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Cost allocation protocols for supply contract design in network situations

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Abstract The class of *Construct and Charge* (*CC*-) rules for minimum cost spanning tree (mcst) situations is considered. *CC*-rules are defined starting from the notion of *charge systems*, which specify particular allocation protocols rooted on the Kruskal algorithm for computing an mcst. These protocols can be easily implemented in practical network situations (for instance, in supply transportation networks), are flexible to changes in the network situation and meet the requirement of continuous monitoring by the agents involved. Special charge systems, that we call *conservative*,

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lead to a subclass of *CC*-rules that coincides with the class of obligation rules for mcst situations.

Keywords Cost allocation · Minimum cost spanning tree games · Cost monotonicity

1 Introduction

Supply contracts are devoted to the explicit specification of relationships between partners within supply chains and networks. Various aspects dealing with supply contracts can be considered (Voß and Schneidereit 2002). For example, the supply contract can include clauses on how the delivery is performed including transport mode selection and the definition of penalties for modifications or late arrivals. Minimum cost spanning tree (mcst) situations (Bird 1976) may be useful to answer realistic questions regarding the implementation of clauses in supply contracts concerning transportation networks and the related cost allocation problem (Voß and Schneidereit 2002; Sharkey 1995).

An mcst situation arises when there is a group of agents (e.g. customer nodes of a supply chain) $N = \{1, 2, \ldots, n\}$ who all want to be connected with a source 0, directly or via other agents, and where connections are costly. To such mcst situations correspond two problems: to construct an mcst which connects all the agents with the source and to divide the cost of constructing this mcst among the agents. Given an mcst situation with a group of agents, Bird (1976) introduced a corresponding cooperative cost game (known as *mcst game*), where the players are the agents and the worth of a coalition is the minimal cost of connecting this coalition to the source via edges between members of the coalition.

To *construct* an mcst two methods are mainly used: the Prim algorithm (Prim 1957) and the Kruskal algorithm (Kruskal 1956). Both algorithms determine an mcst where exactly one edge is constructed in every step of the algorithm. The total number of steps equals n. To divide the cost of an mcst among the agents, both algorithms are suitable to define cost allocation protocols which *charge* the agents with "fractions" of the cost of each edge constructed in each step of the procedure. *Construct* and *Charge* rules, formally introduced in Sect. 4, rely on this idea of allocation protocol. In this paper, the Kruskal algorithm is central. This algorithm works in the following way: in the first step an edge between two nodes in $N \cup \{0\}$ of minimal cost is formed. In every subsequent step, a new edge of minimal cost is formed, under the constraint that no cycles are formed. In summary, a sequence of edges is produced and after n steps an mcst appears. Since some edges may have the same cost, different mcsts may be selected by the Kruskal algorithm, depending on the ordering of the edges with respect to their increasing costs which has been considered in the Kruskal algorithm.

Construct and Charge rules are defined using the Kruskal algorithm and have been studied already in Feltkamp et al. (1994b) for *minimum cost spanning extension* (mcse) situations. These mcse situations are generalized mcst situations in which some network can be present initially, which has to be extended to a network connecting every player to the source. Feltkamp et al. (1994b) proved that the allocations provided by Construct and Charge rules are in the core of the game corresponding to an mcse situation (in case no network is present initially, an mcse situation is an mcst situation, and the game is the corresponding mcst game).

In this paper we study and characterize a subclass of Construct and Charge rules for mcst situations, which we call 'conservative Construct and Charge' rules. An interesting feature of such rules is that different feasible orderings of the edges lead to the same cost allocations. In Theorem 3, it is shown that the subclass of conservative Construct and Charge rules coincides with the class of Obligations rules (Tijs et al. 2006a). Two interesting properties for Obligation rules are stability, i.e. they provide cost allocations that are in the core of the corresponding mcst game, and cost monotonicity, i.e. if some connection costs go up, then no agents will pay less. Stability is an important characteristic for cost allocation protocols applied to supply transportation networks, since it is a necessary condition for any subset of customers not to secede and build their own competing transportation sub-network. But, increasing of transportation costs may occur, and, consequently, other incentives to cooperation are demanded. For instance, supply contracts must take into consideration clauses for having various transport possibilities enabling, e.g., expedited delivery in cases of necessary adjustments in the lead times (Voß and Schneidereit 2002) with corresponding increasing of transportation costs. In this case, cost monotonicity ensures that no customers are motivated to delay the lead times, since according to a cost monotonic allocation protocol no customer will pay less.

We also show that the ERO-rule introduced in Feltkamp et al. (1994b), and rebaptized as the P-value (Branzei et al. 2004), is a conservative Construct and Charge rule and, consequently, does not depend on the mcst selected and provides a unique cost allocation for each mcst situation. Differently, the Proportional rule introduced in Feltkamp et al. (1994a) is a Construct and Charge rule but it is not conservative, and may provide different cost allocations on the same mcst situation, depending on the feasible orderings of the edges with respect to increasing costs. In addition, the Proportional rule does not satisfy monotonicity requirements.

We start introducing some basic notions in the next section. In Sect. 3 the definition of a charge system is introduced, specific examples are given and some basic properties are studied. In Sect. 4 conservative charge systems are introduced and a related concept of potential is discussed. Based on charge systems and orderings of the edges with respect to increasing costs, the definition of a Construct and Charge rule for mcst situations is given in Sect. 5, together with some examples and properties for such rules. In Sect. 6 the connection with Obligation rules is studied.

2 Preliminaries and notations

An (undirected) *graph* is a pair $\langle V, E\rangle$, where V is a set of vertices or nodes and E is a set of edges e of the form $\{i, j\}$ with $i, j \in V$, $i \neq j$. The *complete graph* on a set V of vertices is the graph $\langle V, E_V\rangle$, where $E_V = \{\{i, j\}|i, j \in V \text{ and } i \neq j\}$.

This paper deals with mcst situations, i.e. situations where a set $N = \{1, \ldots, n\}$ of agents is willing to be connected as cheap as possible to a source (i.e. a supplier of a service) denoted by 0, based on a given weight (or cost) system of connection. In the sequel we use also the notation $N' = N \cup \{0\}$, and w for the *weight function*, i.e. a map which assigns to each edge $e \in E_{N'}$ a nonnegative number $w(e)$ representing the weight or cost of edge e. We denote an *mcst situation* with set of users N, source 0, and weight function w by $\langle N', w\rangle$ (or simply w). Further, we denote by $\mathcal{W}^{N'}$ the set of all mcst situations $\langle N', w\rangle$ (or w) with node set N'. Most of the notation on mcst situations and related cooperative games is in line with that used in our papers (Branzei et al. 2004; Tijs et al. 2006a,b). In the following we recall briefly some basic notions and notation.

A *path* between i and j in a graph $\langle N', E\rangle$ is a sequence of nodes $i = i_0, i_1, \ldots, i_k = j$, $k \geq 1$, such that all the edges $\{i_s, i_{s+1}\} \in E$, for each $s \in \{0, \ldots, k-1\}$, are distinct edges. A *cycle* in $\langle N', E\rangle$ is a path from i to i for some $i \in N'$.

Two nodes $i, j \in N'$ are connected in $\langle N', E\rangle$ if $i = j$ or if there exists a path between i and j in E. A *connected component* of N' in a graph $\langle N', E\rangle$ is a maximal subset of N' with the property that any two nodes in this subset are connected in $\langle N', E\rangle$.

The cost of a *network* $\Gamma \subseteq E_{N'}$ is $w(\Gamma) = \sum_{e\in\Gamma} w(e)$. A network Γ is a *spanning network* on $S' = S \cup \{0\}$, with $S \subseteq N$, if for every $e \in \Gamma$ we have $e \in E_{S'}$ and for every $i \in S$ there is a path in $\langle S', \Gamma\rangle$ from i to the source. For any mcst situation $w \in \mathcal{W}^{N'}$ it is possible to determine at least one *spanning tree* on N', i.e. a spanning network without cycles on N', of minimum cost; each spanning tree of minimum cost is called an *mcst* for N' in w or, shorter, an mcst for w.

Next, we recall some basic game theoretical notions. A *cooperative cost game* (or simply a cost game) is a pair (N, c), where N denotes the finite set of *players* and $c : 2^N \to \mathbb{R}$ the *characteristic function*, with $c(\emptyset) = 0$ (here 2^N denotes the power set of player set N). Often we identify a cost game (N, c) with the corresponding characteristic function c. A group of players $T \subseteq N$ is called a *coalition* and $c(T)$ is called the *cost* of this coalition. The class of all cost games with N as set of players is denoted by $\mathcal{G}^N$.

A particular set, possibly empty, of cost allocations of a cost game (N, c) is the *core* of c, which is defined as follows:

$$core(c) = \{x \in \mathbb{R}^N | \sum_{i\in S} x_i \leq c(S)\ \forall S \in 2^N \setminus \{\emptyset\}; \sum_{i\in N} x_i = c(N)\}.$$

Let $\langle N', w\rangle$ be an mcst situation. The *mcst game* (N, c_w) (or simply c_w) corresponding to $\langle N', w\rangle$, introduced by Bird (1976), is defined by

$$c_w(S) = \min\{w(\Gamma)|\Gamma \text{ is a spanning network on } S'\}$$

for every $S \in 2^N \setminus\{\emptyset\}$, with the convention that $c_w(\emptyset) = 0$.

