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ORIGINAL ARTICLE

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A core-allocation family for generalized holding cost games

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Abstract Inventory situations, introduced in Meca et al. (Eur J Oper Res 156: 127–139, 2004), study how a collective of firms can minimize its joint inventory cost by means of co-operation. Depending on the information revealed by the individual firms, they analyze two related cooperative TU games: inventory cost games and holding cost games, and focus on proportional division mechanisms to share the joint cost. In this paper we introduce a new class of inventory games: generalized holding cost games, which extends the class of holding cost games. It turns out that generalized holding cost games are totally balanced.We then focus on the study of a core-allocation family which is called *N-rational solution family.* It is proved that a particular relation of inclusion exists between the former and the core. In addition, an N-rational solution called *minimum square proportional rule* is studied.

Keywords Generalized holding cost games · Core-allocations · Minimum square proportional rule · Inventory situations · Cooperative games

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1 Introduction

During the past decades the interrelation between operations research and game theory has been disclosed many times. We could say that the starting point is

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the famous visit of George Dantzig to von Neumann in 1947 (see Dantzig 1991) where the connection between duality theory and the minimax theorem was set. Moreover, we could point out the relation between linear complementary problems and bimatrix games (Lemke and Howson 1964), Markov processes and stochastic games (Shapley 1953b), and optimal control theory and differential games (Fleming 1961). However, the relationship with cooperative game theory is much more recent, and for TU cooperative games, it could be condensed to studying operations research games; i.e., studying aspects of joint cost allocation in operations research models. These models are designed to optimize the operation of a complex system in which, commonly, several agents are involved. The effects of cooperation and/or competition of the agents who interact in an operations research problem clearly play a prominent role here. In the last years some surveys on this topic have been written, i.e., Curiel (1997) and Borm et al. (2001).

Very recently inventory models have also been approached from this point of view. In particular, cooperation in a news-vendor problem has been treated in Hartman et al. (2000), Slikker et al. (2005), and Müller et al. (2002). Tijs et al. (2005) study a situation where one agent has an amount of storage space available and the other agents have some goods, part of which can be stored generating benefits. A general framework for the study of continuous time decentralized distribution systems is analyzed in Anupindi et al. (1991). The problem of sharing the benefits produced by full cooperation between agents is tackled by introducing a related cooperative game. Minner (2005) analyzes horizontal cooperations between organizations that have the opportunity to jointly replenish material requirements.

In Meca et al. (2004) inventory cost games and holding cost games are introduced and studied. In an inventory cost game, a group of firms dealing with the ordering and holding of a certain commodity (every individual agent's problem being an EOQ problem), decide to cooperate and jointly make their orders. To coordinate the ordering policy of the firms, some revelation of information is needed: the amount of revealed information between the firms is kept as low as possible since they may be competitors on the consumer market. However, in a holding cost game coordination with regard to holding cost is also considered. In this case a full disclosure of information is needed. These kinds of cooperation are not unusual in the economic world: for instance, pharmacies usually form groups that order and share storage space. For both classes of games, Meca et al. (2004) focus on proportional division mechanisms to share the joint cost. They introduce and characterize the SOC-rule (Share the Ordering Costs) as a core-allocation for inventory cost games, and Meca et al. (2003) revisit inventory cost games and the SOC-rule. There it is shown that the wider class of n-person EPQ inventory situations with shortages leads to exactly the same class of cost games. Moreover, an alternative characterization of the SOC-rule is provided there. In addition, Meca et al. (2004) show that holding cost games are permutationally concave. Moreover, the demand proportional rule leads to a core-allocation of the corresponding game that can even be sustained as a population monotonic allocation scheme (Sprumont 1990).

In this paper we complete the study of holding cost games. We present a new class of inventory games inspired by the aforesaid ones. Following the ideas in Meca et al. (2003) we first consider the n-person EPQ inventory model with shortages. However, we take one step further and focus on a more general class of inventory games than the one corresponding to the aforementioned inventory model. It is called *generalized holding cost games.*

We start by introducing definitions and notations in Sect. 2. In Sect. 3 we first give a complete description of the process which leads to define the class of generalized holding cost games. Next it is shown that generalized holding cost games and all their subgames are permutationally concave; hence generalized holding cost games are totally balanced. We then focus on the study of a core-allocation family which is called *N-rational solution family* (Sect. 4). It is proved that a particular relation of inclusion exists between the above family and the core. Finally a new proportional rule called *minimum square proportional rule* is studied, which is an N-rational solution (Sect. 5). Some concluding remarks and directions for future research complete the paper.

2 Preliminaries

Inventory and holding cost games constitute two classes of cooperative games with transferable utility (TU games). A TU cost game is a pair (N, c) where $N = \{1, 2, \ldots, n\}$ is the finite player set and $c : 2^N \to \mathbb{R}(2^N$ is the set of all subsets of $N)$ the characteristic function satisfying $c(\varnothing) = 0$. Define the zero cost game (N, c^0) by $c^0(S) = 0$ for all $S \subseteq N$. The subgame related to coalition S, c^S, is the restriction of mapping c to the subcoalitions of S. We denote by lower case letter s the cardinality of set S, i.e., $\text{card}(S) = s$ for all $S \subseteq N$. A cost-sharing vector will be $x \in \mathbb{R}^n$, and for every coalition $S \subseteq N$ we shall write $x(S) := \sum_{i \in S} x_i$, the cost-sharing to coalition S (where $x(\varnothing) = 0$). The core of the game (N, c) consists of those cost-sharing vectors which allocate the cost of the grand coalition in such a way that every other coalition pays at most its cost by the characteristic function: $C(c) = \{x \in \mathbb{R}^n / x(N) = c(N) \text{ and } x(S) \leq c(S) \text{ for all } S \subset N\}$. Cost-sharing vectors belonging to the core will be called from now on core-allocations. A cost game (N, c) has a non-empty core if and only if it is balanced (see Bondareva 1963 or Shapley 1967). It is a totally balanced game if the core of every subgame is non-empty.

Let K be a bounded convex polyhedron in $\mathbb{R}^n$. We say that $x \in K$ is an extreme point if $y, z \in K$ and $x = \frac{1}{2}y + \frac{1}{2}z$ imply $y = z$. We denote by $\text{Ext}\, K$ the set of extreme points for K from now on. It is well-known that $x \in K$ if and only if x has at least n binding constraints whose coefficients are linearly independent. From the standard classical convex analysis we know the core is a bounded convex polyhedron. As a consequence it has a finite number of extreme points and the core is the convex hull of its set of extreme points. The search of characterizations of the extreme core-allocations is therefore important.

A population monotonic allocation scheme (Sprumont 1990), or pmas, for the game (N, c) is a collection of vectors $y^S \in \mathbb{R}^S$ for all $S \subseteq N$, $S \neq \varnothing$ such that $y^S(S) = c(S)$ for all $S \subseteq N$, $S \neq \varnothing$, and $y_i^S \geq y_i^T$ for all $S \subseteq T \subseteq N$ and $i \in S$. Note that if $(y^S)_{\varnothing \neq S \subseteq N}$ is a pmas for (N, c), then $y^S \in C(c^S)$ for all $S \subseteq N$, $S \neq \varnothing$. Every cost game with pmas is totally balanced. A core-allocation for (N, c), i.e., $x \in C(c)$ is reached through a pmas if there exits a pmas $(y^S)_{\varnothing \neq S \subseteq N}$ for the game (N, c) such that $y_i^N = x_i$ for all $i \in N$. Hence the set of core-elements that can be reached through a pmas is a refinement of the core.

