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On solvability of a two-sided singular control problem

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Abstract We study a two-sided singular control problem in a general linear diffusion setting and provide a set of conditions under which an optimal control exists uniquely and is of singular control type. Moreover, under these conditions the associated value function can be written in a quasi-explicit form. Furthermore, we investigate comparative static properties of the solution with respect to the volatility and control parameters. Lastly we illustrate the results with two explicit examples.

Keywords Singular stochastic control · Two-sided control · Linear diffusion

Mathematics Subject Classification (1991) 93E20 · 60J60

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathbb{F} = \{\mathcal{F}_t \mid t < \infty\}$ a right continuous, completed filtration. Consider the controlled process $Z_t = X_t + U_t - D_t$ where X_t is a general, linear time homogeneous Itô diffusion on $\mathbb{R}_+ := (0, \infty)$ and (U_t, D_t) is a pair of $\mathbb{F}$-adapted, non-decreasing cádlág processes on $\mathbb{R}_+$. We consider the one-dimensional two-sided singular, or reflecting, control problem

$$\sup_{(U_t, D_t)} \mathbb{E}_x \left\{ \int_0^{\zeta_Z} e^{-rs}\pi(Z_s)ds + p\int_0^{\zeta_Z} e^{-rs}dD_s - q\int_0^{\zeta_Z} e^{-rs}dU_s \right\},$$

where $\pi : \mathbb{R}_+ \to \mathbb{R}$ is a revenue function satisfying suitable conditions (given in Sect. 3), $r > 0$ and $q, p \in \mathbb{R}$, $q > p$, are exogenously given constants, ζ_Z =inf$\{t \geq$

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$0 \mid Z_t \notin \mathbb{R}_+\}$ denotes the first exit time from $\mathbb{R}_+$, and the supremum is taken over all admissible controls.

In this study we give sufficient conditions under which the above mentioned problem has a unique two-sided reflecting control as an optimal control. Moreover, under the same conditions, we see that the value function can be written in a (quasi-)explicit form. Further, since we can identify the value function and control boundaries explicitly, we are also able to investigate the comparative static properties of the value function with respect to the volatility and the control coefficients p and q.

Since the pioneering work by Bather and Chernoff (1966) appeared, singular stochastic control problems have been subjected to extensive investigation due to their applicability in various fields. These fields include for example a costly reversible investment problem, or an irreversible one, depending whether $U_t \equiv 0$ or not. In these problems the investor has a chance to purchase capital at price q and sell it with lower price $p < q$. In different specific forms the irreversible case is studied for example in Kobila (1993), Oksendal (2000), Chiarolla and Haussmann (2005) and the costly reversible case in Abel and Eberly (1996), Guo and Pham (2005), Alvarez (2011). Another example is an optimal dividend payments problem combined to obligative reinvestment (see Sethi and Taksar 2002; Paulsen 2008). The company pays dividends to the owners at rate p and on the other hand, the owners are obliged to reinvest if the value of the income process becomes too small. Without the reinvestment possibility, the dividend payments problem has been studied for example in Asmussen and Taksar (2006), Højgaard and Taksar (1999), Alvarez and Virtanen (2006). Further applications include, for example, rational harvesting (see e.g. Lande et al. 1995; Lungu and Oksendal 1997; Alvarez 2000; Alvarez and Koskela 2007), monotone fuel follower problem (Chow et al. 1985; Jacka 2002; Bank 2005), exchange rates (Mundaca and Oksendal 1998), inventory theory (Harrison and Taksar 1983) and controlling a dam (Faddy 1974).

Singular stochastic control problems can be approached in different ways. The one used also in this study is based on the theory of partial differential equations and on variational arguments. In this approach one typically first constructs (by ad hoc methods) a solution to some necessary (e.g. Hamilton–Jacobi–Bellman) conditions and then validates the optimality of the solution by a verification theorem (see Karatzas 1983; Shreve et al. 1984; Chow et al. 1985; Bayraktar 2008; Alvarez and Lempa 2008). Alternatively, it is also possible to rely on probabilistic methods. In Karatzas and Shreve (1984), Karatzas (1985a), and Karatzas and Wang (2001) the existence of an optimal control was proved by showing, leaning on a weak compactness argument, that the optimizing sequence of the considered problem converges to an admissible control. These two approaches could be classified as direct techniques, as the problem is approached straightforwardly. In contrast to this, in an indirect approach the control problem is showed to be equivalent with other type of problem and the latter one is then solved. For example in recent studies (Guo and Tomecek 2008a,b) the authors reveal one-to-one correspondence between a singular control and a switching problem. They then go on to use this relation in a general multidimensional case to find an integral representation for the value function and, moreover, sufficient conditions for the existence of an optimal control.

Although singular control problems have attained lots of attention in general, theory considering two-sided controls is not yet as vast as the theory of one-sided controls. There are some general existence results for a two-sided control problem, e.g. Shreve et al. (1984), Sethi and Taksar (2002), Guo and Tomecek (2008b), and Paulsen (2008), which provide sufficient conditions or verification theorems for the solution in a general diffusion setting. In this paper we also follow this path and give rather easily verifiable sufficient conditions for the optimality, but in addition we can also give a (quasi-) explicit form for the value function. To accomplish this task, we have chosen to combine some existing techniques (from Harrison 1985; Shreve et al. 1984; Alvarez 2008; Lempa 2010) in appropriate way with the classical theory of linear diffusions and r-excessive mappings.

More specifically, we formulate the problem in exact terms in Sect. 2, after which we derive necessary first order optimality conditions for the two-sided singular control in Sect. 3. In Sect. 4, we present our first result, leaning on techniques from Harrison (1985) and Shreve et al. (1984). We prove that if the derived necessary optimality conditions attain a solution, then under a set of weak assumptions this solution is unique and the associated reflecting control is the optimal one among all admissible controls. In Sect. 5 we will find sufficient assumptions under which the above mentioned first order optimality conditions obtain a solution, after which it follows from the first result that this solution must be unique. The solution to the optimality conditions is found by using a fixed point argument, originating from Alvarez and Lempa (2008), and Lempa (2010), which results directly into the verification of the existence of the optimal exercise thresholds. An advantage of this approach is that it simultaneously results into an algorithm for finding the optimal thresholds numerically as a limit of a converging sequence.

The most important results are presented in Sect. 6, where we consider the comparative static properties of the value function. Previously this kind of examination has been done with one-sided controls (e.g. Alvarez 2001), but the author is not aware of similar treatment concerning a general two-sided control problem. We show that the same set of sufficient assumptions as above guarantees that the value function is unambiguously decreasing with respect to the volatility. This in turn decelerates the usage of optimal controls by expanding the inactivity region where exerting the optimal policy is suboptimal. These findings are in line with the previous literature concerning one-sided policies, see e.g. Alvarez (2001). We also demonstrate the sensitiveness with respect to the control parameters, and in particular that the one-sided control problem can be attained as a special case of this two-sided problem when $p \to 0$ or $q \to \infty$. Lastly, we will illustrate our results with two explicit examples in Sect. 7.

2 Problem formulation

2.1 The underlying dynamics

Let $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$ be a complete filtered probability space satisfying the usual conditions (see Borodin and Salminen 2002, p. 2). We assume that the regular linear diffusion process X_t is defined on $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$ and evolves on $\mathbb{R}_+$ according to the dynamics described by the Itô stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \quad X_0 = x, \tag{1}$$

where W_t denotes a standard Brownian motion. We assume that both the drift coefficient $\mu : \mathbb{R}_+ \mapsto \mathbb{R}$ and the volatility coefficient $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ are once continuously differentiable and that $\sigma(x) > 0$ for all $x \in (0, \infty)$. These conditions are sufficient for the existence of a weak solution for the stochastic differential equation (1) (cf. (Karatzas and Shreve, 1988, Section 5.5.B–C)). Moreover, we assume that the boundary ∞ is unattainable (i.e. natural or entrance-not-exit) for the process X_t and that the boundary 0 can, in addition to being unattainable, be also attainable (i.e. exit or regular), and that whenever 0 is regular we assume that it is killing. Further, if 0 is attainable, we assume in addition that the condition $\mu(0+) \leq 0$ holds. It is also worth mentioning here that the assumption that the state space is $\mathbb{R}_+$ is for notational convenience.

We define the differential operator associated to the underlying diffusion process as

$$\mathcal{A} = \tfrac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}.$$

Let us denote, respectively, by ψ and φ the increasing and decreasing fundamental solution of the ordinary differential equation $(\mathcal{A} - r)u = 0$, where $r > 0$ is the discount coefficient (for a complete characterization and basic properties of these minimal r-excessive functions, see Borodin and Salminen 2002, pp. 18–20). We know that

$$BS'(x) = \psi'(x)\varphi(x) - \varphi'(x)\psi(x), \tag{2}$$

where B is the constant Wronskian of the fundamental solutions ψ and φ and

$$S'(x) = \exp\left(-\int^{x} \frac{2\mu(y)}{\sigma^2(y)}dy\right)$$

is the density of the scale function of X_t.

We denote by $\mathcal{L}^1$ the class of measurable mappings $f : \mathbb{R}_+ \to \mathbb{R}$ satisfying the absolute integrability condition $\mathbb{E}_x \int_0^\infty e^{-rs}|f(X_s)|ds < \infty$. For all $f \in \mathcal{L}^1$ write

$$(R_r f)(x) = \mathbb{E}_x \int_0^\infty e^{-rs} f(X_s)ds$$

for the expected cumulative present value of a flow f. It is known from the literature on linear diffusion (e.g. Oksendal 2000, Proposition 4.3) that $(R_r f)(x)$ can be also re-expressed as

$$(R_r f)(x) = B^{-1}\varphi(x)\int_0^x \psi(y)f(y)m'(y)dy$$
$$+B^{-1}\psi(x)\int_x^\infty \varphi(y)f(y)m'(y)dy, \tag{3}$$

where $m'(x) = 2/(\sigma^2(x)S'(x))$ denotes the density of the speed measure of X_t.

2.2 The control and the problem

An admissible control policy is defined as a pair of processes (U_t, D_t) such that both processes are non-negative, non-decreasing, right-continuous, and $\{\mathcal{F}_t\}$-adapted. With admissible control (U_t, D_t), we define the associated controlled process $Z_t = X_t + U_t - D_t$. We associate a unit price p to the downward control D_t and a unit cost $-q$ to the upward control U_t. For example, in a timber harvesting example, D_t represents the cumulative harvest while U_t can be interpreted as the cumulative replanting. In capital theoretic or natural resource management applications of singular stochastic control, the unit price p is typically positive and the unit cost $-q$ is negative. However, there are cases where we may want to use negative values of p as well. For example if we consider controlling a boat in a stormy sea, with the controls as steering left and right, then it is sensible that both of these controls are costly, and so $p < 0$. So in order to grasp the most general aspect of the problem, we only assume $q > p$ without specifying their signs (the opposite inequality would lead easily to an infinite value function).

For an admissible control (U, D) our payoff function gets the form

$$H^{(U,D)}(x) = \mathbb{E}_x\left[\int_0^{\zeta_Z} e^{-rs}(\pi(Z_s)ds + pdD_s - qdU_s)\right], \tag{4}$$

where $\zeta_Z = \inf\{t \geq 0 : Z_t \notin \mathbb{R}_+\}$ denotes the first exit time of the controlled diffusion from its state space and $\pi : \mathbb{R}_+ \to \mathbb{R}$ captures the state dependent cash flow accrued from continuing operation, or it can be also interpreted as an utility function of the controller. Our objective is to solve the problem

$$V(x) = \sup_{(U,D)} H^{(U,D)}(x), \tag{5}$$

where the supremum is taken over all admissible policies (U_t, D_t). Our purpose is to delineate a set of fairly general assumptions under which there exists a well-defined and unique two-sided reflecting control policy for which the supremum (5) is attained.

