Math. Meth. Oper. Res. (2007) 65: 565–579
DOI 10.1007/s00186-006-0135-8

ORIGINAL ARTICLE

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Risk-sensitive capacity control in revenue management

Received: 8 February 2006 / Accepted: 19 October 2006 / Published online: 13 December 2006

Abstract Both the static and the dynamic single-leg revenue management problem are studied from the perspective of a risk-averse decision maker. Structural results well-known from the risk-neutral case are extended to the risk-averse case on the basis of an exponential utility function. In particular, using the closure properties of log-convex functions, it is shown that an optimal booking policy can be characterized by protection levels, depending on the actual booking class and the remaining time. Moreover, monotonicity of the protection levels with respect to the booking class and the remaining time are proven.

Keywords Markov decision processes · Revenue management · Exponential utility · Risk-sensitivity · Log-convex functions

1 Introduction

We consider a single leg flight of an airplane with a capacity of C seats that is to depart after a certain time T. Customers request tickets of a certain booking class $i = 1, \ldots, k$ with associated fare of $\hat{r}_i$. Without loss of generality we assume throughout that $0 < \hat{r}_k < \hat{r}_{k-1} < \cdots < \hat{r}_1$. Each customer requests a single seat, neither cancellations nor no-shows are considered. It is to determine which exogenously arriving requests to accept or reject assuming that customer demand is independent between booking classes and of the controls being applied.

If demand for each booking class arrives in non-overlapping periods, this model is called the static capacity control model, dynamic capacity control models allow passengers to arrive in any order.

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These two models are the textbook capacity control models in revenue management and do not reflect the state-of-the-art in the revenue management literature (see e.g. Talluri and van Ryzin 2004). However, even in the latest literature on capacity control the most widespread optimality criterion used is expected revenue, i.e. a risk-neutral decision maker is modelled. Under this assumption, the above mentioned models have been studied extensively, structural properties of the optimal policy have been proven, heuristics were promoted and various extensions and alternatives were suggested.

But evidently, not all revenue managers are risk-neutral. Most product managers in charge of revenue management policies present some degree of risk-aversion (see Bitran and Caldentey 2003, p. 226; Weatherford 2004, p. 279). Traditional capacity control models fall short of meeting the needs of risk-averse planner, since they do not suggest mechanisms to reduce the chance of unfavorable revenue levels.

The concept of risk-aversion has already been applied to a variety of stopping and inventory models (see among others Müller 2000; Bouakiz and Sobel 1992). But despite the very rich literature on revenue management to the authors' knowledge only very few papers deal with risk-aversion in the revenue management context: in particular the papers of Agrawal and Seshadri (2000), Feng and Xiao (1999), Chen et al. (2005), Lancaster (2003), Mitra and Wang (2003, 2005), Weatherford (2004) and Barz (2006). Agrawal and Seshadri (2000), Feng and Xiao (1999) and Chen et al. (2005) focus on pricing (and inventory) problems, Mitra and Wang investigate a specific traffic engineering model for bandwidth provisioning. Only Weatherford (2004) and Barz (2006) incorporate risk-aversion into the classic seat inventory control problem. Weatherford (2004) extends the expected marginal seat revenue (EMSR)-b heuristic introduced in Belobaba and Weatherford (1996) to a concept called expected marginal seat utility (EMSU) by substituting a ticket's revenue by the utility of its revenue. He finds that his heuristic can have significant impact on the expected utility and revenue performance and increases the probability of hitting certain revenue thresholds. The paper by Barz (2006) is closely related to ours. Assuming constant absolute risk-aversion in the sense of Pratt (1964), Barz (2006) states the optimality equation for the static seat inventory control with a risk-averse decision maker and simulates the effect of varying coefficients of risk-aversion, but no structural results are proven. Furthermore, an extension of the EMSR-b heuristic to account for risk-aversion is given. Lancaster (2003) does not directly incorporate risk-aversion into revenue management models, but using a sensitivity analysis he emphasizes that "revenue managers do have an opportunity to manipulate policy and strategy to achieve more financially stable results" (Lancaster 2003, p. 163).

In the present paper, we extend both the basic static and the dynamic capacity control model in revenue management to introduce risk-sensitivity in case of an exponential utility function. We show that all well-known structural results of the expected revenue maximizing policy hold for the resulting (risk-sensitive) optimal policy as well.

The paper is organized as follows. In Sect. 2, the decision problem of the basic dynamic revenue management model is introduced, reduced to a Markov decision model and extended to a risk-sensitive Markov decision model in the spirit of Howard and Matheson (1972). The ideas of Lautenbacher and Stidham (1999) proving the existence of protection levels are generalized, properties of the protection levels are shown and an example illustrating the effect of risk-sensitivity on the optimal policy is given. Accordingly, Sect. 3 introduces and generalizes the decision problem of the static revenue management model, proves structural results of an optimal policy and demonstrates the impact of risk-sensitivity in an example.

Notation We use $\mathbb{Z}$ ($\mathbb{N}_0$, $\mathbb{N}$) to denote the set of all (nonnegative, positive) integers. A real-valued function v is said to be increasing (decreasing) if $x \leq x'$ implies $v(x) \leq v(x')$ ($v(x) \geq v(x')$).

2 The dynamic model

In the basic dynamic model, the booking horizon is divided into N time periods in such a way that the probability of two or more requests arriving within one period can be neglected. These periods are indexed by n and the indices run backwards in time so that smaller values of n indicate later points in time. Period N corresponds to the beginning of the booking horizon, and period 0 denotes the scheduled departure time. For each period n, the probability of a class i customer request is given by p_{in}. Furthermore, $p_{0n} = 1 - \sum_{i=1}^{k} p_{in}$ denotes the probability of no customer request in period n.

Dynamic models answer the question whether or not to accept a particular reservation request for booking class i in period n given a remaining capacity of c.