A game is said to be subadditive when for all disjoint coalitions S and T, $c(S \cup T) \leq c(S) + c(T)$ holds. In a subadditive game, it will always be beneficial for two disjoint coalitions to cooperate and form a larger coalition. Balanced cost games might not be subadditive but they always satisfy subadditive inequalities involving the grand coalition. However, totally balanced cost games are subadditive. A well-known class of totally balanced and subadditive games is the class of concave games (Shapley 1971). A cost game (N, c) is concave if and only if $c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T)$ for all player $i \in N$ and all pair of coalitions $S, T \subseteq N$ such that $S \subseteq T \subseteq N \backslash \{i\}$.

Another class of balanced and subadditive games is the class of permutationally concave games (Granot and Huberman 1982). Before defining it we first introduce orders. An order σ of N is a bijection $\sigma : N \cup \{0\} \to N \cup \{0\}$ such that $\sigma(0) = 0$. This order is denoted by $\sigma^{-1}(1) \cdots \sigma^{-1}(n)$, where $\sigma(i) = j$ means that with respect to σ, player i is in the jth position. As usual, $\sigma(S) = \{\sigma(i)/i \in S\}$ for all $S \subseteq N$. We denote by $\Pi(N)$ the set of all orders in N. Let (N, c) be a cost game. For any $\sigma \in \Pi(N)$, the marginal vector $m^\sigma(c)$ is defined by $m_i^\sigma(c) := c(\overline{P}_i^\sigma) - c(P_i^\sigma)$ for all $i \in N$ where $P_i^\sigma = \{j \in N/\sigma(j) < \sigma(i)\}$ is the set of predecessors of i with respect to σ excluding i, and $\overline{P}_i^\sigma = \{j \in N/\sigma(j) \leq \sigma(i)\} = P_i^\sigma \cup \{i\}$ is the set of predecessors of i with respect to σ including i. Define for all $\sigma \in \Pi(N)$, $P_0^\sigma = \varnothing$.

A cost game (N, c) is permutationally concave with respect to $\sigma \in \Pi(N)$ if and only if $c(\overline{P}_i^\sigma \cup R) - c(\overline{P}_i^\sigma) \geq c(\overline{P}_j^\sigma \cup R) - c(\overline{P}_j^\sigma)$ for all $i, j \in N \cup \{0\}$ such that $\sigma(i) \leq \sigma(j)$ and all $R \subseteq N \backslash \overline{P}_j^\sigma$. A game is permutationally concave if and only if there exists an order $\sigma \in \Pi(N)$ such that the game is permutationally concave with respect to σ. Granot and Huberman (1982) showed that if the game (N, c) is permutationally concave with respect to $\sigma \in \Pi(N)$ then $m^\sigma(c) \in C(c)$. It is well-known that for all (N, c), if $m^\sigma(c) \in C(c)$ then $m^\sigma(c) \in \operatorname{Ext} C(c)$.

The Weber set for the cost game (N, c) is the convex hull of all marginal vectors; i.e., $W(c) := \operatorname{conv}\{m^\sigma(c)/\sigma \in \Pi(N)\}$. Weber (1978) proved that $C(c) \subseteq W(c)$ for any cost game (N, c). If a cost game is concave, then it follows from Shapley (1971) that all its marginal vectors belong to the core. Hence the core of a concave cost game coincides with the Weber set.

The Shapley value (Shapley 1953a) is a linear operator on the class of all TU games and for a cost game (N, c) is defined as $\Phi(c) = (1/n!) \cdot \sum_{\sigma \in \Pi(N)} m^\sigma(c)$.

The τ-value (Tijs 1981) is an operator on the class of quasibalanced games. A cost game (N, c) is quasibalanced if and only if $m_i(c) \geq M_i(c)$ for all $i \in N$, and $\sum_{j \in N} m_j(c) \geq c(N) \geq \sum_{j \in N} M_j(c)$, where for all $i \in N$,

$$M_i(c) = c(N) - c(N \backslash i), \quad m_i(c) = \min_{S \subseteq N, i \in S} \left\{ c(S) - \sum_{j \in S \backslash i} M_j(c) \right\}.$$

Every balanced game is a quasibalanced one. For a quasibalanced cost game (N, c), the τ-value is defined as $\tau(c) = m(c) + \alpha[M(c) - m(c)]$, where $M(c), m(c) \in \mathbb{R}^n$ are upper and lower vectors, respectively, and $\alpha \in \mathbb{R}$ is such that $\sum_{i \in N} \tau_i(c) = c(N)$.

The proportional rule with respect to $\lambda \in \mathbb{R}^n$ such that $\lambda(N) \neq 0$, or λ -proportional rule, is a linear operator on the class of all TU games, and for a cost game (N, c) is defined as $p(c) = \lambda \cdot c(N)/\lambda(N)$.

We will now introduce four properties for solution rules (operators) on the class of all cost games. Let Ψ be a solution rule on the class of all cost games. Then $\Psi_i(c) \in \mathbb{R}$ denotes the cost allocated to player $i \in N$ according to this rule in the game c and $\Psi(c) = (\Psi_i(c))_{i\in N} \in \mathbb{R}^n$. Let (N, c) and $(N, \overline{c})$ be cost games. The rule Ψ satisfies *efficiency* if $\sum_{i\in N} \Psi_i(c) = c(N)$. It satisfies *symmetry* if $\Psi_i(c) = \Psi_j(c)$ when the players i and j are symmetric, that is, $c(S \cup \{i\}) = c(S \cup \{j\})$ for all $S \subset N\backslash\{i, j\}$. Ψ satisfies *zero-symmetry* if it is symmetric only for the zero cost game. Finally the rule Ψ satisfies *monotonicity* if for all $i \in N$ such that $c(\{i\}) \leq \overline{c}(\{i\})$ it holds that $c(N)\cdot\Psi_i(c) \leq \overline{c}(N)\cdot\Psi_i(\overline{c})$.

Inventory and holding cost games were introduced in Meca et al. (2004) as models for inventory situations. The player set N consists of a group of firms dealing with the ordering and holding of a certain commodity (every individual agent's problem being an EOQ problem). In an inventory cost game, a group of players minimize their total cost by placing their orders together as one big order (paying a fix ordering cost a). To coordinate the ordering policy of the firms, some minimum public information is needed: the optimal number of orders for each player, i.e., m_i for all $i \in N$. Then an inventory cost situation is given by the 3-tuple $\langle N, a, \{m_i\}_{i\in N}\rangle$ with $a > 0$ and $m_i \geq 0$, for all $i \in N$. The corresponding inventory cost game (N, c_v) is defined as follows. For all coalitions $S \subseteq N$, $S \neq \varnothing$,

$$c_v(S) := 2a\sqrt{\sum_{i\in S} m_i^2}. \tag{1}$$

Meca et al. (2004) show that inventory cost games are concave and monotone. Moreover, the c^2-proportional rule with $c^2 = (c_v(i)^2)_{i\in N}$, or SOC-rule, on inventory cost games is a core-allocation which can be reached through a pmas for (N, c_v). In addition, the SOC-rule is the unique rule on the class of inventory cost games satisfying efficiency, symmetry and monotonicity. Meca et al. (2003) revisit inventory cost games and the SOC-rule. They prove that the wider class of n-person EPQ inventory situations with shortages leads to exactly the same class of cost games. Moreover, an alternative characterization of the SOC-rule, based on some kind of additivity property, is provided there.