We call a map $F : \mathcal{W}^{N'} \to \mathbb{R}^N$ assigning to every mcst situation w a unique cost allocation in $\mathbb{R}^N$ a *solution*. A solution F is a *cost monotonic* solution if for all mcst situations $w, w' \in \mathcal{W}^{N'}$ such that $w(\bar{e}) \leq w'(\bar{e})$ for one edge $\bar{e} \in E_{N'}$ and $w(e) = w'(e)$ for each $e \in E_{N'} \setminus \{\bar{e}\}$, it holds that $F(w) \leq F(w')$. We mention that our notion of cost monotonicity is different (see Tijs et al. 2006a) from that introduced and studied by Dutta and Kar (2004). A solution F is *efficient* if $\sum_{i\in N} F_i(w) = w(\Gamma)$ for each $w \in \mathcal{W}^{N'}$, where Γ is a minimum cost spanning network on N' for w.

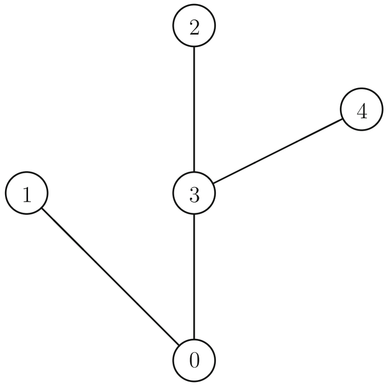

Fig. 1 A spanning tree on $N' = \{0, 1, 2, 3, 4\}$

3 Charge systems

To introduce charge systems we need some additional notations. Let $N = \{1, \ldots, n\}$ and $\Delta(N) = \{x \in \mathbb{R}^N_+ | \sum_{i \in N} x_i = 1\}$. We denote by $\mathcal{E}_{N'}$ the set of n-vectors of edges which form a spanning tree on N', i.e.

$$\mathcal{E}_{N'} = \{(a_1, \ldots, a_n) \in (E_{N'})^n | \{a_1, \ldots, a_n\} \text{ is a spanning network on } N'\}.$$

Note that the number of edges which form a spanning tree on N' is n.

Given an element $\mathbf{a} = (a_1, \ldots, a_n) \in (E_{N'})^n$, we denote by $\mathbf{a}_{|j}$ the restriction of $\mathbf{a}$ to the first j components, that is $\mathbf{a}_{|j} = (a_1, \ldots, a_j)$ for each $j \in N$. Further, for each $j \in N$, we denote by $\Pi(\mathbf{a}_{|j})$ the partition of N' such that

$$\Pi(\mathbf{a}_{|j}) = \{T \subseteq N' | T \text{ is a connected component in } \langle N', \{a_1, \ldots, a_j\}\rangle\}.$$

Example 1 Consider the spanning tree depicted in Fig. 1 on $N' = \{0, 1, 2, 3, 4\}$.

Vectors $\mathbf{a} = (\{2, 3\}, \{0, 1\}, \{3, 4\}, \{0, 3\})$ and $\mathbf{b} = (\{3, 4\}, \{2, 3\}, \{0, 1\}, \{0, 3\})$ are elements of $\mathcal{E}_{\{0,1,2,3,4\}}$. Note that $\mathbf{a}_{|3} = (\{2, 3\}, \{0, 1\}, \{3, 4\})$ and $\mathbf{b}_{|3} = (\{3, 4\}, \{2, 3\}, \{0, 1\})$ implying that $\Pi(\mathbf{a}_{|3}) = \Pi(\mathbf{b}_{|3}) = \{\{0, 1\}, \{2, 3, 4\}\}$.

Summing up, each element $\mathbf{a} \in \mathcal{E}_{N'}$ tells the "history" of the spanning network formation, that is adding the edge a_j to the already formed graph $\mathbf{a}_{|j-1}$, for each $j \in N$. Note that when the first edge a_1 is formed, the already formed graph is $\langle N', \emptyset\rangle$; so, $\Pi(\mathbf{a}_{|0})$ is the singleton partition of N'.

Now, let $\theta \in \Theta(N')$, where $\Theta(N')$ is the family of partitions of N', and let $T \subseteq N'$. If T is a subset of a certain element of the partition θ, we denote this element as $S(\theta, T)$.

Definition 1 A *charge system* $\mathcal{C}$ on N is a set of functions $\mathcal{C} = \{C^1, \ldots, C^n\}$ with $C^j : \{\mathbf{a}_{|j} | \mathbf{a} \in \mathcal{E}_{N'}\} \to \Delta(N)$ for each $j \in N$ satisfying the following properties:

(Connection property): $C_i^j(\mathbf{a}_{|j}) = 0$ for each $i \in S(\Pi(\mathbf{a}_{|j-1}), \{0\})$, each $j \in N$, and each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$;

(Involvement property): $C_i^j(\mathbf{a}_{|j}) = 0$ for each $i \in N \setminus S(\Pi(\mathbf{a}_{|j}), a_j)$ each $j \in N$, and each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$;

(Total aggregation property): $\sum_{j=1}^{n} C_i^j(\mathbf{a}_{|j}) = 1$ for each $i \in N$, and each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$.

A charge system specifies how to charge agents during the construction of a spanning tree. Let $\mathbf{a} \in \mathcal{E}_{N'}$. First, the cost of each edge a_j, for each $j \in N$, should be totally charged among agents as soon as a_j is formed. This requirement makes a charge system promptly adaptable to modified situations, where edges are formed according to different orders (for instance, due to a change in the route of transportation).

The connection property says that agents already connected to the source in $\mathbf{a}_{|j-1}$ should not be charged anymore. This property accounts for the fact that there is no interest for agents already connected to the source in contributing to the construction of other edges in the network.

The involvement property specifies that only agents who are connected to nodes in a_j in the graph $\mathbf{a}_{|j}$ (i.e. agents involved in forming a_j) should be charged with fractions of the cost of a_j. This property is particularly valuable in supply transportation networks, because the continuous control on the charge procedure is simpler for customers which are directly involved in the construction of the edges.

The total aggregation property says that when the construction of the spanning network corresponding to $\mathbf{a}$ is completed, each agent has been charged for a total amount of fractions equal to 1. This property is a natural a priori requirement of fairness in a charge system, since it guarantees that all agents have duties on the same amount of total fractions of edges of an mcst.

The charge systems in Examples 2 and 3 will play a role in Sect. 5 to define special Construct and Charge rules. The intuition behind the charge system of Example 2 is to charge each agent in a connected component according to his 'remaining obligation (RO)'. At the start the RO is 1 for every agent. If in some step of the algorithm the connected component of an agent i is linked to some other connected component, then agent i is charged according to the following rule: if i is linked to a component containing the source, then i is charged by his RO (leaving a RO of 0 for this agent); otherwise, if i is linked to a component not containing the source, then i is charged half of his RO (leaving a RO that is half of his RO in the previous step). The charge system of Example 3 charges the agents involved in forming the edge a_j, for each $j \in N$, taking into account the cardinality of their connected components in the graphs $\mathbf{a}_{|j-1}$ and $\mathbf{a}_{|j}$. As a result of this procedure, at each stage $j \in N$, agents in the same connected component have the same RO.

Example 2 Consider the charge system $\tilde{\mathcal{C}} = \{\tilde{C}^1, \ldots, \tilde{C}^n\}$ on N such that for each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$ and for each $i, j \in N$

$$\tilde{C}_i^j(\mathbf{a}_{|j}) = \begin{cases} \frac{1}{2} r_i^j & \text{if } \; i \in S(\Pi(\mathbf{a}_{|j}), a_j) \\ & \text{and } \; 0 \notin S(\Pi(\mathbf{a}_{|j}), a_j), \\ \\ r_i^j & \text{if } \; \{0, i\} \subseteq S(\Pi(\mathbf{a}_{|j}), a_j) \\ & \text{and } \; 0 \notin S(\Pi(\mathbf{a}_{|j-1}), \{i\}), \\ \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where the *remaining obligation* r^j is defined as

$$r_i^j = 1 - \sum_{k=1}^{j-1} \tilde{C}_i^k(\mathbf{a}_{|k}) \tag{2}$$

for each $j \in N$, $j > 1$, and $r_i^1 = 1$ for each $i \in N$.

The involvement property and the connection property of functions $\tilde{C}^1, \dots, \tilde{C}^n$ are a direct consequence of relation (1). For the total aggregation property of functions $\tilde{C}^1, \dots, \tilde{C}^n$, first note that, by relation (2), for each $i, j \in N$ such that $i \in S(\Pi(\mathbf{a}_{|j}), a_j)$ and $0 \notin S(\Pi(\mathbf{a}_{|j}), a_j)$, the quantity $\sum_{k=1}^{j} \tilde{C}_i^k(\mathbf{a}_{|k}) < 1$.

Then, by relation (1) the total aggregation property follows immediately.

In order to prove that function $\tilde{C}^j$, $j \in N$, takes values in $\Delta(N)$, we first note by relations (1) that $\tilde{C}_i^j \geq 0$, for all $i, j \in N$. Second, we prove by induction to j that the sum $\sum_{i \in N} \tilde{C}_i^j(\mathbf{a}_{|j}) = 1$ for each $j \in N$.

If $j = 1$ we have that $\sum_{i \in N} \tilde{C}_i^1(\mathbf{a}_{|1}) = \sum_{i \in a_1, i \neq 0} \tilde{C}_i^1(\mathbf{a}_{|1}) = 1$.