3 Assumptions and preliminary results

3.1 Barrier policy and associated value function

For two arbitrary barriers z and y satisfying the inequality $0 < z < y < \infty$, we focus on barrier policies which maintain the state between these two barriers at all times. For given boundaries (z, y) we denote the exerted barrier policies, or reflecting controls, as U^z and D^y. If the initial state of the controlled process is between the boundaries, then the barrier policy (U^z, D^y) is obtained by assigning to the X_t the two-sided regulator so that U^z and D^y are continuous and increase only when $Z = z$ and $Z = y$, respectively. Thus, for $x \in (z, y)$, the controlled process evolves according to the diffusion X_t reflected at the boundaries z and y. If $x > y$, then we take $D_0^y = x - y$ resulting into an instantaneous gain $p(x - y)$ and apply the above mentioned regulator to $X - D_0^y$ from thereon. Similarly if $x < z$, we exert the policy $U_0^z = z - x$ resulting into the instantaneous cost $-q(z - x)$ and apply the regulator to $X + U_0^z$ from thereon. We shall see that the optimal control is of this class.

Next we shall write down the associated value function using the following application of Ito's lemma (cf. Harrison 1985, Corollary 5.2.4).

Lemma 3.1 *Let f be a twice continuously differentiable function. Fix $z < x < y$ and consider the barrier policy (U^z, D^y). Then*

$$f(x) = \mathbb{E}_x\left[\int_0^\infty e^{-rs}\left[(r - \mathcal{A})f(Z_s)ds + f'(y)dD_s^y - f'(z)dU_s^z\right]\right].$$

Proof By (generalised) Ito's lemma

$$\begin{aligned} e^{-rt}f(Z_t) &= f(Z_0) + \int_0^t e^{-rs}df(Z_s) - r\int_0^t e^{-rs}f(Z_s)ds \\ &= f(x) + M_t + \int_0^t e^{-rs}\left[(\mathcal{A} - r)f(Z_s)ds - f'(y)dD_s^y + f'(z)dU_s^z\right], \end{aligned} \tag{6}$$

where $M_t = \int_0^t e^{-rs}\sigma(Z_s)f'(Z_s)dW_s$. Since $z < Z_s < y$ for all $s > 0$, we see that both $f(Z_s)$ and $f'(Z_s)$ are bounded and so $\lim_{t\to\infty} e^{-rt}f(Z_t) = 0$ and $\mathbb{E}_x\{M_t\} = 0$. Therefore the claim follows by taking expectation of both sides in (6) and letting $t \to \infty$. □

Fix barriers z and y, let π be an integrable and once continuously differentiable function, and let $H^{(z,y)}$ be the value function associated to the barrier policy (U^z, D^y). For $z < x < y$ we have, by definition,

$$H^{(z,y)}(x) = \mathbb{E}_x \left[\int_0^\infty e^{-rs} \left(\pi(Z_s)ds + p dD_s^y - q dU_s^z \right) \right]. \tag{7}$$

Consider now the function $f(x) = (R_r\pi)(x) + c_1\psi(x) + c_2\varphi(x)$, where $c_1 = c_1(z, y)$ and $c_2 = c_2(z, y)$ are such that $f'(z) = q$ and $f'(y) = p$. This is a twice continuously differentiable function and consequently, by the lemma above, for $x \in (z, y)$, we have

$$f(x) = \mathbb{E}_x \left[\int_0^\infty e^{-rs} \left[\pi(Z_s)ds + p dD_s^y - q dU_s^z \right] \right].$$

Comparing this to (7) we see that, for $x \in (z, y)$, we must have $H^{(z,y)}(x) = f(x)$. Furthermore, it is clear from the definition of barrier policy rule that for $x \geq y$ we have $H^{(z,y)}(x) = p(x - y) + H^{(z,y)}(y)$, and similarly for $x \leq z$ we have $H^{(z,y)}(x) = q(x - z) + H^{(z,y)}(z)$. Hence the proposed class of considered barrier policies (U^z, D^y) leads to the value function

$$H^{(z,y)}(x) = \begin{cases} p(x - y) + H^{(z,y)}(y) & x \geq y, \\ (R_r\pi)(x) + c_1(z, y)\psi(x) + c_2(z, y)\varphi(x) & z < x < y, \\ q(x - z) + H^{(z,y)}(z) & x \leq z, \end{cases} \tag{8}$$

where the z and y-dependent factors c_1 and c_2 are such that

$$\begin{cases} (R_r\pi)'(y) + c_1(z, y)\psi'(y) + c_2(z, y)\varphi'(y) = p, \\ (R_r\pi)'(z) + c_1(z, y)\psi'(z) + c_2(z, y)\varphi'(z) = q. \end{cases}$$

Notice that the value function $H^{(z,y)}$ is once continuously differentiable for all barriers $z < y$.

3.2 The first order optimality conditions

A necessary first order condition for a pair (z, y) to be optimal is that $\frac{dc_1}{dz} = \frac{dc_1}{dy} = 0 = \frac{dc_2}{dz} = \frac{dc_2}{dy}$. Carrying out the computations we see that these conditions are, in fact, equivalent to the smooth pasting requirement that the second derivative of $H^{(z,y)}$ vanishes at z and y, i.e. the requirement that $H^{(z,y)}$ is twice continuously differentiable everywhere. After performing the differentiations, our necessary optimality conditions for the two-sided threshold (z^*, y^*) can be written as

$$\begin{cases} J_q(z^*) - J_p(y^*) = 0, \\ I_q(z^*) - I_p(y^*) = 0, \end{cases} \tag{9}$$

where, for $b = p, q$,

$$J_b(x) := \frac{\big((R_r\pi)'(x) - b\big)\varphi''(x) - (R_r\pi)''(x)\varphi'(x)}{\psi'(x)\varphi''(x) - \varphi'(x)\psi''(x)}$$
$$\text{and} \quad I_b(x) := \frac{\big((R_r\pi)'(x) - b\big)\psi''(x) - (R_r\pi)''(x)\psi'(x)}{\psi'(x)\varphi''(x) - \varphi'(x)\psi''(x)}. \tag{10}$$

If the pair of equations (9) is solvable, then the factors c_1 and c_2 are $-J_q(z^*)$ and $I_q(z^*)$ respectively. Furthermore, provided that sufficient differentiability conditions hold, we get by straight differentiation, and using the harmonicity of $(R_r\pi)$, ψ and φ, that for $b = p, q$

$$J_b'(x) = \frac{\varphi'(x)\big(\pi'(x) + b(\mu'(x) - r)\big)}{rBS'(x)} = \frac{\varphi'(x)\rho_b'(x)}{rBS'(x)}$$
$$\text{and} \quad I_b'(x) = \frac{\psi'(x)\big(\pi'(x) + b(\mu'(x) - r)\big)}{rBS'(x)} = \frac{\psi'(x)\rho_b'(x)}{rBS'(x)}, \tag{11}$$

where $\rho_b(x) = \pi(x) + b(\mu(x) - rx)$.

3.3 Assumptions and auxiliary results

The assumptions presented here are needed to show that the solution is unique and of two-sided reflecting control type. So, throughout the study we will make the following assumptions.

Assumption 3.2 For $b \in [p, q]$, denote $\rho_b(x) := \pi(x) + b(\mu(x) - rx)$. Assume that

(i) $q > p$,
(ii) $\mu(x), \pi(x), \sigma(x) \in C^1(\mathbb{R}_+)$ and $\pi(x), \mu(x), x \in \mathcal{L}^1$,
(iii) $\mu'(x) < r$, and if 0 is attainable, then in addition $\mu(0+) \leq 0$ (these imply that ψ and φ are convex, see Lemma 3.3 below),
(iv) for every $b \in [p, q]$, there is $\tilde{x}_b \in \mathbb{R}_+$ such that $\frac{d}{dx}\rho_b(x) \gtreqless 0$ whenever $x \lesseqgtr \tilde{x}_b$.

Let us make a few remarks on Assumption 3.2. First the differentiability conditions for π in Assumption (ii) could be relaxed, but it would complicate matters without gaining any relevant extra insight.

Assumption (iii) seems a little restricting, but it is justified; in the opposite case ($\mu' > r$) we would easily end up to an infinite value function, implying an ill-posed problem setting. Moreover, often μ is assumed to be Lipschitz continuous, i.e. that for some $C > 0$ we have $\mu' < C$, and hence Assumption (iii) may be seen merely setting an upper bound for the Lipschitz constant. One could try to relax this assumption by assuming that $\mu' > r$ in some bounded subset of $\mathbb{R}_+$, but that would complicate the analysis and possibly lead to a peculiar behaviour (see e.g. Example 5.3 in Shreve et al. 1984).

The three first assumptions are more or less standard assumptions, setting no strict restrictions for the problem. It turns out that the last quasi-concavity assumption (iv), the only restraining assumption needed, is enough to ensure the uniqueness of a well-defined solution (cf. Proposition 4.2 and Theorem 4.4). The function $\rho_b(x)$ itself, for $b = p, q$, can be seen (cf. Alvarez and Lempa 2008) to measure the expected net return from postponing the dividend payments (or reinvestments, depending whether $b = p$ or q) into the future instead of paying out the dividends (or reinvesting) instantaneously.

We close this section by revealing vital monotonicity properties, which shall be used later on several times.

Lemma 3.3 *(A) Let Assumption 3.2 (iii) hold and assume that $x \in \mathcal{L}^1$. Then ψ and φ are convex functions.*
(B) Let Assumption 3.2 hold. Then
(1) for $b = p, q$, $\frac{d}{dx} J_b(x) \lesseqqgtr 0$, whenever $x \lesseqqgtr \tilde{x}_b$. In addition $J_p(x) > J_q(x)$ for all $x \in \mathbb{R}_+$.
(2) for $b = p, q$, $\frac{d}{dx} I_b(x) \gtreqqless 0$, whenever $x \lesseqqgtr \tilde{x}_b$. In addition $I_p(x) > I_q(x)$ for all $x \in \mathbb{R}_+$.

Proof See Appendix A.1. □

4 Uniqueness and optimality of the two-sided reflecting control

4.1 Uniqueness of (z^*, y^*)

Before proving the main proposition about the uniqueness of the solution of (9) we will show that we can restrict the examination to two disjoint sets on positive real line.

Lemma 4.1 *Let Assumption 3.2 hold. Assume further that the necessary condition (9) has a solution (z^*, y^*). Then $(z^*, y^*) \in (0, \tilde{x}_q) \times (\tilde{x}_p, \infty)$, where $\tilde{x}_q \leq \tilde{x}_p$ are as in Assumption 3.2(iv).*

Proof To see that the inequality $\tilde{x}_q \leq \tilde{x}_p$ holds, set $x < \tilde{x}_p$. Then

$$\rho_q'(x) = \pi'(x) + q(\mu'(x) - r) \leq \pi'(x) + p(\mu'(x) - r) = \rho_p'(x) \leq 0$$

by Assumption 3.2(iii) and (i). Thus by Assumption 3.2(iv) we must have $\tilde{x}_q \leq \tilde{x}_p$.

The rest of the proof follows that of Alvarez (2008, Theorem 4.3). For a fixed $y \in \mathbb{R}_+$, consider the functional

$$L_1^y(z) = J_q(z) - J_p(y).$$

By Lemma 3.3(B) we know that $L_1^y(y) < 0$ and that $L_1^y(z)$ is z-decreasing on $(0, \tilde{x}_q)$ and z-increasing on $(\tilde{x}_q, \infty)$. Thus, if there exists a root $z_y^* \in (0, y)$ satisfying the condition $L_1^y(z_y^*) = 0$, it has to be on the interval $(0, \tilde{x}_q)$.

Analogously, for a fixed $z \in \mathbb{R}_+$, consider the functional

$$L_2^z(y) = I_q(z) - I_p(y).$$

By Lemma 3.3(B) we know that $L_2^z(z) < 0$ and that $L_2^z(y)$ is y-decreasing on $(0, \tilde{x}_p)$ and y-decreasing on $(\tilde{x}_p, \infty)$. Thus, if there exists a root $y_z^* \in (z, \infty)$ satisfying the condition $L_2^z(y_z^*) = 0$, it has to be on the interval $(\tilde{x}_p, \infty)$. □

Previous lemma narrows the possible region for the optimal thresholds. We shall use this information in next proposition, which is our main result on the uniqueness of the solution to the necessary conditions (9).

