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ORIGINAL ARTICLE

# The Karush-Kuhn-Tucker optimality conditions for the optimization problem with fuzzy-valued objective function

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**Abstract** The Karush-Kuhn-Tucker (KKT) conditions for an optimization problem with fuzzy-valued objective function are derived in this paper. A solution concept of this optimization problem is proposed by considering an ordering relation on the class of all fuzzy numbers. The solution concept proposed in this paper will follow from the similar solution concept, called non-dominated solution, in the multiobjective programming problem. In order to consider the differentiation of a fuzzy-valued function, we use the Hausdorff metric to define the distance between two fuzzy numbers and the Hukuhara difference to define the difference of two fuzzy numbers. Under these settings, the KKT optimality conditions are elicited naturally by introducing the Lagrange function multipliers.

**Keywords** Hausdorff metric · Hukuhara difference · H-differentiability · Lagrange function multipliers · Karush-Kuhn-Tucker conditions

## 1 Introduction

The occurrence of randomness and fuzziness in the real world is inevitable owing to some unexpected situations. Therefore, imposing the uncertainty upon the conventional optimization problems becomes an interesting research topic. The randomness occurring in the optimization problems is categorized as the stochastic optimization problems, and the fuzziness occurring in the optimization problems is categorized as the fuzzy optimization problems. The books written by Birge and Louveaux (1997), Kall (1976), Prékopa (1995), Stancu-Minasian (1984) and Vajda (1972) give many useful techniques for solving the stochastic optimization problems. Bellman and Zadeh

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(1970) inspired the development of fuzzy optimization by providing the aggregation operators, which combined the fuzzy goals and fuzzy decision space. After this motivation and inspiration, there come out a lot of articles dealing with the fuzzy optimization problems. The earliest interesting works were initiated by Rödder and Zimmermann 1977 and Zimmermann 1976, 1978, 1985 who applied fuzzy sets theory to the linear programming problems and linear multiobjective programming problems by using the aspiration level approach. The collection of papers on fuzzy optimization edited by Słowiński (1998) and Delgado et al. (1994) gives the main stream of this topic. Lai and Hwang 1992, 1994 also give an insightful survey. On the other hand, the book edited by Słowiński and Teghem (1990) gives the comparisons between fuzzy optimization and stochastic optimization for the multiobjective programming problems. Inuiguchi and Ramík (2000) also gives a brief review of fuzzy optimization and a comparison with stochastic optimization in portfolio selection problem.

For the real problems, the data sometimes cannot be recorded or collected precisely under some unexpected situations. For instance, owing to the fluctuation of market from time to time, we cannot exactly know the price of a product. In fact, we can just know that the price of a product is around $p$ dolloars. The phrase "around $p$ dollars" can be described as a fuzzy number $\tilde{p}$. This situation is completely different from assuming $p$ as a random variable. Under this explanation, we now can say that the price of a product is $\tilde{p}$ (by considering the price as a fuzzy number). We consider a simple example to motivate our study for this optimization problem. Suppose that a factory can produce five products $x_1, \ldots, x_5$ subject to some budget constraints. For selling products $x_1, \ldots, x_5$, the factory can earn income around 5, 8, 7, 4, 6, respectively, depending on the market effect. In this case, we can say that, for selling products $x_1, \ldots, x_5$, the factory can earn income $\tilde{5}, \tilde{8}, \tilde{7}, \tilde{4}, \tilde{6}$, respectively. Now we can formulate this problem as folows:

$$
\begin{array}{ll}
\max & \widetilde{5}x_1 \oplus \widetilde{8}x_2 \oplus \widetilde{7}x_3 \oplus \widetilde{4}x_4 \oplus \widetilde{6}x_5 \\
\text{subject to} & 2x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 \leq 20 \\
& 3x_1 + 5x_2 + 4x_3 + 2x_4 + 4x_5 \leq 30 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0,
\end{array}
$$

where "$\oplus$" is the addition between fuzzy numbers, which will be defined in this paper below. In order to solve the above optimization problem with fuzzy-valued objective function, we shall define an ordering among fuzzy numbers. Since this ordering is a partial ordering, we are going to consider the non-dominated solution which follows from the similar notion of the multiobjective optimization problem.

The usual ordering "$\leq$" for real numbers is a total ordering on $\mathbb{R}$. However, there exist no natural orderings among the class of all fuzzy numbers. In this paper, we provide a solution concept for the optimization problem (nonlinear programming problem) with fuzzy-valued objective function by proposing an ordering relationship between two fuzzy numbers. We shall see that this ordering relation is not a total ordering on the class of all fuzzy numbers. Therefore, the solution concept will follow from the similar solution concept (non-dominated solution) in the conventional multiobjective programming problems. Under these settings, we are going to derive the Karush-Kuhn-Tucker optimality conditions in an optimization problem with fuzzy-valued objective function by introducing the Lagrange function multipliers.

In Sect. 2, we introduce some basic properties and arithmetics of fuzzy numbers. In Sect. 3, we use the well-known Hausdorff metric to define the distance between any two fuzzy numbers. Using this metric, we can consider the continuity and limit of a fuzzy-valued function. In Sect. 4, we use the Hukuhara difference to define the difference of any two fuzzy numbers. Using this Hukuhara difference and the concept of limit in fuzzy-valued function which is defined in Sect. 3, we are capable of proposing the differentiation of a fuzzy-valued function. In Sect. 5, we formulate an optimization problem with fuzzy-valued objective function and provide a solution concept for this problem. In the final Sect. 6, we derive the KKT conditions for our problem by introducing the Lagrange function multipliers.

## 2 Arithmetics of fuzzy numbers

Let $U$ be a topological vector space. The fuzzy subset $\tilde{a}$ of $U$ is defined by a function $\xi_{\tilde{a}} : U \to [0, 1]$, which is called a *membership function*. The $\alpha$-level set of $\tilde{a}$, denoted by $\tilde{a}_\alpha$, is defined by $\tilde{a}_\alpha = \{x \in U : \xi_{\tilde{a}}(x) \geq \alpha\}$ for all $\alpha \in (0, 1]$. The 0-level set $\tilde{a}_0$ is defined as the closure of the set $\{x \in U : \xi_{\tilde{a}}(x) > 0\}$, i.e., $\tilde{a}_0 = \text{cl}\,(\{x \in U : \xi_{\tilde{a}}(x) > 0\})$.

**Definition 2.1** We denote by $\mathcal{F}(U)$ the set of all fuzzy subsets $\tilde{a}$ of $U$ with membership function $\xi_{\tilde{a}}$ satisfying the following conditions:

(i) $\tilde{a}$ is normal, i.e., there exists an $x \in U$ such that $\xi_{\tilde{a}}(x) = 1$;
(ii) $\xi_{\tilde{a}}$ is quasi-concave, i.e., $\xi_{\tilde{a}}(\lambda x + (1-\lambda)y) \geq \min\{\xi_{\tilde{a}}(x), \xi_{\tilde{a}}(y)\}$ for all $x, y \in U$ and $\lambda \in [0, 1]$;
(iii) $\xi_{\tilde{a}}$ is upper semicontinuous, i.e., $\{x \in U : \xi_{\tilde{a}}(x) \geq \alpha\} = \tilde{a}_\alpha$ is a closed subset of $U$ for each $\alpha \in (0, 1]$;
(iv) the 0-level set $\tilde{a}_0$ is a compact subset of $U$.

Throughout this paper, the topological vector space $U$ is assumed to be the set of all real numbers $\mathbb{R}$ which is endowed with the usual topology. Then the member $\tilde{a}$ in $\mathcal{F}(\mathbb{R})$ is called a *fuzzy number*. Suppose now that $\tilde{a} \in \mathcal{F}(\mathbb{R})$. Then condition (ii) says that the $\alpha$-level set $\tilde{a}_\alpha$ of $\tilde{a}$ is a convex subset of $\mathbb{R}$ for each $\alpha \in [0, 1]$. Combining this fact with conditions (iii) and (iv), the $\alpha$-level set $\tilde{a}_\alpha$ of $\tilde{a}$ is a compact and convex subset of $\mathbb{R}$ for each $\alpha \in [0, 1]$, i.e., $\tilde{a}_\alpha$ is a closed interval in $\mathbb{R}$ for each $\alpha \in [0, 1]$. Therefore, we also write $\tilde{a}_\alpha = \left[\tilde{a}_\alpha^L, \tilde{a}_\alpha^U\right]$.

**Definition 2.2** Let $\tilde{a}$ be a fuzzy number. We say that $\tilde{a}$ is *nonnegative* if $\tilde{a}_\alpha^L \geq 0$ for all $\alpha \in [0, 1]$. We say that $\tilde{a}$ is *positive* if $\tilde{a}_\alpha^L > 0$ is for all $\alpha \in [0, 1]$.

*Remark 2.1* Let $\tilde{a}$ be a fuzzy number. Then $\tilde{a}_\alpha^L \leq \tilde{a}_\alpha^U$ for all $\alpha \in [0, 1]$. Therefore if $\tilde{a}$ is nonnegative then $\tilde{a}_\alpha^L \geq 0$ and $\tilde{a}_\alpha^U \geq 0$ for all $\alpha \in [0, 1]$, and if $\tilde{a}$ is positive then $\tilde{a}_\alpha^L > 0$ and $\tilde{a}_\alpha^U > 0$ for all $\alpha \in [0, 1]$.

Let "$\odot$" be any binary operations $\oplus$ or $\otimes$ between two fuzzy numbers $\tilde{a}$ and $\tilde{b}$. The membership function of $\tilde{a} \odot \tilde{b}$ is defined by

$$\xi_{\tilde{a}\odot\tilde{b}}(z) = \sup_{x\circ y=z} \min\{\xi_{\tilde{a}}(x), \xi_{\tilde{b}}(y)\}$$

using the extension principle in Zadeh (1975), where the operations $\odot = \oplus$ and $\otimes$ correspond to the operations $\circ = +$ and $\times$, respectively. Then we have the following results.

**Proposition 2.1** *Let $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$. Then we have*

(i) $\tilde{a} \oplus \tilde{b} \in \mathcal{F}(\mathbb{R})$ *and*

$$(\tilde{a} \oplus \tilde{b})_\alpha = \left[\tilde{a}^L_\alpha + \tilde{b}^L_\alpha, \tilde{a}^U_\alpha + \tilde{b}^U_\alpha\right];$$

(ii) $\tilde{a} \otimes \tilde{b} \in \mathcal{F}(\mathbb{R})$ *and*

$$(\tilde{a} \otimes \tilde{b})_\alpha = \Big[\min\left\{\tilde{a}^L_\alpha\tilde{b}^L_\alpha, \tilde{a}^L_\alpha\tilde{b}^U_\alpha, \tilde{a}^U_\alpha\tilde{b}^L_\alpha, \tilde{a}^U_\alpha\tilde{b}^U_\alpha\right\}, \max\left\{\tilde{a}^L_\alpha\tilde{b}^L_\alpha, \tilde{a}^L_\alpha\tilde{b}^U_\alpha, \tilde{a}^U_\alpha\tilde{b}^L_\alpha, \tilde{a}^U_\alpha\tilde{b}^U_\alpha\right\}\Big].$$

Let $A$ and $B$ be two compact and convex subsets of $\mathbb{R}^n$. If there exists a compact and convex subset of $\mathbb{R}^n$, say $C$, such that $A = B + C$, then $C$ is called the *Hukuhara difference* of $A$ and $B$. We also write $C = A \ominus B$ (ref. Banks and Jacobs 1970). Inspired by this concept, we can also define the Hukuhara difference between two fuzzy numbers. Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. If there exists a fuzzy number $\tilde{c}$ such that $\tilde{c} \oplus \tilde{b} = \tilde{a}$ (note that the fuzzy addition is commutative), then $\tilde{c}$ is unique. In this case, $\tilde{c}$ is called the *Hukuhara difference* of $\tilde{a}$ and $\tilde{b}$ and is denoted by $\tilde{a} \ominus_H \tilde{b}$ (ref. Puri and Ralescu 1983). The following proposition is very useful for considering the differentiation of fuzzy-valued function.

