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Hahn–Banach extension theorems for multifunctions revisited

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Abstract Several generalizations of the Hahn–Banach extension theorem to K-convex multifunctions were stated recently in the literature. In this note we provide an easy direct proof for the multifunction version of the Hahn–Banach–Kantorovich theorem and show that in a quite general situation it can be obtained from existing results. Then we derive the Yang extension theorem using a similar proof as well as a stronger version of it using a classical separation theorem. Moreover, we give counterexamples to several extension theorems stated in the literature.

Keywords Hahn–Banach–Kantorovich extension theorem · Yang extension theorem · K-convex multifunction · Intrinsic core

1 Introduction

It is well known the importance of the Hahn–Banach theorem in Functional Analysis. It was not a surprise that the generalization of this theorem to the case when $\mathbb{R}$ is replaced by an ordered linear space having the least upper bound property, the so called Hahn–Banach–Kantorovich theorem, interested many mathematicians. In the last period working with set-valued functions instead of functions became an important tool in analysis, mainly in non-smooth analysis. In this context the interest for having versions of the Hahn–Banach–Kantorovich theorem with the sublinear or convex

Dedicated to Jean-Paul Penot with the occasion of his retirement.

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operators replaced by convex processes or convex multifunctions increased. On the other hand, in $\mathbb{R}$ the logic propositions $\alpha \leq \beta$ and $\alpha \not> \beta$ are equivalent, which is not the case when the order relation $\leq$ on the linear space Z is defined by a convex cone K with non-empty algebraic interior. So having extension results not only for the relation $\leq$, but also for the relation $\not>$ is natural (and maybe more useful). The scope of this paper is to discuss Hahn–Banach type extension theorems involving multifunctions. We point out that several recent results concerning extension theorems of Kantorovich type for convex and affine (like) multifunctions follow easily from known (now classic) results; for this we extend slightly the version of the Hahn–Banach–Kantorovich theorem for multifunctions. Then we reinforce the conclusion of Yang's Hahn–Banach type theorem (Yang 1992) providing two proofs.

In the sequel X, Y, Z are real linear spaces. The class of linear operators from X into Z is denoted by $\mathcal{L}(X, Z)$; we set $X' := \mathcal{L}(X, \mathbb{R})$. We consider $K \subset Z$ a proper (i.e. $\{0\} \neq K \neq Z$), convex cone containing 0. The cone K induces a partial order on Z, denoted $\leq_K$ or simply $\leq$ if there is no risk of confusion. So, for $z_1, z_2 \in Z$ one has $z_1 \leq z_2$ (or equivalently $z_2 \geq z_1$) if $z_2 - z_1 \in K$. We denote by A^i the algebraic interior (or core) of $A \subset X$ and by ${}^iA$ the relative algebraic interior (or intrinsic core) of A, that is ${}^iA$ is the algebraic interior of A with respect to the affine hull aff A of A. Recall that for A a convex set one has

$$a \in {}^iA \Leftrightarrow [\forall x \in A, \exists \lambda \in \mathbb{P} : (1+\lambda)a - \lambda x \in A], \tag{1}$$

where $\mathbb{P} :=]0, \infty[\subset \mathbb{R}$. When $K_0 := K^i \neq \emptyset$, we use $z_1 < z_2$ (or $z_2 > z_1$) when $z_2 - z_1 \in K_0$, and $z_1 \not< z_2$ (or $z_2 \not> z_1$) if $z_2 - z_1 \notin K_0$. We extend Z to $Z^\bullet := Z \cup \{-\infty, +\infty\}$, $\pm\infty \notin Z$, and consider that $-\infty \leq z \leq +\infty$ (even $-\infty < z < +\infty$ if $K_0 \neq \emptyset$) for all $z \in Z$; moreover, $z + (\pm\infty) := \pm\infty$, $0 \cdot (+\infty) := +\infty$, $0 \cdot (-\infty) := 0$, $t \cdot (\pm\infty) := \pm\infty$ for all $z \in Z$ and $t \in \mathbb{P}$. For a function $f : X \to Z^\bullet$, its domain is $\operatorname{dom} f := \{x \in X \mid f(x) \in Z\}$ and its epigraph is $\operatorname{epi} f := \{(x, z) \in X \times Z \mid f(x) \leq z\}$; f is proper if $\operatorname{dom} f \neq \emptyset$ and it does not take the value $-\infty$, while f is convex if epi f is convex. We say that $f : X \to Z^\bullet$ is sublinear if f is proper, $f(0) = 0$, $f(tx) = tf(x)$ and $f(x + x') \leq f(x) + f(x')$ for all $x, x' \in X$ and $t \in \mathbb{P}$; hence epi f is a convex cone containing the origin when f is sublinear.

In the sequel we identify a multifunction $\Gamma : E \rightrightarrows F$ (that is, a function $\Gamma : E \to 2^F$) with its graph $\operatorname{gph} \Gamma := \{(x, y) \in E \times F \mid y \in \Gamma(x)\}$. So, several times in the sequel, having a subset A of $E \times F$ we interpret A as the graph of a multifunction from E to F, and so $A(x) := \{y \in F \mid (x, y) \in A\}$. Of course, for the multifunction $\Gamma : E \rightrightarrows F$ we have $\operatorname{dom} \Gamma = P_E(\Gamma)$ and $\operatorname{Im} \Gamma = P_F(\Gamma)$, where $P_E : E \times F \to E$, $P_E(x, y) := x$ and similarly for P_F. The multifunction $\Gamma : X \rightrightarrows Z$ is said to be K–convex if its epigraph $\operatorname{epi} \Gamma := \operatorname{gph} \Gamma + (\{0\} \times K)$ is convex, that is, the graph of the multifunction $\Gamma_K : X \rightrightarrows Z$ defined by $\Gamma_K(x) := \Gamma(x) + K$ is convex. Of course, for $A, B \subset X$, $x \in X$, $\Lambda \subset \mathbb{R}$ and $\alpha \in \mathbb{R}$ we set $A + B := \{a + b \mid a \in A, b \in B\}$, $x + A := \{x\} + A$, $\Lambda A := \{\lambda x \mid \lambda \in \Lambda, a \in A\}$ and $\alpha A := \{\alpha\}A$ (so $A + \emptyset = \emptyset + A = \emptyset$, $\alpha\emptyset = \emptyset$).

2 The Hahn–Banach–Kantorovich theorem

In this section the proper convex cone $K \subset Z$ is pointed (i.e. $K \cap (-K) = \{0\}$) and (Z, K) has the least upper bound property, i.e. every non-empty and upper bounded set has a least upper bound; of course, in this situation every non-empty and lower bounded set has a greatest lower bound. If the non-empty set $B \subset Z$ is bounded from below (resp. above), we denote by $\inf B$ ($\sup B$) the greatest lower (resp. least upper) bound of B; if B is not bounded below (resp. above) we set $\inf B := -\infty$ (resp. $\sup B := +\infty$). We also use the notation $\inf \emptyset := +\infty$ and $\sup \emptyset := -\infty$.

We first give an extension theorem for convex multifunctions. As we shall see below, the result is equivalent to the classical Hahn–Banach–Kantorovich theorem when the relative algebraic interior of the domain of the involved multifunction is non-empty.

Theorem 1 *Let $\Gamma : X \rightrightarrows Z$ be a K–convex multifunction, $X_0 \subset X$ a linear subspace and $T_0 \in \mathcal{L}(X_0, Z)$. Suppose that $0 \in {}^i(\operatorname{dom}\Gamma - X_0)$ and $T_0 x \leq z$ for all $(x, z) \in \Gamma \cap (X_0 \times Z)$. Then there exists $T \in \mathcal{L}(X, Z)$ such that $T\big|_{X_0} = T_0$ and $Tx \leq z$ for all $(x, z) \in \Gamma$.*

Proof Consider $\overline{X} := \operatorname{aff}(\operatorname{dom}\Gamma - X_0)$. Because $0 \in {}^i(\operatorname{dom}\Gamma - X_0)$, $\overline{X}$ is a linear space.

