of nonempty subcoalitions of S is called *balanced on* S if there are positive weights $\delta = \{\delta_T\}_{T\in\mathcal{B}}$, $\delta_T > 0$ for all $T \in \mathcal{B}$, such that $\sum_{T\in\mathcal{B}} \delta_T e^T = e^S$ or, equivalently, $\sum_{T\in\mathcal{B}:T\ni i} \delta_T = 1$ for all $i \in S$. Here, $e^S \in \mathbb{R}^N$ is the characteristic vector of S and is defined by $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \notin S$. Given a balanced family $\mathcal{B}$, we denote by $\Delta(\mathcal{B})$ the set of positive weights satisfying the balancedness condition. For each $k = 1, \ldots, |N|$, we denote by $\mathcal{F}_k(S)$ the collection of balanced families on S such that for all $\mathcal{B} \in \mathcal{F}_k(S)$ and $R \in \mathcal{B}$, $|R| \leq k$. A *balanced* family $\mathcal{B} \in \mathcal{F}_k(S)$ is *minimal* if $\mathcal{B}' \subseteq \mathcal{B}$ and $\mathcal{B}' \in \mathcal{F}_k(S)$ implies $\mathcal{B}' = \mathcal{B}$. We denote by $\mathcal{F}_k^m(S)$ the collection of minimal balanced families on S. It is well known that a minimal balanced family has a unique vector of balanced weights. Given $\mathcal{B} \in \mathcal{F}_k^m(S)$, we denote by $\{\gamma_T^{\mathcal{B}}\}_{T\in\mathcal{B}}$ the corresponding vector of balanced weights. It is also known that if $\mathcal{B} \in \mathcal{F}_k(S)\setminus\mathcal{F}_k^m(S)$ and $\delta \in \Delta(\mathcal{B})$, then, there exist $\mathcal{B}_1, \ldots, \mathcal{B}_r \in \mathcal{F}_k^m(S)$ with $r \geq 2$ and $t_1, \ldots, t_r \in (0, 1)$ with $\sum_{l=1}^r t_l = 1$ such that $\mathcal{B} = \cup_{l=1}^r \mathcal{B}_l$ and $\delta_R = \sum_{l\in\{1,\ldots,r\}:\mathcal{B}_l\ni R} t_l \gamma_R^{\mathcal{B}_l}$.

A *game* $v \in G^N$ is called *balanced* if for all balanced families $\mathcal{B} \in \mathcal{F}_{|N|}(N)$ and all $\{\delta_S\}_{S\in\mathcal{B}} \in \Delta(\mathcal{B})$, $\sum_{S\in\mathcal{B}} \delta_S v(S) \leq v(N)$. Bondareva (1963) and Shapley (1967) established that a game $v \in G^N$ has a nonempty core if, and only if, it is balanced. In fact, they show that a game has a nonempty core if, and only if, all balancedness inequalities are satisfied for minimal balanced families on N.

3 A family of core catchers

3.1 Utopia and minimal rights games

In this subsection we introduce the notions of the utopia and minimal rights games associated to a TU-game.

Definition 3.1 Let $v \in G^N$. The *dual* or *utopia game*, v_D, is defined by

$$v_D(S) = v(N) - v(N \backslash S) \qquad \text{for all } S \subseteq N.$$

The *minimal rights game*,[3] v_m, is defined by

$$v_m(S) = \max_{T \subseteq N : T \supseteq S} \left\{ v(T) - \max_{\mathcal{B} \in \mathcal{F}^m_{|S|}(T \backslash S)} \sum_{R \in \mathcal{B}} \gamma^{\mathcal{B}}_R v_D(R) \right\} \qquad \text{for all } S \subseteq N.$$

Note that, for each $S \subseteq N$, $v_D(S)$ reflects the marginal contribution of coalition S to the grand coalition N. Therefore, if coalition S asks for a higher share of $v(N)$ than $v_D(S)$, it will be profitable for coalition $N \backslash S$ to avoid cooperation with the players of S. Accordingly, $v_D(S)$ can be interpreted as a utopia value for coalition S. Once the values of the utopia game are known to all players, the question is how to compute the minimal rights game. Following the general idea of the minimal rights of a player, we first have to consider what is left from the value of coalition T, with $S \subseteq T$, once the players in $T \backslash S$ are paid using the utopia game; secondly, coalition S will maximize its benefit over all potential partners $T \backslash S$, with $S \subseteq T$. Clearly, the difficulty encountered when defining the value of a coalition S in the minimal rights game is how to determine the amount that S should concede to the players of $T \backslash S$, with $S \subseteq T$, according to the utopia game. Using a common pessimistic approach, we consider that this quantity is the maximum expected utopia value that coalition $T \backslash S$ can achieve. When the minimal right of a coalition S is computed, the allowed cooperation of the players of $T \backslash S$ is restricted to coalitions of size at most $|S|$. Note that, by only considering the balanced families of cardinality at most $|S|$, we are generalizing the concept of minimal right of a player where only the utopia values of the individual players are taken into account. In fact, given a player i and a coalition T with $i \in T$, player i concedes $\sum_{j \in T \backslash \{i\}} M_j(v)$ to the players in $T \backslash \{i\}$, where $\{\{j\} : j \in T \backslash \{i\}\}$ is the only minimal balanced family in $\mathcal{F}^m_1(T \backslash \{i\})$. To conclude, note that if $|T \backslash S| < |S|$, then, $\mathcal{F}^m_{|T \backslash S|}(T \backslash S) = \mathcal{F}^m_{|S|}(T \backslash S)$ since all coalitions in any minimal balanced family in $\mathcal{F}^m_{|S|}(T \backslash S)$ are contained in $T \backslash S$.

It is easily seen that the maximum expected value for a coalition T over all balanced families of elements with cardinality at most k and over all associated positive weights is achieved in a minimal balanced family. Formally, given $v \in G^N$, $T \subseteq N$, and $k \in \{1, \ldots, |T|\}$, we have

[3] In general, it is computationally hard to obtain $v_m(S)$ even for coalitions S of size 1. In case of convex games, $v_m = v$ as we show subsequently.

Table 1 Utopia and minimal rights games in Example 3.2

S	{1}	{2}	{3}	{4}	{1, 2}	{1, 3}	{1, 4}	{2, 3}	{2, 4}	{3, 4}
$v(S)$	1	1	2	3	2	3	5	3	4	5
$v_D(S)$	2	2	3	4	5	6	7	5	7	8
$v_m(S)$	1	1	2	3	3	4	5	4	5	6
S	{1, 2, 3}	{1, 2, 4}	{1, 3, 4}	{2, 3, 4}	{1, 2, 3, 4}					
$v(S)$	6	7	8	8	10					
$v_D(S)$	7	8	9	9	10					
$v_m(S)$	6	7	8	8	10					

$$\max_{\mathcal{B}\in\mathcal{F}_k(T)} \max_{\delta\in\Delta(\mathcal{B})} \sum_{R\in\mathcal{B}} \delta_R v(R) = \max_{\mathcal{B}\in\mathcal{F}_k^m(T)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v(R). \tag{2}$$

In particular, when applying Eq. (2) to the dual game, we see that the definition of minimal rights game coincides with our informal description.