In a holding cost game a group of players decide to cooperate making their orders jointly and storing in the warehouse of the player with the lowest holding cost. A full disclosure of information is needed now. Each player $i \in N$ reveals its demand d_i and holding cost h_i. Then a holding cost situation is given by $\langle N, a, \{d_i, h_i\}_{i\in N}\rangle$ with $a > 0$, $d_i \geq 0$, $h_i > 0$, for all $i \in N$. The corresponding holding cost game (N, c_h) is defined as follows. For all coalitions $S \subseteq N$, $S \neq \varnothing$,

$$c_h(S) = \sqrt{2a\sum_{i\in S} d_i \cdot h_S}, \tag{2}$$

where $h_S = \min_{j\in S}\{h_j\}$. In Meca et al. (2004) it is shown that holding cost games are permutationally concave. In addition, the d-proportional rule with $d = (d_i)_{i\in N}$, or demand proportional rule, on holding cost games is also a core-allocation which can be reached through a pmas for (N, c_h).

3 Generalized holding cost games

Following the ideas in Meca et al. (2003) we consider a set of agents N making orders of a certain good that they need and the fixed cost of an order is a. Every agent i needs d_i units of the good per time unit and has a holding cost h_i for keeping one unit of the good in stock during one time unit. Besides, every agent i considers the possibility of not fulfilling all the demand in time, but allowing a shortage of the good. The cost of a shortage of one unit of the good for one time unit is $s_i > 0$. When an order is placed, after a deterministic and constant lead time (which can be assumed to be zero, w.l.o.g.), agent i receives the order gradually; more precisely, r_i units of the good are received per time unit. It is assumed that $r_i > d_i$ (otherwise the model makes little sense). We call r_i the *replacement rate* of agent i.

The inventory model we are dealing with for every agent is the Economic Production Quantity (EPQ) with shortages. It is a well-known model in inventory management which generalizes the EOQ model (an EOQ model can be seen as an EPQ model for which the replacement rate and the shortage cost are infinite). The analysis of this model, that we will summarize below, can be found in Tersine (1994). The reader may notice that every player i must choose $\hat{Q}_i$ (order size) and $\hat{M}_i$ (maximum shortage), minimizing his average inventory cost per time unit given by

$$c(Q_i, M_i) = a\frac{d_i}{Q_i} + h_i\frac{\left(Q_i\left(1-\frac{d_i}{r_i}\right)-M_i\right)^2}{2Q_i\left(1-\frac{d_i}{r_i}\right)} + s_i\frac{M_i^2}{2Q_i\left(1-\frac{d_i}{r_i}\right)}.$$

Then it turns out that

$$\hat{Q}_i = \sqrt{\frac{2ad_i}{h_i\left(1-\frac{d_i}{r_i}\right)}\left(\frac{h_i+s_i}{s_i}\right)}, \quad \hat{M}_i = \sqrt{\frac{2ad_ih_i}{s_i(h_i+s_i)}\left(1-\frac{d_i}{r_i}\right)}.$$

It is easy to check that

$$c(\hat{Q}_i, \hat{M}_i) = \sqrt{2ad_ih_i\left(\frac{s_i}{h_i+s_i}\right)\left(1-\frac{d_i}{r_i}\right)}.$$

Now assume that the agents in $S \subseteq N$ decide to make their orders jointly to save part of the order costs. We will consider situations in which there is full disclosure of information. Each agent $i \in S$ reveals its demand d_i, holding cost h_i, shortage cost s_i, replacement rate r_i, its individual optimal order size $\hat{Q}_i$ and maximum shortage $\hat{M}_i$. In addition, if we assume there are no limits to storage capacities, transport costs are equal to zero and deterministic transport times, then we can consider coordination with regard to holding cost. If a member of a coalition S has a very low holding cost then this coalition can reduce its cost if it stores its inventory in the warehouse of this member.

Following the same reasoning in Meca et al. (2004), it can be easily checked that, as in order to minimize the sum of the average inventory costs per time unit, the agents must coordinate their orders so $Q_i^*/d_i = Q_j^*/d_j$ for all $i, j \in N$, Q_i^* and Q_j^* denoting the optimal order sizes for i and j if agents in S cooperate. Moreover, all goods will be stored in the warehouse of the agent with the lowest holding cost.

Define $h_S := \min_{j \in S}\{h_j\}$. Then the total average cost per time unit is given by

$$c(Q_i, (M_j)_{j\in S}) = \frac{ad_i}{Q_i} + \sum_{j\in S} h_S \frac{\left(Q_j\left(1-\frac{d_j}{r_j}\right)-M_j\right)^2}{2Q_j\left(1-\frac{d_j}{r_j}\right)} + \sum_{j\in S} s_j \frac{M_j^2}{2Q_j\left(1-\frac{d_j}{r_j}\right)}$$
$$= \frac{ad_i}{Q_i} + \frac{1}{2}\sum_{j\in S} h_S \left(\frac{d_j}{d_i}Q_i\left(1-\frac{d_j}{r_j}\right) - 2M_j + \frac{d_i M_j^2}{d_j Q_i\left(1-\frac{d_j}{r_j}\right)}\right)$$
$$+ \frac{1}{2}\sum_{j\in S} s_j \frac{d_i M_j^2}{d_j Q_i\left(1-\frac{d_j}{r_j}\right)}.$$

Note in passing that the above average cost is a function depending on coalition $S \subseteq N$, it means $c((Q_i, M_j)_{j\in S})$. However, taking relations between Q_i and Q_j into account, it can be expressed just by $c(Q_i, (M_j)_{j\in S})$.

Applying standard techniques of differential analysis it can be checked that the values $(Q_i^*)_{i\in S}$ and $(M_i^*)_{i\in S}$ which minimize c are given by

$$Q_i^* = \sqrt{\frac{2ad_i^2}{h_S \cdot \sum_{j\in S} d_j \frac{s_j}{h_S+s_j}\left(1-\frac{d_j}{r_j}\right)}}, \quad M_i^* = Q_i^* \cdot \frac{h_S \cdot \left(1-\frac{d_j}{r_j}\right)}{h_S+s_i}$$

for all $i \in S$. From this it follows that the minimal average cost per time unit for coalition S equals

$$c(Q_i^*, (M_j^*)_{j\in S}) = \sqrt{2a\sum_{j\in S} d_j \cdot h_S \left(\frac{s_j}{h_S+s_j}\right)\left(1-\frac{d_j}{r_j}\right)}. \tag{3}$$

At this point we could consider a holding cost situation $\langle N, a, \{d_i, h_i\}_{i\in N}\rangle$ and define the corresponding generalized holding cost game as the one which assigns to coalition $S \subseteq N$, $S \neq \varnothing$ its minimal cost as in (3). However, we take advantage of a common property underlying (2) and (3), specifically the increasing character of their functions in h_S, and focus on a more general class of inventory games which contains the class defined by (3) and the holding cost games one.