The case $\overline{X} = X$. Hence $0 \in (\operatorname{dom}\Gamma - X_0)^i$. Note that if X_1 is a linear subspace of X such that $X_0 \subset X_1$, then $0 \in (\operatorname{dom}\Gamma - X_1)^i$, too. Also note that it is sufficient to show that, if $\overline{x} \in X \setminus X_0$ then there exists $T_1 : X_1 \to Z$, with $X_1 := X_0 + \mathbb{R}\overline{x}$, such that $T_1\big|_{X_1} = T_0$ and $T_1 x \leq z$ for all $(x, z) \in \Gamma \cap (X_1 \times Z)$. If so, by using Zorn's lemma, as in the standard proof of the Hahn–Banach theorem, one gets a maximal T defined on the entire space X.

Let $\overline{x}$ and X_1 be as above. Since $0 \in (\operatorname{dom}\Gamma - X_0)^i$, there exists $\lambda \in \mathbb{P}$ such that $\pm\lambda\overline{x} \in \operatorname{dom}\Gamma - X_0$, and so there exist $x_1 \in X_0$ and $z_1 \in Z$ such that $(x_1 + \lambda\overline{x}, z_1) \in \Gamma$ and $x_2 \in X_0$ and $z_2 \in Z$ such that $(x_2 - \lambda\overline{x}, z_2) \in \Gamma$. Set $A := \operatorname{epi}\Gamma$. It follows that the sets

$$\begin{aligned} B_1 &:= \left\{ \frac{z_1 - T_0 x_1}{\lambda_1} \,\middle|\, x_1 \in X_0, z_1 \in Z, \lambda_1 \in \mathbb{P} : (x_1 + \lambda_1\overline{x}, z_1) \in A \right\} \\ &= \{z_1 - T_0 x_1 \mid x_1 \in X_0, z_1 \in Z : (x_1 + \overline{x}, z_1) \in \mathbb{P}A\}, \end{aligned}$$

and

$$\begin{aligned} B_2 &:= \left\{ \frac{T_0 x_2 - z_2}{\lambda_2} \,\middle|\, x_2 \in X_0, z_2 \in Z, \lambda_2 \in \mathbb{P} : (x_2 - \lambda_2\overline{x}, z_2) \in A \right\} \\ &= \{T_0 x_2 - z_2 \mid x_2 \in X_0, z_2 \in Z : (x_2 - \overline{x}, z_2) \in \mathbb{P}A\}, \end{aligned}$$

are non-empty (and convex) sets. Moreover,

$$\forall b_1 \in B_1, \forall b_2 \in B_2 : b_1 \geq b_2.$$

Indeed, let $b_1 \in B_1$ and $b_2 \in B_2$; then $b_1 = z_1 - T_0 x_1$ and $b_2 = T_0 x_2 - z_2$ with $x_1, x_2 \in X_0, z_1, z_2 \in Z$ and $(x_1 + \overline{x}, z_1), (x_2 - \overline{x}, z_2) \in \mathbb{P}A$. It follows that $(x_1 + x_2, z_1 + z_2) \in$

$\mathbb{P}A$ (because this set is a convex cone), and $x_1 + x_2 \in X_0$. From the hypothesis, we have that $T_0(x_1 + x_2) \leq z_1 + z_2$, which is equivalent to $b_2 \leq b_1$. It follows that B_2 is bounded from above, B_1 is bounded from below, and $\sup B_2 \leq \inf B_1$. Taking $\overline{z} \in Z$ such that $\sup B_2 \leq \overline{z} \leq \inf B_1$, and defining T_1 by $T_1(x + \lambda\overline{x}) := T_0 x + \lambda\overline{z}$ (for $x \in X_0$ and $\lambda \in \mathbb{R}$), we have that $T_1 : X_1 \to Z$ is linear, $T_1 \left|_{X_0}\right. = T_0$ and $T_1 x \leq z$ for all $(x, z) \in \Gamma \cap (X_1 \times Z)$. Let us prove the last assertion. So, take $(x, z) \in \Gamma$ such that $x := x_0 + \lambda\overline{x}$ with $x_0 \in X_0$ and $\lambda \in \mathbb{R}$. If $\lambda = 0$ then $(x, z) = (x_0, z) \in \Gamma \cap (X_0 \times Z)$, and so $T_1 x = T_0 x_0 \leq z$. If $\lambda < 0$ then $\lambda_2 := -\lambda > 0$ and $T_1 x = T_0 x_0 - \lambda_2 \overline{z}$. Setting $b_2 := \lambda_2^{-1}(T_0 x_0 - z)$ we have that $b_2 \in B_2$. But

$$T_1 x \leq z \iff T_0 x_0 - \lambda_2 \overline{z} \leq z \iff \frac{T_0 x_0 - z}{\lambda_2} \leq \overline{z} \iff b_2 \leq \overline{z}.$$

Hence our assertion is true in this case by the choice of $\overline{z}$. The case $\lambda > 0$ is proven similarly.

The case $\overline{X} \neq X$. Of course, $\operatorname{dom}\Gamma \subset \overline{X}$. Taking $x_0 \in X_0 \cap \operatorname{dom}\Gamma$, we have that $X_0 = x_0 - X_0 \subset \overline{X}$. Applying the first case we find $\overline{T} : \overline{X} \to Z$ a linear operator such that $\overline{T} \left|_{X_0}\right. = T_0$ and $\overline{T}x \leq z$ for all $(x, z) \in \Gamma \subset \overline{X} \times Z$. Taking $\overline{Y}$ a linear subspace of X such that $X = \overline{X} \oplus \overline{Y}$ (that is $X = \overline{X} + \overline{Y}$ and $\overline{X} \cap \overline{Y} = \{0\}$) and $T : X \to Z$ defined by $T(\overline{x} + \overline{y}) := \overline{T}(\overline{x})$ for $\overline{x} \in \overline{X}$, $\overline{y} \in \overline{Y}$, T verifies the conclusion. □

Remark 1 In the proof of Theorem 1 we used only the fact that $\mathbb{P}\cdot \operatorname{epi}\Gamma$ is a convex cone and not that $\operatorname{epi}\Gamma$ itself is convex.

Note that it is possible to have convex sets $A, B \subset X$ with ${}^{i}A$ or ${}^{i}B$ empty but ${}^{i}(A - B)$ non-empty. For this take $A := B := \ell_2^+$ or $A := \{(x_n)_{n\geq 1} \in \ell_2 \mid x_1 = x_2\}$ and $B := \ell_2^+$ (in the second case A is a linear space, and so ${}^{i}A = A$). The fact that ${}^{i}(\ell_2^+) = \emptyset$ follows from the equality $\ell_2 = \ell_2^+ - \ell_2^+$; hence $\operatorname{aff} \ell_2^+ = \ell_2$, and so ${}^{i}(\ell_2^+) = (\ell_2^+)^i = \emptyset$. Related to operations with the intrinsic core we mention the following result which is well known in finite dimensional spaces. We provide its proof for reader's convenience.

Lemma 2 (i) *Let $A \subset X$ be a convex set and $T \in \mathcal{L}(X, Y)$. If ${}^{i}A$ is non-empty then ${}^{i}(T(A)) = T({}^{i}A)$.*
(ii) *Let $A \subset X$ and $B \subset Y$ be non-empty sets. Then ${}^{i}(A \times B) = {}^{i}A \times {}^{i}B$.*
(iii) *Let $A, C \subset X$ be convex sets such that ${}^{i}A$ and ${}^{i}C$ are non-empty. Then ${}^{i}(A - C) = {}^{i}A - {}^{i}C$.*

Proof (i) Let first $a \in {}^{i}A$ and take $z \in T(A)$; then $z = Tx$ with $x \in A$. Because $a \in {}^{i}A$, by (1) there exists $\lambda > 0$ such that $(1 + \lambda)a - \lambda x \in A$. It follows that $(1+\lambda)Ta - \lambda z \in T(A)$. Since $T(A)$ is convex, using again (1), we get $Ta \in {}^{i}(T(A))$. Conversely, fix $a \in {}^{i}A$ and take $z \in {}^{i}(T(A))$. By (2) we get $\lambda > 0$ such that $z' := (1 + \lambda)z - \lambda Ta \in T(A)$, and so $z' = Tx'$ with $x' \in A$. Then $z = Tx''$ with $x'' := \frac{1}{1+\lambda}x' + \frac{\lambda}{1+\lambda}a$. Since A is convex, $a \in {}^{i}A$, $x' \in A$ and $1/(1 + \lambda) \in \,]0, 1[$ we have that $x'' \in {}^{i}A$, and so $z \in T({}^{i}A)$.