Now, let $j \in \{2, \dots, n\}$ and suppose that $\sum_{i \in N} \tilde{C}_i^k(\mathbf{a}_{|k}) = 1$ for every $k \in \{1, \dots, j-1\}$. Let $z \in a_j$ be one of the two nodes of edge a_j such that $0 \notin S(\Pi(\mathbf{a}_{|j-1}), \{z\})$ and let $K_z \subseteq \{1, \dots, j-1\}$ be the set of indices k such that a_k is contained in $S(\Pi(\mathbf{a}_{|j-1}), \{z\})$, in formula $K_z = \{k \in \{1, \dots, j-1\} | a_k \subseteq S(\Pi(\mathbf{a}_{|j-1}), \{z\})$. Note that $|K_z| = |S(\Pi(\mathbf{a}_{|j-1}), \{z\})| - 1$, since $|K_z|$ edges are needed to construct a spanning tree on $S(\Pi(\mathbf{a}_{|j-1}), \{z\})$. We have

$$\begin{aligned} &\sum_{i \in S(\Pi(\mathbf{a}_{|j-1}), \{z\})} \sum_{k=1}^{j-1} \tilde{C}_i^k(\mathbf{a}_{|k}) \\ &= \sum_{k=1}^{j-1} \sum_{i \in S(\Pi(\mathbf{a}_{|j-1}), \{z\})} \tilde{C}_i^k(\mathbf{a}_{|k}) \\ &= \sum_{k \in K_z} \sum_{i \in S(\Pi(\mathbf{a}_{|j-1}), \{z\})} \tilde{C}_i^k(\mathbf{a}_{|k}) \\ &= \sum_{k \in K_z} \sum_{i \in N} \tilde{C}_i^k(\mathbf{a}_{|k}) = |K_z|, \end{aligned} \tag{3}$$

where the second equality follows from the involvement property which specifies that $\tilde{C}_i^k(\mathbf{a}_{|k}) = 0$ for each $i \in S(\Pi(\mathbf{a}_{|j-1}), \{z\})$ and $k \in \{1, \dots, j-1\} \setminus K_z$; the third equality follows from the involvement property which specifies that $\tilde{C}_i^k(\mathbf{a}_{|k}) = 0$ for each $i \in N \setminus S(\Pi(\mathbf{a}_{|j-1}), \{z\})$ and $k \in K_z$; finally, the last equality follows from the induction hypothesis. When edge a_j is constructed, a new partition of nodes $\Pi(\mathbf{a}_{|j})$ forms. By the connection property, only nodes which were not yet connected to 0 in $\Pi(\mathbf{a}_{|j-1})$ are charged. Then, we distinguish two cases:

case 1) $a_j = \{u, v\} \in E_{N'}$, $0 \notin S(\Pi(\mathbf{a}_{|j}), \{u\})$, $0 \notin S(\Pi(\mathbf{a}_{|j}), \{v\})$. We have

$$\begin{aligned}
&\sum_{i \in N} \tilde{C}_i^j(\mathbf{a}_{|j}) \\
&\quad = \sum_{i \in S(\Pi(\mathbf{a}_{|j-1}), \{u\})} \frac{1}{2} r_i^j + \sum_{i \in S(\Pi(\mathbf{a}_{|j-1}), \{v\})} \frac{1}{2} r_i^j \\
&\quad = \sum_{i \in S(\Pi(\mathbf{a}_{|j-1}), \{u\})} \frac{1}{2} \left(1 - \sum_{k=1}^{j-1} \tilde{C}_i^k(\mathbf{a}_{|k})\right) \\
&\qquad + \sum_{i \in S(\Pi(\mathbf{a}_{|j-1}), \{v\})} \frac{1}{2} \left(1 - \sum_{k=1}^{j-1} \tilde{C}_i^k(\mathbf{a}_{|k})\right) \\
&\quad = \frac{1}{2} \big(|S(\Pi(\mathbf{a}_{|j-1}), \{u\})| - |K_u|\big) + \frac{1}{2} \big(|S(\Pi(\mathbf{a}_{|j-1}), \{v\})| - |K_v|\big) \\
&\quad = \frac{1}{2} \big(|S(\Pi(\mathbf{a}_{|j-1}), \{u\})| - |S(\Pi(\mathbf{a}_{|j-1}), \{u\})| + 1\big) \\
&\qquad + \frac{1}{2} \big(|S(\Pi(\mathbf{a}_{|j-1}), \{v\})| - |S(\Pi(\mathbf{a}_{|j-1}), \{v\})| + 1\big) = 1.
\end{aligned}$$

where the first equality follows by relation (1) and the involvement property, and the third equality from relation (3).
case 2) $a_j = \{u, v\} \in E_{N'}$, $0 \notin S(\Pi(\mathbf{a}_{|j}), \{u\})$, $0 \in S(\Pi(\mathbf{a}_{|j}), \{v\})$. We have

$$\begin{aligned}
&\sum_{i \in N} \tilde{C}_i^j(\mathbf{a}_{|j}) \\
&\quad = \sum_{i \in S(\Pi(\mathbf{a}_{|j-1}), \{u\})} r_i^j \\
&\quad = \sum_{i \in S(\Pi(\mathbf{a}_{|j-1}), \{u\})} \left(1 - \sum_{k=1}^{j-1} \tilde{C}_i^k(\mathbf{a}_{|k})\right) \\
&\quad = \big(|S(\Pi(\mathbf{a}_{|j-1}), \{u\})| - |K_u|\big) \\
&\quad = \big(|S(\Pi(\mathbf{a}_{|j-1}), \{u\})| - |S(\Pi(\mathbf{a}_{|j-1}), \{u\})| + 1\big) = 1,
\end{aligned}$$

where the first equality follows by relation (1) and the involvement property, and the third equality from relation (3).
We may conclude that $\tilde{C}^1, \ldots, \tilde{C}^n$ constitute a charge system.

Example 3 Consider the set of functions $\hat{\mathcal{C}} = \{\hat{C}^1, \dots, \hat{C}^n\}$ on N such that for each $\mathbf{a} = (a_1, \dots, a_n) \in \mathcal{E}_{N'}$ and for each $j \in N$

$$\hat{C}_i^j(\mathbf{a}_{|j}) = \begin{cases} \frac{1}{|S(\Pi(\mathbf{a}_{|j-1}),\{i\})|} - \frac{1}{|S(\Pi(\mathbf{a}_{|j}),\{i\})|} & \text{if } \ 0 \notin S(\Pi(\mathbf{a}_{|j}), a_j) \\ \frac{1}{|S(\Pi(\mathbf{a}_{|j-1}),\{i\})|} & \text{if } \ \{0, i\} \subseteq S(\Pi(\mathbf{a}_{|j}), a_j) \\ & \text{and } \ 0 \notin S(\Pi(\mathbf{a}_{|j-1}), \{i\}), \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

for each $i \in N$.

In order to check that the functions $\hat{C}^1, \dots, \hat{C}^n$ constitute a charge system, we first show that functions $\hat{C}^1, \dots, \hat{C}^n$ take values in $\Delta(N)$. Note that for each $j \in N$ such that $a_j = \{u, v\} \in E_{N'}$ and $0 \notin S(\Pi(\mathbf{a}_{|j}), a_j)$ we have that

$$\begin{aligned}
&\sum_{i \in N} \hat{C}_i^j(\mathbf{a}_{|j}) \\
&\quad = \sum_{i \in S(\Pi(\mathbf{a}_{|j-1}),\{u\})} \frac{1}{|S(\Pi(\mathbf{a}_{|j-1}), \{u\})|} - \frac{1}{|S(\Pi(\mathbf{a}_{|j}), \{u\})|} \\
&\qquad + \sum_{i \in S(\Pi(\mathbf{a}_{|j-1}),\{v\})} \frac{1}{|S(\Pi(\mathbf{a}_{|j-1}), \{v\})|} - \frac{1}{|S(\Pi(\mathbf{a}_{|j}), \{v\})|} \\
&\quad = \sum_{i \in S(\Pi(\mathbf{a}_{|j-1}),\{u\})} \frac{1}{|S(\Pi(\mathbf{a}_{|j-1}), \{u\})|} + \sum_{i \in S(\Pi(\mathbf{a}_{|j-1}),\{v\})} \frac{1}{|S(\Pi(\mathbf{a}_{|j-1}), \{v\})|} \\
&\qquad - \sum_{i \in S(\Pi(\mathbf{a}_{|j}),a_j)} \frac{1}{|S(\Pi(\mathbf{a}_{|j}), a_j)|} = 1 + 1 - 1 = 1.
\end{aligned}$$

Differently, for each $j \in N$ such that $0 \in S(\Pi(\mathbf{a}_{|j}), a_j)$ we have that

$$\textstyle\sum_{i \in N} \hat{C}_i^j(\mathbf{a}_{|j}) = \sum_{i \in S(\Pi(\mathbf{a}_{|j-1}),\{m\})} \frac{1}{|S(\Pi(\mathbf{a}_{|j-1}),\{m\})|} = 1,$$

where $m \in S(\Pi(\mathbf{a}_{|j}), a_j)$ is such that $0 \notin S(\Pi(\mathbf{a}_{|j-1}), \{m\})$.

The connection property of functions $\hat{C}^1, \dots, \hat{C}^n$ directly follows by relation (4) .