A *generalized holding cost situation* is described by the tuple $\langle N, a, \{d_i, h_i, f_i\}_{i\in N}\rangle$ where $a > 0$, and for all $i \in N, d_i \geq 0, h_i > 0$, and f_i is an increasing function from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$. Given a generalized holding cost situation we can define the corresponding *generalized holding cost game* (N, c) as the game that assigns to coalition $S \subseteq N$, $S \neq \emptyset$ its minimal cost as follows

$$c(S) := \sqrt{2a\sum_{i\in S} d_i f_i(h_S)}. \tag{4}$$

As we have just announced (2) and (3) are particular cases of (4), with $f_i(x) = x$ and $f_i(x) = x \cdot (\frac{s_i}{x+s_i})(1-\frac{d_j}{r_j})$ for all $i \in S \subseteq N$, respectively.

Next we denote by GH^N the class of generalized holding cost games with player set N. The reader may notice that generalized holding cost games are subadditive, but not necessarily concave as Example 1 given in Meca et al. (2004) shows.

The following theorem shows that generalized holding cost games are also permutationally concave games.

Theorem 3.1 *Generalized holding cost games are permutationally concave games.*

Proof Let (N, c) be a generalized holding cost game. Without loss of generality we number all players from 1 to n, $N = \{1, \dots, n\}$, in such a way that the holding cost per time unit of all players forms a non-decreasing sequence, i.e., $h_1 \leq h_2 \leq \cdots \leq h_n$. Take $\sigma \in \Pi(N)$ such that $\sigma(i) = i, \forall i \in N$. Let us see that (N, c) is permutationally concave with respect to σ.

Take $i, j \in N \cup \{0\}$ such that $\sigma(i) \leq \sigma(j)$ and $R \subseteq N \backslash \overline{P_j^\sigma}$. Then $i \leq j$ since $\sigma(k) = k$ for all $k \in N \cup \{0\}$. The game $(N, \overline{c})$ defined by $\overline{c}(S) = \sqrt{\sum_{j \in S} d_j f_j(h_1)}$ for all $S \subseteq N$ is concave, because $\sqrt{x}$ is a concave and monotone increasing function; i.e., for all $S \subseteq T \subseteq N$ and all $U \subseteq N \backslash T$, $\overline{c}(S \cup U) - \overline{c}(S) \geq \overline{c}(T \cup U) - \overline{c}(T)$.

If we take $S = \overline{P_i^\sigma}$, $T = \overline{P_j^\sigma}$ and $U = R$, then $S \subseteq T$ since $\sigma(i) \leq \sigma(j)$, $U \subset N \backslash T$ and

$$\sqrt{\sum_{k \in \overline{P_i^\sigma} \cup R} d_k f_k(h_1)} - \sqrt{\sum_{k \in \overline{P_i^\sigma}} d_k f_k(h_1)} \geq \sqrt{\sum_{k \in \overline{P_j^\sigma} \cup R} d_k f_k(h_1)} - \sqrt{\sum_{k \in \overline{P_j^\sigma}} d_k f_k(h_1)}. \tag{5}$$

To complete the proof we can distinguish three cases:

1. If $i = 0$ and $j = 0$ then $\overline{P_i^\sigma} = \overline{P_j^\sigma} = \emptyset$ and $c(\overline{P_i^\sigma} \cup R) - c(\overline{P_i^\sigma}) = c(R) - c(\emptyset) = c(\overline{P_j^\sigma} \cup R) - c(\overline{P_j^\sigma})$.
2. If $i = 0$ and $j > 0$, then $\overline{P_i^\sigma} = \emptyset$ and $\overline{P_j^\sigma} = \{1, 2, \dots, j\}$. Since $1 \in \overline{P_j^\sigma}$ and $1 \notin R$ it holds $h_{\overline{P_j^\sigma}} = h_{\overline{P_j^\sigma} \cup R} = h_1$. Hence
$f_k(h_{\overline{P_j^\sigma}}) = f_k(h_{\overline{P_j^\sigma} \cup R}) = f_k(h_1) \leq f_k(h_R), \forall k \in \overline{P_j^\sigma} \cup R$, taking into account that f_k is increasing for all $k \in N$ and $h_1 \leq h_R$. By (5) we get
$$\begin{aligned} \sqrt{\sum_{k \in R} d_k f_k(h_R)} - 0 &\geq \sqrt{\sum_{k \in R} d_k f_k(h_1)} - 0 \\ &> \sqrt{\sum_{k \in \overline{P_j^\sigma} \cup R} d_k f_k(h_1)} - \sqrt{\sum_{k \in \overline{P_j^\sigma}} d_k f_k(h_1)}, \end{aligned}$$
then $c(R) - c(\emptyset) > c(\overline{P_j^\sigma} \cup R) - c(\overline{P_j^\sigma})$.
3. If $0 < i \leq j$ then $1 \in \overline{P_i^\sigma}$ and $1 \in \overline{P_j^\sigma}$, hence (5) implies that $c(\overline{P_i^\sigma} \cup R) - c(\overline{P_i^\sigma}) \geq c(\overline{P_j^\sigma} \cup R) - c(\overline{P_j^\sigma})$.

□

The reader may notice that the above proof also works for every subgame of the generalized holding cost game. Then we can conclude that generalized holding cost games are totally balanced.

4 N-rational solution family

Now that we have revisited the class of holding cost games and it has been extended to the generalized holding cost games, we are interested in studying the core of the latter. We introduce a core-allocation family which is called *N-rational solution family.* It is proved that a particular relation of inclusion among the former, the core and the Weber set exists.

Given $c \in GH^N$, we define the set of players with minimum holding cost as $M = \{i \in N \mid h_i = h_N\}$. Players belonging to M are called *minimum players* (*M-players*) and players belonging to $N \backslash M$, are called *complementary players* (*c-players*). By (4) we know that the individual cost for player $i \in N$ is given by $c(\{i\}) = \sqrt{2ad_i f_i(h_i)}$. Likewise, for each $T \subseteq N, T \neq \emptyset$, and $i \in T$, we define the *cost for player i in coalition T* as $c_T(\{i\}) := \sqrt{2ad_i f_i(h_T)}$. Then for each $S \subseteq T \subseteq N, S \neq \varnothing$ the *cost for coalition S in coalition T* is defined by $c_T(S) := \sqrt{\sum_{i \in S} c_T(\{i\})^2}$.

The meaning of this cost is clear and simple: it is the cost generated by the smaller coalition (S) when storing in the optimal place of the bigger (T). Note that $c_R(S) \leq c_T(S)$ for all $T \subseteq R$, and $c_T(S) \leq c_T(R)$ for all $S \subseteq R \subseteq T$. Moreover, $c_N(S) \leq c(S)$ for each $S \subseteq N$; in particular, $c_N(S) = c(S)$ for all $S \subseteq N$ such that $S \cap M \neq \emptyset$, and $c_N(S) < c(S)$ for all $S \subseteq N \backslash M$.

Given $c \in GH^N$ we could consider a new game (N, c_N), which will be called *N-cost game*. It is clear that c_N is a concave game. In fact, c_N could be seen as either an inventory cost game with $a = 1/2$ and $m_i = c_N(\{i\})$ for all $i \in N$, or a generalized holding cost game with $M = N$.