(ii) It is clear (and known) that $\operatorname{aff}(A \times B) = \operatorname{aff} A \times \operatorname{aff} B$. Doing a translation we may assume that $0 \in A$ and $0 \in B$. In this way, replacing if necessary X by $\operatorname{aff} A$ and Y by $\operatorname{aff} B$, the conclusion reduces to $(A \times B)^i = A^i \times B^i$, which is immediate.

(iii) Consider $T \in \mathcal{L}(X \times X, X)$ defined by $T(x, x') := x + x'$. Applying (ii) to A and C, then (i) to T and $A \times C$ we get the conclusion. □

Corollary 3 *Let $p : X \to Z^\bullet$ be a sublinear operator, X_0 be a linear subspace of X and $T_0 \in \mathcal{L}(X_0, Z)$. Suppose that $T_0 x \le p(x)$ for every $x \in X_0$. If $X_0 + \operatorname{dom} p$ is a linear subspace of X then there exists $T \in \mathcal{L}(X, Z)$ such that $T\,|_{X_0} = T_0$ and $Tx \le p(x)$ for every $x \in X$.*

Proof Because $\operatorname{dom} p$ is a convex cone, the condition $X_0 + \operatorname{dom} p$ is a linear subspace is equivalent to $0 \in {}^i(\operatorname{dom} p - X_0)$. In Theorem 1 take $\operatorname{gph} \Gamma = \operatorname{epi} p$; of course, the hypotheses of the theorem hold, so that there exists $T \in \mathcal{L}(X, Z)$ such that $T\,|_{X_0} = T_0$ and $(x, y) \in \Gamma$ implies $Tx \le y$. Taking $y = p(x)$, the conclusion follows. □

When $\operatorname{dom} p = X$ the preceding corollary is the well-known Kantorovich' generalization of the Hahn–Banach extension theorem. Corollary 3 is equivalent to Theorem 3(3) in Malivert et al. (1978), which, at its turn covers Theorem 3(1) in Malivert et al. (1978). It is possible to state a version of Theorem 1 which, when applied to the epigraph of a sublinear operator, yields Theorem 3(2) in Malivert et al. (1978).

We emphasize the importance of the condition "$X_0 + \operatorname{dom} p$ is a linear subspace of X" in Corollary 3. Without this condition the conclusion of Corollary 3 could be false even for $Z = \mathbb{R}$ and $\dim X < \infty$; see Simons (1968) and Anger and Lembcke (1974) for interesting (counter) examples.

Before stating the next result recall that the subdifferential $\partial f(x_0)$ of the proper operator $f : X \to Z^\bullet$ at $x_0 \in \operatorname{dom} f$ is the set of those $T \in \mathcal{L}(X, Z)$ such that

$$\forall x \in X : Tx - Tx_0 \le f(x) - f(x_0).$$

Corollary 4 *Let $f : X \to Z^\bullet$ be a proper convex operator and consider $x_0 \in {}^i(\operatorname{dom} f)$. Then $\partial f(x_0)$ is non-empty.*

Proof Consider $g : X \to Z^\bullet$, $g(x) = f(x_0 + x) - f(x_0)$. Then g is a convex operator with $0 \in {}^i(\operatorname{dom} g)$. Consider now $\operatorname{gph} \Gamma = \operatorname{epi} g$, $X_0 = \{0\}$ and $T_0(0) := 0$; Γ is a convex multifunction. As $\operatorname{dom} \Gamma = \operatorname{dom} g$, we have that $0 \in {}^i(\operatorname{dom} \Gamma - X_0)$, and of course $(x, z) \in \Gamma \cap (X_0 \times Z)$ implies $T_0 x = 0 \le z$. Applying Theorem 1 we get $T \in \mathcal{L}(X, Z)$ such that $Tx \le z$ for every $(x, z) \in \Gamma$. In particular, if $x \in \operatorname{dom} f$ then $(x - x_0, f(x) - f(x_0)) \in \operatorname{epi} g$, whence $T(x - x_0) \le f(x) - f(x_0)$. The proof is complete. □

Note that Corollary 4 can be viewed as a particular case of the next result; just take $g : X \to Z^\bullet$ defined by $g(x_0) := -f(x_0)$ and $g(x) := +\infty$ for $x \in X \setminus \{x_0\}$. The next result is the sandwich theorem proved by Zowe (1978, Theorem 3.1) (see also Zălinescu 1983, Corollary 2.6).

Corollary 5 *Let $f, g : X \to Z^\bullet$ be proper convex operators. Suppose that $0 \in {}^i(\operatorname{dom} f - \operatorname{dom} g)$ and that $f(x) \ge -g(x)$ for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$. Then there exists $T \in \mathcal{L}(X, Z)$ and $z_0 \in Z$ such that*

$$\forall x \in X : -g(x) \le Tx + z_0 \le f(x).$$

Proof Consider $\Gamma : X \rightrightarrows Z$ with

$$\operatorname{gph}\Gamma := \{(x,z) \in X \times Z \mid f(x) \leq z\} + \{(-x',z') \in X \times Z \mid g(x') \leq z'\}.$$

Then Γ is convex and $\operatorname{dom}\Gamma = \operatorname{dom} f - \operatorname{dom} g$. The hypotheses of Theorem 1 hold for $X_0 = \{0\}$ and $T_0(0) := 0$. Indeed, if $(0,z) \in \Gamma$ then $(0,z) = (x,z') + (-x,z'')$, with $f(x) \leq z'$ and $g(x) \leq z''$; it follows that $T_0(0) = 0 \leq f(x)+g(x) \leq z'+z'' = z$. Therefore, there exists $T \in \mathcal{L}(X,Z)$ such that $Tx \leq z$ for $(x,z) \in \Gamma$. In particular, for $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} g$ we get $T(x_1 - x_2) \leq f(x_1) + g(x_2)$, which yields $-g(x_2) - Tx_2 \leq f(x_1) - Tx_1$ for all $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} g$. It follows that

$$\sup_{x_2 \in \operatorname{dom} g} (-g(x_2) - Tx_2) \leq \inf_{x_1 \in \operatorname{dom} f} (f(x_1) - Tx_1).$$

Taking z_0 between these two values, we get the desired conclusion. □

Note that Theorem 1 follows from Corollary 5 when ${}^i(\operatorname{dom}\Gamma)$ is non-empty. Indeed, in this case there exists $x_0 \in X_0 \cap {}^i(\operatorname{dom}\Gamma)$. Setting $f(x) := \inf\{z \mid (x,z) \in \Gamma\} \in Z^\bullet$, $T_0(x_0) \leq f(x_0) < +\infty$, and so $f(x_0) \in Z$. Using Proposition 1.5(i) in Zălinescu (1983) we get $f(x) \in Z$ for every $x \in \operatorname{dom}\Gamma$. (Indeed, take $x \in \operatorname{dom}\Gamma$. Since $x_0 \in {}^i(\operatorname{dom}\Gamma)$, there exists $x' \in \operatorname{dom}\Gamma$ and $\lambda \in]0,1[$ such that $x_0 = (1-\lambda)x' + \lambda x$. Take $z' \in Z$ with $(x',z') \in \Gamma$. Then for every z with $(x,z) \in \Gamma$ we have $(x_0, (1-\lambda)z' + \lambda z) \in \Gamma$, and so $z \geq \lambda^{-1}(f(x_0) - (1-\lambda)z')$.) Taking $g(x) := -T_0 x$ for $x \in X_0$, $g(x) := +\infty$ for $x \in X \setminus X_0$ and applying Corollary 5 we get the conclusion of Theorem 1.

The preceding discussion shows that only the case when the intrinsic core of $\operatorname{dom}\Gamma$ is empty in Theorem 1 is not covered by known results (however see Malivert et al. 1978, Theorem 3). It is worth of observing that under the hypotheses of Theorem 1 we have that $f(x) := \inf\{z \mid (x,z) \in \Gamma\} \in Z$ for every $x \in \operatorname{dom}\Gamma$ and $T_0 x \leq f(x)$ for every $x \in X_0 \cap \operatorname{dom}\Gamma$. For this use the conclusion of Theorem 1. However a direct proof is not so obvious as that in the case $X_0 \cap {}^i(\operatorname{dom}\Gamma) \neq \emptyset$.