**Proposition 2.2** *Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. If the Hukuhara difference $\tilde{c} = \tilde{a} \ominus_H \tilde{b}$ exists, then $\tilde{c}^L_\alpha = \tilde{a}^L_\alpha - \tilde{b}^L_\alpha$ and $\tilde{c}^U_\alpha = \tilde{a}^U_\alpha - \tilde{b}^U_\alpha$ for all $\alpha \in [0, 1]$.*

*Proof* The result follows from Proposition 2.1 (i) immediately. □

**Definition 2.3** Let $\tilde{a}$ be a fuzzy number. We say that $\tilde{a}$ is a *canonical fuzzy number* if the functions $\eta_1(\alpha) = \tilde{a}^L_\alpha$ and $\eta_2(\alpha) = \tilde{a}^U_\alpha$ are continuous on $[0, 1]$, where $[\tilde{a}^L_\alpha, \tilde{a}^U_\alpha] = \tilde{a}_\alpha$.

We denote by $\mathcal{F}_c(\mathbb{R})$ the set of all canonical fuzzy numbers.

*Remark 2.2* Let $\tilde{a}$ be a fuzzy number. Then $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ for all $\alpha \in [0, 1]$. Suppose that its membership function is strictly increasing on the interval $\left[\tilde{a}^L_0, \tilde{a}^L_1\right]$ and strictly decreasing on the interval $\left[\tilde{a}^U_1, \tilde{a}^U_0\right]$. Then, from the fact of strict monotonicity, $\eta_1(\alpha) = \tilde{a}^L_\alpha$ and $\eta_2(\alpha) = \tilde{a}^U_\alpha$ are continuous functions with respect to the variable $\alpha$ on $[0, 1]$. It shows that $\tilde{a}$ is a canonical fuzzy number.

We have the converse result for Proposition 2.2. First of all, we need a useful lemma.

**Lemma 2.1** (Negoita and Ralescu 1975) *Let $A$ be a set and $\{A_\alpha : \alpha \in [0, 1]\}$ be a family of subsets of $A$ such that*

(a) $A_0 = A$
(b) $A_\beta \subseteq A_\alpha$ *for* $\alpha < \beta$
(c) $A_\alpha = \bigcap_{n=1}^{\infty} A_{\alpha_n}$ *for* $\alpha_n \uparrow \alpha$.

*Then the function $\xi : A \to [0, 1]$ defined by*

$$\xi(x) = \sup_{\alpha \in [0,1]} \alpha \cdot 1_{A_\alpha}(x)$$

*has the property that $A_\alpha = \{x : \xi(x) \geq \alpha\}$ for all $\alpha \in [0, 1]$, where $1_{A_\alpha}$ is the indicator function of set $A_\alpha$.*

**Proposition 2.3** *Let $\tilde{a}$ and $\tilde{b}$ be two canonical fuzzy numbers and satisfy $\tilde{a}_\alpha^L - \tilde{b}_\alpha^L \leq \tilde{a}_\alpha^U - \tilde{b}_\alpha^U$, $\tilde{a}_\alpha^L - \tilde{b}_\alpha^L \leq \tilde{a}_\beta^L - \tilde{b}_\beta^L$ and $\tilde{a}_\beta^U - \tilde{b}_\beta^U \leq \tilde{a}_\alpha^U - \tilde{b}_\alpha^U$ for $\alpha < \beta$, then the Hukuhara difference $\tilde{c} = \tilde{a} \ominus_H \tilde{b}$ exists, and $\tilde{c}$ is also a canonical fuzzy number.*

*Proof* Let $A_\alpha = \left[\tilde{a}_\alpha^L - \tilde{b}_\alpha^L, \tilde{a}_\alpha^U - \tilde{b}_\alpha^U\right]$. Then $A_\beta \subseteq A_\alpha$ for $\alpha < \beta$. By Definition 2.3, we have $\bigcap_{n=1}^{\infty} A_{\alpha_n} = A_\alpha$ for $\alpha_n \uparrow \alpha$. From Lemma 2.1, we can induce a canonical fuzzy number $\tilde{c}$ such that $\tilde{c}_\alpha^L = \tilde{a}_\alpha^L - \tilde{b}_\alpha^L$ and $\tilde{c}_\alpha^U = \tilde{a}_\alpha^U - \tilde{b}_\alpha^U$. This completes the proof. □

We say that $\tilde{a}$ is a *crisp number* with value $m$ if its membership function is given by

$$\xi_{\tilde{a}}(r) = \begin{cases} 1 \text{ if } r = m \\ 0 \text{ otherwise.} \end{cases}$$

We also use the notation $\tilde{1}_{\{m\}}$ to represent the crisp number with value $m$. It is easy to see that $(\tilde{1}_{\{m\}})_\alpha^L = (\tilde{1}_{\{m\}})_\alpha^U = m$ for all $\alpha \in [0, 1]$. Let us remark that a real number $m$ can be regarded as a crisp number $\tilde{1}_{\{m\}}$.

## 3 Limit and continuity of fuzzy-valued function

Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$. The *Hausdorff metric* is defined by

$$d_H(A, B) = \max\left\{\sup_{a \in A} \inf_{b \in B} \| a - b \|, \sup_{b \in B} \inf_{a \in A} \| a - b \|\right\}.$$

Let $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$. We define the metric $d_{\mathcal{F}}$ in $\mathcal{F}(\mathbb{R})$ as

$$d_{\mathcal{F}}(\tilde{a}, \tilde{b}) = \sup_{0 \leq \alpha \leq 1} d_H(\tilde{a}_\alpha, \tilde{b}_\alpha).$$

The following result is obvious.

**Proposition 3.1** *Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. Then we have*

$$d_H(\tilde{a}_\alpha, \tilde{b}_\alpha) = \max\left\{\left|\tilde{a}_\alpha^L - \tilde{b}_\alpha^L\right|, \left|\tilde{a}_\alpha^U - \tilde{b}_\alpha^U\right|\right\}$$

*for all $\alpha \in [0, 1]$.*

**Proposition 3.2** *Let $\tilde{a}$ and $\tilde{b}$ be two canonical fuzzy numbers. Then $d_{\mathcal{F}}(\tilde{a}, \tilde{b}) < \epsilon$ implies $|\tilde{a}_\alpha^L - \tilde{b}_\alpha^L| < \epsilon$ and $|\tilde{a}_\alpha^U - \tilde{b}_\alpha^U| < \epsilon$ for all $\alpha \in [0, 1]$.*

*Proof* By definition, $\tilde{a}_\alpha^L$, $\tilde{a}_\alpha^U$, $\tilde{b}_\alpha^L$ and $\tilde{b}_\alpha^U$ are continuous with respect to the variable $\alpha$ on $[0, 1]$. From Proposition 3.1, $\eta(\alpha) = d_H(\tilde{a}_\alpha, \tilde{b}_\alpha)$ is continuous on $[0, 1]$. Therefore, we have

$$\epsilon > d_{\mathcal{F}}(\tilde{a}, \tilde{b}) = \sup_{0\leq\alpha\leq 1} d_H(\tilde{a}_\alpha, \tilde{b}_\alpha) = \max_{0\leq\alpha\leq 1} d_H(\tilde{a}_\alpha, \tilde{b}_\alpha).$$

It shows that $d_H(\tilde{a}_\alpha, \tilde{b}_\alpha) < \epsilon$ for all $\alpha \in [0, 1]$. From Proposition 3.1 again, the proof is complete. □

Let $\tilde{f} : \mathbb{R}^n \to \mathcal{F}_c(\mathbb{R})$ be a fuzzy-valued function defined on $\mathbb{R}^n$, i.e., $\tilde{f}(\mathbf{x})$ is a canonical fuzzy number for each $\mathbf{x} \in \mathbb{R}^n$. For any fixed $\alpha \in [0, 1]$, we can define two real-valued functions $\tilde{f}_\alpha^L(\mathbf{x}) = (\tilde{f}(\mathbf{x}))_\alpha^L$ and $\tilde{f}_\alpha^U(\mathbf{x}) = (\tilde{f}(\mathbf{x}))_\alpha^U$ on $\mathbb{R}^n$.

**Definition 3.1** Let $\tilde{a}$ be a canonical fuzzy number. For $\mathbf{c} \in \mathbb{R}^n$, we write

$$\lim_{\mathbf{x}\to\mathbf{c}} \tilde{f}(\mathbf{x}) = \tilde{a}$$

if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that, for $\| \mathbf{x} - \mathbf{c} \| < \delta$, we have $d_{\mathcal{F}}(\tilde{f}(\mathbf{x}), \tilde{a}) < \epsilon$. We say that $\tilde{f}$ is *continuous* at $\mathbf{c}$ if

$$\lim_{\mathbf{x}\to\mathbf{c}} \tilde{f}(\mathbf{x}) = \tilde{f}(\mathbf{c}).$$

We say that the fuzzy-valued function $\tilde{f}$ is *level-wise continuous* at $\mathbf{c}$ if and only if the real-valued functions $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are continuous at $\mathbf{c}$ for all $\alpha \in [0, 1]$.

We are going to show that the continuity implies the level-wise continuity.

**Proposition 3.3** *Let $\tilde{f} : \mathbb{R}^n \to \mathcal{F}_c(\mathbb{R})$ be a fuzzy-valued function defined on $\mathbb{R}^n$. If $\tilde{f}$ is continuous at $\mathbf{c}$, then $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are continuous at $\mathbf{c}$ for all $\alpha \in [0, 1]$, i.e., $\tilde{f}$ is level-wise continuous at $\mathbf{c}$.*

*Proof* From Proposition 3.2, we see that $d_H(\tilde{f}(\mathbf{x}), \tilde{f}(\mathbf{c})) < \epsilon$ implies $|\tilde{f}_\alpha^L(\mathbf{x}) - \tilde{f}_\alpha^L(\mathbf{c})| < \epsilon$ and $|\tilde{f}_\alpha^U(\mathbf{x}) - \tilde{f}_\alpha^U(\mathbf{c})| < \epsilon$ for all $\alpha \in [0, 1]$. The remaining proof is obvious. □

Suppose now that the fuzzy-valued function $\tilde{f} : \mathbb{R} \to \mathcal{F}_c(\mathbb{R})$ is defined on $\mathbb{R}$. Then we can similarly define the right-hand limit

$$\lim_{x \to c+} \tilde{f}(x).$$

Then, from Proposition 3.2, we can also have the following result.

**Proposition 3.4** *Let $\tilde{f} : \mathbb{R} \to \mathcal{F}_c(\mathbb{R})$ be a fuzzy-valued function defined on $\mathbb{R}$. If*

$$\lim_{x \to c+} \tilde{f}(x) = \tilde{a},$$

*where $\tilde{a}$ is a canonical fuzzy number, then*

$$\lim_{x \to c+} \tilde{f}_\alpha^L(x) = \tilde{a}_\alpha^L \text{ and } \lim_{x \to c+} \tilde{f}_\alpha^U(x) = \tilde{a}_\alpha^U$$

*for all $\alpha \in [0, 1]$.*

## 4 Differentiation of fuzzy-valued function

Using the concept of Hukuhara difference between two fuzzy numbers, we can propose the differentiation of fuzzy-valued function (ref. Puri and Ralescu 1983).