Next we propose a core-allocation solution family for generalized holding cost games, which will allow to know better the core of these games.

Such a family is inspired by the fact that for generalized holding cost games there always exist core-allocations which assign a lower cost to each coalition than its own when all members in such a coalition store in the M-player's warehouse. This makes sense since because of coordination on ordering and holding simultaneously, which produces a bigger reduction in total costs.

Thinking about core restrictions for a generalized holding cost game carefully, we observe that a sufficient condition for being a core-allocation is the following: $x(N) = c(N)$ and no coalition $S \subsetneq N$ exists such that $x(S) > c_N(S)$ since $c_N(S) \leq c_S(S) = c(S)$.

All the above leads us to define a core-allocation solution family for generalized holding cost games which we call *N-rational solution family* , as follows:

$$\mathcal{F}(c) := \left\{x \in \mathbb{R}^n \big/ x(N) = c(N) \text{ and } x(S) \leq c_N(S), \forall S \subsetneq N\right\}.$$

Note that $\mathcal{F}(c) = C(c_N)$. Since c_N is concave, $\mathcal{F}(c) \neq \emptyset$; in fact, $\mathcal{F}(c) = \text{conv}\{m^\sigma(c_N)/\sigma \in \Pi(N)\}$. Besides, since $c_N(S) \leq c(S)$ for all $S \subseteq N$, $\mathcal{F}(c) \subset C(c)$, but in general, $\mathcal{F}(c) \neq C(c)$. Moreover, for each $x \in \mathcal{F}(c)$, $x_i \leq c_N(\{i\})$, $\forall i$

$\in N$; i.e., all players will at most pay their cost in the grand coalition. This is the main reason for the name N-rational solution family.

The following example illustrates the relationship among the core, the Weber set and the N-rational solution family for a generalized holding cost game.

Example 4.1 Consider the generalized holding cost situation given by $N=\{1,2,3\}$, $a = 1/2, d_i = 1, \forall i \in N; h_1 = h_2 = 1, h_3 = 4; f_i(x) = x, \forall x \in \mathbb{R}_{++}, \forall i \in N$.

The generalized holding cost game corresponding to the above situation is

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$c(S)$	1	1	2	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{3}$

Hence,

$$C(c) = \operatorname{conv}\left\{ \begin{array}{c} \begin{pmatrix} 1 \\ \sqrt{2}-1 \\ \sqrt{3}-\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{3}-\sqrt{2} \\ \sqrt{2}-1 \end{pmatrix}, \begin{pmatrix} \sqrt{2}-1 \\ 1 \\ \sqrt{3}-\sqrt{2} \end{pmatrix}, \\ \begin{pmatrix} \sqrt{3}-\sqrt{2} \\ 1 \\ \sqrt{2}-1 \end{pmatrix}, \begin{pmatrix} \sqrt{3}-\sqrt{2} \\ \sqrt{3}-\sqrt{2} \\ 2\sqrt{2}-\sqrt{3} \end{pmatrix} \end{array} \right\},$$

$$W(c) = \operatorname{conv}\left\{ \begin{array}{c} \begin{pmatrix} 1 \\ \sqrt{2}-1 \\ \sqrt{3}-\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{3}-\sqrt{2} \\ \sqrt{2}-1 \end{pmatrix}, \begin{pmatrix} \sqrt{2}-1 \\ 1 \\ \sqrt{3}-\sqrt{2} \end{pmatrix}, \\ \begin{pmatrix} \sqrt{3}-\sqrt{2} \\ 1 \\ \sqrt{2}-1 \end{pmatrix}, \begin{pmatrix} \sqrt{2}-2 \\ \sqrt{3}-\sqrt{2} \\ 2 \end{pmatrix}, \begin{pmatrix} \sqrt{3}-\sqrt{2} \\ \sqrt{2}-2 \\ 2 \end{pmatrix} \end{array} \right\},$$

and

$$\mathcal{F}(c) = \operatorname{conv}\left\{ \begin{array}{c} \begin{pmatrix} 1 \\ \sqrt{2}-1 \\ \sqrt{3}-\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{3}-\sqrt{2} \\ \sqrt{2}-1 \end{pmatrix}, \begin{pmatrix} \sqrt{2}-1 \\ 1 \\ \sqrt{3}-\sqrt{2} \end{pmatrix}, \\ \begin{pmatrix} \sqrt{3}-\sqrt{2} \\ 1 \\ \sqrt{2}-1 \end{pmatrix}, \begin{pmatrix} \sqrt{2}-1 \\ \sqrt{3}-\sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} \sqrt{3}-\sqrt{2} \\ \sqrt{2}-1 \\ 1 \end{pmatrix} \end{array} \right\}.$$

The reader may notice that there are four common extreme points for the core, the Weber set and the N-rational solution family:

$$\begin{pmatrix} 1 \\ \sqrt{2}-1 \\ \sqrt{3}-\sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \sqrt{3}-\sqrt{2} \\ \sqrt{2}-1 \end{pmatrix}, \quad \begin{pmatrix} \sqrt{2}-1 \\ 1 \\ \sqrt{3}-\sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} \sqrt{3}-\sqrt{2} \\ 1 \\ \sqrt{2}-1 \end{pmatrix};$$

these are the marginal vectors related to the orders with an M-player first; i.e., $m^{\sigma}(c)$ with $\sigma \in \Pi(N)$ such that $\exists i \in M = \{1,2\}, \sigma(i) = 1$. Moreover, we find a vector $x = \begin{pmatrix} \sqrt{3}-\sqrt{2} \\ \sqrt{3}-\sqrt{2} \\ 2\sqrt{2}-\sqrt{3} \end{pmatrix} \in C(c)$, but $x \notin \mathcal{F}(c)$, since $x_3 = 2\sqrt{2}-\sqrt{3} > 1 = c_N(\{3\})$.

At this point we wonder if for every generalized holding cost game all aforementioned extreme points are common extreme points for the core, the N-rational solution family and the Weber set.

The following proposition gives an affirmative answer. It shows that a particular relation of inclusion exists among the N-rational solution family, the core and the Weber set of generalized holding cost games. These polyhedra have a special subset of common extreme points whose cardinality is no larger than $\mathrm{card}(M) \cdot (\mathrm{card}(N) - 1)!$: those marginal vectors related to orders with an M-player first and the rest of the players able to order in any which way.

Next we define by $\Pi_1(N) := \{\sigma \in \Pi(N) \mid \exists i \in M, \sigma(i) = 1\}$ the set of all orders with an M-player first. Then $\Pi(N)\backslash\Pi_1(N) = \{\sigma \in \Pi(N) \mid \forall i \in M, \exists j \in N\backslash M$ with $\sigma(j) < \sigma(i)\}$ is the set of all orders with a c-player first.

Proposition 4.2 *For every $c \in GH^N$,*

$$\left\{m^\sigma(c) \big/ \sigma \in \Pi_1(N)\right\} \subseteq \mathrm{Ext}\,\mathcal{F}(c) \cap \mathrm{Ext}\,C(c) \cap \mathrm{Ext}\,W(c).$$

Proof Take $c \in GH^N$ and $\sigma \in \Pi_1(N)$. We know that $m^\sigma(c_N) \in \mathrm{Ext}\,\mathcal{F}(c)$ and $m^\sigma(c) \in \mathrm{Ext}\,W(c)$. Let us see that $m^\sigma(c_N) = m^\sigma(c)$. Then $m^\sigma(c) \in \mathrm{Ext}\,C(c)$ and so the proposition's statement.