Recently several papers appeared which deal with the Hahn–Banach extension theorem for multifunctions; we envisage mainly Peng et al. (2005a,b), where instead of Z one uses a topological linear space Y ordered by the pointed convex cone K such that (Y,K) has the least upper bound property. For example, Theorem 3.1 in Peng et al. (2005b) follows immediately from Theorem 1, applying this for $Z := Y$, $\Gamma = \operatorname{epi} F - (0, y_0)$ and $T_0 = h - y_0$ where $y_0 := h(0)$; because in Theorem 3.1 of Peng et al. (2005b) $X_0 \cap (\operatorname{dom} F)^i \neq \emptyset$, as seen above, this result follows from the Hahn–Banach–Kantorovich theorem; note that the topology of Y is not used at all. Note also that it is not possible to deduce Theorems 2 and 3 in Chen and Craven (1990) from Theorem 3.1 of Peng et al. (2005b)) (as claimed in Peng et al. 2005b, Remark 3.1(a)) because the affine mapping provided by Theorem 3.1 in Peng et al. (2005b) is not continuous (see Remark 3).

Let us point out the following consequence of Theorem 1. Similar to Definition 1.3 in Peng et al. (2005a), we say that $H : X \rightrightarrows Z$ is *affinelike* if there exist $T \in \mathcal{L}(X,Z)$ and a non-empty convex set $M \subset Y$ such that $H(x) = T(x) + M$ for every $x \in X$.

Corollary 6 *Let $F : X \rightrightarrows Z$ be a convex multifunction, $X_0 \subset X$ a linear subspace and $H_0 : X_0 \rightrightarrows Z$ an affinelike multifunction. Suppose that $0 \in {}^i(\operatorname{dom} F - X_0)$ and $F(x) - H_0(x) \subset K$ for every $x \in X_0 \cap \operatorname{dom} F$. Then there exists $H : X \rightrightarrows Z$ an affinelike multifunction such that $H\big|_{X_0} = H_0$ and $F(x) - H(x) \subset K$ for every $x \in X \cap \operatorname{dom} F$.*

Proof Let $T_0 \in \mathcal{L}(X_0, Z)$ and $M \subset Y$ a non-empty convex set such that $H_0(x) = T_0(x) + M$ for every $x \in X_0$. Consider $\Gamma : X \rightrightarrows Z$ with

$$\operatorname{gph} \Gamma := \{(x, z - m + k) \mid z \in F(x), m \in M, k \in K\} = \operatorname{epi} F - \{0\} \times M.$$

One obtains immediately that Γ is convex, $\operatorname{dom} \Gamma = \operatorname{dom} F$ and $T_0(x) \le z$ for all $(x, z) \in \Gamma \cap (X_0 \times Z)$. Applying Theorem 1 we get $T \in \mathcal{L}(X, Z)$ such that $T|_{X_0} = T_0$ and $T(x) \le z$ for all $(x, z) \in \Gamma$. Setting $H(x) := T(x) + M$, the conclusion follows. □

Asking, furthermore, X and Z to be topological vector spaces and the operator T in the definition of an affinelike multifunction to be in $L(X, Z)$, that is, $T : X \to Z$ be linear and continuous, the statement of the preceding corollary becomes Theorem 2.1 in Peng et al. (2005a). However, we have the following remark concerning Theorem 2.1 in Peng et al. (2005a).

Remark 2 (i) The conclusion of in Theorem 2.1 in Peng et al. (2005a) can be false even if $M = \{0\}$ and $Y = \mathbb{R}$. For this take $Y = \mathbb{R}$, $K = \mathbb{R}_+$, X a nontrivial separated topological vector space with topological dual X^* reducing to $\{0\}$, $X_0 = \{0\}$, $H(0) = \{0\}$, $F(x) = \{\varphi(x)\}$ for $x \in C := X$, where $\varphi : X \to \mathbb{R}$ is a non-null linear functional. It is clear that $X_0 \cap \operatorname{core} C = X_0 \cap X = \{0\}$ and $F(x) - H(x) = \{\varphi(x)\} = \{0\} \subset K$ for $x \in X_0 = \{0\}$. The conclusion has to be the existence of $f \in X^*$ and $\emptyset \ne M \subset \mathbb{R}$ such that for $L(x) := f(x) + M$ to have $L(x) = H(x)$ for every $x \in X_0 \cap C$ and $F(x) - L(x) \subset K$ for every $x \in C$. This means that $L(0) = M = H(0) = \{0\}$ and $\varphi(x) - f(x) \ge 0$ for every $x \in X$. Of course, the last relation yields the contradiction $0 \ne \varphi = f \in X^* = \{0\}$.

(ii) The proof of Theorem 2.1 in Peng et al. 2005a is not convincing even for the algebraic case. Indeed, the set $\overline{Y}$ must be exactly the set M in the definition of the affinelikeness of H.

Remark 3 Taking Y, K, X, X_0, φ as in Remark 2(i) and $F = \varphi$ we have a counterexample for Theorems 2 and 3 in Chen and Craven (1990).

We do not treat here the continuous versions of the preceding results. This can be done as in Sect. 4 of Zălinescu (1983).

3 Yang's generalization of the Hahn–Banach theorem

In this section the proper (that is, $\{0\} \ne K \ne Z$) convex cone K has non-empty algebraic interior K^i. Of course $0 \notin K^i$, $K + K^i = K^i + K^i = K^i$ and $\mathbb{P}K^i = K^i$.

Theorem 7 *Let $\Gamma : X \rightrightarrows Z$ be a K–convex multifunction, $X_0 \subset X$ a linear subspace and $T_0 \in \mathcal{L}(X_0, Z)$. Suppose that $0 \in {}^{i}(\operatorname{dom}\Gamma - X_0)$ and $T_0 x \not> z$ for all $(x, z) \in \Gamma \cap (X_0 \times Z)$. Then there exists $T \in \mathcal{L}(X, Z)$ such that $T\big|_{X_0} = T_0$ and $Tx \not> z$ for all $(x, z) \in \Gamma$.*

Proof As in the proof of Theorem 1, assume first that $0 \in (\operatorname{dom}\Gamma - X_0)^i$. Let B_1 and B_2 be defined as in the proof of Theorem 1. These sets are non-empty. We claim that

$$(B_1 + K^i) \cap (B_2 - K^i) = \emptyset.$$

In the contrary case there exist $k_1, k_2 \in K^i$, $x_1, x_2 \in X_0$, $z_1, z_2 \in Z$ such that $(x_1 + \overline{x}, z_1), (x_2 - \overline{x}, z_2) \in \mathbb{P}A$ and $k_1 + z_1 - T_0 x_1 = T_0 x_2 - z_2 - k_2$, whence $T_0(x_1 + x_2) > z_1 + z_2$. But $(x_1 + x_2, z_1 + z_2) \in \mathbb{P}A$ and $x_1 + x_2 \in X_0$, contradicting the hypothesis.

Assume that $(B_1 + K^i) \cup (B_2 - K^i) = Z$. Then $B_1 \subset B_1 + K^i$ and $B_2 \subset B_2 - K^i$. Indeed, if $b_1 \in B_1 \setminus (B_1 + K^i)$, then $b_1 \in B_2 - K^i$, whence $b_1 = b_2 - k$, with $b_2 \in B_2$ and $k \in K^i$. In this situation we get the contradiction $b_1 + k/2 = b_2 - k/2 \in (B_1 + K^i) \cap (B_2 - K^i) = \emptyset$. Fix now $b_1 \in B_1$ and $b_2 \in B_2$ and consider

$$\overline{\gamma} := \sup\{\gamma \in [0, 1] \mid (1 - t)b_1 + tb_2 \in B_1 + K^i \quad \forall t \in [0, \gamma]\}.$$

Set $\overline{b} := (1 - \overline{\gamma})b_1 + \overline{\gamma}b_2$ and suppose first that $\overline{b} \in B_1 + K^i$; of course, in this situation $\overline{\gamma} < 1$. Then $\overline{b} = \overline{b}_1 + \overline{k}$, with $\overline{b}_1 \in B_1$ and $\overline{k} \in K^i$. Since $\overline{k} \in K^i$, there exists $\delta \in \mathbb{P}$ such that $\overline{k} + \mu(b_2 - b_1) \in K^i$ for $\mu \in [-\delta, \delta]$. It follows that

$$\overline{b} + \mu(b_2 - b_1) = \overline{b}_1 + \overline{k} + \mu(b_2 - b_1) \in B_1 + K^i \quad \forall \mu \in [-\delta, \delta].$$