Proposition 4.2 *Let Assumption 3.2 hold. Assume further that the necessary conditions (9) have a solution (z^*, y^*). Then the pair (z^*, y^*) is unique.*

Proof Define a function $K : (0, \tilde{x}_q] \to (0, \tilde{x}_q]$ by $K(x) = \big(\hat{J}_q^{-1} \circ \hat{J}_p \circ \hat{I}_p^{-1} \circ \hat{I}_q\big)(x)$, where $\hat{J}_q = J_q|_{(0,\tilde{x}_q]}$, $\hat{J}_p = J_p|_{[\tilde{x}_p,\infty)}$, $\hat{I}_q = I_q|_{(0,\tilde{x}_q]}$ and $\hat{I}_p = I_p|_{[\tilde{x}_p,\infty)}$.

By Lemma 3.3(B) we know that the functions $\hat{J}_b$ and $\hat{I}_b$, for $b = p, q$, are monotonic in their domains $(0, \tilde{x}_q]$ and $[\tilde{x}_p, \infty)$ and therefore

$$K'(x) = \hat{J}_q^{-1\prime}(\hat{J}_p(\hat{I}_p^{-1}(\hat{I}_q(x)))) \cdot \hat{J}_p'(\hat{I}_p^{-1}(\hat{I}_q(x))) \cdot \hat{I}_p^{-1\prime}(\hat{I}_q(x)) \cdot \hat{I}_q'(x) > 0,$$

for all $x \in (0, \tilde{x}_q)$ and thus K is monotonically increasing.

Moreover, we see at once that if there exists a pair (z^*, y^*) satisfying the necessary conditions (9), then z^* must be a fixed point for K, that is $K(z^*) = z^*$. In order to prove the uniqueness, it suffices to establish that $K'(z^*) < 1$ for any given fixed point z^*. Utilizing the fixed point property $K(z^*) = z^*$ and the monotonicity properties of ψ' and φ' [Lemma 3.3(A)], ordinary differentiation yields

$$K'(z^*) = \frac{\psi'(z^*)}{\psi'(y^*)} \frac{\varphi'(y^*)}{\varphi'(z^*)} < 1.$$

This means that whenever the curve $K(x)$ intersects the diagonal of $\mathbb{R}_+^2$, the intersection is from above. This observation completes the proof. □

Thus, if the first order optimality conditions (9) attain a solution (z^*, y^*), it must be unique under Assumption 3.2. Next we shall concentrate on the optimality of the associated control (U^{z^*}, D^{y^*}).

4.2 Proving the optimality of the barrier policy

The two-sided barrier policy (z^*, y^*), which satisfy the pair of equations (9), leads to the value function [cf. (8)]

$$V(x) = \begin{cases} p(x - y^*) + V(y^*) & x \geq y^*, \\ (R_r\pi)(x) + c_1^*\psi(x) + c_2^*\varphi(x) & z^* < x < y^*, \\ q(x - z^*) + V(z^*) & x \leq z^*, \end{cases}$$

where $c_1^* = -J_q(z^*) = -J_p(y^*)$ and $c_2^* = I_q(z^*) = I_p(y^*)$ with I and J as in (10). Using the expressions $c_1^* = -J_p(y^*)$ and $c_2^* = I_p(y^*)$, applying the harmonicity of

$(R_r\pi)$, ψ and φ, and using the identity (2) we can calculate the limit in the boundary y^* to get

$$V(y^*-) = \frac{\left(p\frac{2\mu(y^*)}{\sigma^2(y^*)} - (R_r\pi)''(y^*) - \frac{2\mu(y^*)}{\sigma^2(y^*)}(R_r\pi)'(y^*) + \frac{2r}{\sigma^2(y^*)}(R_r\pi)(y^*)\right) S'(y^*)B}{\frac{2r}{\sigma^2(y^*)}S'(y^*)B}$$
$$= \frac{1}{r}\left[p\mu(y^*) + \pi(y^*)\right].$$

Similarly, using now the expressions $c_1^* = -J_q(z^*)$ and $c_2^* = I_q(z^*)$, we get

$$V(z^*+) = \frac{1}{r}\left[q\mu(z^*) + \pi(z^*)\right],$$

and so the value function can be written as

$$V(x) = \begin{cases} p(x-y^*) + \frac{1}{r}\left[p\mu(y^*) + \pi(y^*)\right] & x \geq y^*, \\ (R_r\pi)(x) + c_1^*\psi(x) + c_2^*\varphi(x) & z^* < x < y^*, \\ q(x-z^*) + \frac{1}{r}\left[q\mu(z^*) + \pi(z^*)\right] & x \leq z^*. \end{cases} \tag{12}$$

To prove that the two-sided barrier control (U^{z^*}, D^{y^*}) is the optimal control among all admissible controls and that $V(x)$ above is the optimal value function we shall need the following concavity result, which is a slight modification of Shreve et al. (1984, Lemma 4.2).

Lemma 4.3 *Let Assumption 3.2 hold, let (z^*, y^*) be a solution to (9) and let V be as in (12). Then*

(A) $V''(x) \leq 0$ for all $x \in (z^*, y^*)$.
(B) V is an increasing function.

Proof See Appendix A.2. □

Now we are ready to prove the main result about optimality of a reflecting control.

Theorem 4.4 *Let Assumption 3.2 hold and assume in addition that the necessary conditions (9) have a solution (z^*, y^*). Then the barrier policy (U^{z^*}, D^{y^*}) is the unique optimal policy to the problem (5) and the optimal value function $V(x)$ is as in (12).*

Proof Let V^* be the optimal value of the problem (5) and let V be as in (12). Since V is obtained with an admissible control $(U_t^{z^*}, D_t^{y^*})$, we know that $V^* \geq V$. The following properties will be proved to be sufficient for the opposite inequality:

(i) $V \in C^2$;
(ii) $(\mathcal{A} - r)V(x) + \pi(x) \leq 0$ for all $x \in \mathbb{R}_+$;
(iii) $p \leq V'(x) \leq q$ for all $x \in \mathbb{R}_+$.

Let us show that V satisfies these. Firstly the case (i) is valid, since (z^*, y^*) was chosen so that V is twice continuously differentiable. To show that (ii) hold, we get by straight differentiation that

$$(\mathcal{A}-r)V(x)+\pi(x)=\begin{cases}\rho_p(x)-\rho_p(y^*) & \text{if } x\geq y^*,\\ 0 & \text{if } x\in(z^*,y^*),\\ \rho_q(x)-\rho_q(z^*) & \text{if } x\leq z^*.\end{cases}$$

Here the first and the last expressions are non-positive due to Assumption 3.2(iv) and Lemma 4.1, and thus the case (ii) follows. The case (iii) is obtained as soon as we notice that combining the concavity of V from Lemma 4.3(A) with the fact that $V'(z^*+)=q>p=V'(y^*-)$ yields $p\leq V'(x)\leq q$ for $z^*\leq x\leq y^*$ and that $V'(x)=p$ for $x>y^*$ and $V'(x)=q$ for $x<z^*$.

To show that these three properties imply $V\geq V^*$, let (U_t,D_t) be an arbitrary admissible control, fix $T<\infty$ and define

$$U_t^c=U_t-\sum_{0<s\leq t}\Delta U_s \quad\text{and}\quad D_t^c=D_t-\sum_{0<s\leq t}\Delta D_s,$$

where $\Delta U_s=U_s-U_{s-}$ so that U_t^c and D_t^c are the continuous parts of U_t and D_t respectively. Letting $\tau_T=T\wedge\zeta_Z$, which is an almost surely finite stopping time, we apply generalised Ito's lemma to the function $e^{-r\tau_T}V(Z_{\tau_T})$ to get

$$\begin{aligned}E_x\left[e^{-r\tau_T}V(Z_{\tau_T})\right]=V(x)&+E_x\left[\int_0^{\tau_T}e^{-rs}(\mathcal{A}-r)V(Z_s)ds\right]\\ &+E_x\left[\int_0^{\tau_T}e^{-rs}V'(Z_s)(dU_s^c-dD_s^c)\right]\\ &+E_x\left[\sum_{0\leq s\leq\tau_T}e^{-rs}\Delta V(Z_s)\right],\end{aligned}$$

where $\Delta V(Z_s)=V(Z_s)-V(Z_{s-})$.

Let v be the value function corresponding the chosen control (U_t,D_t). Set

$$v_{\tau_T}(x)=E_x\left[\int_0^{\tau_T}e^{-rs}(\pi(Z_s)ds+pdD_s-qdU_s)+e^{-r\tau_T}V(Z_{\tau_T})\right]. \quad (13)$$

This is a compound policy, which follows the arbitrarily chosen policy (U_t,D_t) until time τ_T and thereafter applies the barrier policy (U^{z^*},D^{y^*}) with value function $V(x)$. Using the expression for $E_x\left[e^{-r\tau_T}V(Z_{\tau_T})\right]$ above and utilizing the three properties of the function V above we can calculate that

$$v_{\tau_T}(x)=V(x)+E_x\left[\int_0^{\tau_T}e^{-rs}\left((\mathcal{A}-r)V(Z_s)+\pi(Z_s)\right)ds\right]$$

$$
\begin{aligned}
&+E_x\left[\int_0^{\tau_T} e^{-rs}(V'(Z_s)-q)dU_s^c\right] \\
&+E_x\left[\int_0^{\tau_T} e^{-rs}(p-V'(Z_s))dD_s^c\right]+E_x\left[\sum_{0\le s\le\tau_T}\Delta V(Z_s)-q\Delta U_s+p\Delta D_s\right] \\
&\le V(x)+E_x\left[\sum_{0\le s\le\tau_T}\Delta V(Z_s)-q\Delta U_s+p\Delta D_s\right].
\end{aligned}
$$

Here the last sum is non-positive: assume that $\Delta U_s > 0$ and $\Delta D_s = 0$. Then $\Delta Z_s = \Delta U_s$ and

$$
\begin{aligned}
&\Delta V(Z_s)-q\Delta U_s+p\Delta D_s \\
&\quad = V(Z_s)-V(Z_s-\Delta U_s)-q\Delta U_s \le q\Delta U_s-q\Delta U_s=0,
\end{aligned}
$$

where the inequality follows from the fact that $V'(x) \le q$ for all $x > 0$. Similar arguments apply to the case, where $\Delta U_s = 0$ and $\Delta D_s > 0$ as well as to the case $\Delta U_s > 0$ and $\Delta D_s > 0$. In every case $v_{\tau_T}(x) \le V(x)$. As $V(x)$ is bounded from below, $\lim_{T\to\infty} e^{-rT}V(Z_T) \ge 0$. Letting $T \to \infty$ in (13) we see that $v(x) \le v_{\tau_T}(x) \le V(x)$ for all admissible policies (U_t, D_t). Therefore also $V^* \le V$. Lastly, the uniqueness follows from Proposition 4.2.

The argument in the proof has been used for example in Harrison (1985, Chapter 6), where it is called a policy improvement logic. The theorem itself confirms that if we have already found a solution satisfying the first order optimality conditions (9), then fairly weak conditions ensure it to be unique and the corresponding control to be optimal for the problem (5) and the value function can be written explicitly as in (12). All in all, this is a pleasant result for the applications, since often if a solution to the necessary conditions (9) exists, it can be found numerically without too much difficulty.

Moreover we have seen in Lemma 4.3 that under Assumption 3.2 the marginal value $V'(x)$ is positive but diminishing everywhere. This generalises the known result from one-sided control, e.g. (Alvarez, 2001, Theorem 5), to two-sided ones.

A connection to the Dynkin game is also worth mentioning. There is a strong connection between one-sided singular control and optimal stopping, which is known already from the pioneering work (Bather and Chernoff 1966). It says that a derivative of the value function of a one-sided control problem constitutes the value function of an associated optimal stopping problem, see also Karatzas and Shreve (1984) and Karatzas (1985b) and Alvarez (2001). The two-sided control problem, like ours, is in turn known to have a similar connection with an associated two-player optimal stopping game known as a Dynkin game, see for example Karatzas and Wang (2001) and Boetius (2005).