2.1 The risk-neutral approach

The objective of finding a policy maximizing the expected revenue can be reduced to solving the optimality equation of a finite stage Markov decision model MDP(N, $\mathfrak{X}$, $\mathfrak{A}$, (q_n), (r_n), V_0) with planning horizon N, state space $\mathfrak{X} = \{(c, i) \in \mathbb{Z} \times \mathbb{N}_0 \mid c \leq C,\ i \leq k\}$, where we refer to c as the remaining capacity and to i as the requested booking class with $i = 0$ denoting the artificial class 0 having fare $\hat{r}_0 = 0$, action space $\mathfrak{A} = \{0, 1\} \equiv \{\text{reject, accept}\}$, specifying the sets $A(c, i) = \mathfrak{A}$ for $i > 0$ and $A(c, i) = \{0\}$ of admissible actions in state $(c, i) \in \mathfrak{X}$, transition laws q_n from $\mathfrak{D} := \{(c, i, a) \in \mathfrak{X} \times \mathfrak{A} \mid a \in A(c, i)\}$ into $\mathfrak{X}$, for $n = N, N-1, \ldots, 0$ defined by $q_n((c, i), a, (c - a, j)) = p_{jn}$ and 0 otherwise, one-stage reward functions r_n on $\mathfrak{D}$, $r_n((c, i), a) = a \cdot \hat{r}_i$, and terminal reward function V_0 on $\mathfrak{X}$, $V_0(c, i) = 0$ for $c \geq 0$ and $V_0(c, i) = \bar{r} \cdot c$ for $c < 0$ with $\bar{r} > \max_i\{\hat{r}_i\}$.

A (Markov) policy $\pi = (f_N, f_{N-1}, \ldots, f_1)$ is defined as a sequence f_N, $f_{N-1}, \ldots, f_1$ of decision rules f_n specifying the action $a_n = f_n(c_n, i_n)$ to be taken at stage n in state (c_n, i_n). Let F denote the set of all decision rules and F^N the set of all policies.

Denote by $(X_N, X_{N-1}, \ldots, X_0)$ the state process of the MDP, and introduce $V^*(c, i)$ to be the maximal expected revenue starting with capacity c and request i, i.e.

$$V^*(c, i) = \max_{\pi \in F^N} E_\pi \left[\sum_{n=1}^{N} r_n(X_n, f_n(X_n)) + V_0(X_0) \mid X_N = (c, i) \right], \quad (c, i) \in \mathfrak{X}.$$

It is well known in dynamic programming that $V^* \equiv V_N$ is the unique solution to the optimality equation

$$V_n(c,i) = \max_{a \in A(c,i)} \left\{ a\hat{r}_i + \sum_{j=0}^{k} p_{jn} V_{n-1}(c-a, j) \right\}, \quad (2.1)$$

which can be obtained for $n = 1, \ldots, N$ iteratively, starting with V_0. Moreover, each policy π^* formed by actions $a = f_n^*(c,i)$ each maximizing the right hand side of (2.1) is optimal, i.e. leads to V^*.

For this model, Lee and Hersh (1993), Lautenbacher and Stidham (1999), Liang (1999) and Barz and Waldmann (2006) prove structural results of an optimal policy.

Within this context, a decision rule $f_n \in F$ of a control limit type is known to be a time-dependent protection level rule, if there exist constants $y_{i-1}(n) \in \mathbb{N}_0$ such that for all (c,i) it holds that

$$f_n(c,i) = \begin{cases} 1, & c > y_{i-1}(n) \\ 0, & c \le y_{i-1}(n), \end{cases}$$

which implies that, given a request from customer class $i = 2, \ldots, k$, a number of $y_{i-1}(n)$ seats (so-called time-dependent protection level) is reserved for demand in periods $n-1, \ldots, 1$.

It is widely known (see e.g. Lee and Hersh 1993; Lautenbacher and Stidham 1999 or Talluri and van Ryzin 2004) that for every period n, $f_n^*(c,i)$ is monotone in the remaining capacity c, i.e. the more capacity is available, the more one is willing to sell given a request of booking class i. f_n^* can be shown to be a time-dependent protection level rule with protection levels of

$$y_{i-1}^*(n) = \max \left\{ c \in \mathbb{N}_0 : \hat{r}_i < \sum_{i=0}^{k} p_{in}(V_{n-1}(c,i) - V_{n-1}(c-1,i)) \right\}$$

for $i = 1, \ldots, k$. It is the largest value of c for which the expected marginal seat revenue is higher than the class i fare. The choice of $\bar{r}$ ensures that $y_{i-1}^*(n) \ge 0$ for all i and n. Furthermore, for fixed capacity c, $f_n^*(c,i)$ has been shown to be monotone in the remaining arrival periods n and in the booking class i requested.

For a more comprehensive introduction to traditional dynamic models, see Talluri and van Ryzin (2004, Chap. 2.5).

2.2 A risk-sensitive approach

Next we assume a risk-averse decision maker, who seeks to maximize the expected utility of the revenue $R_\pi := \sum_{n=1}^{N} r_n(X_n, f_n(X_n)) + V_0(X_0)$ based on a von Neumann–Morgenstern utility function $u : \mathbb{R} \to \mathbb{R}$.

According to Howard (1988, p. 689), exponential utility functions "satisfactorily treat a wide range of individual and corporate risk preferences". In addition, Kirkwood (2004) shows that in most cases an appropriately chosen exponential utility function is a very good approximation for general utility functions. This is why, as a first step, we will restrict ourselves to exponential utility functions, i.e.

$$u_\gamma(x) = -\exp(-\gamma x). \tag{2.2}$$

Since a risk-averse decision-maker has a concave utility function, in the following we always suppose positive values of the parameter γ. This parameter determines the degree of constant absolute risk-aversion. Thus, in the sense of Pratt (1964), a decision maker with utility function u_{γ_1} is more risk-averse than one with u_{γ_2}, if γ_1 is larger than γ_2.

The objective of finding a policy $\pi^{*\gamma} = (f_N^{*\gamma}, f_{N-1}^{*\gamma}, \dots, f_1^{*\gamma})$, called γ-optimal, maximizing the expected exponential utility of revenue leads to a modification of the MDP studied in Sect. 2.1.

Let $V^{*\gamma}(c, i)$, $(c, i) \in \mathfrak{X}$, denote the maximal expected exponential utility, i.e.

$$\begin{aligned} V^{*\gamma}(c, i) &\\ = \max_{\pi \in F^N} E_\pi &\left[-\exp\left(-\gamma \cdot \left[\sum_{n=1}^{N} r_n(X_n, f_n(X_n)) + V_0(X_0) \right] \right) | X_N = (c, i) \right]. \end{aligned} \tag{2.3}$$

It can be shown that for $\gamma \to 0$ this criterion approximates a maximization of $E_\pi(R_\pi) - \frac{\gamma}{2}\mathrm{Var}_\pi(R_\pi)$; for $\gamma \to \infty$ it reduces to a worst-case optimization (see Coraluppi 1997, pp. 24, 43, respectively).