Taking $\mu = \min\{\delta, 1 - \overline{\gamma}\}$, we get

$$\big[1 - (\overline{\gamma} + t)\big] b_1 + (\overline{\gamma} + t)b_2 \in B_1 + K^i \quad \forall t \in [0, \mu],$$

contradicting the choice of $\overline{\gamma}$. Suppose now that $\overline{b} \in B_2 - K^i$; of course $\overline{\gamma} > 0$. In this situation $\overline{b} = \overline{b}_2 - \overline{k}$, with $\overline{b}_2 \in B_2$ and $\overline{k} \in K^i$. Since $\overline{k} \in K^i$, there exists $\delta \in \mathbb{P}$ such that $\overline{k} + \mu(b_2 - b_1) \in K^i$ for $\mu \in [-\delta, \delta]$. It follows that

$$\overline{b} - \mu(b_2 - b_1) = \overline{b}_2 - \overline{k} - \mu(b_2 - b_1) \in B_2 - K^i \quad \forall \mu \in [-\delta, \delta].$$

Taking $\mu = \min\{\delta, \overline{\gamma}\}$, we get the contradiction

$$\big[1 - (\overline{\gamma} - \mu)\big] b_1 + (\overline{\gamma} - \mu)b_2 \in B_1 + K^i.$$

Therefore $(B_1 + K^i) \cup (B_2 - K^i) \neq Z$. Taking $\overline{z} \in Z \setminus \big[(B_1 + K^i) \cup (B_2 - K^i)\big]$ and $X_1 = X_0 + \mathbb{R}\overline{x}$, then defining T_1 by $T_1(x + t\overline{x}) = T_0 x + t\overline{z}$, $T_1 : X_1 \to Z$ is a prolongation of T_0 and $(x, z) \in \Gamma \cap (X_1 \times Z)$ implies $T_1 x \not> z$.

Continuing by the standard argument, we obtain a maximal linear operator T which is the desired operator.

When aff (dom $\Gamma - X_0$) $\neq X$, proceeding as in the second case of the proof of Theorem 1 we get the desired conclusion. □

Remark 4 In the proof of Theorem 7 we used only the fact that $\mathbb{P}\cdot$ epi Γ is a convex cone and not that epi Γ itself is convex.

Corollary 8 *Let $p : X \to Z^\bullet$ be a sublinear operator, X_0 be a linear subspace of X and $T_0 \in \mathcal{L}(X_0, Z)$. Suppose that $T_0 x \not> p(x)$ for every $x \in X_0$. If $X_0 + \operatorname{dom} p$ is a linear subspace then there exists $T \in \mathcal{L}(X, Z)$ such that $T\big|_{X_0} = T_0$ and $Tx \not> p(x)$ for every $x \in X$.*

Proof In Theorem 7 take $\Gamma : X \rightrightarrows Z$ with gph Γ = epi p; of course, the hypotheses of the theorem hold, so that there exists $T \in \mathcal{L}(X, Z)$ such that $T\big|_{X_0} = T_0$ and $(x, y) \in \Gamma$ implies $Tx \not> y$. Taking $y = p(x)$, the conclusion follows. □

Before stating the next result recall that the weak subdifferential $\partial^w f(x_0)$ of the proper operator $f : X \to Z^\bullet$ at $x_0 \in \operatorname{dom} f$ is the set of those $T \in \mathcal{L}(X, Z)$ such that

$$\forall x \in X : Tx - Tx_0 \not> f(x) - f(x_0).$$

Corollary 9 *Let $f : X \to Z^\bullet$ be a proper convex operator and consider $x_0 \in {}^i(\operatorname{dom} f)$. Then $\partial^w f(x_0)$ is non-empty.*

Proof Take $g(x) = f(x_0 + x) - f(x)$, gph Γ = epi g, $X_0 = \{0\}$ and $T_0 := 0 \in \mathcal{L}(X_0, Z)$. Of course, if $(x, z) \in \Gamma \cap (X_0 \times Z)$, then $x = 0$ and $z \in K$, whence $T_0 x \not> z$. Using the preceding theorem we obtain $T \in \mathcal{L}(X, Z)$ such that $Tx \not> z$ for $(x, z) \in \Gamma$. Taking $x \in \operatorname{dom} f$, we have that $(x - x_0, f(x) - f(x_0)) \in \Gamma$, so that $T(x - x_0) \not> f(x) - f(x_0)$. □

Corollary 10 *Let $f, g : X \to Z^\bullet$ be proper convex operators. Suppose that $0 \in {}^i(\operatorname{dom} f - \operatorname{dom} g)$ and that $f(x) + g(x) \not< 0$ for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$. Then there exists $T \in \mathcal{L}(X, Z)$ such that*

$$\forall x_1 \in \operatorname{dom} f, \forall x_2 \in \operatorname{dom} g : f(x_1) - Tx_1 \not< -g(x_2) - Tx_2.$$

Proof Consider $\Gamma : X \rightrightarrows Z$ with

$$\operatorname{gph} \Gamma := \{(x, z) \in X \times Z \mid f(x) \leq z\} + \{(-x', z') \in X \times Z \mid g(x') \leq z'\}.$$

Then Γ is convex and dom Γ = dom f − dom g. Take $X_0 = \{0\}$ and $T_0 := 0 \in \mathcal{L}(X_0, Z)$. The hypotheses of the preceding theorem hold. Indeed, if $(0, z) \in \Gamma$ then $(0, z) = (x, z') + (-x, z'')$, with $f(x) \leq z'$ and $g(x) \leq z''$; assuming that $0 = T_0(0) > z$, from $f(x) + g(x) \leq z' + z'' = z$ and the known relation $K + K^i = K^i$ we get the contradiction $f(x) + g(x) < 0$. Hence $T_0(0) \not> z$. Therefore, there exists $T \in \mathcal{L}(X, Z)$ such that $Tx \not> z$ for $(x, z) \in \Gamma$. In particular, for $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} g$ we get $T(x_1 - x_2) \not> f(x_1) + g(x_2)$, which yields $-g(x_2) - Tx_2 \not> f(x_1) - Tx_1$ for all $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} g$. □

Note that applying Corollary 10 for $f = p$ and $g(x) = -T_0 x$ for $x \in X_0$, $g(x) = +\infty$ for $x \notin X_0$, where $p : X \to Z^\bullet$ is a sublinear operator and $T_0 \in \mathcal{L}(X_0, Z)$ we get a weaker variant of Corollary 8.

As mentioned by Yang (1992, Lemma 1), Wang (1986) obtained Theorem 7 for $X_0 = \{0\}$, $T_0 = 0$ and Γ a K–convex multifunction with $(\operatorname{epi}\Gamma)^i \neq \emptyset$ and $0 \in \left(\Gamma^{-1}(z_0)\right)^i$ for some $z_0 \in Z$ (compare also with Lemma 12), that is, a separation theorem; based on results in Wang (1986), Yang (1992, Theorem 1) obtained a weaker form of Theorem 7: the interiority hypothesis is stronger [more precisely, $(\operatorname{epi}\Gamma)^i \neq \emptyset$ and $X_0 \cap (\operatorname{dom}\Gamma)^i \neq \emptyset$ instead of $0 \in {}^i(\operatorname{dom}\Gamma - X_0)$] and the conclusion is weaker [more precisely $0 \not< (T - T_0)(x) \not< 0$ for all $x \in X_0$ instead of $T\big|_{X_0} = T_0$]. Of course, applying Theorem 7 we can obtain other results of Yang (1992) under weaker interiority conditions.

As in the case $Z = \mathbb{R}$, the extension theorems can be used for separating convex sets in product spaces. For example, from Theorem 7 we can deduce the next separation result.