To prove that $\hat{C}^1, \dots, \hat{C}^n$ satisfy the involvement property we note that if for $i, j \in N$ we have that $i \notin S(\Pi(\mathbf{a}_{|j}), a_j)$, then it follows that $S(\Pi(\mathbf{a}_{|j-1}), \{i\}) = S(\Pi(\mathbf{a}_{|j}), \{i\})$, since nothing is changed in the connected component of agent i from stage $j - 1$ to stage j. Consequently, by relation (4), we have that $\hat{C}_i^j(\mathbf{a}_{|j}) = 0$.

Finally, to prove that functions $\hat{C}^1, \dots, \hat{C}^n$ satisfy the total aggregation property, first note that for each $i \in N$, we have that $\sum_{j=1}^n \hat{C}_i^j(\mathbf{a}_{|j}) = \sum_{j=1}^k \hat{C}_i^j(\mathbf{a}_{|j})$, where $k \in N$ is such that $\{0, i\} \subseteq S(\Pi(\mathbf{a}_{|k}), a_k)$ and $0 \notin S(\Pi(\mathbf{a}_{|k-1}), \{i\})$. Consequently, for $k = 1$, by relation (4) we have that

$$\sum_{j=1}^{1} \hat{C}_i^j(\mathbf{a}_{|j}) = \frac{1}{|S(\Pi(\mathbf{a}_{|0}), \{i\})|} = 1.$$

For $k > 1$ we have that

$$\sum_{j=1}^{k} \hat{C}_i^j(\mathbf{a}_{|j}) = \left(\sum_{j=1}^{k-1} \frac{1}{|S(\Pi(\mathbf{a}_{|j-1}), \{i\})|} - \frac{1}{|S(\Pi(\mathbf{a}_{|j}), \{i\})|} \right) + \frac{1}{|S(\Pi(\mathbf{a}_{|k-1}), \{i\})|}$$
$$= \frac{1}{|S(\Pi(\mathbf{a}_{|0}), \{i\})|} - \frac{1}{|S(\Pi(\mathbf{a}_{|k-1}), \{i\})|} + \frac{1}{|S(\Pi(\mathbf{a}_{|k-1}), \{i\})|} = \frac{1}{|S(\Pi(\mathbf{a}_{|0}), \{i\})|} = 1.$$

Next example illustrates a numerical application of the charge systems introduced in Examples 2 and 3.

Example 4 Consider the spanning tree depicted in Fig. 1 of Example 1 and consider the charge systems $\tilde{\mathcal{C}}$ and $\hat{\mathcal{C}}$, respectively introduced in Example 2 and Example 3. In Tables 1 and 2 we show the respective charge systems corresponding to $\mathbf{a}$ and $\mathbf{b}$ of Example 1.

4 Conservative charge systems

In this section, special charge systems, which we call *conservative*, will play a role. Consider a charge system $\mathcal{C} = \{C^1, \ldots, C^n\}$ on N. We define the *aggregate contribution* of the charge system $\mathcal{C}$ on $\mathbf{a}_{|j}$, for each $j \in N$ and for each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$, as the n-vector $\mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j})$ calculated via the following formula

$$\mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}) = \sum_{k=1}^{j} C^k(\mathbf{a}_{|k}). \tag{5}$$

Definition 2 Let $\mathcal{C} = \{C^1, \ldots, C^n\}$ be a charge system on N. We call $\mathcal{C}$ a *conservative charge system* if for all $j \in N$ and for each pair $\mathbf{a}, \mathbf{b} \in \mathcal{E}_{N'}$, with $\Pi(\mathbf{a}_{|j}) = \Pi(\mathbf{b}_{|j})$ we have that

$$\mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}) = \mathrm{A}^{\mathcal{C}}(\mathbf{b}_{|j}). \tag{6}$$

The peculiarity of conservative charge systems is that they preserve the aggregate contribution from the network construction history, i.e. the aggregate contribution corresponding to $\mathbf{a}_{|j}$, for $\mathbf{a} \in \mathcal{E}_{N'}$ and $j \in N$, is only dependent on the partition of N' induced by the connected components in $\langle N', \{a_1, \ldots, a_j\}\rangle$.

Example 5 It is easy to check that the charge system $\tilde{\mathcal{C}}$ of Example 2 is not conservative. Consider, for instance, $\mathrm{A}^{\tilde{\mathcal{C}}}(\mathbf{a}_{|3})$ and $\mathrm{A}^{\tilde{\mathcal{C}}}(\mathbf{b}_{|3})$ in Example 1. As we noted in Example 1, $\Pi(\mathbf{a}_{|3}) = \Pi(\mathbf{b}_{|3})$ but, from Table 1 in Example 4, we have that $\mathrm{A}^{\tilde{\mathcal{C}}}(\mathbf{a}_{|3}) = (1, \frac{3}{4}, \frac{3}{4}, \frac{1}{2})^t \neq (1, \frac{1}{2}, \frac{3}{4}, \frac{3}{4})^t = \mathrm{A}^{\tilde{\mathcal{C}}}(\mathbf{b}_{|3})$.

Table 1 The charge system of Example 2 for **a** and **b** of Example 1

j	1	2	3	4
$\tilde{C}^j(\mathbf{a}_{\vert j})$	$(0,\frac{1}{2},\frac{1}{2},0)^t$	$(1,0,0,0)^t$	$(0,\frac{1}{4},\frac{1}{4},\frac{1}{2})^t$	$(0,\frac{1}{4},\frac{1}{4},\frac{1}{2})^t$
$\tilde{C}^j(\mathbf{b}_{\vert j})$	$(0,0,\frac{1}{2},\frac{1}{2})^t$	$(0,\frac{1}{2},\frac{1}{4},\frac{1}{4})^t$	$(1,0,0,0)^t$	$(0,\frac{1}{2},\frac{1}{4},\frac{1}{4})^t$

Table 2 The charge system of Example 3 for **a** and **b** of Example 1

j	1	2	3	4
$\hat{C}^j(\mathbf{a}_{\vert j})$	$(0,\frac{1}{2},\frac{1}{2},0)^t$	$(1,0,0,0)^t$	$(0,\frac{1}{6},\frac{1}{6},\frac{2}{3})^t$	$(0,\frac{1}{3},\frac{1}{3},\frac{1}{3},)^t$
$\hat{C}^j(\mathbf{b}_{\vert j})$	$(0,0,\frac{1}{2},\frac{1}{2})^t$	$(0,\frac{2}{3},\frac{1}{6},\frac{1}{6})^t$	$(1,0,0,0)^t$	$(0,\frac{1}{3},\frac{1}{3},\frac{1}{3})^t$

Now, consider the charge system $\hat{\mathcal{C}}$ introduced in Example 3. For each $i, j \in N$ and each $\mathbf{a} \in \mathcal{E}_{N'}$ we have that

$$\mathrm{A}_i^{\hat{\mathcal{C}}}(\mathbf{a}_{|j}) = \begin{cases} 1 - \frac{1}{|S(\Pi(\mathbf{a}_{|j}),\{i\})|} & \text{if } \; 0 \notin S(\Pi(\mathbf{a}_{|j}),\{i\}) \\ 1 & \text{otherwise.} \end{cases}$$

Note that $\mathrm{A}_i^{\hat{\mathcal{C}}}(\mathbf{a}_{|j})$ is only dependent on the partition of N' induced by the connected components in $\langle N', \{a_1, \ldots, a_j\}\rangle$, for each $i, j \in N$, i.e. $\hat{\mathcal{C}}$ is a conservative charge system.

Now, let $\mathcal{C}$ be a conservative charge system on N. We introduce the notion of *potential* with respect to $\mathcal{C}$, denoted by $P^{\mathcal{C}}$, which is a function on $2^{N'} \setminus \{\emptyset\}$ with values in $\mathbb{R}^N$.

Definition 3 Let $\mathcal{C} = \{C^1, \ldots, C^n\}$ be a conservative charge system on N. For each $S \in 2^{N'} \setminus \{\emptyset\}$, consider an element $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$ such that $\Pi(\mathbf{a}_{|j}) = \{S, \{i\}_{i \in N' \setminus S}\}$, with $j \in N$.

We define the *potential* of S with respect to the conservative charge system $\mathcal{C}$ as the unique[1] aggregate contribution corresponding to the partition $\{S, \{i\}_{i \in N' \setminus S}\}$, in formula

$$P^{\mathcal{C}}(S) := \mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}).$$

The name of potential is inspired from physics where each conservative vector field has a potential. In a connection situation, an intuitive interpretation of the potential $P^{\mathcal{C}}(S)$, $S \in 2^{N'} \setminus \{\emptyset\}$, is as the level of "connection work" done by nodes in N when $\{S, \{i\}_{i \in N' \setminus S}\}$ is the current set of connected components and the conservative charge system $\mathcal{C}$ is used. Note that at the beginning of the connection process, when no edges are formed and all the connected components are singletons, the level of connection

[1] Let $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathcal{E}_{N'}$, and $S \in 2^{N'} \setminus \{\emptyset\}$ be such that $\Pi(\mathbf{a}_{|j}) = \Pi(\mathbf{b}_{|j}) = \{S, \{i\}_{i \in N' \setminus S}\}$, with $j \in N$. Recall that by Definition 2, we have $\mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}) = \mathrm{A}^{\mathcal{C}}(\mathbf{b}_{|j})$. So, the aggregate contribution corresponding to $\{S, \{i\}_{i \in N' \setminus S}\}$ is unique.

work performed by nodes should be zero. This motivates us to use the convention that $P_i^{\mathcal{C}}(\{j\}) = P_i^{\mathcal{C}}(\{0\}) = 0$ for all $i, j \in N$.