For all $i \in M$ such that $\sigma(i) = 1, m_i^\sigma(c_N) = c_N(\overline{P_i^\sigma}) - c_N(\emptyset) = c(\overline{P_i^\sigma}) = m_i^\sigma(c)$.

On the other hand, for all $j \in N, j \neq i$, it holds that $i \in \overline{P_j^\sigma}$, and $i \in P_j^\sigma$. Hence, $m_j^\sigma(c_N) = c_N(\overline{P_j^\sigma}) - c_N(P_j^\sigma) = c(\overline{P_j^\sigma}) - c(P_j^\sigma) = m_j^\sigma(c)$.

Then we can conclude that $m^\sigma(c_N) = m^\sigma(c)$. □

Next we wonder if for every generalized holding cost game the aforementioned common extreme points for the core and the N-rational solution family are the unique common ones.

It is easy to check that if $M = N$, then c is concave and so $\mathcal{F}(c) = C(c) = W(c)$, but in general the converse is not true. From now on we focus on the case $M \neq N$. Two sufficient conditions for the uniqueness of these common extreme points are obtained. The first shows that for generalized holding cost games corresponding to situations with positive constant demand, and the same strictly increasing function for each player, the unique common extreme points for the N-rational solution family and the core are the marginal vectors related to the orders with an M-player first. The second one generalizes the former to 3-player generalized holding cost games corresponding to situations with positive demands and strictly increasing functions. It is an open question whether the latter can be extended to games with at least four players.

The following technical lemma will be very useful to prove the first condition.

Lemma 4.3 *Let $N = \{1, 2, \ldots, n\}$ a finite set. Then, for each $S \subseteq N$,*

$$\sum_{j \in S}\left(\sqrt{j} - \sqrt{j-1}\right)\begin{cases} = \sqrt{\mathrm{card}(S)} & \text{if } S = \{1, 2, \ldots, \mathrm{card}(S)\} \\ < \sqrt{\mathrm{card}(S)} & \text{any other case.}\end{cases}$$

Proof It is a straightforward consequence of the fact that the function $f(x) = \sqrt{x} - \sqrt{x-1}$ is strictly decreasing on $[1, +\infty)$. □

As we just announced, the next theorem shows that, under some particular conditions, the set of common extreme points for the N-rational solution family and the core of generalized holding cost games is the special subset referred to just before Proposition 4.2.

Theorem 4.4 *Let $\langle N, a, \{d_i, h_i, f_i\}_{i\in N}\rangle$ be a generalized holding cost situation and c the corresponding generalized holding cost game. Then*

$$\operatorname{Ext}\mathcal{F}(c) \cap \operatorname{Ext} C(c) = \left\{m^{\sigma}(c)\big/\sigma \in \Pi_1(N)\right\}, \tag{6}$$

(i) *if $d_i = d > 0$ and $f_i = f$ strictly increasing for all $i \in N$;*
(ii) *if $N = \{1, 2, 3\}$, $d_i > 0$ and f_i a strictly increasing function for all $i \in N$.*

Proof (i) Taking into account that

$$\operatorname{Ext}\mathcal{F}(c) = \left\{m^{\sigma}(c)\big/\sigma \in \Pi_1(N)\right\} \cup \left\{m^{\sigma}(c_N)\big/\sigma \in \Pi(N)\backslash\Pi_1(N)\right\},$$

and Proposition 4.2, it is enough to prove that $m^{\sigma}(c_N) \notin \operatorname{Ext} C(c)$ for all $\sigma \in \Pi(N)\backslash\Pi_1(N)$.

Take $\sigma \in \Pi(N)\backslash\Pi_1(N)$, then $\sigma(i) = j \neq 1$ for all $i \in M$. We prove that $m^{\sigma}(c_N)$ has exactly $n - (k - 1)$ binding constraints in $C(c)$, where $2 \leq k = \min_{i\in M}\{\sigma(i)\} \leq n$: those corresponding to $\overline{P}^{\sigma}_{\sigma^{-1}(k)}, \overline{P}^{\sigma}_{\sigma^{-1}(k+1)}, \ldots, \overline{P}^{\sigma}_{\sigma^{-1}(n)}$.

If $S = \overline{P}^{\sigma}_{\sigma^{-1}(r)}$ (or equivalently $\sigma(S) = \{1, \ldots, r\}$), for an arbitrary $r \in \{1, \ldots, n\}$, then

$$\sum_{i\in S} m_i^{\sigma}(c_N) = \sum_{j=1}^{r} m^{\sigma}_{\sigma^{-1}(j)}(c_N) = c_N\left(\overline{P}^{\sigma}_{\sigma^{-1}(r)}\right).$$

Two cases are considered:

(1) If $r \in \{1, \ldots, k-1\}$, then $\overline{P}^{\sigma}_{\sigma^{-1}(r)} \cap M = \varnothing$, hence $\sum_{i\in S} m_i^{\sigma}(c_N) < c(\overline{P}^{\sigma}_{\sigma^{-1}(r)})$, since f is strictly increasing and $d > 0$.
(2) If $r \in \{k, \ldots, n\}$, then $\overline{P}^{\sigma}_{\sigma^{-1}(r)} \cap M \neq \varnothing$, hence $\sum_{i\in S} m_i^{\sigma}(c_N) = c(\overline{P}^{\sigma}_{\sigma^{-1}(r)})$.

For all $S \neq \overline{P}^{\sigma}_{\sigma^{-1}(r)}, r = 1, \ldots, n$,

$$\begin{aligned}
&\sum_{i\in S} m_i^{\sigma}(c_N) \\
&\quad= \sum_{j\in\sigma(S)} m^{\sigma}_{\sigma^{-1}(j)}(c_N) \\
&\quad= \sum_{j\in\sigma(S)} \left[c_N\left(\left\{\sigma^{-1}(1), \ldots, \sigma^{-1}(j)\right\}\right) - c_N\left(\left\{\sigma^{-1}(1), \ldots, \sigma^{-1}(j-1)\right\}\right)\right] \\
&\quad= \sum_{j\in\sigma(S)} \sqrt{2a \cdot d \cdot f(h_1)}\left(\sqrt{j} - \sqrt{j-1}\right).
\end{aligned}$$

By lemma 4.3, $\sum_{j\in\sigma(S)}(\sqrt{j} - \sqrt{j-1}) < \sqrt{\operatorname{card}(S)}$. Therefore, we can conclude that $\sum_{i\in S} m_i^{\sigma}(c_N) < c_N(S) < c(S)$.

(ii) Let $\langle N, a, \{d_i, h_i, f_i\}_{i\in N}\rangle$ be a generalized holding cost situation with $N = \{1, 2, 3\}$, $d_i > 0$ and f_i a strictly increasing function for all $i \in N$. Let us see that $m^{\sigma}(c_N) \notin \operatorname{Ext} C(c)$ for all $\sigma \in \Pi(N)\backslash\Pi_1(N)$, hence we can conclude that (6) holds.