Example 3.2 Consider the game $v \in G^N$ given in Table 1, where also the values of the utopia and minimal rights games are provided. Next, we illustrate the computation of $v_m(\{1\})$, $v_m(\{1,2\})$, and $v_m(\{1,2,3\})$.

$$\begin{aligned}
v_m(\{1\}) &= \max_{T\subseteq N: T\supseteq\{1\}} \left\{ v(T) - \max_{\mathcal{B}\in\mathcal{F}_1^m(T\setminus S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_D(R) \right\} \\
&= \max\{v(\{1\}), v(\{1,2\}) - v_D(\{2\}), v(\{1,3\}) - v_D(\{3\}), \\
&\quad v(\{1,4\}) - v_D(\{4\}), \\
&\quad v(\{1,2,3\}) - v_D(\{2\}) - v_D(\{3\}), v(\{1,2,4\}) - v_D(\{2\}) - v_D(\{4\}), \\
&\quad v(\{1,3,4\}) - v_D(\{3\}) - v_D(\{4\}), v(N) - v_D(\{2\}) - v_D(\{3\}) - v_D(\{4\})\} \\
&= \max\{1, 2-2, 3-3, 5-4, 6-2-3, 7-2-4, 8-3-4, \\
&\quad 10-2-3-4\} = 1. \\
v_m(\{1,2\}) &= \max_{T\subseteq N: T\supseteq\{1,2\}} \left\{ v(T) - \max_{\mathcal{B}\in\mathcal{F}_2^m(T\setminus S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_D(R) \right\} \\
&= \max\{v(\{1,2\}), v(\{1,2,3\}) - v_D(\{3\}), v(\{1,2,4\}) - v_D(\{4\}), \\
&\quad v(N) - \max\{v_D(\{3\}) + v_D(\{4\}), v_D(\{3,4\})\}\} \\
&= \max\{2, 6-3, 7-4, 10 - \max\{3+4, 8\}\} = 3. \\
v_m(\{1,2,3\}) &= \max_{T\subseteq N: T\supseteq\{1,2,3\}} \left\{ v(T) - \max_{\mathcal{B}\in\mathcal{F}_3^m(T\setminus S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_D(R) \right\} \\
&= \max\{v(\{1,2,3\}), v(N) - v_D(\{4\})\} \\
&= \max\{6, 10-4\} = 6.
\end{aligned}$$

The following proposition gives some straightforward implications of the definitions of utopia and minimal rights games. In fact, statement (a) implies that utopia and minimal rights games generalize the utopia values and minimal rights of players. We recall that a game $v \in G^N$ is *monotone* if $v(S) \leq v(T)$ for every $S \subseteq T \subseteq N$. A game $v \in G^N$ is *convex* (see Shapley 1971) if $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for every $i \in N$ and $S \subseteq T \subseteq N \backslash \{i\}$.

Proposition 3.3 *Let $v \in G^N$. Then,*

(a) $v_D(\{i\}) = M_i(v)$ and $v_m(\{i\}) = m_i(v)$ for every $i \in N$.
(b) $v_D(N) = v_m(N) = v(N)$ and $v_m(N \backslash \{i\}) = v(N \backslash \{i\})$ for all $i \in N$.
(c) $v_m \geq v$.
(d) If v is monotone, then, v_D is non-negative and monotone.
(e) If v is convex, then, $v_m = v$.

Proof The first three items are straightforward.

(d) Let v be monotone. Then, $v_D(S) = v(N) - v(N \backslash S) \geq 0$ for all $S \subseteq N$. If $S \subseteq T$, then, $N \backslash S \supseteq N \backslash T$, and $v_D(S) = v(N) - v(N \backslash S) \leq v(N) - v(N \backslash T) = v_D(T)$; therefore, v_D is monotone.
(e) Let v be convex. First, we show that for all $S \subseteq R \subseteq N$,

$$v(S) \geq v(R) - \sum_{i \in R \backslash S} v_D(\{i\}) \tag{3}$$

or, equivalently, that $v(R) - v(S) \leq \sum_{i \in R \backslash S} v_D(\{i\})$. Let $R \backslash S = \{i_1, \ldots, i_r\}$. Then,

$$\begin{aligned} v(R) - v(S) &= v(S \cup (R \backslash S)) - v(S) \\ &= v(S \cup \{i_1, \ldots, i_r\}) - v(S \cup \{i_1, \ldots, i_{r-1}\}) \\ &\quad + v(S \cup \{i_1, \ldots, i_{r-1}\}) - v(S \cup \{i_1, \ldots, i_{r-2}\}) + \ldots \\ &\quad + v(S \cup \{i_1\}) - v(S) \\ &\leq v(N) - v(N \backslash \{i_r\}) + v(N) - v(N \backslash \{i_{r-1}\}) + \ldots \\ &\quad + v(N) - v(N \backslash \{i_1\}) \\ &= \sum_{i \in R \backslash S} v_D(\{i\}) \end{aligned}$$

where the inequality follows from convexity of v.
Next, we show that for all $S \subseteq R \subseteq N$,

$$\max_{\mathcal{B} \in \mathcal{F}^m_{|S|}(R \backslash S)} \sum_{U \in \mathcal{B}} \gamma^{\mathcal{B}}_U v_D(U) = \sum_{i \in R \backslash S} v_D(\{i\}). \tag{4}$$

Consider $\mathcal{B} \in \mathcal{F}^m_{|S|}(R \backslash S)$. Note that for all $U \in \mathcal{B}$, $v_D(U) \leq \sum_{i \in U} v_D(\{i\})$ as a consequence of applying Inequality (3) to $R = N$ and $S = N \backslash U$. Then,

$$\sum_{U\in\mathcal{B}} \gamma_U^{\mathcal{B}} v_D(U) \le \sum_{U\in\mathcal{B}} \gamma_U^{\mathcal{B}} \sum_{i\in U} v_D(\{i\}) = \sum_{i\in R\setminus S} v_D(\{i\}).$$

Clearly, with $\mathcal{B} = \{\{i\} : i \in R\setminus S\} \in \mathcal{F}^m_{|S|}(R\setminus S)$ and $\gamma_U^{\mathcal{B}} = 1$ for every $U \in \mathcal{B}$, we have $\sum_{U\in\mathcal{B}} \gamma_U^{\mathcal{B}} v_D(U) = \sum_{i\in R\setminus S} v_D(\{i\})$ and (4) is proved.
Then,

$$\begin{aligned} v_m(S) &= \max_{T\subseteq N: T\supseteq S} \left\{ v(T) - \max_{\mathcal{B}\in\mathcal{F}^m_{|S|}(T\setminus S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_D(R) \right\} \\ &= \max_{T\subseteq N: T\supseteq S} \left\{ v(T) - \sum_{i\in T\setminus S} v_D(\{i\}) \right\} \\ &= v(S), \end{aligned}$$

where the last equality follows from Inequality (3) and Proposition 3.3 (c). □

3.2 k-core covers

This subsection introduces the k-core cover of a TU-game $v \in G^N$, where k is a natural number between 0 and $|N|$. The k-core cover is the set of efficient allocations in which every coalition of size less than or equal to k gets an amount no lower than its value of the minimal rights game and no higher than its value of the utopia game. Formally, we have

Definition 3.4 Let $v \in G^N$ and $k \in \{0, 1, 2, \ldots, |N|\}$. The k-core cover of v, $\mathcal{CC}^k(v)$, is defined by

$$\mathcal{CC}^k(v) = \left\{ x \in \mathbb{R}^N \;:\; \begin{array}{l} \sum_{i\in N} x_i = v(N) \text{ and} \\ v_m(S) \le \sum_{i\in S} x_i \le v_D(S) \text{ for every } S \subseteq N \text{ with } |S| \le k \end{array} \right\}.$$

Note that $\mathcal{CC}^0(v) = \left\{x \in \mathbb{R}^N : \sum_{i\in N} x_i = v(N)\right\}$, $\mathcal{CC}^1(v) = \mathcal{CC}(v)$, and $\mathcal{CC}^{|N|-1}(v) = \mathcal{CC}^{|N|}(v)$.

Remark 3.1 Equivalently, the k-core cover can be recursively defined as follows.

1. $\mathcal{CC}^0(v) = \left\{x \in \mathbb{R}^N : \sum_{i\in N} x_i = v(N)\right\}$.
2. For $k = 1, \ldots, |N|$,

$$\mathcal{CC}^k(v) = \left\{ x \in \mathcal{CC}^{k-1}(v) : v_m(S) \le \sum_{i\in S} x_i \le v_D(S) \text{ for every } S \subseteq N \text{ with } |S| = k \right\}.$$

The following result states that any k-core cover is a core catcher.