5 Sufficient conditions for the solution

5.1 Assumptions and auxiliary results

Although one could try to find numerically the solution to the necessary conditions (9), we are nevertheless in a state of uncertainty whether there does exist a solution or not. To make things clearer, in this section we shall provide a set of sufficient conditions under which there exists a unique pair (z^*, y^*) satisfying the first order optimality conditions (9). These conditions are summarised in the following.

Assumption 5.1 Assume that Assumption 3.2 hold, that the boundaries 0 and ∞ are natural and in addition that for $b = p, q$

(v) $\rho_b(\infty) = -\infty$ and that $\rho_b'(0+) > 0$
(vi) $\lim_{x\downarrow 0} - \int_x^{\tilde{x}_b} \varphi'(t)/S'(t)dt = \infty$.

Basically all these additional assumptions aim to dictate the boundary behaviour of the auxiliary functions I and J, so that we can be sure they intersect each other. Of these assumptions, especially (vi) seems a bit bizarre and hard to verify, but it has a clear interpretation; the assumption that 0 is natural means that it is also not-entrance, implying that the scale derivative $-\varphi'(x)/S'(x)$ approaches infinity as x tends to zero. Now, Assumption (vi) requires the scale derivative to be even steeper at zero, namely that also the integral $-\int_x^{\tilde{x}_b} \varphi'(t)/S'(t)dt$ approaches infinity as x tends to zero. So loosely speaking one could say that Assumption (vi) makes zero even more forbidden entrance than the naturality assumption of the boundary. Since this assumption can be troublesome to verify, we shall give in Lemma 5.2 below two different conditions which imply the assumption. Before that we need to introduce the associated diffusion

$$d\hat{X}_t = (\mu(\hat{X}_t) + \sigma'(\hat{X}_t)\sigma(\hat{X}_t))dt + \sigma(\hat{X}_t)dW_t,$$

with killing rate $r - \mu'(x)$. (Its infinitesimal generator $\hat{\mathcal{A}} - \big(r - \mu'(x)\big) = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \big(\mu(x) + \sigma'(x)\sigma(x)\big)\frac{d}{dx} - \big(r - \mu'(x)\big)$ is got by differentiating the generator $\mathcal{A} - r$.)

Lemma 5.2 *Assume that either*

(A) $\mu'(x) < r$ *for all* $x \geq 0$, μ *is concave near zero, and the boundary* 0 *is not-entrance for the associated diffusion* $\hat{X}_t$*; or*
(B) $\psi'(0) = 0$ *and* $(R_r\mathrm{id})'(0+) > 0$.

Then $\lim_{x\downarrow 0} - \int_x^{\tilde{x}_b} \varphi'(t)/S'(t)dt = \infty$ *for* $b = p, q$.

Proof See Appendix A.3. □

In previous lemma the condition (A) can be checked from initial functions, while condition (B) can be convenient, if ψ and $(R_r\mathrm{id})$ can be calculated explicitly. Moving on, in the following lemma we see that I and J from (10) can be written in a tidy integral form.

Lemma 5.3 *Let Assumption 5.1 hold. Then, for $b = p, q$, the functions I and J from (10) can be written as*

$$\begin{cases} J_b(x) = -\dfrac{1}{B}\left(\displaystyle\int_x^\infty \varphi_t(\rho_b(x) - \rho_b(t))m_t' dt\right) \\ I_b(x) = \dfrac{1}{B}\left(\displaystyle\int_0^x \psi_t(\rho_b(x) - \rho_b(t))m_t' dt\right) \end{cases}$$

Proof See Appendix A.4 □

We have previously proved (Lemma 3.3) that these auxiliary functions satisfy certain monotonicity properties, which were adequate for the uniqueness of a solution. But for the existence we also need to know something about their boundary behaviour.

Lemma 5.4 *Let Assumption 5.1 hold. Then*

(A) for $b = p, q$, $J_b(0+) = \infty$ and $J_b(\infty) \le 0$.
(B) for $b = p, q$, $I_b(0+) \ge 0$ and $I_b(\infty) < 0$.

Proof See Appendix A.5. □

5.2 Proving the existence of (z^*, y^*)

We already know from Lemma 4.1 that if there exists a pair (z^*, y^*) satisfying the condition (9), then it must be in the set $(0, \tilde{x}_q) \times (\tilde{x}_p, \infty)$. Now with stricter assumptions, we can shrink this acceptable set into a bounded set.

Lemma 5.5 *Let Assumption 5.1 hold. Assume further that the necessary conditions (9) have a solution (z^*, y^*). Then $(z^*, y^*) \in (x_q^J, \tilde{x}_q) \times (\tilde{x}_p, x_p^I)$, where $x_q^J, x_p^I \in \mathbb{R}_+$ are the unique interior points for which $J_q(x_q^J) = 0$ and $I_p(x_p^I) = 0$ and $\tilde{x}_q, \tilde{x}_p$ are as in Assumption 3.2(iv).*

Proof The proof follows that of Theorem 4.3 in Alvarez (2008). From Lemma 5.4 we get $J_b(0+) > 0$ and $J_b(\infty) \le 0$ for $b = p, q$. Combining these facts with the monotonicity properties (Lemma 3.3) we see that there must exist a unique $x_b^J < \tilde{x}_b$ such that $J_b(x) \gtreqless 0$ for all $x \lesseqgtr x_b^J$. Especially we see that $J_b(x) < 0$ for all $x > \tilde{x}_b$. Analogously we see that there exists a unique $x_b^I > \tilde{x}_b$ such that $I_b(x) \gtreqless 0$ for all $x \lesseqgtr x_b^I$, and especially that $I_b(x) > 0$ for all $x < \tilde{x}_b$.

To prove the new lower boundary for z^*, we notice first that by Lemma 4.1 we have $y^* > \tilde{x}_p$, and thus, since z^* satisfies (9), using the sign results above we get $J_q(z^*) = J_p(y^*) < 0$. Moreover utilizing the sign results above once more we get $z^* > x_q^J$. The new upper boundary for y^* follows similarly.

So the possible region for optimal thresholds is narrowed to a compact region. This information is useful in next theorem, which is our main result on the solvability of the necessary conditions (9).

Theorem 5.6 *Let Assumption 5.1 hold. Then there exists a unique pair* (z^*, y^*) *satisfying the first order optimality conditions* (9).

Proof As in proof of Proposition 4.2, define a function $K : [x_q^J, \tilde{x}_q] \to [x_q^J, \tilde{x}_q]$ by $K(x) = \big(\hat{J}_q^{-1} \circ \hat{J}_p \circ \hat{I}_p^{-1} \circ \hat{I}_q\big)(x)$, where $\hat{J}_q = J_q|_{(0,\tilde{x}_q]}$, $\hat{J}_p = J_p|_{[\tilde{x}_p,\infty)}$, $\hat{I}_q = I_q|_{(0,\tilde{x}_q]}$ and $\hat{I}_p = I_p|_{[\tilde{x}_p,\infty)}$. As before, we notice that K is increasing. Notice that now the domain of K is different.

To ensure that K is well defined, we will show that the endpoints $x_q^J, \tilde{x}_q$ are mapped into the domain of K. Firstly $0 < x_q^J < \tilde{x}_q$, and so $I_q(x_q^J) > 0$. Since $I_p(x) > I_q(x)$ for all $x \in \mathbb{R}_+$, there exists a point $s_1 \in (\tilde{x}_p, x_p^I)$ such that $I_p(s_1) = I_q(x_q^J)$. Moreover, $J_p(s_1) < 0$ and since $J_p(x) > J_q(x)$ for all $x \in \mathbb{R}_+$, there exists a point $s_2 \in (x_q^J, \tilde{x}_q)$ such that $J_p(s_1) = J_q(s_2)$, so especially $K(x_q^J) = s_2 \in (x_q^J, \tilde{x}_q)$. For the upper endpoint, since $I_p(x) > I_q(x)$ for all $x \in \mathbb{R}_+$, we know that there exists $t_1 \in (\tilde{x}_p, x_p^I)$ such that $I_p(t_1) = I_q(\tilde{x}_q)$. Reasoning as above, we get that there exists $t_2 \in (x_q^J, \tilde{x}_q)$ such that $J_p(t_1) = J_q(t_2)$ so in particular $K(\tilde{x}_q) = t_2 \in (x_q^J, \tilde{x}_q)$ and K is well defined.

Let us define a sequence $z_n = K^n(x_q^J)(= (K \circ \cdots \circ K)(x_q^J))$. This sequence converges by induction: It is clear that $z_1 = K(z_0) > z_0$. Because K is an increasing function, we have $K\big(K(z_0)\big) > K(z_0)$. By induction $K^n(z_0) > K^{n-1}(z_0)$. Since the sequence z_n is increasing and bounded from above, it converges.

Writing $z^* = \lim_{n\to\infty} z_n$, we see that z^* is the fixed point of the function K. Defining $y^* = \hat{J}_p^{-1}(\hat{J}_q(z^*))(= \hat{I}_p^{-1}(\hat{I}_q(z^*)))$, we get a pair (z^*, y^*) that satisfies the necessary conditions (9). The uniqueness of such a pair follows directly from Proposition 4.2. □

In the previous theorem we saw that under Assumption 5.1 the unique pair (z^*, y^*) satisfying (9) always exists. Furthermore we saw how it can be found when it is identified as a fixed point. Analogous fixed point argument is used also in Alvarez and Lempa (2008) in an impulse control situation and in Lempa (2010) in a traditional optimal stopping situation. Theorem 5.6 also shows how we can find the pair (z^*, y^*) numerically. First we identify the point x_q^J. After that, we apply the function $K(x) = \big(\hat{J}_q^{-1} \circ \hat{J}_p \circ \hat{I}_p^{-1} \circ \hat{I}_q\big)(x)$ to that point (actually any point in $(0, x_q^J]$ will do) and calculate $K^k(x_q^J)$, where we might for example set a stopping limit $\varepsilon > 0$ and stop at the first step step k, for which $|K^k(x_q^J) - K^{k-1}(x_q^J)| < \varepsilon$. After this we have $z^* \approx K^k(x_q^J)$ and $y^* \approx J_2^{-1}(J_1(K^k(x_q^J)))$.

6 Comparative analysis

Let us next study the sensitiveness of the value function and the optimal barriers, firstly and most importantly with respect to the volatility, and secondly with respect to the control parameters. We shall also compare the differences between the solutions of two-sided and one-sided control problems.

6.1 Volatility sensitiveness

Our main results on the effect of the increased volatility are summarised in the following.

Theorem 6.1 *Let Assumption 3.2 hold and let* (z^*, y^*) *be a solution to* (9). *Then*

(A) $V(x)$ *is non-increasing in* σ.
(B) if we assume further that the inequalities concerning ρ_b' *in Assumption 3.2(iv) are strict, the inactivity region* (z^*, y^*) *widens as* σ *increases.*

Proof Let $\hat{\sigma}(x) \geq \sigma(x)$ for all $x \geq 0$ and let $\hat{\mathcal{A}} = \frac{1}{2}\hat{\sigma}^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$ be the infinitesimal generator, $\hat{V}$ the optimal value function and $(\hat{z}^*, \hat{y}^*)$ the optimal inactivity region with respect to the volatility $\hat{\sigma}$.

(A) We have

$$(\hat{\mathcal{A}} - r)V(x) + \pi(x) = \begin{cases} \rho_p(x) - \rho_p(y^*) \leq 0 & \text{if } x \geq y^* \\ \frac{1}{2}\left(\hat{\sigma}^2(x) - \sigma^2(x)\right) V''(x) \leq 0 & \text{if } x \in (z^*, y^*) \\ \rho_q(x) - \rho_q(z^*) \leq 0 & \text{if } x \leq z^*, \end{cases}$$

the first and the last expressions being non-positive due to Assumption 3.2(iv) and the middle expression due to the concavity of V (Lemma 4.3). Hence V satisfies the property (ii) in the proof of Theorem 4.4 with respect to $\hat{\sigma}$, while the properties (i) and (iii) can be handled as previously. Therefore analysis similar to that in the proof of Theorem 4.4 shows that $V \geq \hat{V}$.