Proposition 11 *Let $A, B \subset X \times Z$ be convex sets. Assume that $0 \in {}^i(P_X(A) - P_X(B))$ and $z_1 \not< z_2$ for all $x \in P_X(A) \cap P_X(B)$ and $z_1 \in A(x)$, $z_2 \in B(x)$. Then there exists $T \in L(X, Z)$ such that*

$$z_1 - Tx_1 \not< z_2 - Tx_2 \quad \forall (x_1, z_1) \in A, \forall (x_2, z_2) \in B. \tag{2}$$

Proof Consider the set $C := A - B \subset X \times Z$. Then $P_X(C) = P_X(A) - P_X(B)$ and if $(0, z) \in C$ then $z = z_1 - z_2$ with $z_1 \in A(x)$, $z_2 \in B(x)$ for some $x \in P_X(A) \cap P_X(B)$; hence $z \not< 0$. Taking $X_0 := \{0\}$ and $T_0(0) := 0$, we can apply Theorem 7 for $\operatorname{gph}\Gamma := C$ and T_0. Therefore, there exists $T \in \mathcal{L}(X, Z)$ such that $z \not< Tx$ for every $(x, z) \in C$. Hence (2) holds. □

Note that Thierfelder (1991a,b) says that A and B are separable by an affine mapping if there exist $T \in \mathcal{L}(X, Z)$ and $z_0 \in Z$ such that

$$z_1 - Tx_1 \not< z_0 \not< z_2 - Tx_2 \quad \forall (x_1, z_1) \in A, \forall (x_2, z_2) \in B. \tag{3}$$

One can ask which is the relationship between (2) and (3). Setting $A_0 := \{z_1 - Tx_1 \mid (x_1, z_1) \in A\}$, $B_0 := \{z_2 - Tx_2 \mid (x_2, z_2) \in B\}$, condition (2) becomes $B_0 \cap (A_0 + K^i) = \emptyset$, or equivalently, $(B_0 - K^i) \cap (A_0 + K^i) = \emptyset$, while condition (3) becomes $z_0 \notin (B_0 - K^i) \cup (A_0 + K^i)$. So, it is quite clear that (3) does not imply (2). (One can take $X := \mathbb{R}$, $Z := \mathbb{R}^2$, $K := \mathbb{R}^2_+$, $A := \{(0, (-1, -1))\}$, $B := \{(0, (1, 1))\}$, $T := 0$ and $z_0 := (-2, 2)$.)

However, when $A, B \subset X \times Z$ are convex and $K^i \neq \emptyset$ (which is the case in Proposition 11) we have that (2) implies (3). Indeed, taking A_0 and B_0 as above we have seen that condition (2) becomes $B_0 \cap (A_0 + K^i) = \emptyset$. Since A_0 and B_0 are convex sets and $A_0 + K^i$ is algebraically open, by the classic (algebraic) separation theorem we get $z^* \in Z'$, $z^* \neq 0$, and $\gamma \in \mathbb{R}$ such that $\langle b, z^* \rangle \leq \gamma \leq \langle a + k, z^* \rangle$ for all $a \in A_0, b \in B_0, k \in K^i$. It follows that

$$\langle k, z^* \rangle > 0 \; \forall k \in K^i \quad \text{and}$$
$$\langle z_1 - Tx_1, z^* \rangle \geq \gamma \geq \langle z_2 - Tx_2, z^* \rangle \forall (x_1, z_1) \in A, \forall (x_2, z_2) \in B.$$

Taking $z_0 \in Z$ such that $\langle z_0, z^* \rangle = \gamma$ we have that (3) holds.

The previous discussion shows that (2) is more adequate as separation property than (3). Another argument is the fact that taking $x_1 = x_2$ in (2) we recover a part of the hypothesis of Proposition 11, which is not the case for (3).

However, using directly the classic separation theorem we obtain a stronger conclusion than that of Theorem 7 under the same hypotheses. To prove this we need the next lemma which is probably known; in fact assertion (ii) for X and Z finite dimensional linear spaces is just (Rockafellar 1970, Theorem 6.8).

Lemma 12 *Let $A \subset X \times Z$ be a convex set (multifunction) and $(x_0, z_0) \in X \times Z$. Then:*

(i) *the following statements are equivalent:* a) $(x_0, z_0) \in A^i$, b) $x_0 \in (P_X(A))^i$ *and* $z_0 \in (A(x_0))^i$, c) $x_0 \in \left(A^{-1}(z_0)\right)^i$ *and* $z_0 \in (A(x_0))^i$;

(ii) *the following statements are equivalent:* a) $(x_0, z_0) \in {}^iA$, b) $x_0 \in {}^i(P_X(A))$ *and* $z_0 \in {}^i(A(x_0))$.

Proof First observe that, doing a translation, we may (and we do) suppose that $(x_0, z_0) = (0, 0)$. Let us first prove (i).

(a) $\Rightarrow$ (c) Let $x \in X$; since $(0, 0) \in A^i$, there exists $\lambda > 0$ such that $\lambda(x, 0) \in A$, whence $\lambda x \in A^{-1}(0)$. Therefore $0 \in \left(A^{-1}(0)\right)^i$. Similarly, $0 \in (A(0))^i$.

(c) $\Rightarrow$ (b) is obvious (because $A^{-1}(0) \subset P_X(A)$).

(b) $\Rightarrow$ (a) Let $(x, z) \in X \times Z$. Since $0 \in (P_X(A))^i$, there exists $\lambda > 0$ such that $\lambda x \in P_X(A)$, and so there exists $z' \in Z$ such that $(\lambda x, z') \in A$. Since $0 \in (A(0))^i$, there exists $\mu > 0$ such that $(0, \mu(\lambda z - z')) \in A$. Since A is convex, it follows that

$$\frac{\mu\lambda}{1+\mu}(x, z) = \frac{1}{1+\mu}\left(0, \mu(\lambda z - z')\right) + \frac{\mu}{1+\mu}\left(\lambda x, z'\right) \in A.$$

Therefore $(0, 0) \in A^i$.

(ii) (a) $\Rightarrow$ (b) Assume that $(0, 0) \in {}^iA$. Consider first $x \in P_X(A)$. Then $(x, z) \in A$ for some $z \in Z$. It follows that $(x', z') := -\lambda(x, z) \in A$ for some $\lambda > 0$, and so $x' = -\lambda x \in P_X(A)$. Hence $0 \in {}^i(P_X(A))$. Let now $z \in A(0)$, that is, $(0, z) \in A$. As before, $-\lambda(0, z) \in A$ for some $\lambda > 0$, whence $-\lambda z \in A(0)$. Hence $0 \in {}^i(A(0))$.

(b) $\Rightarrow$ (a) Let $(x, z) \in A$. Then $x \in P_X(A)$. Since $0 \in {}^i(P_X(A))$, there exists $\lambda > 0$ such that $x' := -\lambda x \in P_X(A)$ for some $\lambda > 0$, and so $(x', z') \in A$ for some $z' \in Z$. Then

$$\left(0, \frac{1}{1+\lambda}z' + \frac{\lambda}{1+\lambda}z\right) = \frac{1}{1+\lambda}(x', z') + \frac{\lambda}{1+\lambda}(x, z) \in A,$$

and so $z'' := \frac{1}{1+\lambda}z' + \frac{\lambda}{1+\lambda}z \in A(0)$. Because $0 \in {}^i(A(0))$, there exists $\mu > 0$ such that $z''' := -\mu z'' \in A(0)$. Taking $\eta := \mu/(1+\lambda+\mu) \in]0, 1[$ we obtain that $\eta x' = -\eta\lambda x$ and

$$\eta z' + (1-\eta)z''' = \eta z' - \frac{\mu(1-\eta)}{1+\lambda}z' - \frac{\lambda\mu(1-\eta)}{1+\lambda}z = -\frac{\lambda\mu}{1+\lambda+\mu}z = -\eta\lambda z.$$

Hence $-\eta\lambda(x, z) = \eta(x', z') + (1-\eta)(0, z''') \in A$, which proves that $(0, 0) \in {}^iA$. □

Note that the implications (a) ⇒ (c) and (c) ⇒ (b) in (i) are valid for arbitrary sets A, but generally (c) ⇒ (a) [and so (b) ⇒ (a)] is not valid if A is not convex; take for example $A := ([-1, 1] \times \{0\}) \cup (\{0\} \times [-1, 1])$. Also note that even for A convex the fact that $0 \in {}^i(A^{-1}(0))$ and $0 \in {}^i(A(0))$ do not imply that $(0, 0) \in {}^iA$; for this take $A := \{(x, x) \mid x \in \mathbb{R}_+\}$. Moreover, the assertion (ii) of the preceding lemma cannot be obtained from (i) because, even for $(0, 0) \in A$, span A is not the product of two linear spaces.