Theorem 3.5 *Let $v \in G^N$. Then,*

$$\emptyset \ne \mathcal{CC}^0(v) \supseteq \mathcal{CC}^1(v) \supseteq \mathcal{CC}^2(v) \supseteq \ldots \supseteq \mathcal{CC}^{|N|-1}(v) = \mathcal{CC}^{|N|}(v) = \mathcal{C}ore(v).$$

Proof Clearly, we only have to show the last equality.

First, we show "⊆". For this, let $x \in \mathcal{CC}^{|N|}(v)$. We show that $x \in \mathcal{C}ore(v)$. Note that $\sum_{i\in N} x_i = v(N)$ and for every $S \subseteq N$, $\sum_{i\in S} x_i \geq v_m(S) \geq v(S)$ where the last inequality follows from Proposition 3.3 (c). Therefore, $x \in \mathcal{C}ore(v)$.

Second, we show "⊇". Let $x \in \mathcal{C}ore(v)$. We show that $x \in \mathcal{CC}^{|N|}(v)$. Note that $\sum_{i\in N} x_i = v(N)$, hence, we only have to show that $v_m(S) \leq \sum_{i\in S} x_i \leq v_D(S)$ for every $S \subseteq N$. Let $S \subseteq N$. To see that $\sum_{i\in S} x_i \leq v_D(S)$, note that

$$\sum_{i\in S} x_i = \sum_{i\in N} x_i - \sum_{i\in N\backslash S} x_i = v(N) - \sum_{i\in N\backslash S} x_i \leq v(N) - v(N\backslash S) = v_D(S). \quad (5)$$

To show that $\sum_{i\in S} x_i \geq v_m(S)$, let $T \subseteq N$ with $S \subseteq T$. Then, for every $\mathcal{B} \in \mathcal{F}^m_{|S|}(T\backslash S)$, we have that

$$v(T) \leq \sum_{i\in T} x_i = \sum_{i\in S} x_i + \sum_{i\in T\backslash S} x_i = \sum_{i\in S} x_i + \sum_{R\in\mathcal{B}} \gamma^{\mathcal{B}}_R \sum_{i\in R} x_i \leq \sum_{i\in S} x_i + \sum_{R\in\mathcal{B}} \gamma^{\mathcal{B}}_R v_D(R),$$

where the second inequality follows from Eq. (5). Hence,

$$\sum_{i\in S} x_i \geq v(T) - \sum_{R\in\mathcal{B}} \gamma^{\mathcal{B}}_R v_D(R) \quad \text{for every } \mathcal{B} \in \mathcal{F}^m_{|S|}(T\backslash S).$$

Consequently, $\sum_{i\in S} x_i \geq \max_{T\subseteq N: T\supseteq S}\{v(T) - \max_{\mathcal{B}\in\mathcal{F}^m_{|S|}(T\backslash S)} \sum_{R\in\mathcal{B}} \gamma^{\mathcal{B}}_R v_D(R)\} = v_m(S)$. □

Using Theorem 3.5, it can be shown that the core of a game coincides with the core of its minimal rights game.

Theorem 3.6 *Let $v \in G^N$ and v_m its minimal rights game. Then, $\mathcal{C}ore(v) = \mathcal{C}ore(v_m)$.*

Proof Note that $\mathcal{C}ore(v_m) \subseteq \mathcal{C}ore(v)$ since for all $S \subseteq N$, $v_m(S) \geq v(S)$ while $v_m(N) = v(N)$. Then, we only have to show that $\mathcal{C}ore(v_m) \supseteq \mathcal{C}ore(v)$. Let $x \in \mathcal{C}ore(v)$. Then, $\mathcal{C}ore(v) = \mathcal{CC}^{|N|}(v)$ by Theorem 3.5 and $\sum_{i\in S} x_i \geq v_m(S)$ for all $S \subseteq N$. Consequently, $x \in \mathcal{C}ore(v_m)$. □

Moreover, it turns out that the core of a game with set of players N coincides with[4] the $\lfloor \frac{|N|}{2} \rfloor$-core cover. From this fact, one can easily derive the coincidence of the core and the 1-core cover for arbitrary 3-player games (cf. Theorem 2.1 (ii)).

Theorem 3.7 *Let $v \in G^N$. Then, $\mathcal{C}ore(v) = \mathcal{CC}^{\lfloor \frac{|N|}{2} \rfloor}(v)$.*

Proof Using Theorem 3.5, it is sufficient to show that $\mathcal{C}ore(v) \supseteq \mathcal{CC}^{\lfloor \frac{|N|}{2} \rfloor}(v)$. Let $x \in \mathcal{CC}^{\lfloor \frac{|N|}{2} \rfloor}(v)$. Clearly, $\sum_{i\in N} x_i = v(N)$. Let $S \subseteq N$ with $|S| \leq \lfloor \frac{|N|}{2} \rfloor$. Then,

[4] For each $r \in \mathbb{R}$, $\lfloor r \rfloor$ denotes the largest integer below or equal to r.

$$\sum_{i \in S} x_i \geq v_m(S) \geq v(S),$$

where the first inequality is a direct consequence of the definition of $\lfloor \frac{|N|}{2} \rfloor$-core cover and the second inequality follows from Proposition 3.3 (c). Next, let $S \subseteq N$ with $|S| > \lfloor \frac{|N|}{2} \rfloor$. Then,

$$\sum_{i \in S} x_i = \sum_{i \in N} x_i - \sum_{i \in N \setminus S} x_i = v(N) - \sum_{i \in N \setminus S} x_i \geq v(N) - v_D(N \setminus S) = v(S)$$

where the inequality is a direct consequence of $x \in \mathcal{CC}^{\lfloor \frac{|N|}{2} \rfloor}(v)$ and $0 < |N \setminus S| \leq \lfloor \frac{|N|}{2} \rfloor$. Consequently, $x \in \mathcal{C}ore(v)$. □

As an immediate consequence of Theorem 3.5 and Theorem 3.7, we have the following result.

Corollary 3.8 *Let $v \in G^N$. Then, $\mathcal{CC}^k(v) = \mathcal{C}ore(v)$ for all $k \geq \lfloor \frac{|N|}{2} \rfloor$.*

Since, by Theorem 3.5, the k-core cover is contained in the l-core cover for every $l < k$, it is useful to define the smallest nonempty k-core cover of a game.

Definition 3.9 For $v \in G^N$, the least core cover, $\mathcal{LCC}(v)$, is defined by $\mathcal{LCC}(v) = \mathcal{CC}^{k^*}(v)$ where $k^* = \max\{k \in \{0, 1, \ldots, \lfloor \frac{|N|}{2} \rfloor\} : \mathcal{CC}^k(v) \neq \emptyset\}$.

Note that the least core cover is a nonempty core catcher. Besides, if v is balanced, then, $\mathcal{LCC}(v) = \mathcal{C}ore(v)$. The following example illustrates the least core cover of a game with an empty core.

Example 3.10 Consider the 6-player game $v \in G^N$ where the characteristic function is given by

$$v(S) = \begin{cases} 0 & \text{if } |S| = 1, \\ 2 & \text{if } |S| = 2, \\ 5 & \text{if } |S| = 3, \\ 4 & \text{if } |S| = 4, \\ 5 & \text{if } |S| = 5, \\ 8 & \text{if } |S| = 6. \end{cases}$$

First, we show that this game has an empty core. For this, suppose that the core is nonempty and let $x \in \mathcal{C}ore(v)$. Then, $x_1 + x_2 + x_3 \geq 5$ and $x_4 + x_5 + x_6 \geq 5$. Adding both inequalities and taking into account that $\sum_{i=1}^{6} x_i = v(N) = 8$ since $x \in \mathcal{C}ore(v)$, we obtain $8 = \sum_{i=1}^{6} x_i \geq 10$, establishing a contradiction. According to Theorem 3.7, $\mathcal{CC}^3(v) = \mathcal{C}ore(v) = \emptyset$. It turns out that the game has a nonempty 2-core cover. We subsequently compute the 1- and 2-core covers.