(B) Let us first prove the ordering for the lower boundaries. Suppose, contrary to our claim, that $z^* < \hat{z}^*$. Now from the value function expression (12) we see that

$$V(z^*) = \frac{1}{r}\rho_q(z^*) + qz^* < \frac{1}{r}\rho_q(\hat{z}^*) + qz^* = \hat{V}(z^*) - q(z^* - \hat{z}^*) + q(z^* - \hat{z}^*) = \hat{V}(z^*),$$

where the inequality follow from strict inequality in Assumption 3.2(iv). This contradicts the fact $V \geq \hat{V}$ (item (A)). The same reasoning applies also to the case $y^* < \hat{y}^*$. □

According to our theorem, increased volatility affects negatively both the optimal policy and its value. Put differently, our theorem shows that increased volatility expands the inactivity region and postpones the usage of singular policies by decreasing the marginal value of the optimal policy. This result generalises previous findings based on one-sided policies (e.g. Theorem 6 in Alvarez 2001) to a two-sided setting.

6.2 Comparing the two-sided and one-sided solutions

It is also of interest to study the relationship between two-sided and one-sided controls. Obviously, since not using a control is an admissible control, the optimal value function is greater in the two-sided case. But are the reflected barriers from these two problems ordered consistently, and if so, how? To this end let Assumption 5.1 hold and let (z^*, y^*) be the optimal reflecting barriers in two-sided control problem.

Consider first the case where the dynamics are controlled only downwards, so that $Z = X - D$. In that case the value reads as $\sup_D \mathbb{E}_x \int_0^{\zeta_Z} e^{-rs}\big(\pi(Z_s)ds + pdD_s\big)$. Under Assumption 5.1 this one-sided control problem is known to have solution (actually, weaker assumptions are sufficient, see Lemma 3.4 in Alvarez and Lempa 2008) and the optimal control is reflecting control with the reflecting barrier at x_p^I (the unique point for which $I_p(x_p^I) = 0$, cf. Lemma 5.5), and we know from Lemma 5.5 that $y^* < x_p^I$. So, in the harvest example, in the absence of a replanting opportunity we harvest later.

Similarly, consider the case where the dynamics are controlled only upwards, so that $Z = X + U$. In this case the value reads as $\sup_U \mathbb{E}_x \int_0^{\zeta_Z} e^{-rs}\big(\pi(Z_s)ds - qdU_s\big)$. Going through the reasoning in Alvarez and Lempa (2008), one could verify that under Assumption 5.1 this one-sided control problem has a solution, where the optimal control is a reflecting control with the reflecting barrier at x_q^J (the unique point for which $J_q(x_q^J) = 0$, cf. Lemma 5.5), and from Lemma 5.5 we know that $z^* > x_q^J$. Now, in a dividend payments problem with obligative reinvestment example, in the absence of dividend payments we reinvest later.

6.3 Sensitiveness on control parameters

Next we shall consider the sensitiveness with respect to the control parameters p and q in the following two propositions.

Proposition 6.2 *Let Assumption 5.1 hold. Then*

(A) $V(x)$ is p-increasing and q-decreasing.
(B) the inactivity region (z^*, y^*) shrinks as p increases and widens as q increases.

Proof Fix $p_1 < p_2 (< q)$ and let $V_i(x) := V(x; p_i)$ and (z_i^*, y_i^*) be the value function and optimal reflecting barriers, respectively, with respect to p_i.

(A) We see that

$$\begin{aligned} V_1(x) &= \mathbb{E}_x\left[\int_0^\infty e^{-rt}(\pi(Z_t)dt + p_1 dD_t^{y_1^*} - qdU_t^{z_1^*})\right] \\ &\le \mathbb{E}_x\left[\int_0^\infty e^{-rt}(\pi(Z_t)dt + p_2 dD_t^{y_1^*} - qdU_t^{z_1^*})\right] \\ &\le \sup_{(D,U)} \mathbb{E}_x\left[\int_0^\infty e^{-rt}(\pi(Z_t)dt + p_2 dD_t - qdU_t)\right] = V_2(x). \end{aligned}$$

Proving that $V(x; q_2) \le V(x; q_1)$ for all $q_2 > q_1$ is analogous.

(B) Let us first study the sensitiveness with respect to p. Fix again $p_1 < p_2 (< q)$ and let (z_i^*, y_i^*) be the optimal reflecting barriers with respect to p_i. Furthermore let $K_i(x) = \left(\hat{J}_q^{-1} \circ \hat{J}_{p_i} \circ \hat{I}_{p_i}^{-1} \circ \hat{I}_q\right)(x)$, for $i = 1, 2$, be as in Theorem 5.6. Since $\psi'', \varphi'' > 0$ by Lemma 3.3(A) we can use the expression (10) to obtain inequalities $I_{p_2}(x) < I_{p_1}(x)$ and $J_{p_2}(x) < J_{p_1}(x)$. Combining these with the monotonicity properties of $\hat{J}$ and $\hat{I}$ yields

$$\begin{aligned}
&\hat{I}_{p_1}^{-1}(\hat{I}_q(z_1^*)) > \hat{I}_{p_2}^{-1}(\hat{I}_q(z_1^*)) \\
&\quad\Longrightarrow \hat{J}_{p_1}(\hat{I}_{p_1}^{-1}(\hat{I}_q(z_1^*))) > \hat{J}_{p_1}(\hat{I}_{p_2}^{-1}(\hat{I}_q(z_1^*))) > \hat{J}_{p_2}(\hat{I}_{p_2}^{-1}(\hat{I}_q(z_1^*))) \\
&\quad\Longrightarrow \hat{J}_q^{-1}(\hat{J}_{p_1}(\hat{I}_{p_1}^{-1}(\hat{I}_q(z_1^*)))) < \hat{J}_q^{-1}(\hat{J}_{p_2}(\hat{I}_{p_2}^{-1}(\hat{I}_q(z_1^*)))),
\end{aligned}$$

where all the inequalities are strict. In other words $K_2(z_1^*) > K_1(z_1^*) = z_1^*$. Now proceeding as in the proof of Theorem 5.6, we can deduce that as a limit of an increasing sequence $z_2^* = \lim_{n\to\infty} K_2^n(z_1^*)\left(= (K_2 \circ \cdots \circ K_2)(z_1^*)\right) > z_1^*$. Moreover by the monotonicity of functions $\hat{I}$ we get

$$y_1^* = \hat{I}_{p_1}^{-1}(\hat{I}_q(z_1^*)) > \hat{I}_{p_2}^{-1}(\hat{I}_q(z_1^*)) > \hat{I}_{p_2}^{-1}(\hat{I}_q(z_2^*)) = y_2^*.$$

Let us then consider the sensitiveness with respect to q. Same arguments as above with slight changes applies to this case. Fix $q_2 > q_1$ and let (z_i^*, y_i^*) be the optimal continuation region with respect to q_i. Now we need to define functions $H_i : [\tilde{x}_p, x_p^I) \to [\tilde{x}_p, x_p^I)$ (for the definitions of $\tilde{x}_p$ and x_p^I see Lemma 5.5) as $H_i(y) = (\hat{I}_p^{-1} \circ \hat{I}_{q_i} \circ \hat{J}_{q_i}^{-1} \circ \hat{J}_p)(y)$, for $i = 1, 2$, so that $H_1(y_1^*) = y_1^*$. Reasoning as above we can deduce that $H_2(y_1^*) > y_1^*$. Now $H_2^n(y_1^*)$ is an bounded increasing sequence and therefore $y_2^* = \lim_{n\to\infty} H_2^n(y_1^*) > y_1^*$. Lastly by monotonicity of functions $\hat{J}$ we get

$$z_1^* = \hat{J}_{q_1}^{-1}(\hat{J}_p(y_1^*)) > \hat{J}_{q_2}^{-1}(\hat{J}_p(y_1^*)) > \hat{J}_{q_2}^{-1}(\hat{J}_p(y_2^*)) = z_2^*.$$

□

Proposition 6.2 verifies intuitively clear facts: increasing the income (p) from using an upper barrier, the value is understandably also increasing and the controller is encouraged to use the controls, thus the inactivity region is narrowing. The contrary is true when the cost q of using control at the lower barrier is increased.

Subsequent questions are the limiting properties, which are considered in the following.

Proposition 6.3 *Let Assumption 5.1 hold. Then*

(A) $z^* \searrow 0$ *and* $y^* \nearrow x_p^I$ *as* $q \nearrow \infty$;
(B) *if in addition* π *is increasing, we have* $z^* \searrow x_q^J$ *and* $y^* \nearrow \infty$ *as* $p \searrow 0$
(C) *the inactivity region* (z^*, y^*) *shrinks arbitrary small as* $q - p \searrow 0$. *Moreover* $\tilde{x}_p - \tilde{x}_q \searrow 0$ *and* $z^* \nearrow \tilde{x}_q$, $y^* \searrow \tilde{x}_q$ *and the value function approaches, from below, a function* $q(x - \tilde{x}_q) + \frac{1}{r}\left(q\mu(\tilde{x}_q) + \pi(\tilde{x}_q)\right)$.

(D) if in addition π is increasing, we have $(z^*, y^*) \searrow (0,0)$ as $q, p \to \infty$ and $(z^*, y^*) \nearrow (\infty, \infty)$ as $q, p \to 0$.

Proof (A) Let $q_1 < q_2$. Then

$$\rho'_{q_1}(x) = \pi'(x) + q_1(\mu'(x) - r) > \pi'(x) + q_2(\mu'(x) - r) = \rho'_{q_2}(x).$$

Therefore we can deduce that $\tilde{x}_{q_1} > \tilde{x}_{q_2}$. Now $\rho'_\infty(x) = -\infty$ for all $x > 0$, and so $\lim_{q\to\infty} \tilde{x}_q = 0$. By Lemma 4.1 we know that $z^* < \tilde{x}_q$, and therefore we can conclude that $z^* \to 0$ as $q \to \infty$.

Since zero was assumed to be natural, the process never reaches the state 0, and it follows that $U_t^{z^*} = U_t^0 \equiv 0$ as $q \to \infty$. And so, when $q \to \infty$, the problem reduces to

$$\sup_D \mathbb{E}_x \int_0^{\zeta_Z} e^{-rs} \left(\pi(Z_s)ds + p dD_s\right).$$

But this is the one-sided control problem introduced in Sect. 6.2, and its optimal policy is known to be $D_t^{x_p^I}$ (see Lemma 3.4 in Alvarez and Lempa (2008)), and so $\lim_{q\to\infty} y^* = x_p^I$. Moreover, from Lemma 5.5 we know that $y^* < x_p^I$ for all q, and so the convergence must be from below.

(B) Let $0 < p_1 < p_2$. Then

$$\rho'_{p_2}(x) = \pi'(x) + p_2(\mu'(x) - r) < \pi'(x) + p_1(\mu'(x) - r) = \rho'_{p_1}(x).$$

Therefore we can deduce that $\tilde{x}_{p_1} > \tilde{x}_{p_2}$, and this holds for all $\pi(x)$. Now if π is increasing, then $\rho'_0(x) = \pi'(x) \geq 0$ for all $x > 0$, and so $\lim_{p\to 0} \tilde{x}_p = \infty$. And since $y^* > \tilde{x}_p$ (by Lemma 5.5), the rest of the reasoning is similar to the one in (A).

(C) First of all, Proposition 6.2(B) implies that the inactivity region (z^*, y^*) shrinks as $q - p \searrow 0$. Moreover, above we saw that $\tilde{x}_q$ is decreasing in q and $\tilde{x}_p$ is increasing in p. Furthermore, since $\rho_b(x)$ is b-continuous, it is clear that as $q - p \searrow 0$, we get in fact $\tilde{x}_p - \tilde{x}_q \searrow 0$ ($\tilde{x}_p \geq \tilde{x}_q$ always by Lemma 4.1).

Without lost of generality, we from now on fix q and let p approach q. For all $p < q$ we know from Theorem 5.6 that there exist $z^*(p) < \tilde{x}_q$ and $y^*(p) > \tilde{x}_q$. Further, $z^*(p)$ is p-increasing and $y^*(p)$ is p-decreasing by Proposition 6.2(B). Moreover from the proof of Proposition 6.2(B) we see that $z^*(p)$ is p-continuous, since the functions I_b, J_b, I_b^{-1}, J_b^{-1}, for $b = p, q$, are. Similarly also $y^*(p)$ is p-continuous.