Then, as already has been shown in Howard and Matheson (1972), $V^{*\gamma} \equiv V_N^\gamma$ is the unique solution of

$$V_n^\gamma(c, i) = \max_{a \in A(c,i)} \left\{ \exp(-\gamma a \hat{r}_i) \cdot \sum_{j=0}^{k} p_{jn} V_{n-1}^\gamma(c - a, j) \right\}, \quad (c, i) \in \mathfrak{X}, \tag{2.4}$$

which can be obtained for $n = 1, \dots, N$ by backward induction starting with $V_0^\gamma(c, i) = -\exp(-\gamma V_0(c, i))$ for $(c, i) \in \mathfrak{X}$. Moreover, each policy $\pi^{*\gamma}$ formed by actions $a^{*\gamma} = f_n^{*\gamma}(c, i)$ each maximizing the right hand side of (2.4) is γ-optimal, i.e. leads to $V^{*\gamma}$.

To simplify the notation, we will often write $L_n v(c)$ in place of $\sum_{j=0}^{k} p_{jn} v(c, j)$ for an arbitrary real-valued function v on $\mathfrak{X}$ in the following.

It easily follows by induction on n that $V_n^\gamma(\cdot, i)$ is increasing in c for all i. Additionally using $\hat{r}_0 = 0$, we finally have $V_n^\gamma(c, 0) = L_n V_{n-1}^\gamma(c) \geq L_n V_{n-1}^\gamma(c-1)$ for all n and c, which allows us to extend $A(c, 0)$ to $\mathfrak{A}$ without loss of generality.

2.3 Structural results of an optimal policy

For proving structural results it is more convenient to work with $G_n^\gamma := -V_n^\gamma$, which is the unique solution of

$$G_n^\gamma(c, i) = \min_{a \in \{0,1\}} \left\{ \exp(-\gamma a \hat{r}_i) \cdot L_n G_{n-1}^\gamma(c - a) \right\} \tag{2.5}$$

$$= L_n G^\gamma_{n-1}(c-1) \cdot \min\left\{\exp(-\gamma \hat{r}_i),\ \frac{L_n G^\gamma_{n-1}(c)}{L_n G^\gamma_{n-1}(c-1)}\right\} \quad (2.6)$$

$$= L_n G^\gamma_{n-1}(c) \cdot \min\left\{1,\ \exp(-\gamma \hat{r}_i) \cdot \frac{L_n G^\gamma_{n-1}(c-1)}{L_n G^\gamma_{n-1}(c)}\right\} \quad (2.7)$$

(with initial value $G^\gamma_0 = -V^\gamma_0$). Note that (2.5) [which immediately follows from (2.4) by multiplication with (-1)] preserves the γ-optimality of a policy.

First observe that it is optimal to accept an arbitrary request, if there is a remaining capacity c and merely $n \le c$ periods remain. Furthermore, an arbitrary request will be rejected in case of $c \le 0$. This is the result of the following proposition.

Proposition 2.1 *For $\gamma > 0$, $n \in \{1, \ldots, N\}$, and $i \in \{1, \ldots, k\}$ we have*

(i) $G^\gamma_n(c,i) = \exp(-\gamma \hat{r}_i) \cdot L_n G^\gamma_{n-1}(c-1) = \exp(-\gamma \hat{r}_i) \cdot \prod_{m=2}^n \sum_{j=0}^k p_{jm} \exp(-\gamma \hat{r}_j),\quad c \ge n.$

(ii) $G^\gamma_n(c,i) = L_n G^\gamma_{n-1}(c) = \exp(-\gamma \bar{r} c),\quad c \le 0.$

Proof (i) and (ii) follow by induction on n, using the inequalities $\exp(-\gamma \hat{r}_i) \le 1 \le \exp(\gamma(\bar{r} - \hat{r}_i))$. □

We call a function $g : \mathbb{Z} \to (0, \infty)$ log-convex, if $\ln g$ is convex or, equivalently, if

$$g(x+1)^2 \le g(x+2) g(x) \quad (2.8)$$

holds for all $x \in \mathbb{Z}$. Log-convex functions have the nice properties of being closed under (1) addition and (2) multiplication. To verify property (1), use can be made of the inequality $a^{1/2}b^{1/2} + c^{1/2}d^{1/2} \le (a+c)^{1/2}(b+d)^{1/2}$, which holds for all positive numbers a, b, c, d (cf. Roberts and Varberg 1973, p. 19), in order to verify $f(x)^{1/2} f(x+2)^{1/2} + g(x)^{1/2} g(x+2)^{1/2} \le [f(x) + g(x)]^{1/2}[f(x+2) + g(x+2)]^{1/2}$. Property (2) is an immediate consequence of (2.8).

Theorem 2.2 *For $\gamma > 0$, $n \in \{1, \ldots, N\}$, and $i \in \{0, \ldots, k\}$ it holds that*

(i) *$L_n G^\gamma_{n-1}(c)$ is log-convex and decreasing in c.*

(ii) *$G^\gamma_n(c,i)$ is log-convex and decreasing in c.*

Proof The assertion follows by induction on n. Fix $\gamma > 0$. For all $1 \le n \le N$ set

$$g_n(c) := L_n G^\gamma_{n-1}(c), \quad c \in \mathbb{Z}.$$

Let $n = 1$. Then, since $-\gamma V_0(\cdot, j)$ is convex, and using closure of log-convex functions with respect to convex combinations, we have log-convexity of g_1,

$$g_1(s) = \sum_j p_{1j} \exp(-\gamma V_0(c, j)).$$

Next we use g_1 to rewrite (2.5) as

$$\ln\ G^\gamma_1(c,i) = \min_{a \in \{0,1\}} \{-a\gamma \hat{r}_i + \ln\ g_1(c-a)\}. \quad (2.9)$$

Finally, by applying Lemma 1 in Stidham (1978) to (2.9), we obtain convexity of $\ln\ G_1^\gamma(\cdot, i), i \in I$. Hence, $G_1^\gamma(\cdot, i)$ is log-convex.