Note that $v_D(\{i\}) = v(N) - v(N \setminus \{i\}) = 3$ for all $i \in N$ and

$$v_m(\{i\}) = \max\{v(\{i\}), \max\{v(S) - \sum_{j \in S \setminus \{i\}} v_D(\{j\}) \ : \ S \subseteq N, \ i \in S\}\} = 0$$

for all $i \in N$. Then, $v_m(\{i\}) \leq v_D(\{i\})$ for all $i \in N$ and $\sum_{i\in N} v_m(\{i\}) \leq v(N) \leq \sum_{i\in N} v_D(\{i\})$. Therefore, $\mathcal{CC}^1(v) \neq \emptyset$ and it is given by[5]

$$\mathcal{CC}^1(v) = con(\{3e^{\{i\}} + 3e^{\{j\}} + 2e^{\{k\}} \ : \ i, j, k \in N, \ |\{i, j, k\}| = 3\}).$$

Next, we compute the 2-core cover. Note that $v_D(\{i, j\}) = v(N) - v(N\backslash\{i, j\}) = 8 - 4 = 4$ for all $i, j \in N$ with $i \neq j$. Moreover, for all $i, j \in N$ with $i \neq j$ and all $S \subseteq N$ with $i, j \in S$ and $|S| \geq 4$, we have that $v(S) - \max_{\mathcal{B}\in\mathcal{F}_2^m(S\backslash\{i,j\})} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_D(R) \leq 0$. Thus, for all $i, j \in N$ with $i \neq j$,

$$v_m(\{i, j\}) = \max\{v(\{i, j\}), \max\{v(\{i, j, k\}) - v_D(\{k\}) \ : \ k \in N\backslash\{i, j\}\}\} = 2.$$

Then, for all $i, j \in N$ with $i \neq j$, $v_m(\{i, j\}) \leq v_D(\{i, j\})$ and

$$\mathcal{CC}^2(v) = con(\{e^N + 2e^{\{i\}} \ : \ i \in N\}) = \mathcal{LCC}(v).$$

4 *k*-compromise admissibility and *k*-compromise stability

4.1 *k*-compromise admissible games

Definition 4.1 For $k \in \{0, 1, \ldots, |N|\}$, a game $v \in G^N$ is *k-compromise admissible* if $\mathcal{CC}^k(v)$ is nonempty.

Note that any game is 0-compromise admissible. Before characterizing k-compromise admissibility, we introduce the concepts of k-core and k-anti core.

Definition 4.2 Let $v \in G^N$ and $k \in \{1, \ldots, |N|\}$. The k-core of v, $\mathcal{C}ore^k(v)$, is the set of efficient allocations that are stable for coalitions of size smaller than or equal to k. Formally,

$$\mathcal{C}ore^k(v) = \left\{x \in \mathbb{R}^N \ : \ \sum_{i\in N} x_i = v(N), \sum_{i\in S} x_i \geq v(S) \text{ for all } S \subseteq N \text{ with } |S| \leq k\right\}.$$

Similarly, the k-anti core of v_D, $\mathcal{AC}ore^k(v_D)$, is defined by

$$\mathcal{AC}ore^k(v_D) = \left\{x \in \mathbb{R}^N \ : \sum_{i\in N} x_i = v_D(N), \sum_{i\in S} x_i \leq v_D(S) \text{ for all } S \subseteq N \text{ with } |S| \leq k\right\}.$$

The following result follows directly from the definitions of k-core and k-anti core.

Proposition 4.3 *Let $v \in G^N$. Then,*

$$\mathcal{C}ore^1(v) \supseteq \mathcal{C}ore^2(v) \supseteq \ldots \supseteq \mathcal{C}ore^{|N|}(v) = \mathcal{C}ore(v) \text{ and}$$
$$\mathcal{AC}ore^1(v_D) \supseteq \mathcal{AC}ore^2(v_D) \supseteq \ldots \supseteq \mathcal{AC}ore^{|N|}(v_D) = \mathcal{C}ore(v).$$

[5] Given a finite set $A \subseteq \mathbb{R}^N$, $con(A)$ denotes the convex hull of A.

As an immediate consequence, we have

Theorem 4.4 *Let $v \in G^N$ and $k \in \{1, \ldots, |N|\}$. Then, $\mathcal{CC}^k(v) = \mathcal{C}ore^k(v_m) \cap \mathcal{AC}ore^k(v_D)$ and $\mathcal{C}ore(v) = \mathcal{C}ore^{\lfloor \frac{|N|}{2} \rfloor}(v_m) \cap \mathcal{AC}ore^{\lfloor \frac{|N|}{2} \rfloor}(v_D)$.*

Next, we introduce the concepts of k-balanced games and k-dual balanced games and show that k-balancedness (k-dual balancedness) is a sufficient and necessary condition for non-emptiness of the k-core (k-anti core).

Definition 4.5 Let $v \in G^N$ and $k \in \{1, \ldots, |N|\}$.

- v is *k-balanced* if for all balanced families $\mathcal{B} \in \mathcal{F}_k(N)$ and all $\{\delta_S\}_{S\in\mathcal{B}} \in \Delta(\mathcal{B})$,

$$\sum_{S\in\mathcal{B}} \delta_S v(S) \leq v(N).$$

- v is *k-dual balanced* if for all balanced families $\mathcal{B} \in \mathcal{F}_k(N)$ and all $\{\delta_S\}_{S\in\mathcal{B}} \in \Delta(\mathcal{B})$,

$$\sum_{S\in\mathcal{B}} \delta_S v(S) \geq v(N).$$

Just like for balanced games, one defines k-minimal balanced and k-minimal dual balanced games based on k-minimal balanced families.

The following theorem extends the characterization of Bondareva-Shapley of non-emptiness of the core (Bondareva 1963; Shapley 1967) to the nonemptiness of the k-core and the k-anti core. The proof follows the same lines as the proof in Shapley (1967) and is, therefore, omitted.

Theorem 4.6 *Let $v \in G^N$ and $k \in \{1, \ldots, |N|\}$. Then,*

(a) $\mathcal{C}ore^k(v) \neq \emptyset$ if, and only if, v is minimally k-balanced.
(b) $\mathcal{AC}ore^k(v_D) \neq \emptyset$ if, and only if, v_D is minimally k-dual balanced.

Example 4.7 Consider the 6-player game $v \in G^N$ where the characteristic function is given by

$$v(S) = \begin{cases} 0 & \text{if } |S| = 1, \\ 3 & \text{if } |S| = 2 \text{ and } 1 \in S, \\ 2 & \text{if } |S| = 2,\ 1 \notin S, \text{ and } 2 \in S, \\ 1 & \text{if } |S| = 2,\ 1 \notin S, \text{ and } 2 \notin S, \\ 0 & \text{if } |S| = 3, \\ 4.75 & \text{if } |S| = 4 \text{ and } 1, 2 \in S, \\ 3.75 & \text{if } |S| = 4,\ 1 \in S, \text{ and } 2 \notin S, \\ 2.75 & \text{if } |S| = 4 \text{ and } 1 \notin S, \\ 3.25 & \text{if } |S| = 5, \\ 6.25 & \text{if } S = N. \end{cases}$$

It turns out that $\mathcal{C}ore^2(v_m) \neq \emptyset$, $\mathcal{AC}ore^2(v_D) \neq \emptyset$, and $\mathcal{CC}^2(v) = \mathcal{C}ore^2(v_m) \cap \mathcal{AC}ore^2(v_D) = \emptyset$. To see this we give the values of the minimal rights game and the utopia game for coalitions of cardinality at most 2. We have that $v_m(S) = v(S)$ for every $S \subseteq N$ with $|S| \leq 2$ and

$$v_D(S) = \begin{cases} 3 & \text{if } |S| = 1, \\ 3.5 & \text{if } |S| = 2 \text{ and } 1 \in S, \\ 2.5 & \text{if } |S| = 2,\ 1 \notin S, \text{ and } 2 \in S, \\ 1.5 & \text{if } |S| = 2 \text{ and } 1, 2 \notin S. \end{cases}$$

Note that $(2.5, 1.5, 0.5, 0.5, 0.5, 0.75) \in \mathcal{C}ore^2(v_m)$ and $(2.75, 0.75, 0.75, 0.75, 0.75, 0.5) \in \mathcal{AC}ore^2(v_D)$. However, $\mathcal{CC}^2(v) = \mathcal{C}ore^2(v_m) \cap \mathcal{AC}ore^2(v_D) = \emptyset$. To see this, suppose that the 2-core cover is nonempty and let $x \in \mathcal{CC}^2(v)$. Then, $x_1 + x_3 \geq 3$, $x_1+x_4 \geq 3$, $x_2+x_5 \geq 2$, $x_2+x_6 \geq 2$ and $x_1+x_2 \leq 3.5$. Adding the first four inequalities and subtracting the last one, and taking into account that $\sum_{i=1}^{6} x_i = v(N) = 6.25$, we obtain $6.25 = \sum_{i=1}^{6} x_i \geq 6.5$.