It follows that there exist $Z^* = \lim_{p\nearrow q} z^*(p)$ and $Y^* = \lim_{p\nearrow q} y^*(p)$, which satisfy the fixed point properties in the proof of Theorem 5.6 at the limit $p \nearrow q$; i.e. properties $K(Z^*) = Z^*$, $Y^* = (I_q^{-1}|_{[\tilde{x}_q,\infty)} \circ I_q|_{[x_q^J,\tilde{x}_q]})(Z^*)$ and $K'(Z^*) < 1$. But now since the pair $(\tilde{x}_q, \tilde{x}_q)$ also satisfies these properties at the limit $p \nearrow q$, and the fixed point is unique, we must have $(Z^*, Y^*) = (\tilde{x}_q, \tilde{x}_q)$.

The value $V(x)$ is p-increasing by Proposition 6.2(A). Moreover, from the value function expression (12), we see that since $z^*, y^* \to \tilde{x}_q$ as $p \to q$, we have limit $\lim_{p\to q} V(x) = q(x - \tilde{x}_q) + \frac{1}{r}\left(q\mu(\tilde{x}_q) + \pi(\tilde{x}_q)\right)$ for $x \geq y^*$ and $x \leq z^*$. And since $z^* - y^* \to 0$, as $p \to q$, this expression holds everywhere.

(D) Consider first the case $q, p \to \infty$. We have already shown that $z^* \searrow 0$ as $q \to \infty$, so we are left to prove that $y^* \searrow 0$ as $p \to \infty$. Since $y^* < x_p^I$ (by Lemma 5.5), it is sufficient to show that $\lim_{p\to\infty} x_p^I = 0$. Now

$$I_p(x) = \frac{1}{B}\int_0^x \psi_t(\rho_p(x) - \rho_p(t))m_t' dt,$$

and since $\lim_{p\to\infty} \tilde{x}_p = 0$, we know that $\rho_p(x) - \rho_p(t) < 0$ for all $t < x$ as $p \to \infty$. Hence $I_p(x) < 0$ for all $x > 0$ at the limit $p \to \infty$. Consequently $x_p^I \to 0$.

Let us then turn to the case $q, p \to 0$. We already know that $\lim_{p\to 0} y^* = \infty$, and thus it remains to prove that $\lim_{q\to 0} z^* = \infty$. Since $z^* > x_q^J$ (by Lemma 5.5), it is sufficient to show that $\lim_{q\to 0} x_q^J = \infty$. Now

$$J_q(x) = -\frac{1}{B}\int_x^\infty \varphi_t(\rho_q(x) - \rho_q(t))m_t' dt,$$

and since $\lim_{q\to 0} \tilde{x}_q \to \infty$, we know that $\rho_q(x) - \rho_q(t) < 0$ for all $t > x$ as $q \to 0$. Hence $J_q(x) > 0$ for all $x > 0$ at the limit $q \to 0$, and consequently $x_p^J \to \infty$.

In Proposition 6.3(A)–(B) we see that at the limits $q \to \infty$ and $p \to 0$ we get the solutions of the associated one-sided control problems (cf. Sect. 6.2), so that the theory presented in this paper can be seen as a natural generalisation of the one-sided problem. Moreover, we see that the upper boundary x_p^I is approached from below and the lower boundary x_q^J from above. It is also worth stressing that in Proposition 6.3(B) the requirement that π is increasing is necessary; we shall see an example in Sect. 7.2 where a concave revenue function π enables the upper threshold y^* to be finite even with negative values of p.

From case (C) we see that as p and q approach each other, the inactivity region (z^*, y^*) becomes arbitrarily small. Noteworthy is that, although technically at the limit $p \nearrow q$ we get reflecting barriers $(z^*, y^*) = (\tilde{x}_q, \tilde{x}_q)$, the corresponding pair of controls $(U^{\tilde{x}_q}, D^{\tilde{x}_q})$ are no longer admissible policies.

In the last case (D) we see that when both control parameters are set to the same limit, either 0 or ∞, we, respectively, either raise both of the thresholds z^* and y^* up toward infinity, or lower them down toward zero. Noteworthy is that in the limit neither the control U^∞ nor D^0 are admissible, since they usher the diffusion to the state ∞ or 0, respectively, which are not in the state space.

6.4 Stationary distribution

The controlled process $Z_t = X_t + U_t^{z^*} - D_t^{y^*}$ is well defined on the finite interval $[z^*, y^*]$, and so it follows that $M := m(z^*, y^*) = \int_{z^*}^{y^*} m'(u)du < \infty$. Moreover, since the boundaries of the controlled process are reflecting, we can define a stationary probability distribution for controlled process Z_t as $\eta(x) := m'(x)/M$. Now, for every Borel-measurable bounded function $f : [z^*, y^*] \to \mathbb{R}$ we have (see Borodin and Salminen 2002, p. 37)

$$\lim_{t\to\infty} \mathbb{E}_x [f(Z_t)] = \int_{x^*}^{y^*} f(u)\eta(u)du.$$

7 Examples

7.1 Geometric Brownian motion

To illustrate our results explicitly, assume that the underlying uncontrolled diffusion evolves as geometric Brownian motion, i.e.

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t),$$

where $\sigma \in \mathbb{R}_+$, $\mu \in (-\infty, r)$ are exogenously given constants. Furthermore, assume that the revenue flow is $\pi(x) = x^a - c$, with $a \in (0, 1)$ and $c \in \mathbb{R}$, so that

$$(R_r\pi)(x) = \frac{x^a}{r + \frac{1}{2}\sigma^2\left(a - a^2\right) - a\mu} - \frac{c}{r}.$$

It is worth mentioning that with linear payoff function ($a = 1$), there would not emerge a two-sided reflecting barrier as an optimal rule due to invalidity of Assumption 3.2(iv). Furthermore let us still assume that $q > p$.

With geometric Brownian motion our fundamental solutions of the ordinary differential equation $(\mathcal{A} - r)u = 0$ are $\psi(x) = x^{\gamma^+}$ and $\varphi(x) = x^{\gamma^-}$, where

$$\gamma^\pm = \frac{1}{\sigma^2}\left(\frac{1}{2}\sigma^2 - \mu \pm \sqrt{(\tfrac{1}{2}\sigma^2 - \mu)^2 + 2\sigma^2 r}\right) \tag{14}$$

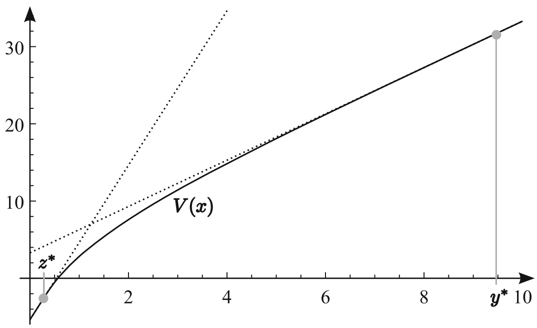
are the solutions of the characteristic equation $\frac{1}{2}\sigma^2\gamma(\gamma - 1) + \mu\gamma - r = 0$. Especially we see that $\gamma^+ > 1$ since $\mu < r$.

7.1.1 Solution to the problem

Let us check that this setup satisfies Assumption 5.1. Now the boundaries are natural and Assumption (i) is already assumed to hold, and clearly conditions in (ii) are satisfied. Furthermore, we assumed $\mu < r$ and so (iii) holds. By straight differentiation $\rho_b'(x) = ax^{a-1} + b(\mu - r)$, which satisfies assumption (iv) since $0 < a < 1$. Furthermore $\rho_b'(0+) = \infty$ and $\rho_b(\infty) = -\infty$, thus (v) is valid. Lastly $\psi'(0) = 0$ since $\gamma^+ > 1$, and $(R_r\mathrm{id})(x) = x/(r - \mu)$ so that $(R_r\mathrm{id})'(0) > 0$ and therefore we can conclude by Lemma 5.2 that also Assumption (vi) is valid.

Hence the results from Sect. 5 can be applied, so especially the optimal solution to (5) is a two-sided reflected control. The optimal reflecting barriers (z^*, y^*) are the

Fig. 1 Optimal value function for a control problem, *dashed lines* are tangents at the points z^* and y^*



unique solution to the necessary conditions (9), which can now be written as

$$\begin{cases} z^{-\gamma^-}\left[2az^a(a-\gamma^-)+Aqz(\gamma^- - 1)\right] = y^{-\gamma^+}\left[2ay^a(a-\gamma^-)+Apy(\gamma^- - 1)\right] \\ z^{-\gamma^-}\left[2az^a(a-\gamma^+)+Aqz(\gamma^+ - 1)\right] = y^{-\gamma^-}\left[2ay^a(a-\gamma^+)+Apy(\gamma^+ - 1)\right], \end{cases}$$

where $A = 2r + a(\sigma^2(1-a) - 2\mu)$. Unfortunately this seems impossible to solve explicitly, but we shall illustrate the optimal barriers numerically below.

With optimal barriers, the value function gets the form

$$V(x) = \begin{cases} p(x-y^*) + \frac{1}{r}\left[p\mu y^* + y^{*a} - b)\right] & x \geq y^*, \\ \frac{x^a}{r+\frac{1}{2}\sigma^2(a-a^2)-a\mu} - \frac{c}{r} - J_q(z^*)x^{\gamma^+} + I_q(z^*)x^{\gamma^-} & z^* < x < y^*, \\ q(x-z^*) + \frac{1}{r}\left[q\mu z^* + z^{*a} - b\right] & x \leq z^*, \end{cases}$$

where J_q and I_q are as in (10).

7.1.2 Numerical illustration

Let us illustrate numerically the results under the parameter configuration $\mu = 0.05$, $\sigma = 0.2$ $r = 0.08$, $a = 1/3$, $c = 1$, $p = 3$ and $q = 10$. With these choices $(z^*, y^*) \approx (0.28, 9.45)$, and the value function is drawn in Fig. 1. As was shown in Lemma 4.3, $V(x)$ is concave.

In Fig. 2 we see how the thresholds are altered, when we change parameter values. By increasing a we increase the payoff function π (for $x > 1$), so that it is sensible that the upper barrier y^* increases. As was proved in Theorem 6.1, higher volatility (σ) leads to a wider inactivity region. Moreover the impact of a change in p and q affects as proved in Propositions 6.2 and 6.3 (now $(x_q^J, \tilde{x}_q, x_p^I) \approx (0.26, 1.17, 9.453)$). What is not seen from those propositions though, is the exceptional rapid widening of the interval (z^*, y^*) with respect to q, when q is near p: With $p = q = 3$, we have $z^* = y^*$, but already with $q = 3.02$, we have $y^* - z^* \approx 3.0$ and with $q = 3.1$, $y^* - z^* \approx 4.8$. Consequently q reaches its upper barrier x_p^I rather quickly. On the other hand, a change in p does not affect the boundaries so strongly. This suggests that the optimal policy is more sensitive with respect to changes in costs than in revenues.