Therefore suppose $G_n^\gamma(\cdot, i)$ to be log-convex for some $1 \le n < N$. Then

$$g_{n+1}(s) = \sum_j p_{n+1,j} G_n^\gamma(c, j)$$

is log-convex as a convex combination of log-convex functions, and, by applying Stidham's lemma to

$$\ln\ G_{n+1}^\gamma(c, i) = \min_{a \in \{0,1\}} \{-a\gamma \hat{r}_i + \ln\ g_{n+1}(c - a)\},$$

we finally get the desired log-convexity of $G_{n+1}^\gamma(\cdot, i), i \in I$.

The monotonicity of $L_n G_{n-1}^\gamma(\cdot, i)$ and $G_n^\gamma(\cdot, i)$ follows by standard arguments in dynamic programming. □

By Theorem 2.2(i), $L_n G_{n-1}^\gamma(c)$ is log-convex in c. Thus $L_n G_{n-1}^\gamma(c)$ is a positive function and the ratio

$$\frac{L_n G_{n-1}^\gamma(c)}{L_n G_{n-1}^\gamma(c-1)}$$

is increasing in c. Together with Proposition 2.1 we may then define constants $y_{i-1}^{*\gamma}(n)$,

$$y_{i-1}^{*\gamma}(n) = \max\left\{ c \in \{0, \ldots, n-1\}\ :\ \exp(-\gamma \hat{r}_i) > \frac{L_n G_{n-1}^\gamma(c)}{L_n G_{n-1}^\gamma(c-1)} \right\} \quad (2.10)$$

such that $\pi^{*\gamma} = (f_N^{*\gamma}, f_{N-1}^{*\gamma}, \ldots, f_1^{*\gamma})$, defined by

$$f_n^\gamma(c, i) = \begin{cases} 1, & c > y_{i-1}^{*\gamma}(n) \\ 0, & c \le y_{i-1}^{*\gamma}(n), \end{cases}$$

is γ-optimal. Note that the (so-called) protection levels $y_{i-1}^{*\gamma}(n)$ allow for a similar interpretation as in the risk-neutral setting. It is the largest value of c for which the utility of a class i request is lower than the expected utility gain of an additional seat. We are now in a position to show the following theorem.

Theorem 2.3 *The protection levels $y_{i-1}^{*\gamma}(n)$ (of an γ-optimal policy) satisfy*

(i) $y_{i-1}^{*\gamma}(n)$ *is increasing in* $i = 1, \ldots, k$ *for all* n *and* γ,
(ii) $y_{i-1}^{*\gamma}(n-1) \le y_{i-1}^{*\gamma}(n) \le y_{i-1}^{*\gamma}(n-1) + 1$ *for* $n = 2, \ldots, N$ *and all* i *and* γ.

Proof (i) follows directly from the definition of $y_{i-1}^{*\gamma}(n)$, Theorem 2.2(i) and $\hat{r}_i \ge \hat{r}_{i+1}$.

To prove (ii) introduce

$$H_n^\gamma(c) := \frac{L_n G_{n-1}^\gamma(c)}{L_n G_{n-1}^\gamma(c-1)}$$

in order to obtain

$$H_n^\gamma(c) = H_{n-1}^\gamma(c-1) \cdot \frac{\sum_{j=0}^k p_{jn-1} \cdot \min\{e^{-\gamma \hat{r}_j}, H_{n-1}^\gamma(c)\}}{\sum_{j=0}^k p_{jn-1} \cdot \min\{e^{-\gamma \hat{r}_j}, H_{n-1}^\gamma(c-1)\}} \geq H_{n-1}^\gamma(c-1),$$

using (2.6) and Theorem 2.2(i), and

$$H_n^\gamma(c) = H_{n-1}^\gamma(c) \cdot \frac{\sum_{j=0}^k p_{jn-1} \cdot \min\{1,\ e^{-\gamma \hat{r}_j} \cdot (1/H_{n-1}^\gamma(c))\}}{\sum_{j=0}^k p_{jn-1} \cdot \min\{1,\ e^{-\gamma \hat{r}_j} \cdot (1/H_{n-1}^\gamma(c-1))\}} \leq H_{n-1}^\gamma(c),$$

using (2.7) and again Theorem 2.2(i). Hence

$$H_{n-1}^\gamma(c-1) \leq H_n^\gamma(c) \leq H_{n-1}^\gamma(c).$$

Now, by definition of the protection levels, for $c = y_{i-1}^{*\gamma}(n)$,

$$\exp(-\gamma \hat{r}_i) > H_n^\gamma(y_{i-1}^{*\gamma}(n)) \geq H_{n-1}^\gamma(y_{i-1}^{*\gamma}(n) - 1),$$

which implies $y_{i-1}^{*\gamma}(n-1) \geq y_{i-1}^{*\gamma}(n) - 1$. Analogously, for $c = y_{i-1}^{*\gamma}(n-1)$,

$$\exp(-\gamma \hat{r}_i) > H_{n-1}^\gamma(y_{i-1}^{*\gamma}(n-1)) \geq H_n^\gamma(y_{i-1}^{*\gamma}(n-1)),$$

which implies $y_{i-1}^{*\gamma}(n) \geq y_{i-1}^{*\gamma}(n-1)$. Thus (ii) holds completing the proof. □

Finally, we can conclude that in the dynamic model all structural properties of the optimal policy that are well-known in the risk-neutral case also hold for exponential, risk-averse utility functions.

2.4 A numerical example

To illustrate our structural results, we take up an example given in Lee and Hersh (1993): they consider four booking classes with fares $\hat{r}_1 = 200, \hat{r}_2 = 150, \hat{r}_3 = 120, \hat{r}_4 = 80$. The capacity of the airplane is $C = 10$, the request probabilities are listed in Table 1.

The left-hand side of Fig. 1 shows the time-dependent values $y_{i-1}^*(n)$ of the optimal protection levels in case of risk-neutrality. The right-hand side shows optimal protection levels in the risk-sensitive model with $\gamma = 0.002$. Recall that $n = 0$ corresponds to the flight departure. In correspondence with Theorem 2.3 the protection levels are increasing in n and i with jumps of a height of 1.