Next, we characterize the class of k-compromise admissible games. Note that k-compromise admissibility, for $k \geq \lfloor \frac{|N|}{2} \rfloor$, is equivalent to balancedness by Theorem 3.7 which is equivalent to balancedness of the minimal rights game by Theorem 3.6. Therefore, we restrict our attention to k-compromise admissibility for $k \in \{1, \ldots, \lfloor \frac{|N|}{2} \rfloor - 1\}$.We denote by $\mathcal{F}_{k,|N|-k}(N)$ the set of balanced families on N whose elements have cardinality at most k, or at least $|N| - k$. We denote by $\mathcal{F}^m_{k,|N|-k}(N)$ the corresponding set of minimal balanced families. Theorem 4.8 can be shown following the same lines as the proof of in Shapley (1967) using the Duality Theorem for linear programming problems or, alternatively, making use of Farkas' Lemma. For this reason, the proof is omitted.

Theorem 4.8 *Let $v \in G^N$ and $k \in \{1, \ldots, \lfloor \frac{|N|}{2} \rfloor - 1\}$. Then, v is k-compromise admissible if, and only if, the following condition is satisfied:*

$$\sum_{\substack{R \in \mathcal{B} \\ 0<|R|\leq k}} \gamma_R^{\mathcal{B}} v_m(R) - \sum_{\substack{R \in \mathcal{B} \\ |N|-k\leq|R|<|N|}} \gamma_R^{\mathcal{B}} v_D(N \backslash R) \leq \left(1 - \sum_{\substack{R \in \mathcal{B} \\ |N|-k\leq|R|<|N|}} \gamma_R^{\mathcal{B}}\right) v(N)$$

for every $\mathcal{B} \in \mathcal{F}^m_{k,|N|-k}(N)$, $\mathcal{B} \neq \{N\}$.

Remark 4.1 (i) Observe that Theorem 4.8 generalizes the characterization of 1-compromise stability given in Eq. (1). Recall that, by Proposition 3.3 (a), $m_i(v) = v_m(\{i\})$ and $M_i(v) = v_D(\{i\})$ for every $i \in N$. Note that $\mathcal{F}^m_{1,|N|-1}(N)$ consists of the families: $\{\{i\} : i \in N\}$, $\{\{i\}, N\backslash\{i\}\}$ for every $i \in N$, $\{N\backslash\{i\} : i \in N\}$ and $\{N\}$.
If $\mathcal{B} = \{\{i\}, N\backslash\{i\}\}$ with $i \in N$, then, $\gamma^{\mathcal{B}}_{\{i\}} = \gamma^{\mathcal{B}}_{N\backslash\{i\}} = 1$ and the condition in Theorem 4.8 becomes $m_i(v) = v_m(\{i\}) \leq v_D(\{i\}) = M_i(v)$.
If $\mathcal{B} = \{\{i\} : i \in N\}$, then, $\gamma^{\mathcal{B}}_{\{i\}} = 1$ for every $i \in N$ and the condition in Theorem 4.8 becomes $\sum_{i\in N} m_i(v) = \sum_{i\in N} v_m(\{i\}) \leq v(N)$.

If $\mathcal{B} = \{N\backslash\{i\} : i \in N\}$, then, $\gamma^{\mathcal{B}}_{N\backslash\{i\}} = \frac{1}{|N|-1}$ for every $i \in N$ and the condition in Theorem 4.8 becomes $\sum_{i\in N} M_i(v) = \sum_{i\in N} v_D(\{i\}) \geq v(N)$.

(ii) It can be easily seen that the conditions (a) $v_m(S) \leq v_D(S)$ for all $S \subseteq N$ such that $|S| \leq k$ and (b) $\sum_{S\in\mathcal{B}} \gamma^{\mathcal{B}}_S v_m(S) \leq v(N) \leq \sum_{S\in\mathcal{B}} \gamma^{\mathcal{B}}_S v_D(S)$ for all $\mathcal{B} \in \mathcal{F}^m_k(N)$ are necessary for k-compromise admissibility. They are however not sufficient for $k > 1$ as we see below.

Note that the game in Example 4.7 satisfies $v_m(S) \leq v_D(S)$ for all $S \subseteq N$ such that $|S| \leq 2$ and $\sum_{S\in\mathcal{B}} \gamma^{\mathcal{B}}_S v_m(S) \leq v(N) \leq \sum_{S\in\mathcal{B}} \gamma^{\mathcal{B}}_S v_D(S)$ for all $\mathcal{B} \in \mathcal{F}^m_2(N)$. However, $\mathcal{CC}^2(v) = \emptyset$. To see this, we take $\mathcal{B} = \{\{1,3\},\{1,4\},\{2,5\},\{2,6\},\{3,4,5,6\}\} \in \mathcal{F}^m_{2,|N|-2}(N)$ where $\gamma^{\mathcal{B}}_S = \frac{1}{2}$ for every $S \in \mathcal{B}$. Then,

$$\sum_{\substack{R\in\mathcal{B}\\ 0<|R|\leq 2}} \gamma^{\mathcal{B}}_R v_m(R) - \sum_{\substack{R\in\mathcal{B}\\ |N|-2\leq|R|<|N|}} \gamma^{\mathcal{B}}_R v_D(N\backslash R)$$
$$= \frac{1}{2}(v_m(\{1,3\})+v_m(\{1,4\})+v_m(\{2,5\})+v_m(\{2,6\})-v_D(N\backslash\{3,4,5,6\}))$$
$$= \frac{1}{2}(3+3+2+2-3.5) = \frac{6.5}{2}$$
$$> \frac{6.25}{2} = \left(1-\frac{1}{2}\right)6.25 = \left(1 - \sum_{\substack{R\in\mathcal{B}\\ |N|-2\leq|R|<|N|}} \gamma^{\mathcal{B}}_R\right) v(N).$$

4.2 k-compromise stable games

Quant et al. (2005) introduce and characterize the class of compromise stable games, which are those games that are compromise admissible and for which the core and the core cover coincide. Here, we perform a similar analysis for so called k-compromise stable games.

Definition 4.9 A k-compromise admissible game $v \in G^N$ is called *k-compromise stable* if $\mathcal{CC}^k(v) = \mathcal{C}ore(v)$.

Note that a k-compromise stable game is k-compromise admissible and that, consequently, the k-core cover, which coincides with the core, is nonempty and the game is balanced. The game in Example 3.10 has an empty core and, therefore, it is not k-compromise stable for any k.

Example 4.10 Reconsider the game $v \in G^N$ in Example 3.2. It is readily checked that this game is 1-compromise stable since $\mathcal{C}ore(v) = \mathcal{CC}^1(v) = con(\{(2,2,3,3),(2,2,2,4),(2,1,3,4),(1,2,3,4)\})$.

Note that for each balanced game, the game is $\lfloor\frac{|N|}{2}\rfloor$-compromise stable. Our main theorem in this section will provide necessary and sufficient conditions for a balanced game to be k-compromise stable with $k \in \{1,\ldots,\lfloor\frac{|N|}{2}\rfloor\}$. The next lemma provides necessary and sufficient conditions for $\mathcal{C}ore^k(v_m) = \mathcal{C}ore(v)$ and $\mathcal{AC}ore^k(v_D) = \mathcal{C}ore(v)$.