**Definition 4.1** Let $X$ be an open subset of $\mathbb{R}$. A fuzzy-valued function $\tilde{f} : X \to \mathcal{F}_c(\mathbb{R})$ is called *level-wise differentiable* at $\bar{x}$ if and only if $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are differentiable at $\bar{x}$ for all $\alpha \in [0, 1]$. We say that the fuzzy-valued function $\tilde{f}$ is *H-differentiable* at $\bar{x}$ if there exists a canonical fuzzy number $\tilde{f}'(\bar{x})$ such that the limits

$$\lim_{h \to 0+} \tilde{1}_{\{\frac{1}{h}\}} \otimes \left[\tilde{f}(\bar{x} + h) \ominus_H \tilde{f}(\bar{x})\right] \text{ and } \lim_{h \to 0+} \tilde{1}_{\{\frac{1}{h}\}} \otimes \left[\tilde{f}(\bar{x}) \ominus_H \tilde{f}(\bar{x} - h)\right]$$

both exist and are equal to $\tilde{f}'(\bar{x})$, where $\tilde{1}_{\{\frac{1}{h}\}}$ is a crisp number with value $\frac{1}{h}$. In this case, $\tilde{f}'(\bar{x})$ is called the *H-derivative* of $\tilde{f}$ at $\bar{x}$.

We are going to show that the H-differentiability implies the level-wise differentiability.

**Proposition 4.1** *Let $X$ be an open subset of $\mathbb{R}$. If a fuzzy-valued function $\tilde{f} : X \to \mathcal{F}_c(\mathbb{R})$ is H-differentiable at $\bar{x}$ with H-derivative $\tilde{f}'(\bar{x})$, then the real-valued functions $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are differentiable at $\bar{x}$ for all $\alpha \in [0, 1]$, i.e., $\tilde{f}$ is level-wise differentiable at $\bar{x}$. Moreover, we have $(\tilde{f}_\alpha^L)'(\bar{x}) = (\tilde{f}'(\bar{x}))_\alpha^L$ and $(\tilde{f}_\alpha^U)'(\bar{x}) = (\tilde{f}'(\bar{x}))_\alpha^U$ for all $\alpha \in [0, 1]$.*

*Proof* From Propositions 2.1, 2.2 and 3.4, we see that

$$\lim_{h \to 0+} \tilde{1}_{\{\frac{1}{h}\}} \otimes \left[\tilde{f}(\bar{x} + h) \ominus_H \tilde{f}(\bar{x})\right] = \tilde{f}'(\bar{x}) = \lim_{h \to 0+} \tilde{1}_{\{\frac{1}{h}\}} \otimes \left[\tilde{f}(\bar{x}) \ominus_H \tilde{f}(\bar{x} - h)\right]$$

implies

$$\lim_{h\to 0+}\frac{\tilde{f}^L_\alpha(\bar{x}+h)-\tilde{f}^L_\alpha(\bar{x})}{h}=(\tilde{f}'(\bar{x}))^L_\alpha=\lim_{h\to 0+}\frac{\tilde{f}^L_\alpha(\bar{x})-\tilde{f}^L_\alpha(\bar{x}-h)}{h}$$

and

$$\lim_{h\to 0+}\frac{\tilde{f}^U_\alpha(\bar{x}+h)-\tilde{f}^U_\alpha(\bar{x})}{h}=(\tilde{f}'(\bar{x}))^U_\alpha=\lim_{h\to 0+}\frac{\tilde{f}^U_\alpha(\bar{x})-\tilde{f}^U_\alpha(\bar{x}-h)}{h}$$

for all $\alpha\in[0,1]$. This shows that the real-valued functions $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ are differentiable at $\bar{x}$, and $(\tilde{f}^L_\alpha)'(\bar{x})=(\tilde{f}'(\bar{x}))^L_\alpha$ and $(\tilde{f}^U_\alpha)'(\bar{x})=(\tilde{f}'(\bar{x}))^U_\alpha$ for all $\alpha\in[0,1]$. □

Now we consider the $n$-dimensional case of differentiability. Let $X$ be an open subset of $\mathbb{R}^n$. We are going to consider the fuzzy-valued function $\tilde{f}$ defined on $X$, i.e., $\tilde{f}(\mathbf{x})=\tilde{f}(x_1,\ldots,x_n)$ is a canonical fuzzy number for each $\mathbf{x}=(x_1,\ldots,x_n)\in X$. Therefore, we have the corresponding real-valued functions

$$\tilde{f}^L_\alpha(\mathbf{x})=\tilde{f}^L_\alpha(x_1,\ldots,x_n)=(\tilde{f}(x_1,\ldots,x_n))^L_\alpha \text{ and } \tilde{f}^U_\alpha(\mathbf{x})=\tilde{f}^U_\alpha(x_1,\ldots,x_n)$$
$$=(\tilde{f}(x_1,\ldots,x_n))^U_\alpha$$

defined on $X$ for all $\alpha\in[0,1]$.

**Proposition 4.2** (Apostol 1974, Theorem 12.11) *Let $f$ be a real-valued function defined on $\mathbb{R}^n$. Assume that one of the partial derivatives $\partial f/\partial x_1,\ldots,\partial f/\partial x_n$ exists at $\bar{\mathbf{x}}$ and that the remaining $n-1$ partial derivatives exist on some neighborhoods of $\bar{\mathbf{x}}$ and are continuous at $\bar{\mathbf{x}}$. Then $f$ is differentiable at $\bar{\mathbf{x}}$.*

Inspired by the above Proposition 4.2, we propose the following definition.

**Definition 4.2** Let $\tilde{f}:X\to\mathcal{F}_c(\mathbb{R})$ be a fuzzy-valued function defined on an open subset $X$ of $\mathbb{R}^n$ and $\bar{\mathbf{x}}=(\bar{x}_1,\ldots,\bar{x}_n)\in X$ be fixed.

(i) We say that the fuzzy-valued function $\tilde{f}$ is *level-wise differentiable* at $\bar{\mathbf{x}}$ if and only if the real-valued functions $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ are differentiable at $\bar{\mathbf{x}}$ for all $\alpha\in[0,1]$ (which imply that all of the partial derivatives $\partial\tilde{f}^L_\alpha/\partial x_i$ and $\partial\tilde{f}^U_\alpha/\partial x_i$ exist at $\bar{\mathbf{x}}$ for all $\alpha\in[0,1]$ and all $i=1,\ldots,n$).

(ii) If the fuzzy-valued function $\tilde{g}(x_i)=\tilde{f}(\bar{x}_1,\ldots,\bar{x}_{i-1},x_i,\bar{x}_{i+1},\ldots,\bar{x}_n)$ is $H$-differentiable at $\bar{x}_i$ with H-derivative $\tilde{g}'(\bar{x}_i)$, then we say that $\tilde{f}$ has the $i$th *partial H-derivative* at $\bar{\mathbf{x}}$. We also write $\tilde{g}'(\bar{x}_i)$ as $(\partial\tilde{f}/\partial x_i)(\bar{\mathbf{x}})$.

(iii) We say that the fuzzy-valued function $\tilde{f}$ is *H-differentiable* at $\bar{\mathbf{x}}$ if one of the partial H-derivatives $\partial\tilde{f}/\partial x_1,\ldots,\partial\tilde{f}/\partial x_n$ exists at $\bar{\mathbf{x}}$ and the remaining $n-1$ partial H-derivatives exist on some neighborhoods of $\bar{\mathbf{x}}$ and are continuous at $\bar{\mathbf{x}}$ (in the sense of fuzzy-valued function).

**Proposition 4.3** *Let $\tilde{f} : X \to \mathcal{F}_c(\mathbb{R})$ be a fuzzy-valued function defined on an open subset $X$ of $\mathbb{R}^n$. If $\tilde{f}$ is H-differentiable at $\bar{\mathbf{x}} \in X$, then $\tilde{f}$ is also level-wise differentiable at $\bar{\mathbf{x}}$.*

*Proof* The result follows from Propositions 3.3, 4.1 and 4.2 immediately. □

**Definition 4.3** Let $\tilde{f} : X \to \mathcal{F}_c(\mathbb{R})$ be a fuzzy-valued function defined on an open subset $X$ of $\mathbb{R}^n$ and $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_n) \in X$ be fixed.

(i) We say that the fuzzy-valued function $\tilde{f}$ is *level-wise continuously differentiable* at $\bar{\mathbf{x}}$ if and only if the real-valued functions $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are continuously differentiable at $\bar{\mathbf{x}}$ for all $\alpha \in [0, 1]$ (i.e., all of the partial derivatives $\partial \tilde{f}_\alpha^L / \partial x_i$ and $\partial \tilde{f}_\alpha^U / \partial x_i$ exist on some neighborhoods of $\bar{\mathbf{x}}$ and are continuous at $\bar{\mathbf{x}}$ for all $\alpha \in [0, 1]$ and all $i = 1, \ldots, n$).

(ii) We say that the fuzzy-valued function $\tilde{f}$ is *continuously H-differentiable* at $\bar{\mathbf{x}}$ if all of the partial H-derivatives $\partial \tilde{f} / \partial x_i$, $i = 1, \ldots, n$, exist on some neighborhoods of $\bar{\mathbf{x}}$ and are continuous at $\bar{\mathbf{x}}$ (in the sense of fuzzy-valued function).

Using Propositions 3.3, 4.1 and 4.2 again, we also have the following result.

**Proposition 4.4** *Let $\tilde{f} : X \to \mathcal{F}_c(\mathbb{R})$ be a fuzzy-valued function defined on an open subset $X$ of $\mathbb{R}^n$. If $\tilde{f}$ is continuously H-differentiable at $\bar{\mathbf{x}} \in X$, then $\tilde{f}$ is also level-wise continuously differentiable at $\bar{\mathbf{x}}$.*

Let $\tilde{f}$ be H-differentiable at $\bar{\mathbf{x}}$. Then the *H-gradient* of $\tilde{f}$ at $\bar{\mathbf{x}}$ is denoted by

$$\nabla \tilde{f}(\bar{\mathbf{x}}) = \left( \frac{\partial \tilde{f}}{\partial x_1}(\bar{\mathbf{x}}), \ldots, \frac{\partial \tilde{f}}{\partial x_n}(\bar{\mathbf{x}}) \right)^T, \tag{1}$$

where each $(\partial \tilde{f} / \partial x_i)(\bar{\mathbf{x}})$ is a canonical fuzzy number for $i = 1, \ldots, n$. The $\alpha$-level set of $\nabla \tilde{f}(\bar{\mathbf{x}})$ is defined and denoted by

$$\left( \nabla \tilde{f}(\bar{\mathbf{x}}) \right)_\alpha = \left( \left( \frac{\partial \tilde{f}}{\partial x_1}(\bar{\mathbf{x}}) \right)_\alpha, \ldots, \left( \frac{\partial \tilde{f}}{\partial x_n}(\bar{\mathbf{x}}) \right)_\alpha \right)^T,$$

where

$$\left( \frac{\partial \tilde{f}}{\partial x_i}(\bar{\mathbf{x}}) \right)_\alpha = \left[ \frac{\partial \tilde{f}_\alpha^L}{\partial x_i}(\bar{\mathbf{x}}), \frac{\partial \tilde{f}_\alpha^U}{\partial x_i}(\bar{\mathbf{x}}) \right] \tag{2}$$

is a closed interval, since Proposition 4.1 shows that

$$\left( \frac{\partial \tilde{f}}{\partial x_i}(\bar{\mathbf{x}}) \right)_\alpha^L = \frac{\partial \tilde{f}_\alpha^L}{\partial x_i}(\bar{\mathbf{x}}) \text{ and } \left( \frac{\partial \tilde{f}}{\partial x_i}(\bar{\mathbf{x}}) \right)_\alpha^U = \frac{\partial \tilde{f}_\alpha^U}{\partial x_i}(\bar{\mathbf{x}}) \tag{3}$$

for all $\alpha \in [0, 1]$. In other words, $(\nabla \tilde{f}(\bar{\mathbf{x}}))_\alpha$ is an $n$-vector whose components are closed intervals as shown in (2) for all $\alpha \in [0, 1]$.