Other elementary properties of $P^{\mathcal{C}} : 2^{N'} \setminus \{\emptyset\} \rightarrow \mathbb{R}_+^N$ are collected in the following lemma, which will play a role in Sect. 6 to prove Theorem 1.

Lemma 1 *Let $\mathcal{C} = \{C^1, \ldots, C^n\}$ be a conservative charge system on N, let $P^{\mathcal{C}}$ be the potential w.r.t. $\mathcal{C}$ and let $S \in 2^{N'} \setminus \{\emptyset\}$. Then,*

(c.1) *if $0 \in S$ then $P^{\mathcal{C}}(S) = e^{S\setminus\{0\}}$;*
(c.2) $\sum_{i \in S\setminus\{0\}} P_i^{\mathcal{C}}(S) = \sum_{i\in N} P_i^{\mathcal{C}}(S) = |S| - 1$;
(c.3) *if $S \subseteq T \subseteq N'$, then $P^{\mathcal{C}}(S) \leq P^{\mathcal{C}}(T)$.*

[Here $e^{S\setminus\{0\}} \in \mathbb{R}_+^N$ is such that $e_i^{S\setminus\{0\}} = 1$ for each $i \in S \setminus \{0\}$ and $e_i^{S\setminus\{0\}} = 0$ for each $i \in N \setminus S$.]

Proof (c.1) Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$ and $j \in N$ be such that $\Pi(\mathbf{a}_{|j}) = \{S, \{i\}_{i\in N'\setminus S}\}$. Then, for each $i \in N \cap S$

$$P_i^{\mathcal{C}}(S) = \mathrm{A}_i^{\mathcal{C}}(\mathbf{a}_{|j}) = \sum_{k=1}^{j} C_i^k(\mathbf{a}_{|k}) = 1 - \sum_{k=j+1}^{n} C_i^k(\mathbf{a}_{|k}) = 1,$$

where the third equality follows from the total aggregation property of $\mathcal{C}$ and the fourth equality follows from the connection property of $\mathcal{C}$. From the involvement property, we have $P_i^{\mathcal{C}}(S) = 0$ for each $i \in N \setminus S$, which finally proves property (c.1).

(c.2) If $0 \in S$ then property (c.2) follows directly from property (c.1).
Now, consider the case $0 \notin S$. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$ and $j \in N$ be such that $\Pi(\mathbf{a}_{|j}) = \{S, \{i\}_{i\in N'\setminus S}\}$. First, note that since $0 \notin S$, $j = |S| - 1$. Then,

$$\begin{aligned}\sum_{i\in S} P_i^{\mathcal{C}}(S) &= \sum_{i\in S} \mathrm{A}_i^{\mathcal{C}}(\mathbf{a}_{|j}) = \sum_{i\in S}\sum_{k=1}^{j} C_i^k(\mathbf{a}_{|k}) \\ &= \sum_{k=1}^{j}\sum_{i\in S} C_i^k(\mathbf{a}_{|k}) = \sum_{k=1}^{j} 1 = |S| - 1,\end{aligned}$$

where the fourth equality follows from the involvement property. By the involvement property it follows too that $P_i^{\mathcal{C}}(S) = 0$ for each $i \in N \setminus S$, which finally proves property (c.2).

(c.3) Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$ and $j, l \in N$ with $l \geq j$ be such that $\Pi(\mathbf{a}_{|j}) = \{S, \{i\}_{i\in N'\setminus S}\}$ and $\Pi(\mathbf{a}_{|l}) = \{T, \{i\}_{i\in N'\setminus T}\}$. Then,

$$\begin{aligned}P^{\mathcal{C}}(S) &= \mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}) = \sum_{k=1}^{j} C^k(\mathbf{a}_{|k}) \\ &\leq \sum_{k=1}^{j} C^k(\mathbf{a}_{|k}) + \sum_{k=j+1}^{l} C^k(\mathbf{a}_{|k}) \\ &= \sum_{k=1}^{l} C^k(\mathbf{a}_{|k}) = \mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|l}) = P^{\mathcal{C}}(T),\end{aligned}$$

which concludes the proof of property (c.3). □

Proposition 1 *Let $\mathcal{C} = \{C^1, \ldots, C^n\}$ be a conservative charge system on N. Let $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$ and $j \in N$ be such that $\Pi(\boldsymbol{a}_{|j}) = \{S_1, S_2, \ldots, S_m\}$, with $S_1, S_2, \ldots, S_m \subset N'$ and $m \leq n$. Then,*

$$\mathrm{A}^{\mathcal{C}}(\boldsymbol{a}_{|j}) = \sum_{r=1}^{m} P^{\mathcal{C}}(S_r).$$

Proof Let $r \in \{1, 2, \ldots, m\}$. Determine $b^r(1), \ldots, b^r(p^r) \in \{1, \ldots, j\}$ such that $\Pi(a_{b^r(1)}, a_{b^r(2)}, \ldots, a_{b^r(p^r)}) = \{S_r, \{i\}_{i \in N' \setminus S_r}\}$ where $p^r = |S_r| - 1$.

Then, for each $i \in N \setminus S_r$, by the involvement property of $\mathcal{C}$

$$P_i^{\mathcal{C}}(S_r) = \mathrm{A}_i^{\mathcal{C}}(a_{b^r(1)}, a_{b^r(2)}, \ldots, a_{b^r(p^r)}) = 0,$$

whereas for each $i \in N \cap S_r$

$$\begin{aligned} P_i^{\mathcal{C}}(S_r) &= \mathrm{A}_i^{\mathcal{C}}(a_{b^r(1)}, a_{b^r(2)}, \ldots, a_{b^r(p^r)}) \\ &= \mathrm{A}_i^{\mathcal{C}}(a_{b^r(1)}, \ldots, a_{b^r(p^r)}, (a_s)_{s \in \{1,\ldots,j\} \setminus \{b^r(1),\ldots,b^r(p^r)\}}) \\ &= \mathrm{A}_i^{\mathcal{C}}(a_1, a_2, \ldots, a_j) = \mathrm{A}_i^{\mathcal{C}}(\mathbf{a}_{|j}), \end{aligned}$$

where the second equality follows from the involvement property in the edge sequence $(a_{b^r(1)}, a_{b^r(2)}, \ldots, a_{b^r(j)})$ and the third equality follows from the fact that $\mathcal{C}$ is conservative. Consequently, $\sum_{r=1}^{m} P^{\mathcal{C}}(S_r) = \mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j})$. □

5 Construct and charge rules

We first recall some notions and results from Branzei et al. (2004) and Tijs et al. (2006a). We define the set $\Sigma_{E_{N'}}$ of *linear orders* on $E_{N'}$ as the set of all bijections $\sigma : \{1, \ldots, |E_{N'}|\} \to E_{N'}$, where $|E_{N'}|$ is the cardinality of the set $E_{N'}$. For each mcst situation $\langle N', w \rangle$ there exists at least one linear order $\sigma \in \Sigma_{E_{N'}}$ such that $w(\sigma(1)) \leq w(\sigma(2)) \leq \cdots \leq w(\sigma(|E_{N'}|))$.

For any $\sigma \in \Sigma_{E_{N'}}$ we define the set

$$K^{\sigma} = \{w \in \mathbb{R}_+^{E_{N'}} \mid w(\sigma(1)) \leq w(\sigma(2)) \leq \cdots \leq w(\sigma(|E_{N'}|))\}.$$

The set K^{σ} is a cone in $\mathbb{R}_+^{E_{N'}}$, which we call the *Kruskal cone with respect to* σ.

Let $w \in \mathcal{W}^{N'}$ and let $\sigma \in \Sigma_{E_{N'}}$ be such that $w \in K^{\sigma}$. We can consider a sequence of precisely $|E_{N'}| + 1$ graphs $\langle N', F^{\sigma,0} \rangle, \langle N', F^{\sigma,1} \rangle, \ldots, \langle N', F^{\sigma,|E_{N'}|} \rangle$ such that $F^{\sigma,0} = \emptyset$, $F^{\sigma,k} = F^{\sigma,k-1} \cup \{\sigma(k)\}$ for each $k \in \{1, \ldots, |E_{N'}|\}$.

For each graph $\langle N', F^{\sigma,k} \rangle$, with $k \in \{0, 1, \ldots, |E_{N'}|\}$, let $\pi^{\sigma,k}$ be the partition of N' consisting of the connected components of N' in $\langle N', F^{\sigma,k} \rangle$.

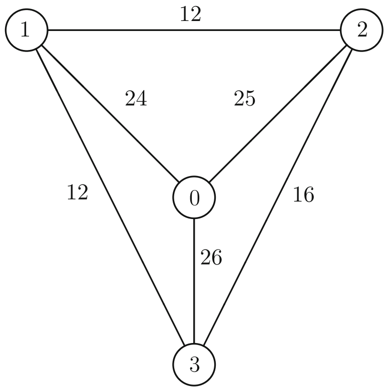

Fig. 2 An mcst situation with three agents

Remark 1 For each $k \in \{1, \ldots, |E_{N'}|\}$, $\pi^{\sigma,k}$ is either equal to $\pi^{\sigma,k-1}$ or obtained from $\pi^{\sigma,k-1}$ by taking the union of two elements of $\pi^{\sigma,k-1}$.