We can distinguish two cases:

Case 1 $M = \{1, 2\}$. Then $\Pi(N)\backslash\Pi_1(N) = \{312, 321\}$ and

$$m^{312}(c_N) = \begin{pmatrix} c(\{1,3\}) - c_N(\{3\}) \\ c(N) - c(\{1,3\}) \\ c_N(\{3\}) \end{pmatrix}.$$

It can be easily checked that $m^{312}(c_N)$ has exactly two binding constraints in $C(c)$: those corresponding to $\overline{P}^{\sigma}_{\sigma^{-1}(2)=1} = \{1, 3\}$, $\overline{P}^{\sigma}_{\sigma^{-1}(3)=2} = N$; i.e., $m^{312}_1(c_N) + m^{312}_3(c_N) = c(\{1, 3\})$ and $m^{312}_1(c_N) + m^{312}_2(c_N) + m^{312}_3(c_N) = c(N)$. Then by the characterization of extreme points of a bounded convex polyhedron in $\mathbb{R}^n$ given in Sect. 2, $m^{312}(c_N) \notin \operatorname{Ext} C(c)$. A similar argument shows that $m^{321}(c_N) \notin \operatorname{Ext} C(c)$.

Case 2 $M = \{1\}$. Then $\Pi(N)\backslash\Pi_1(N) = \{321, 213, 231, 312\}$ and

$$m^{321}(c_N) = \begin{pmatrix} c(N) - c_N(\{2,3\}) \\ c_N(\{2,3\}) - c_N(\{3\}) \\ c_N(\{3\}) \end{pmatrix}, \quad m^{213}(c_N) = \begin{pmatrix} c(\{1,2\}) - c_N(\{2\}) \\ c_N(\{2\}) \\ c(N) - c(\{1,2\}) \end{pmatrix}.$$

It can be easily checked again that $m^{321}(c_N)$ has just one binding constraint in $C(c)$. Moreover, $m^{213}(c_N)$ has exactly 2 binding constraints in $C(c)$: $\overline{P}^{\sigma}_{\sigma^{-1}(2)=1} = \{1, 2\}$, $\overline{P}^{\sigma}_{\sigma^{-1}(3)=3} = N$. Then $m^{321}(c_N), m^{213}(c_N) \notin \operatorname{Ext} C(c)$. Similar arguments show that $m^{231}(c_N), m^{312}(c_N) \notin \operatorname{Ext} C(c)$. □

The reader may notice that for holding cost games with positive constant demand, the set of common extreme points for the N-rational solution family and the core consists of those marginal vectors related to the orders with an M-player first.

5 Minimum square proportional rule

To complete the study of generalized holding cost games, we wonder if any of the well-known solutions for cost games are either N-rational solutions or core-allocations for generalized holding cost games. The following example shows that neither the Shapley value nor the τ-value are N-rational solutions, although the τ-value is a core-allocation [Driessen and Tijs (1985) show that it is always a core-allocation for quasibalanced games with two and three players] but the Shapley value is not.

Example 5.1 Consider the generalized holding cost situation given by $N = \{1, 2, 3\}, a = 1/2, d_1 = 4, d_2 = 1, d_3 = 144; h_1 = 1, h_2 = 4, h_3 = 16; f_i(x) = x, \forall x \in \mathbb{R}_{++}, \forall i \in N$.

The generalized holding cost game and the N-cost game corresponding to the above situation are

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$c(S)$	2	2	48	$\sqrt{5}$	$\sqrt{148}$	$\sqrt{580}$	$\sqrt{149}$
$c_N(S)$	2	1	12	$\sqrt{5}$	$\sqrt{148}$	$\sqrt{145}$	$\sqrt{149}$

The Shapley value is then given by $\Phi(c) = (-9.22, -3.26, 24.69)$. However, $\Phi_1(c) + \Phi_3(c) = 15.47 > c(\{1,3\}) = 12.16$. Hence, $\Phi(c) \notin C(c)$ and so $\Phi(c) \notin \mathcal{F}(c)$.

On the other hand, taking into account that $M(c) = (-11.88, 0.04, 9.97)$, $m(c) = (2, 2, 24.04)$, and then $\alpha = \frac{c(N)-\sum_{j\in N} m_j(c)}{\sum_{j\in N}[M_j(c)-m_j(c)]} = \frac{-15.83}{-29.91} = 0.52$, the τ-value is given by $\tau(c) = (-5.34, 0.96, 16.59)$. However, $\tau_3(c) = 16.59 > 12 = c_N(\{3\})$; so, $\tau(c) \notin \mathcal{F}(c)$.

We realize thinking about the nature of generalized holding cost games that they are similar to inventory cost games [compare (4) with (1)]. This leads us to think of choosing a core-allocation following the idea of proportional allocations. The question that arises immediately is what proportional factor should be chosen?

Since demand plays an important role on the class of generalized holding cost games, a possible allocation rule could be the demand proportional rule. However, the following example shows that the demand proportional rule on generalized holding cost games is not necessarily a core-allocation.

Example 5.2 Consider the generalized holding cost situation given by $N=\{1,2,3\}$, $a = 1/2$, $d_1 = d_2 = 1$, $d_3 = 1/2$, $h_1 = 4$, $h_2 = h_3 = 9$; $f_i(x) = x \cdot (\frac{s_i}{x+s_i})$, $\forall x \in \mathbb{R}_{++}$, $\forall i \in N$, where $s_1 = s_2 = 1$, $s_3 = 4$.

The generalized holding cost game corresponding to the above situation is

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$c(S)$	$\frac{2}{\sqrt{5}}$	$\frac{3}{\sqrt{10}}$	$\frac{6}{\sqrt{13}}$	$\sqrt{\frac{8}{5}}$	$\frac{3}{\sqrt{5}}$	$\sqrt{\frac{297}{130}}$	$\sqrt{\frac{13}{5}}$

Then the demand proportional rule is $p(c) = (\frac{2}{5}\sqrt{\frac{13}{5}}, \frac{2}{5}\sqrt{\frac{13}{5}}, \frac{1}{5}\sqrt{\frac{13}{5}})$. Note that $p_1(c) + p_2(c) = \frac{4}{5}\sqrt{\frac{13}{5}} > \sqrt{\frac{8}{5}} = c(\{12\})$. Hence, $p(c) \notin C(c)$.

Following the ideas adopted for inventory cost games, a new proportional rule is proposed: *minimum square proportional rule.* This is an N-rational solution.

Definition 5.3 *The* minimum square proportional rule *on the class of generalized holding cost games is a map $\mathcal{P} : GH^N \to \mathbb{R}$ such that, for every $c \in GH^N$ and all $i \in N$*

$$\mathcal{P}_i(c) := \begin{cases} \frac{c_N(\{i\})^2}{\sum_{j\in N} c_N(\{j\})^2} c(N) & c \neq c^0 \\ 0 & c = c^0 \end{cases}.$$

Note that $c_N(\{i\})^2 = \min_{S\subset N/i\in S}\{c_S(\{i\})^2\}$, $\forall i \in N$. This is the main reason for the name minimum square proportional rule. The difference between the proportional rule (on inventory cost games) and the minimum square proportional rule (on generalized holding cost games) is the proportionality factor: now $c_N(\{i\})^2$, instead of $c_v(i)^2$, which reflects both kinds of coordination considered: ordering and holding.