## 5 Solution concept

Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two closed intervals in $\mathbb{R}$. We write $B \leq A$ if and only if $b^L \leq a^L$ and $b^U \leq a^U$, and $B < A$ if and only if the following conditions are satisfied

$$\begin{cases} b^L < a^L \\ b^U \leq a^U \end{cases} \text{or} \begin{cases} b^L \leq a^L \\ b^U < a^U \end{cases} \text{or} \begin{cases} b^L < a^L \\ b^U < a^U \end{cases}.$$

Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. Then $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ and $\tilde{b}_\alpha = [\tilde{b}^L_\alpha, \tilde{b}^U_\alpha]$ are two closed intervals in $\mathbb{R}$ for all $\alpha \in [0, 1]$. We write $\tilde{b} \preceq \tilde{a}$ if and only if $\tilde{b}_\alpha \leq \tilde{a}_\alpha$ for all $\alpha \in [0, 1]$, or equivalently, $\tilde{b}^L_\alpha \leq \tilde{a}^L_\alpha$ and $\tilde{b}^U_\alpha \leq \tilde{a}^U_\alpha$ for all $\alpha \in [0, 1]$. It is easy to see that "$\preceq$" is a partial ordering on $\mathcal{F}(\mathbb{R})$.

Now we consider the following optimization problem with fuzzy-valued objective function

$$\begin{aligned} \text{(FOP1)} \quad \min \quad & \tilde{f}(\mathbf{x}) = \tilde{f}(x_1, \ldots, x_n) \\ \text{subject to } & \mathbf{x} = (x_1, x_2, \ldots, x_n) \in X \subseteq \mathbb{R}^n, \end{aligned}$$

where the feasible set $X$ is assumed to be a convex subset of $\mathbb{R}^n$. For instance, the fuzzy-valued objective function $\tilde{f}(\mathbf{x})$ can be taken as the linear-type objective function

$$\tilde{f}(\mathbf{x}) = \tilde{f}(x_1, \ldots, x_n) = \left(\tilde{a}_1 \otimes \tilde{1}_{\{x_1\}}\right) \oplus \left(\tilde{a}_2 \otimes \tilde{1}_{\{x_2\}}\right) \oplus \cdots \oplus \left(\tilde{a}_n \otimes \tilde{1}_{\{x_n\}}\right),$$

where each $\tilde{a}_i$ is a canonical fuzzy number and each $\tilde{1}_{\{x_i\}}$ is a crisp number with value $x_i$ for $i = 1, \ldots, n$.

We need to interprete the meaning of minimization in problem (FOP1). Since "$\preceq$" is a partial ordering, not a total ordering, on $\mathcal{F}(\mathbb{R})$, we may follow the similar solution concept (the non-dominated solution) used in multiobjective programming problem to interprete the meaning of minimization in problem (FOP1).

Now we write $\tilde{a} \prec \tilde{b}$ if and only if $\tilde{a}_\alpha \leq \tilde{b}_\alpha$ for all $\alpha \in [0, 1]$ and there exists an $\alpha^* \in [0, 1]$ such that $\tilde{a}_{\alpha^*} < \tilde{b}_{\alpha^*}$, i.e.,

$$\begin{cases} \tilde{a}^L_{\alpha^*} < \tilde{b}^L_{\alpha^*} \\ \tilde{a}^U_{\alpha^*} \leq \tilde{b}^U_{\alpha^*} \end{cases} \text{or} \begin{cases} \tilde{a}^L_{\alpha^*} \leq \tilde{b}^L_{\alpha^*} \\ \tilde{a}^U_{\alpha^*} < \tilde{b}^U_{\alpha^*} \end{cases} \text{or} \begin{cases} \tilde{a}^L_{\alpha^*} < \tilde{b}^L_{\alpha^*} \\ \tilde{a}^U_{\alpha^*} < \tilde{b}^U_{\alpha^*} \end{cases}. \tag{4}$$

Therefore, we see that $\tilde{a} \prec \tilde{b}$ means $\tilde{a} \preceq \tilde{b}$ and $\tilde{a} \neq \tilde{b}$. For the minimization problem (FOP1), we propose the following definition.

**Definition 5.1** Let $\mathbf{x}^*$ be a feasible solution, i.e., $\mathbf{x}^* \in X$.

(i) We say that $\mathbf{x}^*$ is a *non-dominated solution* of problem (FOP1) if there exists no $\bar{\mathbf{x}} \in X \setminus \{\mathbf{x}^*\}$ such that $\tilde{f}(\bar{\mathbf{x}}) \prec \tilde{f}(\mathbf{x}^*)$.

(ii) We say that $\mathbf{x}^*$ is a *strongly non-dominated solution* of problem (FOP1) if there exists no $\bar{\mathbf{x}} \in X \setminus \{\mathbf{x}^*\}$ such that $\tilde{f}(\bar{\mathbf{x}}) \preceq \tilde{f}(\mathbf{x}^*)$.

*Remark 5.1* It is easy to see that if $\mathbf{x}^*$ is a strongly non-dominated solution of problem (FOP1), then it is also a non-dominated solution of problem (FOP1).

In the sequel, we are going to provide the Karush-Kuhn-Tucker optimality conditions for (strongly) non-dominated solution of problem (FOP1).

## 6 The Karush-Kuhn-Tucker optimality conditions

Let $X$ be a convex subset of $\mathbb{R}^n$ and $f$ be a real-valued function defined on $X$. We recall that $f$ is convex at $\mathbf{x}^*$ if

$$f(\lambda\mathbf{x}^* + (1-\lambda)\mathbf{x}) \leq \lambda f(\mathbf{x}^*) + (1-\lambda)f(\mathbf{x})$$

for each $\lambda \in (0, 1)$ and each $\mathbf{x} \in X$.

Let $f$ and $g_j$, $j = 1, \ldots, m$, be real-valued functions defined on $\mathbb{R}^n$. Then we consider the following (conventional) optimization problem

$$\begin{aligned} \text{(P)} \quad \min \quad & f(\mathbf{x}) = f(x_1, \ldots, x_n) \\ \text{subject to } & g_j(\mathbf{x}) \leq 0,\ j = 1, \ldots, m. \end{aligned}$$

Suppose that the constraint functions $g_j$ are convex on $\mathbb{R}^n$ for each $j = 1, \ldots, m$. Then the feasible set $X = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \leq 0, j = 1, \ldots, m\}$ is a convex subset of $\mathbb{R}^n$. The well-known Karush-Kuhn-Tucker optimality conditions for problem (P) (e.g., Horst et al. 2000 or Bazarra et al. 1993) is stated below.

**Theorem 6.1** *Assume that the constraint functions $g_j : \mathbb{R}^n \to \mathbb{R}$ are convex on $\mathbb{R}^n$ for $j = 1, \ldots, m$. Let $X = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \leq 0, i = 1, \ldots, m\}$ be a feasible set and a point $\mathbf{x}^* \in X$. Suppose that the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is convex at $\mathbf{x}^*$, and $f$, $g_j$, $j = 1, \ldots, m$, are continuously differentiable at $\mathbf{x}^*$. If there exist* (*Lagrange*) *multipliers $0 \leq \mu_j \in \mathbb{R}$, $j = 1, \ldots, m$, such that*

(i) $\nabla f(\mathbf{x}^*) + \sum_{j=1}^{m} \mu_j \nabla g_j(\mathbf{x}^*) = \mathbf{0}$;
(ii) $\mu_j g_j(\mathbf{x}^*) = 0$ *for all* $j = 1, \ldots, m$,
*then $\mathbf{x}^*$ is an optimal solution of problem* (P).

### 6.1 KKT conditions for level-wise differentiable case

In this subsection, the fuzzy-valued objective function $\tilde{f}$ is assumed as level-wise (continuously) differentiable at a feasible solution $\mathbf{x}^*$. Therefore, from Propositions 4.3 and 4.4, we see that all of the results presented in this subsection also hold true for (continuously) H-differentiable case. We shall also present the KKT conditions in the fuzzy-valued form for continuously H-differentiable case in the next subsection. First of all, we introduce the concept of convexity for fuzzy-valued functions.

**Definition 6.1** Let $X$ be a nonempty convex subset of $\mathbb{R}^n$ and $\tilde{f}$ be a fuzzy-valued function defined on $X$. We say that $\tilde{f}$ is *convex* at $\mathbf{x}^*$ if

$$\tilde{f}(\lambda\mathbf{x}^* + (1-\lambda)\mathbf{x}) \preceq \left(\tilde{1}_{\{\lambda\}} \otimes \tilde{f}(\mathbf{x}^*)\right) \oplus \left(\tilde{1}_{\{1-\lambda\}} \otimes \tilde{f}(\mathbf{x})\right)$$

for each $\lambda \in (0, 1)$ and each $\mathbf{x} \in X$, where $\tilde{1}_{\{\lambda\}}$ and $\tilde{1}_{\{1-\lambda\}}$ are crisp numbers with values $\lambda$ and $1-\lambda$, respectively.

**Proposition 6.1** *Let $X$ be a nonempty convex subset of $\mathbb{R}^n$ and $\tilde{f}$ be a fuzzy-valued function defined on $X$. Then $\tilde{f}$ is convex at $\mathbf{x}^*$ if and only if $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are convex at $\mathbf{x}^*$ for all $\alpha \in [0, 1]$.*

*Proof* The result follows from Proposition 2.1 immediately. □

Now we consider the following constrained minimization problem with fuzzy-valued objective function

$$\begin{array}{lll}(\text{FOP2}) & \min & \tilde{f}(\mathbf{x}) \\ & & \text{subject to } g_j(\mathbf{x}) \leq 0,\ j = 1, \ldots, m,\end{array}$$

where the real-valued constraint functions $g_j : \mathbb{R}^n \to \mathbb{R}$ are convex on $\mathbb{R}^n$ for $j = 1, \ldots, m$. We see that problem (FOP2) follows from problem (FOP1) by taking the convex set $X$ as $X = \{\mathbf{x} : g_j(\mathbf{x}) \leq 0, j = 1, \ldots, m\}$.

Now we are in a position to present the Karush-Kuhn-Tucker optimality conditions for non-dominated solutions of problem (FOP2).

**Theorem 6.2** *Assume that the real-valued constraint functions $g_j : \mathbb{R}^n \to \mathbb{R}$ are convex on $\mathbb{R}^n$ for $j = 1, \ldots, m$. Let $X = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \leq 0, j = 1, \ldots, m\}$ be the feasible set of problem* (FOP2) *and a point $\mathbf{x}^* \in X$. Suppose that the fuzzy-valued objective function $\tilde{f} : \mathbb{R}^n \to \mathcal{F}_c(\mathbb{R})$ is convex and level-wise continuously differentiable at $\mathbf{x}^*$, and the real-valued constraint functions $g_j : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable at $\mathbf{x}^*$ for $j = 1, \ldots, m$. If there exist nonnegative real-valued functions $\mu_j$ (nonnegative Lagrange function multipliers) for $j = 1, \ldots, m$ defined on* $[0, 1]$ *such that*

(i) $\nabla\tilde{f}_\alpha^L(\mathbf{x}^*) + \nabla\tilde{f}_\alpha^U(\mathbf{x}^*) + \sum_{j=1}^m \mu_j(\alpha) \cdot \nabla g_j(\mathbf{x}^*) = \mathbf{0}$ *for all* $\alpha \in [0, 1]$;
(ii) $\mu_j(\alpha) g_j(\mathbf{x}^*) = 0$ *for all* $\alpha \in [0, 1]$ *and all* $j = 1, \ldots, m$,
*then $\mathbf{x}^*$ is a non-dominated solution of problem* (FOP2).