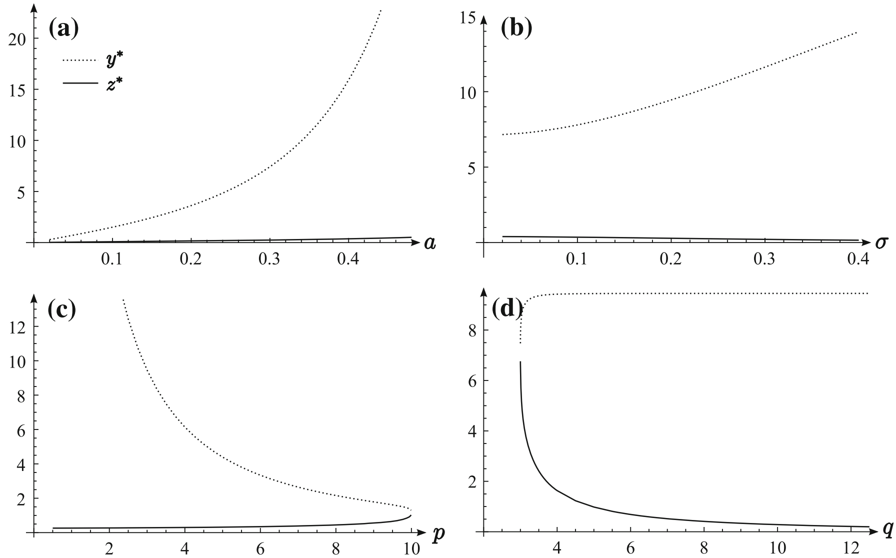


Fig. 2 Sensitivity of the inactivity region with respect to the parameters **a** a; **b** σ; **c** p; **d** q

Furthermore now $m'(x) = \frac{2}{\sigma^2} x^{2\left(\frac{\mu}{\sigma^2}-1\right)}$. Thus, if $\mu \neq \frac{1}{2}\sigma^2$, the stationary probability distribution is

$$\eta(x) = \frac{2\mu - \sigma^2}{\sigma^2 \left(y^{*\frac{2\mu}{\sigma^2}-1} - z^{*\frac{2\mu}{\sigma^2}-1} \right)} x^{2\left(\frac{\mu}{\sigma^2}-1\right)}.$$

Using Sect. 6.4, we can calculate that, with the chosen numerical values, $\lim_{t\to\infty} \mathbb{E}[Z_t] = 5.70$ (the midpoint of the interval (z^*, y^*) is 4.9), and that the variance of the long run stationary state is $\lim_{t\to\infty} \mathrm{Var}(Z_t) = 6.00$. Moreover, choosing $A = [6.4, y^*]$ (the upper third of the interval $[z^*, y^*]$), we get $\lim_{t\to\infty} \mathbb{E}[\mathbf{1}_A(Z_t)] = \lim_{t\to\infty} \mathbb{P}(Z_t \in A) = 0.45$. All this advocates that, in the long run, the controlled process spends more time near the upper threshold y^* than near the lower threshold z^*.

7.2 Mean reverting diffusion

As a slightly more challenging setting, consider that without a control the underlying diffusion X_t follows a mean reverting diffusion:

$$dX_t = \mu X_t(1 - \beta X_t)dt + \sigma X_t dW_t, \quad X_0 = x,$$

where $\mu > 0$ is exogenous constant and $\beta > 0$ is the degree of the mean-reversion and $\sigma > 0$ is the volatility coefficient. In this subsection we shall demonstrate a case, where the "gain" p from downward control can also be negative. An example, where this kind of behaviour might arouse is the following.

Let us consider a house owner who wants to control the inside temperature of her home and dislikes both cold and hot temperature, so that her temperature dependent utility function, represented by π, is a concave function. The house owner can naturally control the temperature of her home either by heating or cooling, by paying a fixed cost q and p for it, respectively. Since both heating and cooling are costly operations, we must have $q > 0 > p$.

To carry on to a more specific analysis, fix $q > 0 > p$ and the utility function $\pi(x) = -x^2 + ax$, where $a > 0$ is an exogenously given constant. Let us next check that this structure satisfies Assumption 5.1(i)–(vi). We notice that Assumption (i) is already assumed and that the smoothness conditions in Assumption (ii) are valid. To see that the integrability Assumption (ii) holds, observe first that by Itô's Lemma

$$X_t^2 = x^2 + \int_0^t 2\left(\sigma^2 + \mu\left(1 - \beta X_s\right)\right) X_s^2 ds + \int_0^t 2\sigma X_s^2 dW_s.$$

It is now straightforward to show that

$$2\left(\sigma^2 + \mu\left(1 - \beta X_s\right)\right) X_s^2 \leq \frac{2(\sigma^2 + \mu)}{3\mu\beta}, \quad \text{and thus} \quad \mathbb{E}_x[X_t^2] \leq x^2 + \frac{2(\sigma^2 + \mu)t}{3\mu\beta}.$$

Thus, it follows that $\mathbb{E}_x \int_0^\infty e^{-rt} X_t^2 dt = \int_0^\infty e^{-rt} \mathbb{E}_x\left[X_t^2\right] dt < \infty$, and consequently $\pi, \mu(x), x \in \mathcal{L}^1$ and assumption (ii) holds.

By straight calculations, Assumption (iii)–(v) hold under the sufficient conditions $\mu < r, q < \frac{a}{r-\mu}$ and $p > -\frac{1}{\mu\beta}$. Finally, since Assumption (iii) is valid and the drift $\mu x(1 - \beta x)$ is concave, the last Assumption (vi) follows from Lemma 5.2 if 0 is non-entrance for the associated diffusion $\hat{X}_t$, which in this case is

$$d\hat{X}_t = (\mu(1 - \beta\hat{X}_t) + \sigma^2)\hat{X}_t dt + \sigma\hat{X}_t dW_t$$

and we observe that 0 is non-entrance for it.

It follows that under the above mentioned conditions, the results from Sect. 5 can be applied. Unfortunately, due to complicated nature of ψ and φ in this case (see Section 6.5 in Dayanik and Karatzas 2003), we cannot solve explicitly any results, but an illustrative numerical solution is seen in Fig. 3.

Furthermore, in this case the speed density is

$$m'(x) = \frac{2}{\sigma^2} x^{2\left(\frac{\mu}{\sigma^2} - 1\right)} e^{-\frac{2\mu\beta}{\sigma^2} x},$$

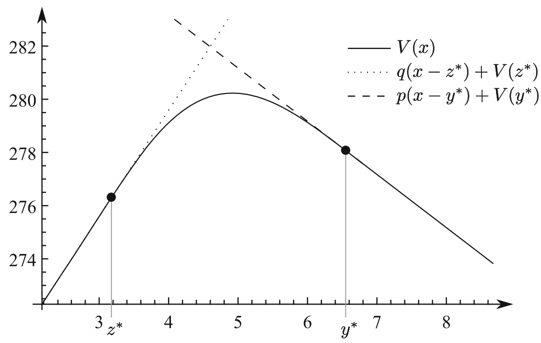


Fig. 3 A numerical illustration of the solution to (5) with the mean reverting set up, introduced above, with the parameter configuration $\mu = 0.04$, $\beta = 0.05$, $\sigma = 0.3$, $r = 0.08$, $a = 10$, $q = 4$ and $p = -2$

and thus the stable stationary distribution on (z^*, y^*) is

$$\eta(x) = \frac{x^{2\left(\frac{\mu}{\sigma^2}-1\right)} e^{-\frac{2\mu\beta}{\sigma^2}x}}{\left(\frac{\sigma^2}{2\mu\beta}\right)^{\frac{2\mu}{\sigma^2}-1}\left(\Gamma(\frac{2\mu}{\sigma^2}-1, \frac{2\mu\beta}{\sigma^2}z^*) - \Gamma(\frac{2\mu}{\sigma^2}-1, \frac{2\mu\beta}{\sigma^2}y^*)\right)},$$

where $\Gamma(s, x) = \int_x^\infty t^{s-a} e^{-t} dt$ is the upper incomplete gamma function.

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Appendix A: Omitted proofs

Firstly, we introduce the following general integral representation result (Corollary 3.2 in Alvarez 2004), which will be referred later on.

Lemma 7.1 *A Assume that* $f \in C^2(\mathbb{R}_+)$, *that* $\lim_{x \to 0+} |f(x)| < \infty$ *and that* $(\mathcal{A} - r) f(x) \in \mathcal{L}^1$. *Then*

$$\frac{f'(x)\psi(x)}{S'(x)} - \frac{\psi'(x) f(x)}{S'(x)} = \int_0^x \psi(t)\big((\mathcal{A} - r) f\big)(t) m'(t) dt - \delta,$$

where $\delta = 0$ *if* 0 *is unattainable and* $\delta = Bf(0)/\varphi(0)$ *if* 0 *is attainable for* X_t.

B *Assume that* $f \in C^2(\mathbb{R}_+)$, *that* $\lim_{x\to\infty} f(x)/\psi(x) = 0$, *and that* $(\mathcal{A}-r)f(x) \in \mathcal{L}^1$. *Then*

$$\frac{f'(x)\varphi(x)}{S'(x)} - \frac{\varphi'(x)f(x)}{S'(x)} = -\int_x^\infty \varphi(t)\big((\mathcal{A}-r)f\big)(t)m'(t)dt$$

Proof of Lemma 3.3

(A) This follows directly from Corollary 1 in Alvarez (2003), if the so called transversality condition $\lim_{t\to\infty} \mathbb{E}_x\left[e^{-rt}X_t; t < \tau_0\right] = 0$ holds. Here $\tau_0 = \inf\{t \geq 0 \mid X_t \notin \mathbb{R}+\}$. But we assumed that $x \in \mathcal{L}^1$, meaning that $\mathbb{E}_x\left[\int_0^\infty e^{-rt}X_t dt\right] < \infty$, and so the transversality condition must hold.

(B) The derivative properties follow from the derivative form (11) by using Assumption 3.2 (iv) together with the facts $\varphi' < 0$ and $\psi' > 0$. Furthermore, from straight calculation we get

$$J_p(x) - J_q(x) = \frac{(q-p)\varphi''(x)\sigma^2(x)}{2rBS'(x)},$$

which is positive due to the fact $q > p$ [Assumption 3.2(i)] and convexity of φ (item (A) of this lemma). Similarly, from straight calculation we get $I_p(x) - I_q(x) = \frac{(q-p)\psi''(x)\sigma^2(x)}{2rBS'(x)}$, which is positive due to the fact $q > p$ [Assumption 3.2(i)] and convexity of ψ (item (A) of this lemma). □

Proof of Lemma 4.3

(A) The function V satisfies the differential equation $(\mathcal{A}-r)V + \pi(x) = 0$ on the interval (z^*, y^*). Differentiating this we obtain

$$\frac{1}{2}\sigma^2(x)V'''(x) = (r - \mu'(x))V'(x) - \big(\mu(x) + \sigma(x)\sigma'(x)\big)V''(x) - \pi'(x).$$

We begin by proving the claim in the case $\mu(x)+\sigma(x)\sigma'(x) \equiv 0$. Since the necessary conditions (9) hold, V is twice continuously differentiable, and $V''(z^*) = V''(y^*) = 0$ and $V'(z^*) = q > p = V'(y^*)$, so

$$\frac{1}{2}\sigma^2(z^*)V'''(z^*) = (r - \mu'(z^*))V'(z^*) - \pi'(z^*) = -\rho_q'(z^*) < 0$$

$$\text{and} \quad \frac{1}{2}\sigma^2(y^*)V'''(y^*) = (r - \mu'(y^*))V'(y^*) - \pi'(y^*) = -\rho_p'(y^*) > 0,$$

where the inequalities follow from the facts that $z^* < \tilde{x}_q$ and $y^* > \tilde{x}_p$ (Lemma 4.1). Therefore $V''(x) \leq 0$ for all x in the neighbourhoods of z^* and y^*. Let $\bar{y} = \sup\{y \in (z^*, y^*) \mid V'''(x) < 0 \text{ for all } z^* < x < y\}$. Then, since $V''(x) < 0$ for all $x < \bar{y}$, we have $V'(x) < q$ for all $x < \bar{y}$. Further, since for all $x \leq \tilde{x}_q$ and

$b < q$ we have $0 < \rho_q'(x) = (\mu'(x) - r)q + \pi'(x) < (\mu'(x) - r)b + \pi'(x)$, and $V'''(x) = -\frac{2}{\sigma^2(x)}\rho_{V'(x)}'(x)$, we must have $\bar{y} > \tilde{x}_q$.

If $V'' \leq 0$ for all $x \in (z^*, y^*)$, then the lemma is proved. So consider for a moment, contrary to our claim, that there exists at least one point for which $V'' > 0$ and let $w_1 < y^*$ be the supremum of such points and let $w_2 < w_1$ be the supremum of the points for which V'' intersects x-axis from below. In other words $V''(w_2) = 0$, $V'''(w_2) > 0$ and $V''(w_1) = 0$, $V'''(w_1) < 0$ and $V'(w_1) > p$. In fact we also have $V'(w_1) \leq q$; If this would not be true, we would have $0 < -\frac{1}{2}\sigma^2(w_1)V'''(w_1) = \rho_{V'(w_1)}'(w_1) < \rho_q'(w_1)$, which contradicts Assumption 3.2(iv), since above we have shown that $\tilde{x}_q < \bar{y} < w_1$.