Now we define recursively the function $\rho^\sigma : N' \to \{0, 1, \ldots, |E_{N'}|\}$ by

- $\rho^\sigma(0) = 0$
- $\rho^\sigma(j) = \min\{k \in \{\rho^\sigma(j-1)+1, \ldots, |E_{N'}|\} | \pi^{\sigma,k} \neq \pi^{\sigma,\rho^\sigma(j-1)}\}$

for each $j \in N$.

Note that $\pi^{\sigma,\rho^\sigma(i)} \neq \pi^{\sigma,\rho^\sigma(j)}$ for each $i, j \in N$ with $i \neq j$, and $\sigma(\rho^\sigma(1)), \ldots, \sigma(\rho^\sigma(n))$ correspond to the n accepted edges in the Kruskal procedure based on the ordering σ.

Example 6 Consider the mcst situation $\langle N', w\rangle$ with $N' = \{0, 1, 2, 3\}$ and w as depicted in Fig. 2. Note that $w \in K^\sigma$, with $\sigma(1) = \{1, 2\}$, $\sigma(2) = \{1, 3\}$, $\sigma(3) = \{2, 3\}$, $\sigma(4) = \{1, 0\}$, $\sigma(5) = \{2, 0\}$, $\sigma(6) = \{3, 0\}$.

The sequence of seven graphs $\langle N', F^{\sigma,k}\rangle$ and the corresponding sequence of partitions $\pi^{\sigma,k}$ are shown in the following table

k	$F^{\sigma,k}$	$\pi^{\sigma,k}$
0	$\emptyset$	$\{\{0\}, \{1\}, \{2\}, \{3\}\}$
1	$\{\{1, 2\}\}$	$\{\{0\}, \{1, 2\}, \{3\}\}$
2	$\{\{1, 2\}, \{1, 3\}\}$	$\{\{0\}, \{1, 2, 3\}\}$
3	$\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$	$\{\{0\}, \{1, 2, 3\}\}$
4	$\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 0\}\}$	$\{N'\}$
5	$\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 0\}, \{2, 0\}\}$	$\{N'\}$
6	$\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 0\}, \{2, 0\}, \{3, 0\}\}$	$\{N'\}$

Then $\rho^\sigma(0) = 0$, $\rho^\sigma(1) = 1$, $\rho^\sigma(2) = 2$, $\rho^\sigma(3) = 4$.

Definition 4 Let $\mathcal{C} = \{C^1, \ldots, C^n\}$ be a charge system on N. Let $\sigma \in \Sigma_{E_{N'}}$ and let K^σ be the Kruskal cone w.r.t. σ. The *Construct and Charge* (CC-) *rule* w.r.t. $\mathcal{C}$ and σ

is the map $\chi^{\mathcal{C},\sigma} : K^\sigma \to \mathbb{R}^N$ given by

$$\chi^{\mathcal{C},\sigma}(w) = \sum_{r=1}^{n} w(\sigma(\rho^\sigma(r)))C^r(\sigma(\rho^\sigma(1)), \dots, \sigma(\rho^\sigma(r))). \quad (7)$$

for each mcst situation w in the cone K^σ. If $\mathcal{C}$ is conservative, we say that $\chi^{\mathcal{C},\sigma}$ is a conservative CC-rule.

Note that the CC-rule $\chi^{\tilde{\mathcal{C}},\sigma}$, where $\tilde{\mathcal{C}}$ is the charge system of Example 2, corresponds to the Proportional rule introduced in Feltkamp et al. (1994a). Moreover, the P-value (Branzei et al. 2004; Feltkamp et al. 1994b) is a CC-rule. In fact $\chi^{\hat{\mathcal{C}},\sigma}(w) = P(w)$ for each mcst situation w in the cone K^σ, where $\hat{\mathcal{C}}$ is the charge system of Example 3.

The following example illustrates these two CC-rules.

Example 7 Consider the mcst situation $\langle N', w\rangle$ with $N' = \{0, 1, 2, 3\}$ and w as depicted in Fig. 2. Let σ be as in Example 6 and $\sigma'(1) = \{1, 3\}$, $\sigma'(2) = \{1, 2\}$, $\sigma'(3) = \{2, 3\}$, $\sigma'(4) = \{1, 0\}$, $\sigma'(5) = \{2, 0\}$, $\sigma'(6) = \{3, 0\}$. Now, we apply Definition 4 to the charge systems introduced in Examples 2 and 3 to calculate the allocations provided by the corresponding CC-rules on $\langle N', w\rangle$.

The charge system $\tilde{\mathcal{C}}$ of Example 2 leads to

$$\begin{aligned}&\chi^{\tilde{\mathcal{C}},\sigma'}(w)\\&= 12 * (\tfrac{1}{2}, 0, \tfrac{1}{2})^t + 12 * (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4})^t + 24 * (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4})^t\\&= (15, 18, 15)^t\end{aligned}$$

and

$$\begin{aligned}&\chi^{\tilde{\mathcal{C}},\sigma}(w)\\&= 12 * (\tfrac{1}{2}, \tfrac{1}{2}, 0)^t + 12 * (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2})^t + 24 * (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2})^t\\&= (15, 15, 18)^t.\end{aligned}$$

Note that $\chi^{\tilde{\mathcal{C}},\sigma}(w) \neq \chi^{\tilde{\mathcal{C}},\sigma'}(w)$.

The charge system $\hat{\mathcal{C}}$ of Example 3 leads to

$$\begin{aligned}&\chi^{\hat{\mathcal{C}},\sigma'}(w)\\&= 12 * (\tfrac{1}{2}, 0, \tfrac{1}{2})^t + 12 * (\tfrac{1}{6}, \tfrac{2}{3}, \tfrac{1}{6})^t + 24 * (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3})^t\\&= (16, 16, 16)^t\end{aligned}$$

and

$$\begin{aligned}&\chi^{\hat{\mathcal{C}},\sigma}(w)\\&= 12 * (\tfrac{1}{2}, \tfrac{1}{2}, 0)^t + 12 * (\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{2}{3})^t + 24 * (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3})^t\\&= (16, 16, 16)^t.\end{aligned}$$

Note that $\chi^{\hat{\mathcal{C}},\sigma}(w) = \chi^{\hat{\mathcal{C}},\sigma'}(w)$.

From Examples 5 and 7 it seems that conservative CC-rules do not depend on the mcst obtained from the Kruskal algorithm, while CC-rules which are not conservative do depend on the selected mcst. In next section, we show that this holds in general.

6 Conservative CC-rules and Obligation rules

The main results in this section are derived from the relation between Obligation rules (Tijs et al. 2006a) and conservative CC-rules.

We first recall some definitions from Tijs et al. (2006a). A function $o : 2^N \setminus \{\emptyset\} \to \mathbb{R}^N_+$ is called an *obligation function* if the following two properties hold for each $S \in 2^N \setminus \{\emptyset\}$:

o.1) $o(S) \in \Delta(S)$,
o.2) for each $T \in 2^N \setminus \{\emptyset\}$ with $S \subseteq T$: $o_i(S) \geq o_i(T)$ for all $i \in S$,

where the sub-simplex $\Delta(S)$ of $\Delta(N) = \{x \in \mathbb{R}^N_+ | \sum_{i \in N} x_i = 1\}$ is given by $\Delta(S) = \{x \in \Delta(N) | \sum_{i \in S} x_i = 1\}$.

Given an obligation function o, the *obligation map* $\hat{o} : \Theta(N') \to \mathbb{R}^N$ is defined by

$$\hat{o}(\theta) = \sum_{S \in \theta, 0 \notin S} o(S) \tag{8}$$

for each $\theta \in \Theta(N')$, with the convention $\hat{o}_i(\theta) = 0$ for each $i \in N$ and $\theta = \{N'\}$.

Let $\hat{o}$ be an obligation map on $\Theta(N')$ and let $\sigma, \sigma' \in \Sigma_{E_{N'}}$. The map $\phi^{\sigma,\hat{o}} : K^\sigma \to \mathbb{R}^N$ defined for each $w \in K^\sigma$ by

$$\phi^{\sigma,\hat{o}}(w) = \sum_{r=1}^{|E_{N'}|} w(\sigma(r)) \left(\hat{o}(\pi^{\sigma,r-1}) - \hat{o}(\pi^{\sigma,r})\right) \tag{9}$$

or, alternatively, by

$$\phi^{\sigma,\hat{o}}(w) = \sum_{r=1}^{n} w(\sigma(\rho^\sigma(r))) \left(\hat{o}(\pi^{\sigma,\rho^\sigma(r-1)}) - \hat{o}(\pi^{\sigma,\rho^\sigma(r)})\right) \tag{10}$$

is used in Tijs et al. (2006a) to prove that

$$\phi^{\sigma,\hat{o}}(w) = \phi^{\sigma',\hat{o}}(w) \tag{11}$$

for all $w \in K^\sigma \cap K^{\sigma'}$, leading to the notion of *Obligation rule* as the map $\phi^{\hat{o}} : \mathcal{W}^{N'} \to \mathbb{R}^N$ defined by

$$\phi^{\hat{o}}(w) = \phi^{\sigma,\hat{o}}(w) \tag{12}$$

for each $w \in \mathcal{W}^{N'}$, where $\sigma \in \Sigma_{E_{N'}}$ is such that $w \in K^\sigma$.