*Proof* We are going to prove this result by contradiction. Suppose that conditions (i) and (ii) are satisfied and $\mathbf{x}^*$ is not a non-dominated solution. Then there exists an $\bar{\mathbf{x}} \in X$ such that $\tilde{f}(\bar{\mathbf{x}}) \prec \tilde{f}(\mathbf{x}^*)$, i.e., from (4),

$$\begin{cases} \tilde{f}_{\alpha^*}^L(\bar{\mathbf{x}}) < \tilde{f}_{\alpha^*}^L(\mathbf{x}^*) \\ \tilde{f}_{\alpha^*}^U(\bar{\mathbf{x}}) \leq \tilde{f}_{\alpha^*}^U(\mathbf{x}^*) \end{cases} \text{or} \begin{cases} \tilde{f}_{\alpha^*}^L(\bar{\mathbf{x}}) \leq \tilde{f}_{\alpha^*}^L(\mathbf{x}^*) \\ \tilde{f}_{\alpha^*}^U(\bar{\mathbf{x}}) < \tilde{f}_{\alpha^*}^U(\mathbf{x}^*) \end{cases} \text{or} \begin{cases} \tilde{f}_{\alpha^*}^L(\bar{\mathbf{x}}) < \tilde{f}_{\alpha^*}^L(\mathbf{x}^*) \\ \tilde{f}_{\alpha^*}^U(\bar{\mathbf{x}}) < \tilde{f}_{\alpha^*}^U(\mathbf{x}^*) \end{cases} \quad (5)$$

for some $\alpha^* \in [0, 1]$. We now define a real-valued function

$$f(\mathbf{x}) = \tilde{f}^L_{\alpha^*}(\mathbf{x}) + \tilde{f}^U_{\alpha^*}(\mathbf{x}). \tag{6}$$

Combining (5) and (6), we see that

$$f(\bar{\mathbf{x}}) < f(\mathbf{x}^*) \tag{7}$$

Since $\tilde{f}$ is convex and level-wise continuously differentiable at $\mathbf{x}^*$, i.e., $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ are convex and continuously differentiable at $\mathbf{x}^*$ for all $\alpha \in [0, 1]$, we see that $f$ is also convex and continuously differentiable at $\mathbf{x}^*$. Furthermore, from (6), we have

$$\nabla f(\mathbf{x}) = \nabla \tilde{f}^L_{\alpha^*}(\mathbf{x}) + \nabla \tilde{f}^U_{\alpha^*}(\mathbf{x}). \tag{8}$$

Since conditions (i) and (ii) are satisfied for all $\alpha \in [0, 1]$, according to (8), we can obtain the following two new conditions for any fixed $\alpha^* \in [0, 1]$:

(i') $\nabla \tilde{f}^L_{\alpha^*}(\mathbf{x}^*)+\nabla \tilde{f}^U_{\alpha^*}(\mathbf{x}^*)+\sum_{j=1}^m \mu_{j\alpha^*}\nabla g_j(\mathbf{x}^*)=\nabla f(\mathbf{x}^*)+\sum_{j=1}^m \mu_{j\alpha^*}\nabla g_j(\mathbf{x}^*)=\mathbf{0}$;

(ii') $\mu_{j\alpha^*} g_j(\mathbf{x}^*) = 0$ for all $j = 1, \ldots, m$,

where $\mu_{j\alpha^*} = \mu_j(\alpha^*) \geq 0$ for $j = 1, \ldots, m$. We consider the following constrained optimization problem

$$\begin{aligned} &\min \quad f(\mathbf{x}) \\ &\text{subject to } g_j(\mathbf{x}) \leq 0,\ j = 1, \ldots, m, \end{aligned}$$

where $f$ is defined in (6). Then this problem has the same constraints of problem (FOP2). Using Theorem 6.1, conditions (i') and (ii') are the KKT conditions for this optimization problem. Therefore, we conclude that $\mathbf{x}^*$ is an optimal solution of the real-valued objective function $f$, i.e., $f(\mathbf{x}^*) \leq f(\bar{\mathbf{x}})$, which contradicts (7). This completes the proof. □

*Remark 6.1* The Lagrange function multipliers $\mu_j$ for $j = 1, \ldots, m$ can be constructed as follows. For any fixed $\alpha \in [0, 1]$, if there exists $0 \leq \mu_{j\alpha} \in \mathbb{R}$ such that the following conditions are satisfied:

(a) $\nabla \tilde{f}^L_\alpha(\mathbf{x}^*) + \nabla \tilde{f}^U_\alpha(\mathbf{x}^*) + \sum_{j=1}^m \mu_{j\alpha} \cdot \nabla g_j(\mathbf{x}^*) = \mathbf{0}$;
(b) $\mu_{j\alpha} g_j(\mathbf{x}^*) = 0$ for all $j = 1, \ldots, m$.

Then we can define the nonnegative real-valued functions $\mu_j(\alpha) = \mu_{j\alpha}$ for all $\alpha \in [0, 1]$ and all $j = 1, \ldots, m$. Therefore, if the above conditions (a) and (b) are satisfied for all $\alpha \in [0, 1]$, then $\mathbf{x}^*$ is a non-dominated solution of problem (FOP2) by constructing the Lagrange function multipliers as described above.

In the sequel, we are going to relax the convexity assumption by considering the pseudoconvexity. Let $X$ be a nonempty feasible set and $\mathbf{x}^* \in \text{cl}X$ (the closure of $X$). The cone of feasible directions of $X$ at $\mathbf{x}^*$, denoted by $D$, is defined by

$$D = \{\mathbf{d} \in \mathbb{R}^n : \mathbf{d} \neq \mathbf{0}, \text{ there exists a } \delta > 0 \text{ such that } \mathbf{x}^* + \eta\mathbf{d} \in X \text{ for all } \eta \in (0, \delta)\}. \tag{9}$$

Each $\mathbf{d}$ of $D$ is called a *feasible direction* of $X$. The following proposition, from Bazarra et al. (1993), is very useful.

**Proposition 6.2** (Bazarra et al. 1993, Lemma 4.2.4) *Let* $X = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \leq 0, j = 1, \ldots, m\}$ *be a feasible set and a point* $\mathbf{x}^* \in X$. *Assume that* $g_j$ *are differentiable at* $\mathbf{x}^*$ *for all* $j = 1, \ldots, m$. *Let* $I = \{j : g_j(\mathbf{x}^*) = 0\}$ *be the index set for the active constraints. Then*

$$D \subseteq \{\mathbf{d} \in \mathbb{R}^n : \nabla g_j(\mathbf{x}^*)^T \mathbf{d} \leq 0 \text{ for each } j \in I\}$$

(*note that this proposition still hold true if we just assume that* $g_j$ *are continuous at* $\mathbf{x}^*$ *instead of differentiable at* $\mathbf{x}^*$ *for* $j \notin I$).

Let $f$ be a differentiable real-valued function defined on a nonempty open convex subset $X$ of $\mathbb{R}^n$. Then $f$ is convex at $\mathbf{x}^*$ if and only if $f(\mathbf{x}) \geq f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*)$ for $\mathbf{x} \in X$ (ref. Bazarra et al. 1993, Theorem 3.3.3), i.e., $f(\mathbf{x}) - f(\mathbf{x}^*) \geq \nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*)$. Therefore, we see that if $f(\mathbf{x}) \leq f(\mathbf{x}^*)$ then $\nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \leq 0$. Also, if $f(\mathbf{x}) < f(\mathbf{x}^*)$ then $\nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) < 0$. Let us also recall that $f$ is pseudoconvex at $\mathbf{x}^*$ if $f(\mathbf{x}) < f(\mathbf{x}^*)$ then $\nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) < 0$ for $\mathbf{x} \in X$, and $f$ is strictly pseudoconvex at $\mathbf{x}^*$ if $f(\mathbf{x}) \leq f(\mathbf{x}^*)$ then $\nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) < 0$ for $\mathbf{x} \in X$. It is well-known that the strict convexity implies the strict pseudoconvexity. Inspired by Proposition 6.1, we propose the following definition.

**Definition 6.2** Let $\tilde{f} : X \to \mathcal{F}_c(\mathbb{R})$ be a fuzzy-valued function defined on a convex set $X \subseteq \mathbb{R}^n$. We say that $\tilde{f}$ is *pseudoconvex* at $\mathbf{x}^*$ if and only if the real-valued functions $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are pseudoconvex at $\mathbf{x}^*$ for all $\alpha \in [0, 1]$.

The Tucker's theorem of the alternative states that, given matrices $A$ and $C$, exactly one of the following system has a solution:

System I: $A\mathbf{x} \leq \mathbf{0}$, $A\mathbf{x} \neq \mathbf{0}$, $C\mathbf{x} \leq \mathbf{0}$ for some $\mathbf{x} \in \mathbb{R}^n$;

System II: $A^T\boldsymbol{\lambda} + C^T\boldsymbol{\mu} = \mathbf{0}$ for some $(\boldsymbol{\lambda}, \boldsymbol{\mu})$, $\boldsymbol{\lambda} > \mathbf{0}$, $\boldsymbol{\mu} \geq \mathbf{0}$.

We are going to use the Tucker's theorem to refine the KKT conditions when some mild conditions are imposed upon the fuzzy-valued objective function.

**Theorem 6.3** *Let* $X = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \leq 0, j = 1, \ldots, m\}$ *be the feasible set of problem* (FOP2). *Assume that* $X$ *is a convex subset of* $\mathbb{R}^n$ *and a point* $\mathbf{x}^* \in X$. *Suppose that the fuzzy-valued objective function* $\tilde{f} : \mathbb{R}^n \to \mathcal{F}_c(\mathbb{R})$ *is level-wise differentiable and pseudoconvex at* $\mathbf{x}^*$, *and the real-valued constraint functions* $g_j : \mathbb{R}^n \to \mathbb{R}$ *are differentiable at* $\mathbf{x}^*$ *for* $j = 1, \ldots, m$. *If there exist nonnegative real-valued functions* $\mu_j^L$ *and* $\mu_j^U$ *for* $j = 1, \ldots, m$ *defined on* $[0, 1]$ *such that*

(i) $\nabla \tilde{f}_\alpha^L(\mathbf{x}^*) + \sum_{j=1}^m \mu_j^L(\alpha) \cdot \nabla g_j(\mathbf{x}^*) = \mathbf{0}$ *for all* $\alpha \in [0, 1]$;

(ii) $\nabla \tilde{f}_\alpha^U(\mathbf{x}^*) + \sum_{j=1}^m \mu_j^U(\alpha) \cdot \nabla g_j(\mathbf{x}^*) = \mathbf{0}$ *for all* $\alpha \in [0, 1]$;

(iii) $\mu_j^L(\alpha) \cdot g_j(\mathbf{x}^*) = 0 = \mu_j^U(\alpha) \cdot g_j(\mathbf{x}^*)$ *for all* $\alpha \in [0, 1]$ *and all* $j = 1, \ldots, m$, *then* $\mathbf{x}^*$ *is a non-dominated solution of problem* (FOP2).