Lemma 4.11 *Let $v \in G^N$ be a balanced game and $k \in \{1, \ldots, \lfloor \frac{|N|}{2} \rfloor\}$. Then,*

(a) $\mathcal{C}ore^k(v_m) = \mathcal{C}ore(v)$ if, and only if, $v(S) \leq \max_{\mathcal{B}\in\mathcal{F}_k^m(S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_m(R)$ for all $S \subseteq N$.
(b) $\mathcal{AC}ore^k(v_D) = \mathcal{C}ore(v)$ if, and only if, $v(S) \leq v(N) - \min_{\mathcal{B}\in\mathcal{F}_k^m(N\setminus S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_D(R)$ for all $S \subseteq N$.
(c) $\mathcal{C}ore^k(v_m) = \mathcal{C}ore(v) = \mathcal{AC}ore^k(v_D)$ if, and only if, $v(S) \leq \min\left\{\max_{\mathcal{B}\in\mathcal{F}_k^m(S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_m(R), v(N) - \min_{\mathcal{B}\in\mathcal{F}_k^m(N\setminus S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_D(R)\right\}$ for all $S \subseteq N$.

Proof We prove (a) in detail. The proof of (b) can be done following similar arguments and (c) is an immediate consequence of (a) and (b) combined. First, we show the "if" part. Let

$$v(S) \leq \max_{\mathcal{B}\in\mathcal{F}_k^m(S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_m(R) \tag{6}$$

for all $S \subseteq N$. We show that $\mathcal{C}ore^k(v_m) = \mathcal{C}ore(v)$. By Proposition 4.3 and Theorem 3.6, we know that $\mathcal{C}ore^k(v_m) \supseteq \mathcal{C}ore(v_m) = \mathcal{C}ore(v)$. Therefore, we only have to prove that $\mathcal{C}ore^k(v_m) \subseteq \mathcal{C}ore(v)$. Let $x \in \mathcal{C}ore^k(v_m)$. Then, $\sum_{i\in N} x_i = v(N)$. Let $S \subseteq N$ be such that $k < |S|$. We show that $\sum_{i\in S} x_i \geq v(S)$. Note that $v_m(R) \leq \sum_{i\in R} x_i$ for every $R \subseteq N$ with $|R| \leq k$ and for all $\mathcal{B} \in \mathcal{F}_k^m(S)$,

$$\sum_{i\in S} x_i = \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} \sum_{i\in R} x_i \geq \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_m(R). \tag{7}$$

Therefore, using (6), $\sum_{i\in S} x_i \geq \max_{\mathcal{B}\in\mathcal{F}_k^m(S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_m(R) \geq v(S)$.
Next, we show the "only if" part. Let $\mathcal{C}ore^k(v_m) = \mathcal{C}ore(v)$. First, note that $\mathcal{C}ore^k(v_m)$ is nonempty and can be obtained as the set of optimal solutions of the linear programming problem (P_1):

$$(P_1) \qquad \min \sum_{i\in N} x_i$$
$$\sum_{i\in R} x_i \geq v_m(R) \quad \text{for every } R \subseteq N,\ |R| \leq k,$$
$$\sum_{i\in N} x_i \geq v(N).$$

Let $S \subseteq N$. If $1 \leq |S| \leq k$, then, clearly,

$$v(S) \leq v_m(S) \leq \max_{\mathcal{B}\in\mathcal{F}_k^m(S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_m(R)$$

where the second inequality follows from the fact that $\{S\} \in \mathcal{F}_k^m(S)$ with $\gamma_S^{\{S\}} = 1$. Take now $|S| \geq k+1$. Since $\mathcal{C}ore^k(v_m) = \mathcal{C}ore(v) = \mathcal{C}ore(v_m)$, we know that

$\sum_{i\in S} x_i \geq v_m(S)$ for every $x \in \mathcal{C}ore^k(v_m)$. Therefore, we can now modify problem (P_1) into problem (P_2)

$$
\begin{aligned}
(P_2) \quad & \min \sum_{i\in N} x_i \\
& \sum_{i\in R} x_i \geq v_m(R) \quad \text{for every } R \subseteq N,\ |R| \leq k, \\
& \sum_{i\in S} x_i \geq v_m(S) \quad \text{for every } S \subseteq N,\ |S| \geq k+1, \\
& \sum_{i\in N} x_i \geq v(N)
\end{aligned}
$$

where for each $S \subseteq N$ with $|S| \geq k+1$, the inequality constraint $\sum_{i\in S} x_i \geq v_m(S)$ is redundant. As a consequence, for every $S \subseteq N$ with $|S| \geq k+1$, there exists a non-negative linear combination of the constraint inequalities in (P_1) such that the linear combination makes the inequality $\sum_{i\in S} x_i \geq v_m(S)$ redundant. That is, there exists $\{\delta_R\}_{R\subseteq S,\,|R|\leq k}$ with $\delta_R \geq 0$ for each $R \subseteq S$ with $|R| \leq k$ and $\sum_R \delta_R = 1$, or equivalently, there is $\mathcal{B} \in \mathcal{F}_k(S)$ and $\{\delta_R\}_{R\in\mathcal{B}} \in \Delta(\mathcal{B})$ such that

$$\sum_{i\in S} x_i = \sum_{R\in\mathcal{B}} \delta_R \sum_{i\in R} x_i \geq \sum_{R\in\mathcal{B}} \delta_R v_m(R) \geq v_m(S).$$

Then, $v(S) \leq v_m(S) \leq \max_{\mathcal{B}\in\mathcal{F}_k(S)} \max_{\{\delta_R\}_{R\in\mathcal{B}}\in\Delta(\mathcal{B})} \sum_{R\in\mathcal{B}} \delta_R v_m(R) = \max_{\mathcal{B}\in\mathcal{F}_k^m(S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_m(R)$, where the first inequality is a direct consequence of Proposition 3.3 (c) and the last equality follows from Eq. (2) applied to v_m. □

Note that Lemma 4.11 (c) provides a sufficient condition for k-compromise admissibility. The following result provides a full characterization of the class of k-compromise stable games.

Theorem 4.12 *Let $v \in G^N$ be a balanced game and $k \in \{1, \ldots, \lfloor \frac{|N|}{2} \rfloor\}$. Then, v is k-compromise stable if, and only if, for every $S \subseteq N$,*

$$v(S) \leq \max\left\{ \max_{\mathcal{B}\in\mathcal{F}_k^m(S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_m(R),\ v(N) - \min_{\mathcal{B}\in\mathcal{F}_k^m(N\setminus S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_D(R) \right\}. \quad (8)$$

Proof We start showing the "only if" part. Let $k \in \{1, \ldots, \lfloor \frac{|N|}{2} \rfloor\}$ and assume that $\mathcal{CC}^k(v) = \mathcal{C}ore(v)$. We show that (8) is satisfied.

Let $S \subseteq N$ be such that $|S| \leq k$. Then,

$$v(S) \leq v_m(S) \leq \max_{\mathcal{B}\in\mathcal{F}_k^m(S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_m(R)$$

where the second inequality follows by considering $\mathcal{B} = \{S\}$ and $\gamma_S^{\mathcal{B}} = 1$. Therefore, (8) follows.