The fact that protection levels given risk-sensitivity are smaller than under risk-neutrality is not surprising. A more risk-averse decision maker values the chance of making revenue from reserving a seat less than a risk-neutral decision-maker. Thus, protection levels are smaller.

Table 1 Request probabilities p_{in}

i	n				
	$1 \leq n \leq 4$	$5 \leq n \leq 11$	$12 \leq n \leq 18$	$19 \leq n \leq 25$	$26 \leq n \leq 30$
1	0.15	0.14	0.10	0.06	0.08
2	0.15	0.14	0.10	0.06	0.08
3	0	0.16	0.10	0.14	0.14
4	0	0.16	0.10	0.14	0.14

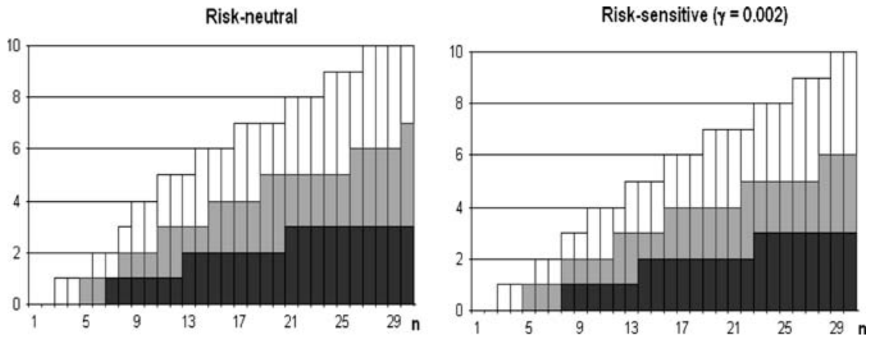

Fig. 1 Protection levels of an optimal policy in case of risk-neutrality and risk aversion. $y_0^{*\gamma}$, $y_1^{*\gamma}$ and $y_2^{*\gamma}$ are indicated in *black*, *grey*, and *white*, respectively

3 The static model

In the basic static model, the demand of each booking class $i \in \{1, \ldots, k\}$ is supposed to arrive during a single contiguous time segment. In this case, the booking period can be divided into periods with booking requests belonging to the same fare class. At the time the total demand d of a booking class i is known, one has to determine the amount $a \in \{0, \ldots, d\}$ of demand to be accepted in order to maximize the expected (utility of) revenue of that flight.

The total demands $D_1, \ldots, D_k$ of the booking classes $i = 1, \ldots, k$ are assumed to be independent random variables on $\mathbb{N}_0$ with counting densities $P(D_i = d) = p_i(d)$, $d \in \mathbb{N}_0$, say. Additionally, it is often assumed that customer requests for tickets arrive in increasing fare order, i.e. the class willing to pay the fare $\hat{r}_k$ before $\hat{r}_{k-1}$, etc. We stick to this assumption in the following. Since there is a one-to-one correspondence between periods and classes, we index both by i.

3.1 The risk-neutral approach

The static yield management model with two fare classes was introduced by Littlewood (1972) and extended heuristically to more than two fare classes by Belobaba (1987a) and Belobaba and Weatherford (1996). An exact solution was found by Curry (1990), Wollmer (1992) and Brumelle and McGill (1993). Li and Oum (2002) discuss the equivalence of the three solutions. Robinson (1995) relaxed the assumption of arrivals in increasing fare order. Lautenbacher and Stidham (1999) and Barz (2006) stressed the underlying Markov decision process.

The objective of finding a policy maximizing the expected revenue in the static model can be reduced to solving the optimality equation of a finite stage Markov decision model MDP($k, \mathfrak{X}, \mathfrak{A}, (q_i), (r_i), V_0$) with planning horizon k, state space $\mathfrak{X} = \{(c, d) \in \mathbb{Z} \times \mathbb{N}_0 \mid c \leq C\}$, where we refer to c as the remaining capacity and to d as the demand observed for the actual booking class, action space $\mathfrak{A} = \mathbb{N}_0$, where action a denotes the number of requests to be accepted, with sets $A(c, d) = \{0, \ldots, d\}$ of admissible actions in states $(c, d) \in \mathfrak{X}$, transition laws q_i from $\mathfrak{D} := \{(c, d, a) \in \mathfrak{X} \times \mathfrak{A} \mid a \in A(c, d)\}$ into $\mathfrak{X}$, defined by $q_i((c, d), a, (c - a, d')) = p_{i-1}(d')$ and 0 otherwise (with $p_0(\cdot)$ arbitrary), one-stage reward functions r_i on $\mathfrak{D}$, $r_i((c, d), a) = a \cdot \hat{r}_i$ (with $\hat{r}_0 = 0$), and terminal reward function V_0 on $\mathfrak{X}$, $V_0((c, d)) = 0$ for $c \geq 0$ and $V_0((c, d)) = \bar{r} \cdot c$ for $c < 0$ with $\bar{r} > \max_i\{\hat{r}_i\}$.

Thus, for booking classes $i = k, k-1, \ldots, 1$, given the residual capacity c_i and demand d_i, we have to determine the number $a_i = f_i(c_i, d_i) \in \{0, \ldots, d_i\}$ of seats to be accepted.

A (Markov) policy $\pi = (f_k, f_{k-1}, \ldots, f_1)$ is then defined as a sequence $f_k, f_{k-1}, \ldots, f_1$ of decision rules f_i specifying the action $a_i = f_i(c_i, d_i)$ to be taken at stage i in state (c_i, d_i). Let F denote the set of all decision rules and F^k the set of all policies.

Within this context, a decision rule $f_i \in F$ is called a protection level rule, if there exists a constant y_{i-1} such that

$$f_i(c, d) = \begin{cases} \min\{d, c - y_{i-1}\}, & c > y_{i-1} \\ 0, & c \leq y_{i-1}, \end{cases}$$

which implies that, given request d from booking class i, a number of y_{i-1} seats, the so called protection level, is reserved for future (higher-value) demand.