*Proof* We are going to prove this result by contradiction. Suppose that conditions (i–iii) are satisfied and $\mathbf{x}^*$ is not a non-dominated solution. Then there exists an $\bar{\mathbf{x}} \in X$ such that $\tilde{f}(\bar{\mathbf{x}}) \prec \tilde{f}(\mathbf{x}^*)$, i.e., from (5), $\tilde{f}_{\alpha^*}^L(\bar{\mathbf{x}}) < \tilde{f}_{\alpha^*}^L(\mathbf{x}^*)$ or $\tilde{f}_{\alpha^*}^U(\bar{\mathbf{x}}) < \tilde{f}_{\alpha^*}^U(\mathbf{x}^*)$ for some $\alpha^* \in [0, 1]$. Since $\tilde{f}$ is level-wise differentiable and pseudoconvex at $\mathbf{x}^*$, we see that the real-valued functions $\tilde{f}_{\alpha^*}^L$ and $\tilde{f}_{\alpha^*}^U$ are differentiable and pseudoconvex at $\mathbf{x}^*$ by definition. By the definition of pseudoconvexity, we have $\nabla \tilde{f}_{\alpha^*}^L(\mathbf{x}^*)^T(\bar{\mathbf{x}} - \mathbf{x}^*) < 0$ or $\nabla \tilde{f}_{\alpha^*}^U(\mathbf{x}^*)^T(\bar{\mathbf{x}} - \mathbf{x}^*) < 0$. First of all, we consider the case of

$$\nabla \tilde{f}_{\alpha^*}^L(\mathbf{x}^*)^T(\bar{\mathbf{x}} - \mathbf{x}^*) < 0. \tag{10}$$

Let $\mathbf{d} = \bar{\mathbf{x}} - \mathbf{x}^*$. Since $X$ is a convex set and $\bar{\mathbf{x}}, \mathbf{x}^* \in X$, for $\eta \in (0, 1)$, we have

$$\mathbf{x}^* + \eta\mathbf{d} = \mathbf{x}^* + \eta(\bar{\mathbf{x}} - \mathbf{x}^*) = \eta\bar{\mathbf{x}} + (1 - \eta)\mathbf{x}^* \in X.$$

This shows that $\mathbf{d} \in D$ as presented in (9). From Proposition 6.2, we conclude that

$$\nabla g_j(\mathbf{x}^*)^T\mathbf{d} \leq 0 \text{ for each } j \in I, \tag{11}$$

where $I$ is the index set for the active constraints. Since conditions (i) and (iii) are satisfied for all $\alpha \in [0, 1]$, we obtain the following two new conditions:

(a) $\nabla \tilde{f}_{\alpha^*}^L(\mathbf{x}^*) + \sum_{j=1}^m \mu_{j\alpha^*} \cdot \nabla g_j(\mathbf{x}^*) = \mathbf{0}$;

(b) $\mu_{j\alpha^*} \cdot g_j(\mathbf{x}^*) = 0$ for all $j = 1, \ldots, m$,

where $\mu_{j\alpha^*} = \mu_j(\alpha^*) \geq 0$ for all $j = 1, \ldots, m$. Let $A = \nabla \tilde{f}_{\alpha^*}^L(\mathbf{x}^*)^T$ and $C$ be the matrix whose rows are $\nabla g_j(\mathbf{x}^*)^T$ for $j \in I$. We consider the following two systems:

System I: $A\mathbf{d} \leq \mathbf{0}$, $A\mathbf{d} \neq \mathbf{0}$, $C\mathbf{d} \leq \mathbf{0}$ for some $\mathbf{d} \in \mathbb{R}^n$;

System II: $A^T\boldsymbol{\lambda} + C^T\boldsymbol{\mu} = \mathbf{0}$ for some $(\boldsymbol{\lambda}, \boldsymbol{\mu})$, $\boldsymbol{\lambda} > \mathbf{0}$, $\boldsymbol{\mu} \geq \mathbf{0}$.

Then from (10) to (11), System I has a solution $\mathbf{d} = \bar{\mathbf{x}} - \mathbf{x}^*$. According to the Tucker's theorem of the alternative, system II will have no solutions. That is, there exist no multipliers $0 < \lambda \in \mathbb{R}$ and $0 \leq \mu_j \in \mathbb{R}$, $j \in I$, such that

$$\lambda\nabla \tilde{f}_{\alpha^*}^L(\mathbf{x}^*) + \sum_{j \in I} \mu_j \nabla g_j(\mathbf{x}^*) = \mathbf{0};$$

or, equivalently, there exist no multipliers $0 \leq \eta_j \in \mathbb{R}$, $j \in I$, such that

$$\nabla \tilde{f}_{\alpha^*}^L(\mathbf{x}^*) + \sum_{j \in I} \eta_j \nabla g_j(\mathbf{x}^*) = \mathbf{0}, \tag{12}$$

where $\eta_j = \mu_j/\lambda$. Since $I$ is the index set of active constraints, we have $g_j(\mathbf{x}^*) \neq 0$ for $j \notin I$. Therefore, if $\eta_j g_j(\mathbf{x}^*) = 0$ for all $j = 1, \ldots, m$, then $\eta_j = 0$ for $j \notin I$, i.e.,

$$\sum_{j \in I} \eta_j \nabla g_j(\mathbf{x}^*) = \sum_{j=1}^m \eta_j \nabla g_j(\mathbf{x}^*). \tag{13}$$

In other words, from (12) and (13), there exist no multipliers $0 \leq \eta_j \in \mathbb{R}$ such that

(a') $\nabla \tilde{f}^L_{\alpha^*}(\mathbf{x}^*) + \sum_{j=1}^m \eta_j \nabla g_j(\mathbf{x}^*) = \mathbf{0}$;
(b') $\eta_j \cdot g_j(\mathbf{x}^*) = 0$ for all $j = 1, \ldots, m$.

However, conditions (a') and (b') violate the previous conditions (a) and (b) for the existence of multipliers $\mu_{j\alpha^*} \geq 0$ for all $j = 1, \ldots, m$. Similarly, for the case of $\nabla \tilde{f}^U_{\alpha^*}(\mathbf{x}^*)^T(\bar{\mathbf{x}} - \mathbf{x}^*) < 0$, conditions (ii) and (iii) of this theorem will be violated. We complete the proof. □

Next, we are going to present the KKT conditions for strongly non-dominated solutions.

**Definition 6.3** Let $\tilde{f} : X \to \mathcal{F}_c(\mathbb{R})$ be a fuzzy-valued function defined on a convex set $X \subseteq \mathbb{R}^n$. We say that $\tilde{f}$ is *strictly upper-pseudoconvex* (resp. *strictly lower-pseudoconvex*) at $\mathbf{x}^*$ if each real-valued function $\tilde{f}^U_\alpha$ (resp. $\tilde{f}^L_\alpha$) is strictly pseudoconvex at $\mathbf{x}^*$ for all $\alpha \in [0, 1]$.

**Theorem 6.4** *Let* $X = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \leq 0, j = 1, \ldots, m\}$ *be the feasible set of problem* (FOP2). *Assume that* $X$ *is a convex subset of* $\mathbb{R}^n$ *and a point* $\mathbf{x}^* \in X$. *Suppose that the fuzzy-valued objective function* $\tilde{f} : \mathbb{R}^n \to \mathcal{F}_c(\mathbb{R})$ *is level-wise differentiable and strictly lower-pseudoconvex* (*resp. strictly upper-pseudoconvex*) *at* $\mathbf{x}^*$, *and the real-valued constraint functions* $g_j : \mathbb{R}^n \to \mathbb{R}$ *are differentiable at* $\mathbf{x}^*$ *for* $j = 1, \ldots, m$. *If there exist* $\alpha^* \in [0, 1]$ *and* $0 \leq \mu_j \in \mathbb{R}$ *for* $j = 1, \ldots, m$ *such that*

(i) $\nabla \tilde{f}^L_{\alpha^*}(\mathbf{x}^*) + \sum_{j=1}^m \mu_j \cdot \nabla g_j(\mathbf{x}^*) = \mathbf{0}$
(*resp.* $\nabla \tilde{f}^U_{\alpha^*}(\mathbf{x}^*) + \sum_{j=1}^m \mu_j \cdot \nabla g_j(\mathbf{x}^*) = \mathbf{0}$);
(ii) $\mu_j \cdot g_j(\mathbf{x}^*) = 0$ *for all* $j = 1, \ldots, m$,
*then* $\mathbf{x}^*$ *is a strongly non-dominated solution of problem* (FOP2).

*Proof* We are going to prove this result by contradiction. Suppose that conditions (i) and (ii) are satisfied and $\mathbf{x}^*$ is not a strongly non-dominated solution. Then there exists an $\bar{\mathbf{x}} \neq \mathbf{x}^*$ such that $\tilde{f}(\bar{\mathbf{x}}) \preceq \tilde{f}(\mathbf{x}^*)$, i.e., $\tilde{f}^L_\alpha(\bar{\mathbf{x}}) \leq \tilde{f}^L_\alpha(\mathbf{x}^*)$ (resp. $\tilde{f}^U_\alpha(\bar{\mathbf{x}}) \leq \tilde{f}^U_\alpha(\mathbf{x}^*)$) for all $\alpha \in [0, 1]$. Since $\tilde{f}$ is level-wise differentiable and strictly lower-pseudoconvex (resp. strictly upper-pseudoconvex) at $\mathbf{x}^*$, i.e., $\tilde{f}^L_\alpha$ (resp. $\tilde{f}^U_\alpha$) is differentiable and strictly pseudoconvex at $\mathbf{x}^*$ by definition for all $\alpha \in [0, 1]$, we have $\nabla \tilde{f}^L_\alpha(\mathbf{x}^*)^T(\bar{\mathbf{x}} - \mathbf{x}^*) < 0$ (resp. $\nabla \tilde{f}^U_\alpha(\mathbf{x}^*)^T(\bar{\mathbf{x}} - \mathbf{x}^*) < 0$) for all $\alpha \in [0, 1]$, i.e., $\nabla \tilde{f}^L_{\alpha^*}(\mathbf{x}^*)^T(\bar{\mathbf{x}} - \mathbf{x}^*) < 0$ (resp. $\nabla \tilde{f}^U_{\alpha^*}(\mathbf{x}^*)^T(\bar{\mathbf{x}} - \mathbf{x}^*) < 0$). Using the similar arguments in the proof of Theorem 6.3, there exist no multipliers $0 \leq \eta_j \in \mathbb{R}$ such that
(a') $\nabla \tilde{f}^L_{\alpha^*}(\mathbf{x}^*) + \sum_{j=1}^m \eta_j \nabla g_j(\mathbf{x}^*) = \mathbf{0}$
(resp. $\nabla \tilde{f}^U_{\alpha^*}(\mathbf{x}^*) + \sum_{j=1}^m \eta_j \nabla g_j(\mathbf{x}^*) = \mathbf{0}$);
(b') $\eta_j \cdot g_j(\mathbf{x}^*) = 0$ for all $j = 1, \ldots, m$.
This shows that conditions (i) and (ii) are violated. We complete the proof. □

## 6.2 KKT conditions for H-differentiable case

We are going to present the KKT conditions in the fuzzy-valued form. For notational convenience, we denote by $\tilde{0}$ the crisp number $\tilde{1}_{\{0\}}$ with value 0. We also write

$\tilde{\mathbf{0}} = (\tilde{0}, \ldots, \tilde{0})^T$. Let $\mathbf{x}$ be an $n$-vector in $\mathbb{R}^n$. Then the crisp vector $\tilde{1}_{\{\mathbf{x}\}}$ is defined as $\tilde{1}_{\{\mathbf{x}\}} = \left(\tilde{1}_{\{x_1\}}, \tilde{1}_{\{x_2\}}, \ldots, \tilde{1}_{\{x_n\}}\right)$. Let $\mathbf{a} = (a_1, \ldots, a_n)$ be an $n$-vector. We say that $\mathbf{a}$ has the same sign if and only if $a_i \geq 0$ for all $i = 1, \ldots, n$ simultaneously, or $a_i < 0$ for all $i = 1, \ldots, n$ simultaneously (i.e., the components of vector $\mathbf{a}$ have the same sign). Or, equivalently, $\mathbf{a}$ has the same sign if and only if $\mathbf{a} \geq \mathbf{0}$ or $\mathbf{a} < \mathbf{0}$.