The following proposition shows that obligation maps may be used to define charge systems.

Proposition 2 *Let $\hat{o}$ be an obligation map on $\Theta(N')$ and let $\mathcal{C} = \{C^1, \ldots, C^n\}$ be a set of functions, such that*

$$C^j(\mathbf{a}_{|j}) = \hat{o}(\Pi(\mathbf{a}_{|j-1})) - \hat{o}(\Pi(\mathbf{a}_{|j})) \tag{13}$$

for each $\mathbf{a} \in \mathcal{E}_{N'}$ and $j \in N$. Then, $\mathcal{C}$ is a charge system.

Proof It is easy to see, via relation (8), that $\mathcal{C}$ satisfies the connection property and the involvement property. By relation (13), we have

$$\begin{aligned}
&\sum_{j=1}^{n} C_i^j(\mathbf{a}_{|j}) \\
&\quad = \sum_{j=1}^{n} \hat{o}_i(\Pi(\mathbf{a}_{|j-1})) - \hat{o}_i(\Pi(\mathbf{a}_{|j})) \\
&\quad = \hat{o}_i(\Pi(\mathbf{a}_{|0})) - \hat{o}_i(\Pi(\mathbf{a}_{|n})) = 1 - 0
\end{aligned} \tag{14}$$

for each $i \in N$, which proves that $\mathcal{C}$ satisfies the total aggregation property as well. As a consequence, $\mathcal{C}$ is a charge system on N. □

To study whether a given charge system may be obtained as a difference of obligation maps it makes sense to introduce for charge systems the following property.

Definition 5 Let $\mathcal{C} = \{C^1, \ldots, C^n\}$ be a charge system on N. We say that $\mathcal{C}$ has the *obligation property* if there exists an obligation map $\hat{o}$ on $\Theta(N')$ such that

$$C^j(\mathbf{a}_{|j}) = \hat{o}(\Pi(\mathbf{a}_{|j-1})) - \hat{o}(\Pi(\mathbf{a}_{|j})) \tag{15}$$

for each $\mathbf{a} \in \mathcal{E}_{N'}$ and $j \in N$.

Remark 2 Let $\chi^{\mathcal{C},\sigma}$ be a CC-rule w.r.t. a charge system $\mathcal{C}$ with the obligation property and an ordering $\sigma \in K^\sigma$. Let $\hat{o}$ be an obligation map on $\Theta(N')$ such that relation (15) holds on $\mathcal{C}$. Then, by relation (10) we have

$$\begin{aligned}
\phi^{\sigma,\hat{o}}(w) &= \sum_{r=1}^{n} w(\sigma(\rho^\sigma(r))) \left(\hat{o}(\pi^{\sigma,\rho^\sigma(r-1)}) - \hat{o}(\pi^{\sigma,\rho^\sigma(r)})\right) \\
&= \sum_{r=1}^{n} w(\sigma(\rho^\sigma(r))) C^r(\sigma(\rho^\sigma(1)), \ldots, \sigma(\rho^\sigma(r))) = \chi^{\mathcal{C},\sigma}(w),
\end{aligned} \tag{16}$$

for all $\sigma \in \Sigma_{E_{N'}}$ and $w \in K^\sigma$.

In the following theorem, we give a sufficient condition for charge systems to satisfy the obligation property.

Theorem 1 *Let $\mathcal{C} = \{C^1, \ldots, C^n\}$ be a conservative charge system on N. Then $\mathcal{C}$ has the obligation property.*

Proof Let $P^{\mathcal{C}}(S)$ be the potential of S with respect to the conservative charge system $\mathcal{C}$ for each $S \in 2^N \setminus \{\emptyset\}$. Consider the map $o^{\mathcal{C}} : 2^N \setminus \{\emptyset\} \to \mathbb{R}^N_+$ defined by

$$o^{\mathcal{C}}(S) = e^S - P^{\mathcal{C}}(S) \tag{17}$$

for each $S \in 2^N \setminus \{\emptyset\}$, where $e^S \in \mathbb{R}^N_+$ is such that $e^S_i = 1$ for each $i \in S$ and $e^S_i = 0$ for each $i \in N \setminus S$. Note that for each $j \in N$, we have

$$\begin{aligned}
&\hat{o}^{\mathcal{C}}(\Pi(\mathbf{a}_{|j-1})) - \hat{o}^{\mathcal{C}}(\Pi(\mathbf{a}_{|j}))\\
&= \sum_{S\in\Pi(\mathbf{a}_{|j-1}),0\notin S} o^{\mathcal{C}}(S) - \sum_{S\in\Pi(\mathbf{a}_{|j}),0\notin S} o^{\mathcal{C}}(S)\\
&= \sum_{S\in\Pi(\mathbf{a}_{|j-1}),0\notin S} \left(e^S - P^{\mathcal{C}}(S)\right) - \sum_{S\in\Pi(\mathbf{a}_{|j}),0\notin S} \left(e^S - P^{\mathcal{C}}(S)\right)\\
&= \sum_{S\in\Pi(\mathbf{a}_{|j-1})} \left(e^{S\setminus\{0\}} - P^{\mathcal{C}}(S)\right) - \sum_{S\in\Pi(\mathbf{a}_{|j})} \left(e^{S\setminus\{0\}} - P^{\mathcal{C}}(S)\right)\\
&= \sum_{S\in\Pi(\mathbf{a}_{|j})} P^{\mathcal{C}}(S) - \sum_{S\in\Pi(\mathbf{a}_{|j-1})} P^{\mathcal{C}}(S)\\
&\quad + \sum_{S\in\Pi(\mathbf{a}_{|j-1})} e^{S\setminus\{0\}} - \sum_{S\in\Pi(\mathbf{a}_{|j})} e^{S\setminus\{0\}}\\
&= \sum_{S\in\Pi(\mathbf{a}_{|j})} P^{\mathcal{C}}(S) - \sum_{S\in\Pi(\mathbf{a}_{|j-1})} P^{\mathcal{C}}(S)\\
&= A^{\mathcal{C}}(\mathbf{a}_{|j}) - A^{\mathcal{C}}(\mathbf{a}_{|j-1})\\
&= \sum_{k=1}^{j} C^k(\mathbf{a}_{|k}) - \sum_{k=1}^{j-1} C^k(\mathbf{a}_{|k}) = C^j(\mathbf{a}_{|j}),
\end{aligned}$$

where the third equality follows from Lemma 1. (c.1), the fifth equality follows from the fact that $\sum_{S\in\theta} e^{S\setminus\{0\}} = e^N$ for each $\theta \in \Theta(N')$ and the sixth equality from Proposition 1.

We want to prove that $o^{\mathcal{C}}$ is an obligation function, i.e. $o^{\mathcal{C}}$ satisfies the properties (o.1) and (o.2).

By definition, it follows directly that $o^{\mathcal{C}}_i(S) = 0$ for each $i \in N \setminus S$, and $o^{\mathcal{C}}_i(S) \geq 0$ for each $i \in S$ and for each $S \in 2^N \setminus \{\emptyset\}$. Moreover, from condition (c.2) in Lemma 1, it follows that

$$\sum_{i\in N} o^{\mathcal{C}}_i(S) = \sum_{i\in S} \left(1 - P^{\mathcal{C}}_i(S)\right) = |S| - (|S| - 1) = 1,$$

for each $S \in 2^N \setminus \{\emptyset\}$, implying that condition (o.1) holds.

Finally, by condition (c.3) in Lemma 1, we have that for each $S \subseteq T \subseteq N$, $S \neq \emptyset$, and each $i \in S$

$$o_i^{\mathcal{C}}(S) = 1 - P_i^{\mathcal{C}}(S) \geq 1 - P_i^{\mathcal{C}}(T) = o_i^{\mathcal{C}}(T), \tag{18}$$

which proves that condition (o.2) holds, too. □

Remark 3 Since the P-value (Branzei et al. 2004) and the P^τ-values, with $\tau \in \Sigma_N$, introduced in Norde et al. (2004) and studied in Branzei et al. (2004), are Obligation rules, one can obtain the corresponding charge systems using relation (13). It is easy to check, for example, that $\chi^{\hat{\mathcal{C}}}$, where $\hat{\mathcal{C}}$ is the charge system of Example 3, is in fact the P-value, since $\hat{\mathcal{C}}$ is obtained by relation (13) for the obligation maps which define the P-value (see Tijs et al. 2006a).

Next theorem answers the question introduced at the end of Sect. 5, i.e. conservative CC-rules do not depend on the mcst obtained from Kruskal algorithm.

Theorem 2 *Let $\mathcal{C} = \{C^1, \ldots, C^n\}$ be a charge system on N. The following statements are equivalent:*

(i) $\chi^{\mathcal{C},\sigma_1}(w) = \chi^{\mathcal{C},\sigma_2}(w)$ for all $\sigma_1, \sigma_2 \in \Sigma_{E_{N'}}$ and $w \in K^{\sigma_1} \cap K^{\sigma_2}$.
(ii) $\mathcal{C}$ is conservative.

Proof First, we prove the implication (i)⇒(ii).

Suppose that (i) holds and $\mathcal{C}$ is not conservative. Then, we can find a $j \in N$ and a pair $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n) \in \mathcal{E}_{N'}$, with $\Pi(\mathbf{a}_{|j}) = \Pi(\mathbf{b}_{|j})$ and $\mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}) \neq \mathrm{A}^{\mathcal{C}}(\mathbf{b}_{|j})$.