Denote by $(X_k, X_{k-1}, \ldots, X_0)$ the state process of the MDP and introduce

$$V^*(c, d) = \max_{\pi \in F^k} E_\pi \left[\sum_{i=1}^{k} r_i(X_i, f_i(X_i)) + V_0(X_0) \mid X_k = (c, d) \right], \quad (c, d) \in \mathfrak{X},$$

to be the maximal expected revenue. Then, in analogy to Sect. 2, $V^* \equiv V_k$ is the unique solution to the optimality equation

$$V_i(c, d) = \max_{a \in \{0, \ldots, d\}} \left\{ a\hat{r}_i + \sum_{d'=0}^{\infty} p_{i-1}(d') V_{i-1}(c - a, d') \right\}, \quad (c, d) \in \mathfrak{X}, \quad (3.1)$$

which can be obtained for $i = 1, \ldots, k$ by backward induction starting with V_0. Moreover, each policy π^* formed by actions $a^* = f_i^*(c, d)$ each maximizing the right hand side of (3.1) is optimal.

It is widely known (see e.g. Wollmer 1992; Lautenbacher and Stidham 1999; Talluri and van Ryzin 2004) that each $f_i^*(\cdot, d)$ (of an optimal π^*) is monotone in the remaining capacity c, i.e. the more capacity is available, the more one is willing to sell. Furthermore, it can be shown that f_i^* is a protection level rule with protection levels y_{i-1}^* of

$$y_{i-1}^* = \max\left\{x \in \mathbb{N}_0 : \hat{r}_i < \sum_{d'=0}^{\infty} p_{i-1}(d')(V_{i-1}(c,d') - V_{i-1}(c-1,d'))\right\},$$

where $i = 2, \ldots, k$ and $y_0^* = 0$. The interpretation is the same as in the dynamic model. Finally, for fixed capacity c and observed demand d, the optimal policy π^* has been shown to be monotone in the remaining arrival periods i.

For a more comprehensive introduction to traditional static models, see Talluri and van Ryzin (2004, Chap. 2.2) or Phillips (2005, Chap. 7).

3.2 A risk-sensitive approach

Next we assume a risk-averse decision maker, who seeks to maximize the expected exponential utility of the revenue $\sum_{i=1}^{k} r_i(X_i, f_i(X_i)) + V_0(X_0)$, which results in determining a policy $\pi^{*\gamma} = (f_k^{*\gamma}, f_{k-1}^{*\gamma}, \ldots, f_1^{*\gamma})$, called γ-optimal, which realizes

$$V^{*\gamma}(c,d) = \max_{\pi \in F^k} E_\pi\left[-\exp\left(-\gamma \cdot \left[\sum_{i=1}^{k} r_i(X_i, f_i(X_i)) + V_0(X_0)\right]\right) | X_k = (c,d)\right].$$

The corresponding optimality equation reads as follows

$$V_i^\gamma(c,d) = \max_{a\in\{0,\ldots,d\}}\left\{\exp(-\gamma a\hat{r}_i) \cdot \sum_{d'=0}^{\infty} p_{i-1}(d') V_{i-1}^\gamma(c-a,d')\right\}, \quad (3.2)$$

where $V_0^\gamma(c,d) = -\exp(-\gamma V_0(c,d))$.

Similar to the dynamic model, $V^{*\gamma} \equiv V_k^\gamma$ and each policy $\pi^{*\gamma}$ formed by actions $f_i^{*\gamma}(c,d)$ each maximizing the right hand side of (3.2) is γ-optimal.

3.3 Structural results of an optimal policy

To simplify the notation, we often write $L_i v(c)$ in place of $\sum_{d'=0}^{\infty} p_i(d')v(c,d')$ for an arbitrary real-valued function v on $\mathfrak{X}$ in the following.

As in Sect. 2, it is more convenient to work with $G_i^\gamma := -V_i^\gamma$, which is the unique solution of

$$G_i^\gamma(c,d) = \min_{a\in\{0,\ldots,d\}}\left\{\exp(-\gamma a\hat{r}_i) \cdot L_{i-1}G_{i-1}^\gamma(c-a)\right\} \quad (3.3)$$

(with initial value $G_0^\gamma = -V_0^\gamma$), and preserves the γ-optimality of a policy.

Lemma 3.1 *For $\gamma > 0$, $d \in \mathbb{N}_0$, and $i \in \{1, \ldots, k\}$ it holds that*

(i) $L_{i-1}G^{\gamma}_{i-1}(c)$ *is log-convex and decreasing in c.*

(ii) $G^{\gamma}_{i}(c, d)$ *is log-convex and decreasing in c.*

Proof The assertions follow by essentially the same arguments as given in the proof of Theorem 2.2. □

We are now in a position to prove the main result of this section.

Theorem 3.2 *For $\gamma > 0$ there exists a γ-optimal policy $\pi^{*\gamma} = (f^{*\gamma}_k, f^{*\gamma}_{k-1}, \ldots, f^{*\gamma}_1)$ such that*

$$f^{*\gamma}_i(c, d) = \begin{cases} \min\{d, c - y^{*\gamma}_{i-1}\}, & c > y^{*\gamma}_{i-1} \\ 0, & c \le y^{*\gamma}_{i-1}, \end{cases}$$

where the constants

$$y^{*\gamma}_{i-1} = \sup\left\{ c \in \mathbb{N}_0 : \exp(-\gamma \hat{r}_i) > \frac{L_{i-1}G^{\gamma}_{i-1}(c)}{L_{i-1}G^{\gamma}_{i-1}(c-1)} \right\}$$

additionally fulfill

$$0 = y^{*\gamma}_0 \le y^{*\gamma}_1 \le \cdots \le y^{*\gamma}_{k-1}.$$

Proof Fix $\gamma > 0$. Introduce $\tilde{G}^{\gamma}_i(c, d) := \exp(\gamma c \hat{r}_i) \cdot G^{\gamma}_{i-1}(c, d)$ in order to rewrite (3.3) as

$$\tilde{G}^{\gamma}_i(c, d) = \min_{c-d \le \tilde{a} \le c} \left\{ \exp(\gamma \tilde{a}(\hat{r}_i - \hat{r}_{i-1})) \cdot L_{i-1}\tilde{G}^{\gamma}_{i-1}(\tilde{a}) \right\}.$$

Observe that $\tilde{a} = c - a$ has no longer the interpretation of the number of customers to be accepted but the remaining capacity after accepting a customers.