**Theorem 6.5** *Assume that the real-valued constraint functions $g_j : \mathbb{R}^n \to \mathbb{R}$ are convex on $\mathbb{R}^n$ for $j = 1, \ldots, m$. Let $X = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \leq 0, j = 1, \ldots, m\}$ be the feasible set of problem* (FOP2) *and a point $\mathbf{x}^* \in X$. Suppose that the fuzzy-valued objective function $\tilde{f} : \mathbb{R}^n \to \mathcal{F}_c(\mathbb{R})$ is convex and continuously H-differentiable at $\mathbf{x}^*$, and the real-valued constraint functions $g_j : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable at $\mathbf{x}^*$ for $j = 1, \ldots, m$. We also assume that each $\nabla g_j(\mathbf{x}^*)$ has the same sign for $j = 1, \ldots, m$. If there exist nonnegative fuzzy numbers* (*nonnegative fuzzy Lagrange multipliers*) *$\tilde{\mu}_j \in \mathcal{F}(\mathbb{R})$, $j = 1, \ldots, m$, such that*

(i) $\nabla \tilde{f}(\mathbf{x}^*) \oplus \left[\bigoplus_{j=1}^m \left(\tilde{\mu}_j \otimes \tilde{1}_{\{\nabla g_j(\mathbf{x}^*)\}}\right)\right] = \tilde{\mathbf{0}}$;
(ii) $\tilde{\mu}_j \otimes \tilde{1}_{\{g_j(\mathbf{x}^*)\}} = \tilde{0}$ *for all* $j = 1, \ldots, m$,

*then $\mathbf{x}^*$ is a non-dominated solution of problem* (FOP2).

*Proof* Let $I_+ \subset \{1, \ldots, m\}$ and $I_- \subset \{1, \ldots, m\}$ be the index sets defined by

$$I_+ = \{j : \nabla g_j(\mathbf{x}^*) \geq \mathbf{0}\} \text{ and } I_- = \{j : \nabla g_j(\mathbf{x}^*) < \mathbf{0}\}.$$

Since $\tilde{f}$ is H-differentiable at $\mathbf{x}^*$, the H-gradient $\nabla \tilde{f}(\mathbf{x}^*)$ exists as shown in (1). Since

$$\tilde{1}_{\{\nabla g_j(\mathbf{x}^*)\}} = \left(\tilde{1}_{\left\{\frac{\partial g_j}{\partial x_1}(\mathbf{x}^*)\right\}}, \tilde{1}_{\left\{\frac{\partial g_j}{\partial x_2}(\mathbf{x}^*)\right\}}, \ldots, \tilde{1}_{\left\{\frac{\partial g_j}{\partial x_n}(\mathbf{x}^*)\right\}}\right)^T,$$

from (1), the $i$th component of the formula in condition (i) is given by

$$\frac{\partial \tilde{f}}{\partial x_i}(\mathbf{x}^*) \oplus \left[\bigoplus_{j=1}^m \left(\tilde{\mu}_j \otimes \tilde{1}_{\left\{\frac{\partial g_j}{\partial x_i}(\mathbf{x}^*)\right\}}\right)\right] = \tilde{0}. \tag{14}$$

Taking the $\alpha$-level set of (14) by using (3) and Proposition 2.1, we have

$$\begin{aligned} &\frac{\partial \tilde{f}_\alpha^L}{\partial x_i}(\mathbf{x}^*) + \sum_{j \in I_+} (\tilde{\mu}_j)_\alpha^L \cdot \frac{\partial g_j}{\partial x_i}(\mathbf{x}^*) + \sum_{j \in I_-} (\tilde{\mu}_j)_\alpha^U \cdot \frac{\partial g_j}{\partial x_i}(\mathbf{x}^*) = 0 \\ &= \frac{\partial \tilde{f}_\alpha^U}{\partial x_i}(\mathbf{x}^*) + \sum_{j \in I_+} (\tilde{\mu}_j)_\alpha^U \cdot \frac{\partial g_j}{\partial x_i}(\mathbf{x}^*) + \sum_{j \in I_-} (\tilde{\mu}_j)_\alpha^L \cdot \frac{\partial g_j}{\partial x_i}(\mathbf{x}^*) \end{aligned}$$

for all $\alpha \in [0, 1]$ and all $i = 1, \ldots, n$, where $(\tilde{\mu}_j)_\alpha^L$ and $(\tilde{\mu}_j)_\alpha^U$ are nonnegative real numbers by Remark 2.1 for all $\alpha \in [0, 1]$ and all $j = 1, \ldots, m$. Equivalently, in the vector form, we have

$$\begin{aligned}\nabla \tilde{f}^L_\alpha(\mathbf{x}^*) + \sum_{j\in I_+} (\tilde{\mu}_j)^L_\alpha \cdot \nabla g_j(\mathbf{x}^*) + \sum_{j\in I_-} (\tilde{\mu}_j)^U_\alpha \cdot \nabla g_j(\mathbf{x}^*) = \mathbf{0}\\ = \nabla \tilde{f}^U_\alpha(\mathbf{x}^*) + \sum_{j\in I_+} (\tilde{\mu}_j)^U_\alpha \cdot \nabla g_j(\mathbf{x}^*) + \sum_{j\in I_-} (\tilde{\mu}_j)^L_\alpha \cdot \nabla g_j(\mathbf{x}^*)\end{aligned}$$

for all $\alpha \in [0, 1]$, which also implies, by adding them together,

$$\nabla \tilde{f}^L_\alpha(\mathbf{x}^*) + \nabla \tilde{f}^U_\alpha(\mathbf{x}^*) + \sum_{j=1}^{m} \mu_{j\alpha} \cdot \nabla g_j(\mathbf{x}^*) = \mathbf{0} \tag{15}$$

for all $\alpha \in [0, 1]$, where $\mu_{j\alpha} = (\tilde{\mu}_j)^L_\alpha + (\tilde{\mu}_j)^U_\alpha$ is a nonnegative real number for all $\alpha \in [0, 1]$ and all $j = 1, \ldots, m$. We are going to prove this theorem by contradiction. Suppose that $\mathbf{x}^*$ is not a non-dominated solution. Then there exists an $\bar{\mathbf{x}} \in X$ such that $\tilde{f}(\bar{\mathbf{x}}) \prec \tilde{f}(\mathbf{x}^*)$, i.e., (5) is satisfied for some $\alpha^* \in [0, 1]$. We now define a real-valued function

$$f(\mathbf{x}) = \tilde{f}^L_{\alpha^*}(\mathbf{x}) + \tilde{f}^U_{\alpha^*}(\mathbf{x}). \tag{16}$$

From Propositions 4.4 and 6.1, we see that the real-valued functions $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ are convex and continuously differentiable at $\mathbf{x}^*$ for all $\alpha \in [0, 1]$. Therefore, $f$ is also convex and continuously differentiable at $\mathbf{x}^*$. From condition (ii) of this theorem and Proposition 2.1, since $g_j(\mathbf{x}^*) \leq 0$ for all $j = 1, \ldots, m$, we see that

$$(\tilde{\mu}_j)^U_\alpha \cdot g_j(\mathbf{x}^*) = \left(\tilde{\mu}_j \otimes \tilde{1}_{\{g_j(\mathbf{x}^*)\}}\right)^L_\alpha = 0 = \left(\tilde{\mu}_j \otimes \tilde{1}_{\{g_j(\mathbf{x}^*)\}}\right)^U_\alpha = (\tilde{\mu}_j)^L_\alpha \cdot g_j(\mathbf{x}^*)$$

for all $\alpha \in [0, 1]$ and all $j = 1, \ldots, m$, which implies

$$0 = (\tilde{\mu}_j)^L_\alpha \cdot g_j(\mathbf{x}^*) + (\tilde{\mu}_j)^U_\alpha \cdot g_j(\mathbf{x}^*) = \mu_{j\alpha} \cdot g_j(\mathbf{x}^*) \tag{17}$$

for all $\alpha \in [0, 1]$ and all $j = 1, \ldots, m$. From (16), we have

$$\nabla f(\mathbf{x}) = \nabla \tilde{f}^L_{\alpha^*}(\mathbf{x}) + \nabla \tilde{f}^U_{\alpha^*}(\mathbf{x}),$$

According to equations (15) and (17), we obtain the following new conditions

(i)' $\nabla f(\mathbf{x}^*) + \sum_{j=1}^m \mu_{j\alpha^*} \nabla g_j(\mathbf{x}^*) = \mathbf{0}$ [note that equation (15) is satisfied for all $\alpha \in [0, 1]$];

(ii)' $\mu_{j\alpha^*} \cdot g_j(\mathbf{x}^*) = 0$ for all $j = 1, \ldots, m$ [note that equation (17) is satisfied for all $\alpha \in [0, 1]$].

Using Theorem 6.1, we see that $\mathbf{x}^*$ is an optimal solution of the real-valued objective function $f$ subject to the same constraints of problem (FOP2), i.e.,

$$f(\mathbf{x}^*) \leq f(\bar{\mathbf{x}}). \tag{18}$$

From (16) to (5), we see that $f(\bar{\mathbf{x}}) < f(\mathbf{x}^*)$, which contradicts (18). This completes the proof. □

## 6.3 Numerical examples

Some examples are provided to illustrate the applications.

*Example 6.1* Now we introduce the concept of triangular fuzzy number. The membership function of a triangular fuzzy number $\tilde{a}$ is defined by

$$\xi_{\tilde{a}}(r) = \begin{cases} (r - a^L)/(a - a^L) & \text{if } a^L \leq r \leq a \\ (a^U - r)/(a^U - a) & \text{if } a < r \leq a^U \\ 0 & \text{otherwise,} \end{cases}$$

which is denoted by $\tilde{a} = (a^L, a, a^U)$. The $\alpha$-level set (a closed interval) of $\tilde{a}$ is then

$$\tilde{a}_\alpha = [(1-\alpha)a^L + \alpha a, (1-\alpha)a^U + \alpha a];$$

that is,

$$\tilde{a}^L_\alpha = (1-\alpha)a^L + \alpha a \text{ and } \tilde{a}^U_\alpha = (1-\alpha)a^U + \alpha a. \tag{19}$$

It is easy to see that the triangular fuzzy number is also a canonical fuzzy numbers. Now we consider the following optimization problem

$$\begin{aligned} \min \quad & \left(\widetilde{-5} \otimes \tilde{1}_{\{x_1\}}\right) \oplus \left(\widetilde{-8} \otimes \tilde{1}_{\{x_2\}}\right) \oplus \left(\widetilde{-7} \otimes \tilde{1}_{\{x_3\}}\right) \oplus \left(\widetilde{-4} \otimes \tilde{1}_{\{x_4\}}\right) \\ & \oplus \left(\widetilde{-6} \otimes \tilde{1}_{\{x_5\}}\right) \\ \text{subject to } & 2x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 \leq 20 \\ & 3x_1 + 5x_2 + 4x_3 + 2x_4 + 4x_5 \leq 30 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0, \end{aligned}$$

where

$$\begin{aligned} &\widetilde{-5} = (-6, -5, -3),\ \widetilde{-8} = (-9, -8, -6),\ \widetilde{-7} = (-8, -7, -4), \\ &\widetilde{-4} = (-5, -4, -1),\ \widetilde{-6} = (-7, -6, -5) \end{aligned}$$

are triangular fuzzy numbers. That is,

$$\begin{aligned} \tilde{f}(x_1, \ldots, x_5) &= \left(\widetilde{-5} \otimes \tilde{1}_{\{x_1\}}\right) \oplus \left(\widetilde{-8} \otimes \tilde{1}_{\{x_2\}}\right) \oplus \left(\widetilde{-7} \otimes \tilde{1}_{\{x_3\}}\right) \oplus \left(\widetilde{-4} \otimes \tilde{1}_{\{x_4\}}\right) \\ &\quad \oplus \left(\widetilde{-6} \otimes \tilde{1}_{\{x_5\}}\right) \\ g_1(x_1, \ldots, x_5) &= 2x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 - 20 \\ g_2(x_1, \ldots, x_5) &= 3x_1 + 5x_2 + 4x_3 + 2x_4 + 4x_5 - 30 \\ g_j(x_1, \ldots, x_5) &= -x_{j-2} \text{ for } j = 3, \ldots, 7. \end{aligned}$$