Since $V''(x) \geq 0$ for all $w_2 < x < w_1$, we have $V'(w_2) < V'(w_1)$. Thus we can calculate that $0 > -\frac{1}{2}\sigma^2(w_2)V'''(w_2) = \rho_{V'(w_2)}(w_2) > \rho_{V'(w_1)}(w_2)$, but above we chose w_1 so that $0 < -\frac{1}{2}\sigma^2(w_1)V'''(w_1) = \rho_{V'(w_1)}'(w_1)$. Since $V'(w_1) \in (p, q)$, this contradicts Assumption 3.2(iv), since $w_2 < w_1$. Therefore we must have $V''(x) \leq 0$ for all $x \in (z^*, y^*)$.

We now turn to the case $\delta(x) := \mu(x) + \sigma(x)\sigma'(x) \not\equiv 0$. Let us introduce a change of variable $f(x) = \int_0^x \exp(\int_0^u \delta(v)dv)du$ and define a function $l'(y) = (V' \circ f^{-1})(y)$. Then by straight derivation

$$\frac{1}{2}(\sigma^2 \circ f^{-1})(y)l'''(y) = \frac{\left(r - (\mu' \circ f^{-1})(y)\right) l'(y) - (\pi' \circ f^{-1})(y)}{(f' \circ f^{-1})^2(y)}.$$

Since $l''(f(x)) = V''(x)/f'(x)$, we see that $l''(f(x))$ has the same sign as $V''(x)$ and thus the claimed property of V follows from that of l.

(B) From (12) we see that $V'(x) > 0$ for all $x \leq z^*$ and $x \geq y^*$. Since, by item (A) $V''(x) \leq 0$ in between, we must also have $V'(x) > 0$ for $x \in (z^*, y^*)$. □

Proof of Lemma 5.2

(A) Let $\tilde{y}_b \in (0, \tilde{x}_b)$ be such that $\mu(x)$ is concave for all $0 < x \leq \tilde{y}_b$ and let $x < \tilde{y}_b$. We can write

$$-\int_x^{\tilde{x}_b} \frac{\varphi'(t)}{S'(t)}dt = -\int_x^{\tilde{y}_b} \frac{\varphi'(t)}{S'(t)}dt - \int_{\tilde{y}_b}^{\tilde{x}_b} \frac{\varphi'(t)}{S'(t)}dt.$$

Here the latter integral in the right-hand side is finite, so we need to show that former one tends to infinity when x tends to zero. To that end let us inspect more closely the associated diffusion $\hat{X}_t$. Straight calculation shows the density of the scale function and the density of the speed measure to be $\hat{S}'(x) = S'(x)/\sigma^2(x)$ and $\hat{m}'(x) = 2/S'(x)$. Moreover, by convexity of φ [Lemma 3.3(A)], we can verify the decreasing fundamental solution to be $\hat{\varphi}(x) = -\varphi'(x)$. Utilizing these together with the concavity of μ allows us to write

$$-\int_x^{\tilde{y}_b} \frac{\varphi'(v)}{S'(v)} dv = \frac{1}{2} \int_x^{\tilde{y}_b} \frac{\mu'(v) - r}{\mu'(v) - r} \hat{\varphi}(v) \hat{m}'(v) dv$$

$$< \frac{1}{2(\mu'(0) - r)} \int_x^{\tilde{y}_b} (\mu'(v) - r) \hat{\varphi}(v) \hat{m}'(v) dv.$$

We can now use Lemma 7.1(B) for the diffusion $\hat{X}_t$ to obtain

$$\frac{1}{2(\mu'(0) - r)} \int_x^{\tilde{y}_b} (\mu'(v) - r) \hat{\varphi}(v) \hat{m}'(v) dv = \frac{1}{2(\mu'(0) - r)} \left(\frac{\hat{\varphi}'(x)}{\hat{S}'(x)} - \frac{\hat{\varphi}'(\tilde{y}_b)}{\hat{S}'(\tilde{y}_b)} \right).$$

Assumed boundary behaviour for $\hat{X}_t$ at 0 and the fact that $\mu'(0) - r < 0$ guarantee that this approach to infinity as x approach to zero, which was desired.

(B) Derivating (3) we get

$$B(R_r \mathrm{id})'(x) = \varphi'(x) \int_0^x \psi(t) t m'(t) dt + \psi'(x) \int_x^\infty \varphi(t) t m'(t) dt.$$

We know that $\lim_{x \downarrow 0} \varphi'(x) \int_0^x \psi(t) t m'(t) dt \leq 0$, so we must have $\lim_{x \downarrow 0} \psi'(x) \int_x^\infty \varphi(t) t m'(t) dt > 0$, for otherwise $(R_r \mathrm{id})'(0)$ cannot be positive. But $\psi'(0+) = 0$, so $\lim_{x \downarrow 0} \int_x^\infty \varphi(t) t m'(t) dt = \infty$. The proof is completed by showing that this integral is smaller than the claimed one. To see this, apply Fubini's Theorem:

$$\int_x^\infty \varphi(t) t m'(t) dt = \int_{t=x}^\infty \int_{v=0}^t \varphi(t) m'(t) dt dv \leq \lim_{u \to 0} \int_{v=u}^\infty \int_{t=v}^\infty \varphi(t) m'(t) dt dv$$

$$= \lim_{u \to 0} -\frac{1}{r} \int_u^\infty \frac{\varphi'(v)}{S'(v)} dv,$$

where the last equality follows from Lemma 7.1 (B). Now, since

$$\int_u^\infty \frac{\varphi'(v)}{S'(v)} dv = \int_u^{\tilde{x}_b} \frac{\varphi'(v)}{S'(v)} dv + \int_{\tilde{x}_b}^\infty \frac{\varphi'(v)}{S'(v)} dv$$

and the last integral in the right hand side is finite, this completes the proof. □

Proof of Lemma 5.3

Let us first prove the integral form for the function $J_b(x)$. Since φ satisfies the differential equation $(\mathcal{A}-r)\varphi=0$ and $(R_r\pi)(x)$ satisfies $(\mathcal{A}-r)(R_r\pi)=-\pi$, we can write J_b from (10) as

$$J_b(x)=\frac{1}{rBS'(x)}\left[\frac{1}{2}b\sigma^2(x)\varphi''(x)-\pi(x)\varphi'(x)+r\big((R_r\pi)(x)\varphi'(x)-(R_r\pi)'(x)\varphi(x)\big)\right]$$
$$=\frac{1}{Br}\left[\frac{1}{2}b\sigma^2(x)\frac{\varphi''(x)}{S'(x)}+r\int_x^\infty\varphi(t)\big(\pi(x)-\pi(t)\big)m'(t)dt\right],$$

where the integral representation follows from Lemma 7.1. Now to cope with the first term observe that since $(\mathcal{A}-r)\varphi=0$, we can write

$$\frac{1}{2}\sigma^2(x)\frac{\varphi_x''}{S_x'}=r\frac{\varphi_x-x\varphi_x'}{S'}-(\mu_x-rx)\frac{\varphi_x'}{S_x'}=r\frac{\varphi_x-x\varphi_x'}{S'}+(\mu_x-rx)r\int_x^\infty\varphi_t m_t'dt$$

so that we need an integral form to $\frac{\varphi_x-x\varphi_x'}{S'}$. But choosing $f(x)=x$ in Lemma 7.1, we get $\frac{\varphi_x-x\varphi_x'}{S'}=-\int_x^\infty\varphi_t(\mu_t-rt)m_t'dt$. Combining all these forms together gives the desired integral representation for $J_b(x)$. The proof for $I_b(x)$ is similar. □

Proof of Lemma 5.4

(A) To calculate the value at the upper boundary, let $x>\tilde{x}_b$, for $b=p,q$. Using the integral representation from Lemma 5.3 we can calculate that

$$\lim_{x\to\infty}J_b(x)=\lim_{x\to\infty}-\frac{1}{B}\left(\int_x^\infty\varphi_t(\rho_b(x)-\rho_b(t))m_t'dt\right)\le\lim_{x\to\infty}-\frac{1}{B}\left(\int_x^\infty\varphi_t(\rho_b(t)-\rho_b(t))m_t'dt\right)=0,$$

where the inequality follows from Assumption 3.2(iv).

To calculate the value at the lower boundary, let $\tilde{y}_b\in(0,\tilde{x}_b)$ and $\varepsilon>0$ be such that $\rho_b'(x)>\varepsilon$ for all $0\le x\le\tilde{y}_b$. This is possible for some constant ε since $\rho_b'(0+)>0$ [Assumption 5.1(v)]. Let $x<\tilde{y}_b$ and apply Fubini's Theorem, Lemma 7.1(B) and inequality $\rho_b'(x)>\varepsilon$ to get

$$J_b(x)=\frac{1}{B}\int_{t=x}^\infty\int_{v=x}^t\varphi(t)m'(t)\rho_b'(v)dvdt=\frac{1}{B}\int_{v=x}^\infty\int_{t=v}^\infty\varphi(t)m'(t)\rho_b'(v)dtdv$$

$$= -\frac{1}{Br}\int_x^\infty \frac{\varphi'(v)}{S'(v)}\rho_b'(v)dv = -\frac{1}{Br}\int_x^{\tilde{y}_b} \frac{\varphi'(v)}{S'(v)}\rho_b'(v)dv - \frac{1}{Br}\int_{\tilde{y}_b}^\infty \frac{\varphi'(v)}{S'(v)}\rho_b'(v)dv$$

$$> -\frac{\varepsilon}{Br}\int_x^{\tilde{y}_b} \frac{\varphi'(v)}{S'(v)}dv - \frac{1}{Br}\int_{\tilde{y}_b}^\infty \frac{\varphi'(v)}{S'(v)}\rho_b'(v)dv.$$

Here the last integral term is finite and $\lim_{x\downarrow 0} -\int_x^{\tilde{y}_b} \frac{\varphi'(v)}{S'(v)}dv = \infty$ by Assumption 5.1(vi), so $J_b(0+) = \infty$.

(B) To calculate the value at the lower boundary, let $x < \tilde{x}_b$, for $b = p, q$. Using the integral representation from Lemma 5.3 we can calculate that

$$\lim_{x\to 0} I_b(x) = \lim_{x\to 0} \frac{1}{B}\left(\int_0^x \psi_t(\rho_b(x) - \rho_b(t))m_t' dt\right)$$

$$\geq \lim_{x\to 0} \frac{1}{B}\left(\int_0^x \psi_t(\rho_b(t) - \rho_b(t))m_t' dt\right) = 0,$$

where the inequality follows from Assumption 3.2(iv).

To calculate the value at the upper boundary, let $x > \tilde{x}_b$. We can write

$$I_b(x) = \frac{1}{B}\int_0^{\tilde{x}_b} \psi_t\big(\rho_b(x) - \rho_b(t)\big)m_t' dt + \frac{1}{B}\int_{\tilde{x}_b}^x \psi_t\big(\rho_b(x) - \rho_b(t)\big)m_t' dt$$

$$= \frac{1}{Br}\big(\rho_b(x) - \rho_b(\eta)\big)\frac{\psi'(\tilde{x}_b)}{S'(\tilde{x}_b)} + \frac{1}{B}\int_{\tilde{x}_b}^x \psi_t\big(\rho_b(x) - \rho_b(t)\big)m_t' dt,$$

for some $\eta \in (0, \tilde{x}_b)$ by mean value theorem for integrals. The last term in the last row is always negative, since $\rho_b(x) < \rho_b(t)$ for all $t > x > \tilde{x}_b$ and the first term in the last row tends to minus infinity as x tends to infinity since by Assumption 5.1(v) $\rho_b(\infty) = -\infty$. Hence $I_b(\infty) = -\infty$. □

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