Since $G_i(\cdot, d)$ is log-convex by Lemma 3.1(ii), we also have that $\tilde{a} \to L_i\tilde{G}^{\gamma}_i(\tilde{a})$ is log-convex. Thus, $J^{\gamma}_i(\tilde{a})$ is log-convex, where

$$J^{\gamma}_i(\tilde{a}) := \exp[\gamma \tilde{a}(\hat{r}_i - \hat{r}_{i-1})] \cdot L_{i-1}\tilde{G}^{\gamma}_{i-1}(\tilde{a}), \quad \tilde{a} \in \mathbb{Z}.$$

Hence, $J^{\gamma}_i(\tilde{a})$ is monotone or there exists some $\tilde{a}^{*\gamma}_{i-1} \in \mathbb{Z}$ for which $J^{\gamma}_i(\tilde{a})$ becomes minimal. $\tilde{a}^{*\gamma}_{i-1}$ may be characterized as the maximum of all $\tilde{a}$, for which $J^{\gamma}_i(\tilde{a}-1) > J^{\gamma}_i(\tilde{a})$ or, equivalently,

$$\exp(-\gamma(\hat{r}_i - \hat{r}_{i-1})) > \frac{L_{i-1}\tilde{G}^{\gamma}_{i-1}(\tilde{a})}{L_{i-1}\tilde{G}^{\gamma}_{i-1}(\tilde{a}-1)} \tag{3.4}$$

holds. If $J^{\gamma}_i(\tilde{a})$ is increasing (resp. decreasing), we formally set $\tilde{a}^{*\gamma}_{i-1} = -\infty$ (resp. $\tilde{a}^{*\gamma}_{i-1} = +\infty$). In particular, the policy $\tilde{\pi}^{\gamma} = (\tilde{f}^{\gamma}_k, \tilde{f}^{\gamma}_{k-1}, \ldots, \tilde{f}^{\gamma}_1)$, defined by

$$\tilde{f}^{\gamma}_i(c, d) = \begin{cases} c, & c \le \tilde{a}^{*\gamma}_{i-1} \\ \tilde{a}^{*\gamma}_{i-1}, & c - d \le \tilde{a}^{*\gamma}_{i-1} < c \\ c - d, & \tilde{a}^{*\gamma}_{i-1} < c - d, \end{cases} \tag{3.5}$$

is γ-optimal.

Next we show that $0 = \tilde{a}_0^{*\gamma} \leq \tilde{a}_1^{*\gamma} \leq \cdots \leq \tilde{a}_{k-1}^{*\gamma}$. First, for $\tilde{a} \leq 0$, we have $J_1^\gamma(\tilde{a}) = e^{\gamma \tilde{a} \hat{r}_1} e^{-\gamma \tilde{a} \bar{r}} \geq 1 = J_1^\gamma(0)$. Hence $\tilde{a}_1^{*\gamma} \geq 0$. Furthermore, since $J_1^\gamma(\tilde{a}) = e^{\gamma \tilde{a} \hat{r}_1} \geq 1 = J_1^\gamma(0)$ for $\tilde{a} \geq 0$, we have $\tilde{a}_1^{*\gamma} \leq 0$ and, finally, $\tilde{a}_1^{*\gamma} = 0$.

Now we show that $J_{i+1}^\gamma(\tilde{a}_{i-1}^{*\gamma} - 1) > J_{i+1}^\gamma(\tilde{a}_{i-1}^{*\gamma})$ holds, which implies $\tilde{a}_i^{*\gamma} \geq \tilde{a}_{i-1}^{*\gamma}$. Indeed, using $\exp[-\gamma(\hat{r}_{i+1} - \hat{r}_i)] > 1$, we get

$$\begin{aligned} \frac{J_{i+1}^\gamma(\tilde{a}_{i-1}^{*\gamma} - 1)}{J_{i+1}^\gamma(\tilde{a}_{i-1}^{*\gamma})} &= \exp[-\gamma(\hat{r}_{i+1} - \hat{r}_i)] \frac{L_i \tilde{G}_i^\gamma(\tilde{a}_{i-1}^{*\gamma} - 1)}{L_i \tilde{G}_i^\gamma(\tilde{a}_{i-1}^{*\gamma})} \\ &> \frac{\sum_{d'=0}^\infty p_i(d') \min\{J_i^\gamma(\tilde{a}) \mid \tilde{a}_{i-1}^{*\gamma} - 1 - d' \leq \tilde{a} \leq \tilde{a}_{i-1}^{*\gamma} - 1\}}{\sum_{d'=0}^\infty p_i(d') \min\{J_i^\gamma(\tilde{a}) \mid \tilde{a}_{i-1}^{*\gamma} - d' \leq \tilde{a} \leq \tilde{a}_{i-1}^{*\gamma}\}} \\ &= \frac{J_i^\gamma(\tilde{a}_{i-1}^{*\gamma} - 1)}{J_i^\gamma(\tilde{a}_{i-1}^{*\gamma})} > 1. \end{aligned}$$

Finally, rewriting (3.5) in the original terms with $a = c - \tilde{a}$, and using that $\tilde{a}_i^{*\gamma} \geq 0$ and that (3.4) is equivalent to

$$\exp(-\gamma \hat{r}_i) > \frac{L_{i-1} G_{i-1}^\gamma(c)}{L_{i-1} G_{i-1}^\gamma(c-1)},$$

the proof is complete. □

Thus, we can conclude that in the static model all structural properties of the optimal policy that are well-known in the risk-neutral case also hold for exponential, risk-averse utility functions.

3.4 A numerical example

For an illustration consider the following data taken from Belobaba (1987b): there are four fare classes with fare prices of $\hat{r}_1 = 105 \geq \hat{r}_2 = 83 \geq \hat{r}_3 = 57 \geq \hat{r}_4 = 39$. The total capacity is $C = 107$. The demand is normally distributed (rounded to integer values). Table 2 shows the associated expectations and standard deviations.

The protection levels of the optimal policy in the risk-neutral setting [obtained by solving (3.1)] read: $y_3^* = 77$, $y_2^* = 49$, and $y_1^* = 13$. This means that e.g. 77 seats are protected for classes 1, 2 and 3 and at most $107 - 77 = 30$ seats would be sold to class 4 customers.