Using Proposition 2.1 and (19), we obtain

$$\begin{aligned}\tilde{f}_\alpha^L(x_1,\ldots,x_5) &= x_1(-6+\alpha)+x_2(-9+\alpha)+x_3(-8+\alpha)+x_4(-5+\alpha)\\ &\quad +x_5(-7+\alpha)\\ \tilde{f}_\alpha^U(x_1,\ldots,x_5) &= x_1(-3-2\alpha)+x_2(-6-2\alpha)+x_3(-4-3\alpha)+x_4(-1-3\alpha)\\ &\quad +x_5(-5-\alpha)\end{aligned}$$

for $\alpha \in [0, 1]$. We can also obtain

$$\nabla \tilde{f}_\alpha^L(\mathbf{x}) = \begin{bmatrix} -6+\alpha \\ -9+\alpha \\ -8+\alpha \\ -5+\alpha \\ -7+\alpha \end{bmatrix}, \nabla \tilde{f}_\alpha^U(\mathbf{x}) = \begin{bmatrix} -3-2\alpha \\ -6-2\alpha \\ -4-3\alpha \\ -1-3\alpha \\ -5-\alpha \end{bmatrix}, \nabla g_1(\mathbf{x}) = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$
$$\text{and } \nabla g_2(\mathbf{x}) = \begin{bmatrix} 3 \\ 5 \\ 4 \\ 2 \\ 4 \end{bmatrix}.$$

Let us first solve the equations $g_1(\mathbf{x}) = 0 = g_2(\mathbf{x})$. Then we obtain

$$\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) = (0, 5, 0, 2.5, 0).$$

From condition (ii) in Theorem 6.2, we see that $\mu_4(\alpha) = 0 = \mu_6(\alpha)$, since $g_4(\mathbf{x}^*) = -5$ and $g_6(\mathbf{x}^*) = -2.5$. Now applying condition (i) at this $\mathbf{x}^*$, we obtain

$$\begin{aligned}&\nabla \tilde{f}_\alpha^L(\mathbf{x}^*) + \nabla \tilde{f}_\alpha^U(\mathbf{x}^*) + \sum_{j=1}^{7} \mu_j(\alpha)\cdot \nabla g_j(\mathbf{x}^*)\\ &\quad = \begin{bmatrix} -9-\alpha+2\mu_1(\alpha)+3\mu_2(\alpha)-\mu_3(\alpha) \\ -15-\alpha+3\mu_1(\alpha)+5\mu_2(\alpha) \\ -12-2\alpha+3\mu_1(\alpha)+4\mu_2(\alpha)-\mu_5(\alpha) \\ -6-2\alpha+2\mu_1(\alpha)+2\mu_2(\alpha) \\ -12+2\mu_1(\alpha)+4\mu_2(\alpha)-\mu_7(\alpha) \end{bmatrix} = \mathbf{0}.\end{aligned}$$

After some algebraic calculations, we obtain the nonnegative real-valued functions

$$\mu_1(\alpha) = 2\alpha,\ \mu_2(\alpha) = 3-\alpha \text{ and } \mu_j(\alpha) = 0 \text{ for } j = 3,\ldots,7 \text{ and } \alpha \in [0, 1].$$

Therefore, using Theorem 6.2, $\mathbf{x}^* = (0, 5, 0, 2.5, 0)$ is a non-dominated solution.

*Example 6.2* Let us consider the following optimization problem

$$\begin{array}{ll} \min & \left[\left(\tilde{1}_{\{x_1\}} \oplus \widetilde{-3}\right) \otimes \left(\tilde{1}_{\{x_1\}} \oplus \widetilde{-3}\right)\right] \oplus \left[\left(\tilde{1}_{\{x_2\}} \oplus \widetilde{-5}\right) \otimes \left(\tilde{1}_{\{x_2\}} \oplus \widetilde{-5}\right)\right] \\ \text{subject to} & x_1 + x_2 \geq 10 \\ & x_1 \geq 4 \\ & x_2 \geq 6, \end{array}$$

where $\widetilde{-3} = (-4, -3, -2)$ and $\widetilde{-5} = (-6, -5, -4)$ are triangular fuzzy numbers. Then we have

$$\begin{aligned} \tilde{f}(x_1, x_2) &= \left[\left(\tilde{1}_{\{x_1\}} \oplus \widetilde{-3}\right) \otimes \left(\tilde{1}_{\{x_1\}} \oplus \widetilde{-3}\right)\right] \oplus \left[\left(\tilde{1}_{\{x_2\}} \oplus \widetilde{-5}\right) \otimes \left(\tilde{1}_{\{x_2\}} \oplus \widetilde{-5}\right)\right] \\ g_1(x_1, x_2) &= -x_1 - x_2 + 10 \\ g_2(x_1, x_2) &= -x_1 + 4 \\ g_3(x_1, x_2) &= -x_2 + 6. \end{aligned}$$

Using Proposition 2.1 and (19), we obtain

$$\begin{aligned} &\left(\tilde{1}_{\{x_1\}} \oplus \widetilde{-3}\right)_\alpha^L = x_1 - 4 + \alpha \geq 0,\ \left(\tilde{1}_{\{x_1\}} \oplus \widetilde{-3}\right)_\alpha^U = x_1 - 2 - \alpha \geq 0 \\ &\left(\tilde{1}_{\{x_2\}} \oplus \widetilde{-5}\right)_\alpha^L = x_2 - 6 + \alpha \geq 0,\ \left(\tilde{1}_{\{x_2\}} \oplus \widetilde{-5}\right)_\alpha^U = x_2 - 4 - \alpha \geq 0 \end{aligned}$$

since $x_1 \geq 4, x_2 \geq 6$ and $\alpha \in [0, 1]$. Therefore, using Proposition 2.1 again, we obtain

$$\begin{aligned} \tilde{f}_\alpha^L(x_1, x_2) &= \left[\left(\tilde{1}_{\{x_1\}} \oplus \widetilde{-3}\right)_\alpha^L\right]^2 + \left[\left(\tilde{1}_{\{x_2\}} \oplus \widetilde{-5}\right)_\alpha^L\right]^2 \\ &= (x_1 - 4 + \alpha)^2 + (x_2 - 6 + \alpha)^2 \\ \tilde{f}_\alpha^U(x_1, x_2) &= \left[\left(\tilde{1}_{\{x_1\}} \oplus \widetilde{-3}\right)_\alpha^U\right]^2 + \left[\left(\tilde{1}_{\{x_2\}} \oplus \widetilde{-5}\right)_\alpha^U\right]^2 \\ &= (x_1 - 2 - \alpha)^2 + (x_2 - 4 - \alpha)^2. \end{aligned}$$

For any fixed $\alpha \in [0, 1]$, we see that $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are strictly convex, i.e., strictly pseudoconvex. It says that $\tilde{f}$ is both strictly lower-pseudoconvex and strictly upper-pseudoconvex. Therefore we are going to apply Theorem 6.4 to obtain the strongly non-dominated solution by considering $\tilde{f}$ as a strictly lower-pseudoconvex. Now we have

$$\begin{aligned} &\nabla \tilde{f}_\alpha^L(\mathbf{x}) = \begin{bmatrix} 2 \cdot (x_1 - 4 + \alpha) \\ 2 \cdot (x_2 - 6 + \alpha) \end{bmatrix}, \nabla \tilde{f}_\alpha^U(\mathbf{x}) = \begin{bmatrix} 2 \cdot (x_1 - 2 - \alpha) \\ 2 \cdot (x_2 - 4 - \alpha) \end{bmatrix}, \\ &\nabla g_1(\mathbf{x}) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \nabla g_2(\mathbf{x}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \text{ and } \nabla g_3(\mathbf{x}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \end{aligned}$$

According to conditions (i) and (ii) in Theorem 6.4, we need to solve the following system of equations

$$\begin{cases} 2x_1 - 8 + 2\alpha^* - \mu_1 - \mu_2 = 0 \\ 2x_2 - 12 + 2\alpha^* - \mu_1 - \mu_3 = 0 \\ \mu_1 \cdot (-x_1 - x_2 + 10) = 0 \\ \mu_2 \cdot (-x_1 + 4) = 0 \\ \mu_3 \cdot (-x_2 + 6) = 0. \end{cases}$$

Then we obtain

$$(x_1, x_2) = (4, 6) \text{ and } \alpha^* = \mu_1 = \mu_2 = \mu_3 = 0.5.$$

This says that $(x_1^*, x_2^*) = (4, 6)$ is a strongly non-dominated solution. From Remark 5.1, $(x_1^*, x_2^*) = (4, 6)$ is also a non-dominated solution.

## References

Apostol TM (1974) Mathematical analysis 2nd edn, Addison–Wesley Publishing Company, Reading

Banks HT, Jacobs MQ (1970) A differential calculus for multifunctions. J Math Anal Appl 29:246–272

Bazarra MS, Sherali HD, Shetty CM (1993) Nonlinear programming. Wiley, New York

Bellman RE, Zadeh LA (1970) Decision making in a fuzzy environment. Manage Sci 17:141–164

Birge JR, Louveaux F (1997) Introduction to stochastic programming. Physica-Verlag, New York

Delgado M, Kacprzyk J, Verdegay J-L, Vila MA (eds.) (1994) Fuzzy optimization: recent advances. Physica-Verlag, New York

Horst R, Pardalos PM, Thoai NV (2000) Introduction to global optimization, 2nd edn. Kluwer, Boston

Inuiguchi M, Ramík J (2000) Possibilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem. Fuzzy Sets Syst 111:3–28

Kall P (1976) Stochastic linear programming. Springer, New York

Lai Y-J, Hwang C-L (1992) Fuzzy mathematical programming: methods and applications. Lecture notes in economics and mathematical systems, vol 394. Springer, New York

Lai Y-J, Hwang C-L (1994) Fuzzy multiple objective decision making: methods and applications. Lecture notes in economics and mathematical systems, vol 404. Springer, New York

Negoita CV, Ralescu DA (1975) Applications of fuzzy sets to systems analysis. Wiley, New York

Prékopa A (1995) Stochastic programming. Kluwer, Boston

Puri ML, Ralescu DA (1983) Differentials of fuzzy function. J Math Anal Appl 91:552–558

Rödder W, Zimmermann H-J (1977) Analyse, Beschreibung und Optimierung von unscharf formulierten Problemen (German). Z Oper Res Ser A–B 21(1):A1-A18

Słowiński R (ed.) (1998) Fuzzy sets in decision analysis. Operations research and statistics. Kluwer, Boston

Słowiński R, Teghem J (eds) (1990) Stochastic versus fuzzy approaches to multiobjective mathematical programming under uncertainty. Kluwer, Boston

Stancu-Minasian IM (1984) Stochastic programming with multiple objective functions. D. Reidel Publishing Company, Dordrecht

Vajda S (1972) Probabilistic programming. Academic, New York

Zadeh LA (1975) The concept of linguistic variable and its application to approximate reasoning I, II and III. Inf Sci 8 (1975), 199–249, 8 (1975), 301–357 and 9:43–80

Zimmermann H-J (1976) Description and optimization of fuzzy systems. Int J Gen Syst 2:209–215

Zimmermann H-J (1978) Fuzzy programming and linear programming with several objective functions. Fuzzy Sets Syst 1:45–55

Zimmermann H-J (1985) Applications of fuzzy set theory to mathematical programming. Inf Sci 36:29–58