Let $S \subseteq N$ be such that $|S| > k$. If $v(S) \leq \max_{\mathcal{B}\in\mathcal{F}_k^m(S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_m(R)$, then (8) follows. Therefore, assume that

$$v(S) > \max_{\mathcal{B}\in\mathcal{F}_k^m(S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_m(R). \tag{9}$$

Note that Theorem 3.5 and the assumption $\mathcal{CC}^k(v) = \mathcal{C}ore(v)$ imply $\mathcal{CC}^k(v) = \mathcal{CC}^l(v) = \mathcal{C}ore(v)$ for all $l > k$. Therefore, the inequalities $\sum_{i\in R} x_i \geq v_m(R)$, with $|R| > k$, are redundant in the description of $\mathcal{CC}^l(v)$ for every $l \geq |R|$ and, in particular, $\sum_{i\in S} x_i \geq v_m(S)$ is also redundant in the description of $\mathcal{CC}^l(v)$ for every $l \geq |S|$. Now, suppose that we can derive $\sum_{i\in S} x_i \geq v_m(S)$ as a positive linear combination of inequalities of the form $\sum_{i\in R} x_i \geq v_m(R)$ with $R \subseteq N$, $|R| \leq k$. Then, there exists $\mathcal{B}_1 \in \mathcal{F}_k^m(S)$ such that

$$\sum_{i\in S} x_i = \sum_{R\in\mathcal{B}_1} \gamma_R^{\mathcal{B}_1} \sum_{i\in R} x_i \geq \sum_{R\in\mathcal{B}_1} \gamma_R^{\mathcal{B}_1} v_m(R) \geq v_m(S) \geq v(S),$$

establishing a contradiction with (9). Therefore, there must exist a linear combination of inequalities of the form $\sum_{i\in R} x_i \leq v_D(R)$ with $R \subseteq N$, $|R| \leq k$, that makes $\sum_{i\in S} x_i \geq v_m(S)$ redundant. Then, there exists $\mathcal{B}_2 \in \mathcal{F}_k^m(N\backslash S)$ such that

$$\begin{aligned}\sum_{i\in S} x_i &= \sum_{i\in N} x_i - \sum_{i\in N\backslash S} x_i = v(N) - \sum_{R\in\mathcal{B}_2} \gamma_R^{\mathcal{B}_2} \sum_{i\in R} x_i \geq v(N) - \sum_{R\in\mathcal{B}_2} \gamma_R^{\mathcal{B}_2} v_D(R)\\ &\geq v_m(S) \geq v(S).\end{aligned}$$

Hence, it follows that $v(S) \leq v(N) - \min_{\mathcal{B}\in\mathcal{F}_k^m(N\backslash S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_D(R)$ and (8) follows. To conclude, we show the "if" part. Assume that

$$v(S) \leq \max\left\{ \max_{\mathcal{B}\in\mathcal{F}_k^m(S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_m(R),\ v(N) - \min_{\mathcal{B}\in\mathcal{F}_k^m(N\backslash S)} \sum_{R\in\mathcal{B}} \gamma_R^{\mathcal{B}} v_D(R) \right\}$$

for every $S \subseteq N$. We have to show that $\mathcal{CC}^k(v) = \mathcal{C}ore(v)$. By Theorem 3.5, it suffices to prove that $\mathcal{CC}^k(v) \subseteq \mathcal{C}ore(v)$. Let $x \in \mathcal{CC}^k(v)$, then,

$$\sum_{i\in N} x_i = v(N) \tag{10}$$

and $v_m(R) \leq \sum_{i\in R} x_i \leq v_D(R)$ for every $R \subseteq N$ with $|R| \leq k$. We have to show that $\sum_{i\in S} x_i \geq v(S)$ for every $S \subseteq N$.

First, let $S \subseteq N$ with $|S| \leq k$, then,

$$\sum_{i\in S} x_i \geq v_m(S) \geq v(S). \tag{11}$$

Second, let $S \subseteq N$ with $|S| \geq |N| - k$. Then, $|N \backslash S| \leq k$ and

$$\sum_{i \in S} x_i = \sum_{i \in N} x_i - \sum_{i \in N \backslash S} x_i = v(N) - \sum_{i \in N \backslash S} x_i \geq v(N) - v_D(N \backslash S) = v(S) \quad (12)$$

where the first inequality follows from the fact that $\sum_{i \in R} x_i \leq v_D(R)$ for every $R \subseteq N$ with $|R| \leq k$.

Third, let $S \subseteq N$ with $k < |S| < |N| - k$. We distinguish between two situations:

1. $\max_{\mathcal{B} \in \mathcal{F}_k^m(S)} \sum_{R \in \mathcal{B}} \gamma_R^{\mathcal{B}} v_m(R) \geq v(N) - \min_{\mathcal{B} \in \mathcal{F}_k^m(N \backslash S)} \sum_{R \in \mathcal{B}} \gamma_R^{\mathcal{B}} v_D(R)$.
   By Condition (8), we have that $v(S) \leq \max_{\mathcal{B} \in \mathcal{F}_k^m(S)} \sum_{R \in \mathcal{B}} \gamma_R^{\mathcal{B}} v_m(R)$. Then, for $\bar{\mathcal{B}} \in \arg\max_{\mathcal{B} \in \mathcal{F}_k^m(S)} \sum_{R \in \mathcal{B}} \gamma_R^{\mathcal{B}} v_m(R)$, it follows that

$$\sum_{i \in S} x_i = \sum_{R \in \bar{\mathcal{B}}} \gamma_R^{\bar{\mathcal{B}}} \sum_{i \in R} x_i \geq \sum_{R \in \bar{\mathcal{B}}} \gamma_R^{\bar{\mathcal{B}}} v_m(R) = \max_{\mathcal{B} \in \mathcal{F}_k^m(S)} \sum_{R \in \mathcal{B}} \gamma_R^{\mathcal{B}} v_m(R) \geq v(S), \quad (13)$$

   where the inequality follows from the fact that $\sum_{i \in R} x_i \geq v_m(R)$ for every $R \subseteq N$ with $|R| \leq k$.
2. $v(N) - \min_{\mathcal{B} \in \mathcal{F}_k^m(N \backslash S)} \sum_{R \in \mathcal{B}} \gamma_R^{\mathcal{B}} v_D(R) > \max_{\mathcal{B} \in \mathcal{F}_k^m(S)} \sum_{R \in \mathcal{B}} \gamma_R^{\mathcal{B}} v_m(R)$.
   By Condition (8), we have that $v(S) \leq v(N) - \min_{\mathcal{B} \in \mathcal{F}_k^m(N \backslash S)} \sum_{R \in \mathcal{B}} \gamma_R^{\mathcal{B}} v_D(R)$. Then, for $\bar{\mathcal{B}} \in \arg\min_{\mathcal{B} \in \mathcal{F}_k^m(N \backslash S)} \sum_{R \in \mathcal{B}} \gamma_R^{\mathcal{B}} v_D(R)$, it follows that

$$\begin{aligned}\sum_{i \in S} x_i &= \sum_{i \in N} x_i - \sum_{i \in N \backslash S} x_i = v(N) - \sum_{R \in \bar{\mathcal{B}}} \gamma_R^{\bar{\mathcal{B}}} \sum_{i \in R} x_i \\ &\geq v(N) - \sum_{R \in \bar{\mathcal{B}}} \gamma_R^{\bar{\mathcal{B}}} v_D(R) = v(N) - \min_{\mathcal{B} \in \mathcal{F}_k^m(N \backslash S)} \sum_{R \in \mathcal{B}} \gamma_R^{\mathcal{B}} v_D(R) \geq v(S),\end{aligned} \quad (14)$$

   where the inequality follows from the fact that $\sum_{i \in R} x_i \leq v_D(R)$ for every $R \subseteq N$ with $|R| \leq k$. □

Note that, for the case $k = 1$, we have that the characterization in Theorem 4.12 boils down to Theorem 2.2.