Table 2 Parameters of the normally distributed demands

Fare class i	$E[D_i]$	$\sigma[D_i]$
1	20.3	8.6
2	33.4	15.1
3	19.3	9.2
4	29.7	13.1

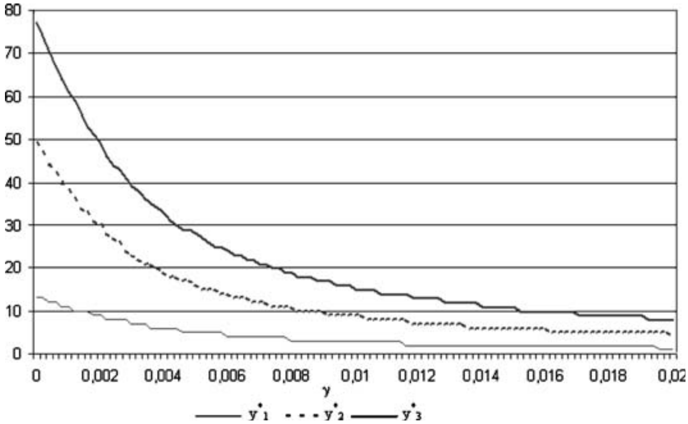

Fig. 2 Protection levels of different γ-optimal policies

If the decision maker is risk-averse, he is more likely to prefer a lower, certain revenue (now) compared to future uncertain revenue. Accordingly, in the risk-averse formulation, by solving (3.2), optimal protection levels can be calculated to be $y_3^{*0.001} = 61$, $y_2^{*0.001} = 38$, and $y_1^{*0.001} = 11$, if $\gamma = 0.001$, and $y_3^{*0.002} = 49$, $y_2^{*0.002} = 30$, and $y_1^{*0.002} = 9$, if $\gamma = 0.002$.

Figure 2 shows the values of the optimal protection levels given different values of γ.

References

Agrawal V, Seshadri S (2000) Impact of uncertainty and risk aversion on price and order quantity in the newsvendor problem. Manuf Serv Oper Manage 2(4):410–423

Barz C (2006) How does risk aversion affect optimal revenue management policies? A simulation study. In: Mattfeld D, Suhl L (eds) DSOR contributions to information systems, vol 4, pp 161–172

Barz C, Waldmann K-H (2006) An application of Markov decision processes to the seat inventory control problem. In: Morlock M, Schwindt C, Trautmann N, Zimmermann J (eds) Perspectives on operations research—essays in honor of Klaus Neumann. Deutscher Universitäts-Verlag, Wiesbaden, pp 113–128

Belobaba PP (1987a) Airline yield management. Transp Sci 21(2):63–73

Belobaba PP (1987b) Air travel demand and airline seat inventory management. PhD Thesis, Massachusetts Institute of Technology

Belobaba PP, Weatherford LR (1996) Comparing decision rules that incorporate customer diversion in perishable asset revenue management situations. Decis Anal 27:343–363

Bitran GR, Caldentey R (2003) An overview of pricing models for revenue management. Manuf Serv Oper Manage 5(3):203–229

Bouakiz M, Sobel MJ (1992) Inventory control with an exponential utility criterion. Oper Res 40(3):603–608

Brumelle SL, McGill JI (1993) Airline seat allocation with multiple nested fare classes. Oper Res 41(1):127–137

Chen X, Sim M, Simchi-Levi D, Sun P (2005) Risk aversion in inventory management. Working paper, Massachusetts Institute of Technology, Cambridge, MA

Coraluppi SP (1997) Optimal control of Markov decision processes for performance and robustness. PhD Thesis, University of Maryland

Curry RE (1990) Optimal airline seat allocation with fare classes nested by origins and destinations. Transp Sci 24(3):193–204
Feng Y, Xiao B (1999) Maximizing revenues of perishable assets with a risk factor. Oper Res 47(2):337–341
Howard R (1988) Decision analysis: practice and promise. Manage Sci 34(6):679–695
Howard R, Matheson JE (1972) Risk-sensitive Markov decision processes. Manage Sci 18(7):356–369
Kirkwood CW (2004) Approximating risk aversion in decision analysis applications. Decis Anal 1(1):55–72
Lancaster J (2003) The financial risk of airline revenue management. J Revenue Pricing Manage 2:158–165
Lautenbacher CJ, Stidham S Jr (1999) The underlying Markov decision process in the single-leg airline yield management problem. Transp Sci 33(2):136–146
Lee TC, Hersh M (1993) A model for dynamic airline seat inventory control with multiple seat bookings. Transp Sci 27(3):252–265
Li MZF, Oum TH (2002) A note on the single leg, multifare seat allocation problem. Transp Sci 36(3):349–353
Liang Y (1999) Solution to the continuous time dynamic yield management model. Transp Sci 33(1):117–123
Littlewood K (1972) Forecasting and control of passangers. In: 12th AGIFORS symposium proceedings, pp 95–117
Mitra D, Wang Q (2003) Stochastic traffic engineering, ith applications to network revenue management. In: IEEE INFOCOM 2003—the conference on computer communications, vol 22(1), pp 396–405
Mitra D, Wang Q (2005) Stochastic traffic engineering for demand uncertainty and risk-aware network revenue management. IEEE/ACM Trans Netw 13(2):221–233
Müller A (2000) Expected utility maximization of optimal stopping problems. Eur J Oper Res 122(1):101–114
Phillips RL (2005) Pricing and revenue optimization. Stanford Business Books, Stanford
Pratt JW (1964) Risk aversion in the small and in the large. Econometrica 32(1/2):122–136
Roberts AW, Varberg DE (1973) Convex functions. Academic, London
Robinson LW (1995) Optimal and approximate control policies for airline booking with sequential nonmonotonic fare classes. Oper Res 43(2):252–263
Stidham S Jr (1978) Socially and individually optimal control of arrivals to a GI/M/1 queue. Manage Sci 24(15):1598–1610
Talluri KT, van Ryzin GJ (2004b) The theory and practice of revenue management. Kluwer Academic Publishers, Dordrecht
Weatherford LR (2004) EMSR versus EMSU: revenue or utility? J Revenue Pricing Manage 3(3):277–284
Wollmer RD (1992) An airline management model for a single leg route when lower fare classes book first. Oper Res 40(1):26–37