In Thierfelder (1991a, Lemma 2.1) and Thierfelder (1991b) the implication (b) ⇒ (a) of (i) is given in a weaker form: if $(P_X(A))^i \neq \emptyset$ and $(A(x))^i \neq \emptyset$ for every $x \in (P_X(A))^i$ then $A^i \neq \emptyset$.

Theorem 13 *Let $\Gamma : X \rightrightarrows Z$ be a K–convex multifunction, $X_0 \subset X$ a linear subspace and $T_0 \in \mathcal{L}(X_0, Z)$. Suppose that $0 \in {}^i(\operatorname{dom}\Gamma - X_0)$ and $T_0x \not> z$ for all $(x, z) \in \Gamma \cap (X_0 \times Z)$. Then there exists $z^* \in Z'$ and $T \in \mathcal{L}(X, Z)$ such that*

$$T\,\big|_{X_0} = T, \quad \langle z, z^*\rangle > 0\ \forall\, z \in K^i \quad \text{and} \quad \langle Tx, z^*\rangle \leq \langle z, z^*\rangle\ \forall\,(x, z) \in \Gamma.$$

In particular $Tx \not> z$ for very $(x, z) \in \Gamma$. Moreover, T can be defined by $Tx = T_0x_0 - \langle x_1, x^*\rangle\,\overline{z}$, where $\overline{z} \in Z$, $x^* \in X'$ and the linear subspace $X_1 \subset X$ with $X = X_0 \oplus X_1$ are fixed, and $x = x_0 + x_1$ with $x_0 \in X_0$, $x_1 \in X_1$.

Proof Assume first that $0 \in (\operatorname{dom}\Gamma - X_0)^i$. Let $B = \operatorname{epi}\Gamma - \operatorname{gph} T_0$. It is obvious that $P_X(B) = \operatorname{dom}\Gamma - X_0$, and so $0 \in (P_X(B))^i$. Then $B(0) \neq \emptyset$ and $z \in B(0) \Rightarrow 0 \not> z$. Indeed, if $z \in B(0)$ then $(0, z) = (x, z') - (x_0, T_0x_0) + (0, k')$ for some $(x, z') \in \Gamma$, $x_0 \in X_0$ and $k' \in K$. It follows that $x = x_0$ and $z = z' + k' - T_0x_0 \not< 0$. Since $K^i \neq \emptyset$, it is clear that $(B(0))^i \neq \emptyset$. Therefore, by Lemma 12(i), $B^i \neq \emptyset$. Moreover, $(0, 0) \notin B^i$. In the contrary case, again by Lemma 12(i), $0 \in (B(0))^i$, contradicting the fact observed above that $z \not< 0$ for every $z \in B(0)$.

Using an algebraic separation theorem, we get $(x^*, z^*) \in X' \times Z' \setminus \{(0, 0)\}$ such that

$$\langle x - x_0, x^*\rangle + \langle z - T_0x_0 + k, z^*\rangle \geq 0 \quad \forall\,(x, z) \in \Gamma,\ \forall x_0 \in X_0,\ \forall k \in K. \tag{4}$$

First notice that, because $0 \in (\operatorname{dom}\Gamma - X_0)^i$, if $z^* = 0$ then $x^* = 0$, a contradiction; therefore, $z^* \neq 0$. Next, from (4), it follows that $\langle k, z^*\rangle \geq 0$ for every $k \in K$,

$$\langle x_0, x^*\rangle + \langle T_0x_0, z^*\rangle = 0 \quad \forall x_0 \in X_0, \tag{5}$$

$$\langle x, x^*\rangle + \langle z, z^*\rangle \geq 0 \quad \forall (x, z) \in \Gamma. \tag{6}$$

Of course, $\langle k, z^*\rangle > 0$ for every $k \in K^i$. Fix $\overline{z} \in Z$ such that $\langle \overline{z}, z^*\rangle = 1$, $X_1 \subset X$ a linear subspace such that $X_0 \oplus X_1 = X$ and $T : X \to Z$ defined by $Tx := T_0x_0 - \langle x_1, x^*\rangle \overline{z}$ for $x = x_0 + x_1$ with $x_0 \in X_0$ and $x_1 \in X_1$. Then, of course, $T\big|_{X_0} = T_0$ and for $(x, z) \in \Gamma$, taking $x_0 \in X_0$ and $x_1 \in X_1$ with $x = x_0 + x_1$, we have

$$\langle Tx, z^*\rangle = \langle T_0x_0 - \langle x_1, x^*\rangle \overline{z}, z^*\rangle = \langle T_0x_0, z^*\rangle - \langle x_1, x^*\rangle = -\langle x, x^*\rangle \le \langle z, z^*\rangle .$$

If $\overline{X} := \text{aff}(\text{dom}\,\Gamma - X_0) = \text{span}(\text{dom}\,\Gamma - X_0) \neq X$ we find first $x^* \in \mathcal{L}(\overline{X}, \mathbb{R})$ and $z^* \in Z'$ as above and then extend x^* to an element of X'. The proof is complete. □

Notice that using the same hypotheses as in Theorem 7, the conclusion of Theorem 13 is stronger, not only because $\langle Tx, z^*\rangle \le \langle z, z^*\rangle$ implies $Tx \not> z$, but also because we have a very special expression for T. This is quite surprising because in the proof of Theorem 13 we utilized the usual separation theorem from the scalar case which is equivalent to the classical Hahn–Banach theorem, while in the proof of Theorem 7 we utilized a similar technique to that used in the proof of Theorem 1, that is, for the proof of the Hahn–Banach–Kantorovich theorem. Furthermore, the technique utilized in the proof of Theorem 13 is more adequate for obtaining continuous versions of this theorem. However, we hope the technique used in the proof of Theorem 7 could be useful in other situations.

A continuous version of Theorem 13 is the next result in which, as mentioned before Remark 2, $L(X, Z)$ denotes the linear space of continuous linear operators from X into Z and X^* denotes the topological dual of X, that is, $L(X, \mathbb{R})$.

Theorem 14 *Let X, Z be separated locally convex spaces, $\Gamma : X \rightrightarrows Z$ be a K–convex multifunction, $X_0 \subset X$ a linear subspace and $T_0 \in L(X_0, Z)$. Suppose that* $\text{int}(\text{epi}\,\Gamma) \neq \emptyset$*, $X_0 \cap \text{int}(\text{dom}\,\Gamma) \neq \emptyset$, and $T_0x \not> z$ for all* $(x, z) \in \Gamma \cap (X_0 \times Z)$. *If either* (a) *$X_0$ has a topological supplement, or* (b) *$T_0x = \langle x, x_0^*\rangle z_0$ for every $x \in X_0$ with fixed $x_0^* \in X^*$ and $z_0 \in Z$, then there exists $z^* \in Z^*$ and $T \in L(X, Z)$ such that*

$$T\big|_{X_0} = T, \quad \langle z, z^*\rangle > 0\, \forall\, z \in K^i \quad \text{and} \quad \langle Tx, z^*\rangle \le \langle z, z^*\rangle\ \forall\, (x, z) \in \Gamma.$$

In particular $Tx \not> z$ for very $(x, z) \in \Gamma$. Moreover, in case (a) T can be defined by $Tx = T_0x_0 - \langle x_1, x^*\rangle \overline{z}$, where $\overline{z} \in Z$, $x^* \in X^*$ are fixed and $x = x_0 + x_1$ with $x_0 \in X_0$, $x_1 \in X_1$, while in case (b) T can be defined by $Tx := \langle x, x_0^*\rangle (z_0 - \langle z_0, z^*\rangle z_1) - \langle x, x^*\rangle z_1$ for $x \in X$ with fixed $x^* \in X^*$ and $z_1 \in Z$ such that $\langle z_1, z^*\rangle = 1$.

Proof In the proof of Theorem 13, because $\text{int}(\text{epi}\,\Gamma) \neq \emptyset$, we have that $\text{int}\, B\ (= B^i)$ is non-empty; it follows that $x^* \in X^*$ and $z^* \in Z^*$. In case (a), because the projections on X_0 and X_1 are continuous, taking into account the construction of T we obtain that T is continuous.