Example 4.13 Consider the 6-player game $v \in G^N$ given by

$$v(S) = \begin{cases} 0 & \text{if } |S| = 1, \\ 2 & \text{if } |S| = 2, \\ 2 & \text{if } |S| = 3, \\ 5 & \text{if } |S| = 4, \\ 15 & \text{if } |S| = 5, \\ 20 & \text{if } |S| = 6. \end{cases}$$

Table 2 Game, utopia game, and minimal rights game in Example 4.13

$\lvert S\rvert$	1	2	3	4	5	6
$v(S)$	0	2	2	5	15	20
$v_D(S)$	5	15	18	18	20	20
$v_m(S)$	0	2	2	10	15	20
$\max_{\mathcal{B}\in\mathcal{F}_2^m(S)}\sum_{R\in\mathcal{B}}\gamma_R^{\mathcal{B}}v_m(R)$	0	2	3	4	5	8
$v(N)-\min_{\mathcal{B}\in\mathcal{F}_2^m(N\setminus S)}\sum_{R\in\mathcal{B}}\gamma_R^{\mathcal{B}}v_D(R)$	−5	0	5	10	15	20

It turns out that this game is 2-compromise stable, but not 1-compromise stable, that is, $\mathcal{C}ore(v)\neq\mathcal{CC}^1(v)$ while $\mathcal{C}ore(v)=\mathcal{CC}^2(v)=\mathcal{C}ore^2(v_m)\cap\mathcal{AC}ore^2(v_D)$. In Table 2, we provide the values of v, v_D, v_m, together with the values of $\max_{\mathcal{B}\in\mathcal{F}_2^m(S)}\sum_{R\in\mathcal{B}}\gamma_R^{\mathcal{B}}v_m(R)$ and $v(N)-\min_{\mathcal{B}\in\mathcal{F}_2^m(N\setminus S)}\sum_{R\in\mathcal{B}}\gamma_R^{\mathcal{B}}v_D(R)$ for every $S\subseteq N$. It follows, by Theorem 4.12, that $\mathcal{C}ore(v)=\mathcal{CC}^2(v)$. However, by Lemma 4.11 (a) and (b), we have that $\mathcal{C}ore^2(v_m)\neq\mathcal{C}ore(v)$ and $\mathcal{AC}ore^2(v_D)\neq\mathcal{C}ore(v)$.

5 Assignment games and 2-compromise stability

This section shows that assignment games as introduced in Shapley (1967) are 2-compromise stable. In an assignment situation there is a two sided market with finite and disjoint set of buyers, M, and of sellers, M'. We denote $m=|M|$, $m'=|M'|$, and $|N|=m+m'$. The worth obtained when one buyer $i\in M$ and one seller $j\in M'$ decide to cooperate is $a_{ij}\geq 0$. These values can be represented in an $m\times m'$ matrix A. Following the notation in Núñez and Rafels (2002), a *matching* between coalitions $S\subseteq M$ and $T\subseteq M'$ is a subset μ of $S\times T$ such that each player belongs to at most one pair in μ. Given two coalitions $S\subseteq M$ and $T\subseteq M'$, we denote the set of matchings between S and T by $\mathcal{M}(S,T)$; then, the maximum value that $S\cup T$ can obtain from cooperation is $\max_{\mu\in\mathcal{M}(S,T)}\sum_{(i,j)\in\mu}a_{ij}$.

Given an assignment situation $((M,M'),A)$, the associated assignment game $(M\cup M',v)$ is defined by

$$v(S\cup T)=\max_{\mu\in\mathcal{M}(S,T)}\sum_{(i,j)\in\mu}a_{ij}.$$

Given an optimal matching $\mu\in\mathcal{M}(M,M')$ for M and M', Shapley and Shubik (1972) show that the nonempty core of the assignment game v is given by

$$\mathcal{C}ore(v)=\left\{(x,y)\in\mathbb{R}^{M\times M'}\;\middle|\;\begin{array}{l}x_i\geq 0\text{ for all }i\in M,\ y_j\geq 0\text{ for all }j\in M',\\ x_i+y_j=a_{ij}\text{ if }(i,j)\in\mu,\\ x_i+y_j\geq a_{ij}\text{ if }(i,j)\notin\mu,\\ x_i=0\text{ if }i\text{ is not assigned by }\mu,\ \ y_j=0\text{ if }j\text{ is not assigned by }\mu\end{array}\right\}$$

Theorem 5.1 *Assignment games are 2-compromise stable.*

Proof Let $((M, M'), A)$ be an assignment situation and let $(M \cup M', v)$ be the associated assignment game. It suffices to show that $\mathcal{C}ore(v) = \mathcal{CC}^2(v)$. By Theorem 3.5, we know that $\mathcal{C}ore(v) \subseteq \mathcal{CC}^2(v)$. Therefore, we only need to show that $\mathcal{C}ore(v) \supseteq \mathcal{CC}^2(v)$. Let $\mu \in \mathcal{M}(M, M')$ be an optimal matching for M and M', let $(x, y) \in \mathcal{CC}^2(v)$, and let $i \in M$ and $j \in M'$. Clearly, $v_m(\{i\}) \leq x_i \leq v_D(\{i\})$ and $v_m(\{j\}) \leq y_j \leq v_D(\{j\})$ for every $i \in M$ and $j \in M'$. Note that $v_m(\{i\}) \geq v(\{i\}) = 0$, and $v_m(\{j\}) \geq v(\{j\}) = 0$, thus,

$$x_i \geq 0 \quad \text{for all } i \in M \quad \text{and} \quad y_j \geq 0 \quad \text{for all } j \in M'.$$

First, let $(i, j) \in \mu$. Then,

$$\begin{aligned} v_D(\{i, j\}) &= v(M \cup M') - v((M\backslash\{i\}) \cup (M'\backslash\{j\})) = a_{ij} = v(\{i, j\}) \leq v_m(\{i, j\}) \\ &\leq v_D(\{i, j\}) \end{aligned}$$

where the second equality follows because μ is optimal for M and M' and $(i, j) \in \mu$. Therefore, all inequalities are equalities. Since $(x, y) \in \mathcal{CC}^2(v)$, we have that $a_{ij} = v_m(\{i, j\}) \leq x_i + y_j \leq v_D(\{i, j\}) = a_{ij}$ and

$$x_i + y_j = a_{ij}.$$

Second, let $(i, j) \notin \mu$. Then, $v_m(\{i, j\}) \geq v(\{i, j\}) = a_{ij}$. Since $x_i + x_j \geq v_m(\{i, j\})$, we have that

$$x_i + y_j \geq a_{ij}.$$

Third, let $i \in M$ be not assigned by μ. Then,

$$v_D(\{i\}) = v(M \cup M') - v((M\backslash\{i\}) \cup M') = 0 = v(\{i\}) \leq v_m(\{i\}) \leq v_D(\{i\})$$

where the second equality follows because μ is optimal for M and M' and i is not assigned by μ. Therefore, all inequalities are equalities. Since $(x, y) \in \mathcal{CC}^2(v)$, we have that $0 = v_m(\{i\}) \leq x_i \leq v_D(\{i\}) = 0$ and

$$x_i = 0.$$

Analogously, if $j \in M'$ is not assigned by μ, then, $v_m(\{j\}) = 0 = v_D(\{j\})$ and

$$y_j = 0.$$

Therefore, $(x, y) \in \mathcal{C}ore(v)$. □

Remark 5.1 In fact, it can be seen that the core of an assignment game coincides with the 2-core of the corresponding minimal rights game and with the 2-anti core of the utopia game. Therefore, the class of assignment games satisfies the conditions in Lemma 4.11 (c).

The following example illustrates that assignment games can be 1-compromise stable.

Example 5.2 Consider $M = \{1, 2, 3\}$ and $M' = \{4\}$ and the assignment matrix $A = (1, 1, 1)^t$. It is easy to check that $\mathcal{C}ore(v) = \{(0, 0, 0, 1)\} = \mathcal{CC}^1(v)$. This game is both 1- and 2-compromise stable.
Consider $M = \{1, 2\}$ and $M' = \{3, 4\}$ and the assignment matrix

$$A = \begin{pmatrix} 5 & 3 \\ 4 & 3 \end{pmatrix}.$$

Then, $\mathcal{C}ore(v) = con\{(0, 0, 5, 3), (1, 0, 4, 3), (3, 3, 2, 0), (4, 3, 1, 0)\} = \mathcal{CC}^2(v)$ while $\mathcal{C}ore(v) \neq \mathcal{CC}^1(v)$ since $(1, 3, 1, 3) \in \mathcal{CC}^1(v) \backslash \mathcal{C}ore(v)$. This game is 2-compromise stable, but not 1-compromise stable.

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