Suppose $\Pi(\mathbf{a}_{|j}) = \{S_1, S_2, \ldots, S_m\}$ and take $w \in \mathcal{W}^{N'}$ such that

$$w(\{i, j\}) = \begin{cases} 0 & \text{if there exists } r \in \{1, \ldots, m\} \text{ s.t. } i, j \in S_r, \\ 1 & \text{otherwise,} \end{cases}$$

for each $\{i, j\} \in E_{N'}$. Let $\sigma_1 \in \Sigma_{E_{N'}}$ be such that $\sigma_1(\rho^{\sigma_1}(k)) = a_k$ for each $k \in \{1, \ldots, j\}$ and $\sigma_1(\rho^{\sigma_1}(l)) = d_l$ for each $l \in \{j+1, \ldots, n\}$, with $(a_1, \ldots, a_j, d_{j+1}, \ldots, d_n) \in \mathcal{E}_{N'}$.

Let $\sigma_2 \in \Sigma_{E_{N'}}$ be such that $\sigma_2(\rho^{\sigma_2}(k)) = b_k$ for each $k \in \{1, \ldots, j\}$ and $\sigma_2(\rho^{\sigma_2}(l)) = d_l$ for each $l \in \{j+1, \ldots, n\}$, with $(b_1, \ldots, b_j, d_{j+1}, \ldots, d_n) \in \mathcal{E}_{N'}$.

In addition, σ_1 and σ_2 can be chosen such that $w \in K^{\sigma_1} \cap K^{\sigma_2}$. We have

$$\begin{aligned}
&\chi^{\mathcal{C},\sigma_1}(w) \\
&\quad = \sum_{r=1}^{j} w(a_r) C^r(\mathbf{a}_{|r}) + \sum_{r=j+1}^{n} w(d_r) C^r(a_1, \ldots, a_j, d_{j+1}, \ldots, d_r) \\
&\quad = \sum_{r=j+1}^{n} C^r(a_1, \ldots, a_j, d_{j+1}, \ldots, d_r) \\
&\quad = e^N - \sum_{r=1}^{j} C^r(\mathbf{a}_{|r}) \\
&\quad = e^N - \mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}),
\end{aligned}$$

where the third equality follows from the total aggregation property.

Similarly,

$$\chi^{\mathcal{C},\sigma_2}(w) = e^N - \sum_{r=1}^{j} C^r(\mathbf{b}_{|r}) = e^N - \mathrm{A}^{\mathcal{C}}(\mathbf{b}_{|j}).$$

So, $\chi^{\mathcal{C},\sigma_1}(w) \neq \chi^{\mathcal{C},\sigma_2}(w)$, which yields a contradiction with the fact that (i) holds.

Now, we prove the implication (ii)⇒(i).

From Theorem 1, we have that a conservative charge system has the obligation property. Consequently, there exists an obligation map $\hat{o}$ on $\Theta(N')$ such that

$$C^j(\mathbf{a}_{|j}) = \hat{o}(\Pi(\mathbf{a}_{|j-1})) - \hat{o}(\Pi(\mathbf{a}_{|j}))$$

for each $\mathbf{a} \in \mathcal{E}_{N'}$ and $j \in N$. Then, by relation (16), for all $\sigma \in \Sigma_{E_{N'}}$ and $w \in K^\sigma$ we have $\phi^{\sigma,\hat{o}}(w) = \chi^{\mathcal{C},\sigma}(w)$, where $\phi^{\sigma,\hat{o}}$ is the map introduced by (10) and $\chi^{\mathcal{C},\sigma}$ is the CC-rule w.r.t. $\mathcal{C}$ and σ. As a consequence, by relation (11) we have $\chi^{\mathcal{C},\sigma_1}(w) = \phi^{\sigma_1,\hat{o}}(w) = \phi^{\sigma_2,\hat{o}}(w) = \chi^{\mathcal{C},\sigma_2}(w)$ for all $\sigma_1, \sigma_2 \in \Sigma_{E_{N'}}$ and $w \in K^{\sigma_1} \cap K^{\sigma_2}$, which concludes the proof. □

Next theorem is the main result of this section.

Theorem 3 *For each charge system $\mathcal{C} = \{C^1, \ldots, C^n\}$ on N the following statements are equivalent:*

(i) $\mathcal{C}$ is a conservative charge system;
(ii) $\mathcal{C}$ satisfies the obligation property.

Proof (i)⇒(ii). By Theorem 1, if $\mathcal{C}$ is a conservative charge system then $\mathcal{C}$ satisfies the obligation property.

(ii)⇒(i). If $\mathcal{C}$ satisfies the obligation property, then by relation (16) there exists an obligation map $\hat{o}$ on $\Theta(N')$ such that $\chi^{\mathcal{C},\sigma}(w) = \phi^{\sigma,\hat{o}}(w)$ for all $\sigma \in \Sigma_{E_{N'}}$ and $w \in K^\sigma$, where $\phi^{\sigma,\hat{o}}$ is the map introduced by (10) and $\chi^{\mathcal{C},\sigma}$ is the CC-rule w.r.t. $\mathcal{C}$ and σ. By relation (11) we have $\chi^{\mathcal{C},\sigma_1}(w) = \phi^{\sigma_1,\hat{o}}(w) = \phi^{\sigma_2,\hat{o}}(w) = \chi^{\mathcal{C},\sigma_2}(w)$ for all $\sigma_1, \sigma_2 \in \Sigma_{E_{N'}}$ and $w \in K^{\sigma_1} \cap K^{\sigma_2}$. Consequently, by Theorem 2, we have that $\mathcal{C}$ is conservative. □

From Remark 2 and Theorem 3 we conclude that the class of conservative CC-rules coincides with the class of Obligation rules, and Tijs et al. (2006a) proved that Obligation rules are cost monotonic rules.

By Theorem 2, we have that non-conservative CC-rules may provide different cost allocations, which depend on the mcst obtained from Kruskal algorithm. So, non-conservative CC-rules are *multi*-solutions for mcst situations (Tijs et al. 2006b). Note that the definition of cost monotonicity studied in the present paper applies to solutions for mcst situations. For mcst situations, Tijs et al. (2006b) introduced a concept of cost monotonicity for multi-solutions which generalizes the concept of cost monotonicity for solutions. However, cost monotonicity for multi-solutions is not satisfied in general by non-conservative CC-rules, as it is shown in Example 8, dealing with specific mcst situations where the optimal tree is unique.

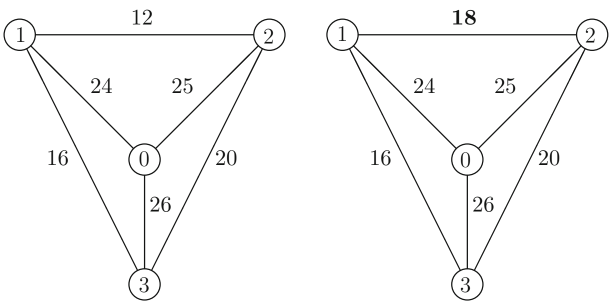

Fig. 3 Two mcst situations w (*left side*) and w' (*right side*)

Example 8 Consider the mcst situation $\langle N', w\rangle$ with $N' = \{0, 1, 2, 3\}$ and w as depicted in Fig. 3 (left side). Note that there exists a unique $\sigma \in \Sigma_{N'}$ with $w \in K^{\sigma}$, where σ is such that $\sigma(1) = \{1, 2\}$, $\sigma(2) = \{1, 3\}$, $\sigma(3) = \{2, 3\}$, $\sigma(4) = \{1, 0\}$, $\sigma(5) = \{2, 0\}$, $\sigma(6) = \{3, 0\}$.

We apply Definition 4 to the charge systems $\tilde{\mathcal{C}}$ introduced in Example 2 to calculate the allocations provided by the corresponding CC-rules on $\langle N', w\rangle$. We have

$$\begin{aligned}
&\chi^{\tilde{\mathcal{C}},\sigma}(w) \\
&= 12 * \left(\frac{1}{2}, \frac{1}{2}, 0\right)^t + 16 * \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)^t + 24 * \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)^t \\
&= (16, 16, 20)^t.
\end{aligned}$$

Now, consider the mcst situation $\langle N', w'\rangle$ with w' as depicted in Fig. 3 (right side), where $w'(e) = w(e)$ for all $e \in E_{N'} \setminus \{1, 2\}$ and $w'(\{1, 2\}) > w(\{1, 2\})$. Note that also for this mcst situation there exists a unique $\sigma' \in \Sigma_{N'}$ with $w' \in K^{\sigma}$, where σ' is such that $\sigma'(1) = \{1, 3\}$, $\sigma'(2) = \{1, 2\}$, $\sigma'(3) = \{2, 3\}$, $\sigma'(4) = \{1, 0\}$, $\sigma'(5) = \{2, 0\}$, $\sigma'(6) = \{3, 0\}$. We have

$$\begin{aligned}
&\chi^{\tilde{\mathcal{C}},\sigma'}(w') \\
&= 16 * \left(\frac{1}{2}, 0, \frac{1}{2}\right)^t + 18 * \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)^t + 24 * \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)^t \\
&= (18.5, 21, 18.5)^t.
\end{aligned}$$

Agent 3 is better off in w', where the cost of edge $\{1, 2\}$ is larger.

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