Assume now that $T_0x = \langle x, x_0^*\rangle z_0$ for every $x \in X_0$ with fixed $x_0^* \in X^*$ and $z_0 \in Z$. From (5) we obtain that $\langle x, x^* + \langle z_0, z^*\rangle x_0^*\rangle = 0$ for every $x \in X_0$. Taking $z_1 \in Z$ with $\langle z_1, z^*\rangle = 1$ and $Tx := \langle x, x_0^*\rangle (z_0 - \langle z_0, z^*\rangle z_1) - \langle x, x^*\rangle z_1$ for $x \in X$, we have that T is continuous and $Tx = T_0 x$ for $x \in X_0$. Moreover, for $(x, z) \in \Gamma$ we have

$$\begin{aligned}\langle Tx, z^* \rangle &= \left\langle \langle x, x_0^* \rangle z_0 - \left(\langle z_0, z^* \rangle \langle x, x_0^* \rangle - \langle x, x^* \rangle\right) z_1, z^* \right\rangle \\ &= \langle x, x_0^* \rangle \langle z_0, z^* \rangle - \langle z_0, z^* \rangle \langle x, x_0^* \rangle - \langle x, x^* \rangle \\ &= -\langle x, x^* \rangle \le \langle z, z^* \rangle,\end{aligned}$$

the last inequality being obtained using (6). □

Thierfelder (1991b, Theorem 2.5) obtains similar conclusions for Theorem 13 in the following two situations: (1) there exists z^* strictly positive on $K \setminus \{0\}$ such that $\langle z, z^* \rangle \ge \langle T_0 x_0, z^* \rangle$ for $(x, z) \in \Gamma \cap (X_0 \times Z)$ and (2) there exists an algebraically open convex cone P such that $K \setminus \{0\} \subset P$ and $\Gamma(0) + P \subset \Gamma(0)$. Taking $P = \{z \in Z \mid \langle z, z^* \rangle > 0\}$ in case 1), in each situation we have that $\Gamma(0) \cap (-P) = \emptyset$, and so the conclusion is obtained applying Theorem 13 for K replaced by $P \cup \{0\}$.

Remark 5 Meng (1998, Theorem 2.1) obtained Theorem 14 for T as in (b) with $z_0 \in \operatorname{int} K$ without the condition $X_0 \cap \operatorname{int}(\operatorname{dom} \Gamma) \neq \emptyset$. (Note that when $z_0 \in K^i$ we can take $z_1 := \langle z_0, z^* \rangle^{-1} z_0$.) However, the hypothesis $X_0 \cap \operatorname{int}(\operatorname{dom} \Gamma) \neq \emptyset$ is essential. For this take $X := Z := \mathbb{R}$, $\Gamma := \left\{(x, z) \in X \times Z \mid x \ge 0, z \ge -\sqrt{x}\right\}$, $K := \mathbb{R}_+$, $X_0 := \{0\}$, $T_0 := 0$; in this case all the hypotheses of Theorem 14, but $X_0 \cap \operatorname{int}(\operatorname{dom} \Gamma) \neq \emptyset$, are verified. However, the conclusion of Theorem 14 (and of Theorem 2.2 in Meng (1998)) does not hold. Note also that Theorem 2.2 in Meng (1998) is false; for this take $X := \mathbb{R}^2$, $Z := \mathbb{R}$, $K := \mathbb{R}_+$, $\psi(x) := \{-\sqrt{v^2 - u^2}\}$ for $x := (u, v)$ with $v \in \mathbb{R}_+$ and $u \in [-v, v]$, $\psi(x) := \emptyset$ otherwise. Then ψ is K-sublinear (that is $\psi(\lambda x) = \lambda \psi(x)$, $\psi(x) + \psi(x') \subset \psi(x + x') + K$ for $x, x' \in X$, $\lambda \in \mathbb{P}$) and $\operatorname{int}(\operatorname{epi} \psi) \neq \emptyset$. The hypothesis of (Meng 1998, Theorem 2.2) is verified for $x_0 := (1, 1)$ and $q_0^* := 1$, but for $p := 1 \in \operatorname{int} K$ the conclusion does not hold for any $x^* \in X^* = \mathbb{R}^2$.

Note that if $\Gamma : X \rightrightarrows Z$ is upper semicontinuous at $x_0 \in \operatorname{int}(\operatorname{dom} \Gamma)$, $\operatorname{int} K \neq \emptyset$ and $\Gamma(x_0)$ is bounded above then $\operatorname{int}(\operatorname{epi} \Gamma) \neq \emptyset$. Concerning the affine and affinelike multifunctions we mention the following result. The equivalence of (a), (b$''$) and (c) (in an slightly different form) is mentioned in Thierfelder (1991a,b).

Proposition 15 *Let $A \subset X \times Z$. Consider the following assertions.*

(a) *A is an affine manifold;*
(b) *$\lambda A(x) + (1 - \lambda) A(x') \subset A\left(\lambda x + (1 - \lambda) x'\right)$ for all $x, x' \in X$ and $\lambda \in \mathbb{R}$;*
(b$'$) *$\lambda A(x) + (1 - \lambda) A(x') \subset A\left(\lambda x + (1 - \lambda) x'\right)$ for all $x, x' \in P_X(A)$ and $\lambda \in \mathbb{R}$;*
(b$''$) *$\lambda A(x) + (1 - \lambda) A(x') = A\left(\lambda x + (1 - \lambda) x'\right)$ for all $x, x' \in P_X(A)$ and $\lambda \in \mathbb{R}$;*
(c) *there exist linear subspaces $X_0 \subset X$, $Z_0 \subset Z$, a linear map $T_0 : X_0 \to Z$ and $(x_0, z_0) \in X \times Z$ such that $P_X(A) = x_0 + X_0$ and $A(x) = T_0(x - x_0) + z_0 + Z_0$ for every $x \in P_X(A)$;*
(d) *A is an affinelike multifunction on a linear subspace $X_0 \subset X$.*

Then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (b$'$) $\Leftrightarrow$ (b$''$) $\Leftrightarrow$ (c)*; if $X_0 := P_X(A)$ is a linear space then* (a) $\Rightarrow$ (d).

Proof We may (and we do) assume that $(0, 0) \in A$; otherwise replace A by $A-(x_0, z_0)$ with $(x_0, z_0) \in A$.

The fact that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (b′) $\Leftarrow$ (b″) is simple verification, (b) being a rewriting of (b′) taking into account the fact that $B + \emptyset = \emptyset + B = \emptyset$ for $B \subset Z$.

(a) $\Rightarrow$ (c) Because A is a linear subspace, $X_0 := P_X(A)$ and $Z_0 := A(0) = P_Z(A \cap (\{0\} \times Z))$ are linear spaces. Take $Z_1 \subset Z$ a linear subspace such that $Z = Z_0 \oplus Z_1$. Observe first that for $(x, z), (x, z') \in A$ we have $(0, z - z') \in A$, and so $z-z' \in Z_0$. Hence, taking $T_0 : X_0 \to Z$ defined by $T_0(x) := z_1$, where $z = z_0+z_1 \in A(x)$ with $z_0 \in Z_0$, $z_1 \in Z_1$, T_0 is well defined. Since A is a linear subspace it follows immediately that T_0 is a linear operator and $A(x) = T_0(x) + Z_0$ for every $x \in X_0$.

(c) $\Rightarrow$ (b″) follows by a simple verification.

(c) $\Rightarrow$ (d) is obvious when $X_0 := P_X(A)$ is a linear space. □

Observe that (d) $\Rightarrow$ (a) if and only if the set M in the definition of an affinelike multifunction is an affine set.

Note also that the spaces X_0 and Z_0 are uniquely determined by A; in fact X_0 is the parallel subspace of $P_X(A)$ and Z_0 is the parallel subspace of $A(x_0)$ for some $x_0 \in P_X(A)$. The equivalences of (a), (b″) and (c) (the last one presented in the form there exist an affine mapping $T_0 : P_X(A) \to Z$ and a linear subspace Z_0 of Z such that $A(x) = T_0(x) + Z_0$ for every $x \in P_X(A)$) are provided in Thierfelder (1991b). Note that when Z is ordered by the convex cone K, in Thierfelder (1991b) one says that the affine multifunction $A \subset X \times Z$ is non-vertical when every two distinct elements from $A(x)$ are not comparable for each $x \in X$, which is equivalent, by Lemma 3.1 in Thierfelder (1991b), to $Z_0 \cap K = \{0\}$ where Z_0 is provided by Proposition 15(d).

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