Taking into account that $c(N)^2 = \sum_{j\in N} c_N(\{j\})^2$, the above proportional rule can be rewritten as follows: for all $c \neq c^0$ and all $i \in N$,

$$\mathcal{P}_i(c) = \frac{2ad_i f_i(h_N)}{\sqrt{\sum_{j\in N} 2ad_j f_j(h_N)}} = \frac{c_N(\{i\})^2}{c(N)}. \tag{7}$$

Moreover, for generalized holding cost games coming from a generalized holding cost situation with $f_i = f, \forall i \in N$ (in particular, holding cost games), it equals the demand proportional rule.

Note that the above rule is determined by the parameters of the generalized holding cost situation that leads to the generalized holding cost game; i.e., it depends on the fixed ordering cost a, demands $\{d_i\}_{i \in N}$, minimum holding cost h_N and functions $\{f_i\}_{i \in N}$. It is not necessary to know the complete characteristic function of such a game and so, it is easy to calculate.

Example 5.4 Consider the generalized holding cost game given in Example 5.2. The minimum square proportional rule is $\mathcal{P}(c) = (\frac{4}{13}\sqrt{\frac{13}{5}}, \frac{4}{13}\sqrt{\frac{13}{5}}, \frac{5}{13}\sqrt{\frac{13}{5}})$. Taking into account that the corresponding N-cost game is

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$c_N(S)$	$\frac{2}{\sqrt{5}}$	$\frac{3}{\sqrt{10}}$	1	$\sqrt{\frac{8}{5}}$	$\frac{3}{\sqrt{5}}$	$\frac{3}{\sqrt{5}}$	$\sqrt{\frac{13}{5}}$

it can be easily checked that $\mathcal{P}(c) \in \mathcal{F}(c)$; it is a core-allocation.

The reader may notice that the minimum square proportional rule satisfies the efficiency and zero-symmetry properties. The following theorem shows that the minimum square proportional rule is always a N-rational solution, hence a core-allocation. Besides it gives some other nice properties for such a rule: for generalized holding cost games coming from situations with $f_i = f, \forall i \in N$, it can be reached through a pmas.

Theorem 5.5 *Let $c \in GH^N$. Then*

(i) $\mathcal{P}(c) \in \mathcal{F}(c)$.
(ii) *If c comes from a generalized holding cost situation with $f_i = f, \forall i \in N$, then $\mathcal{P}(c)$ can be reached through a pmas for c.*
(iii) *$\mathcal{P}(c)$ can be reached through a pmas for c_N.*

Proof (i) We just have to prove that for all non-empty coalitions S in N, $\sum_{i \in S} \mathcal{P}_i(c) \leq c_N(S)$. Take $S \subset N$, then

$$\sum_{i \in S} \mathcal{P}_i(c) = \sum_{i \in S} \frac{2ad_i f_i(h_N)}{\sqrt{\sum_{j \in N} 2ad_j f_j(h_N)}} \leq \sum_{i \in S} \frac{2ad_i f_i(h_N)}{\sqrt{\sum_{i \in S} 2ad_i f_i(h_N)}}$$

$$= \sqrt{\sum_{i \in S} 2ad_i f_i(h_N)} = c_N(S).$$

(ii) Take $c \in GH^N$ such that $f_i = f, \forall i \in N$. For all players $i \in N$

$$\mathcal{P}_i(c) = \frac{d_i}{\sum_{j \in N} d_j} \sqrt{2af(h_N) \sum_{j \in N} d_j}.$$

Now for each $i \in S$, $S \subset N$, $S \neq \emptyset$ we define $y_i^S = \frac{d_i}{\sum_{j \in S} d_j} \sqrt{2af(h_S) \sum_{j \in S} d_j}$. Then $\forall S \subset N$, $S \neq \emptyset$, $y^S(S) = \sqrt{2af(h_S) \sum_{j \in S} d_j} = c(S)$.

Moreover, for all $S, T \subset N$, $S, T \neq \emptyset$, such that $S \subset T$ and all player $i \in S$

$$y_i^S = \frac{d_i\sqrt{2af(h_S)}}{\sqrt{\sum_{j\in S} d_j}} \geq \frac{d_i\sqrt{2af(h_T)}}{\sqrt{\sum_{j\in T} d_j}} = y_i^T,$$

since $h_T \leq h_S$ and f is an increasing function.

(iii) Take $c \in GH^N$. We know that the corresponding N-cost game (N, c_N) is concave. Since $\mathcal{P}_i(c) \in \mathcal{F}(c) = C(c_N)$, it follows from Sprumont (1990) that each of its core-elements, or equivalently each element of $\mathcal{F}(c)$, can be reached through a pmas for c_N. □

We will now introduce a monotonicity property for allocation rules on the class of generalized holding cost games. When thinking of a monotonicity property for a proportional rule on generalized holding cost games, we could follow the ideas adopted when studying the proportional rule on inventory cost games (see Meca et al. 2004). However, we should take into account the fact that now we are also considering coordination on holding. So we focus on the cost for a player in the grand coalition instead of its own individual cost and we propose the following property, which is called the N-monotonicity property.

Let (N, c) and $(N, \overline{c})$ be generalized holding cost games corresponding to generalized holding cost situations $\langle N, a, \{d_i, h_i, f_i\}_{i\in N}\rangle$ and $\langle N, \overline{a}, \{\overline{d_i}, \overline{h_i}, \overline{f_i}\}_{i\in N}\rangle$, respectively. The solution rule Ψ satisfies *N-monotonicity* if for all $i \in N$ such that $c_N(\{i\}) \geq \overline{c}_N(\{i\})$ it holds that $c(N)\cdot\Psi_i(c) \geq \overline{c}(N)\cdot\Psi_i(\overline{c})$.

Together with efficiency and zero-symmetry, the N-monotonicity property characterizes the minimum square proportional rule on the class of generalized holding cost games as the next theorem [closely related to theorem 1 in Meca et al. (2004)] shows.

Theorem 5.6 *A unique rule exists on the class of generalized holding cost games satisfying efficiency, zero-symmetry and N-monotonicity. It is the minimum square proportional rule.*

Proof It is clear that the minimum square proportional rule satisfies efficiency, zero-symmetry and N-monotonicity.

To show the converse, we take a rule Ψ on the class of generalized holding cost games that satisfies efficiency, zero-symmetry and N-monotonicity. Note that N-monotonicity implies that for all generalized holding cost games (N, c) and $(N, \overline{c})$

$$c_N(\{i\}) = \overline{c}_N(\{i\}) \;\Rightarrow\; c(N)\cdot\Psi_i(c) = \overline{c}(N)\cdot\Psi_i(\overline{c}). \tag{8}$$

By efficiency and zero-symmetry it follows that $\Psi_i(c^0) = 0$ for all $i \in N$. Take a generalized holding cost game (N, c). If for some $i \in N$ it holds that $c_N(\{i\}) = 0$ then $c_N(\{i\}) = c^0(\{i\})$. When $c(N) = 0$ then $c = c^0$ (since $d_i = 0$ for all $i \in N$), and so $\Psi_i(c) = 0$. Otherwise, when $c(N) > 0$ then it follows from (8) that $c(N)\cdot\Psi_i(c) = c^0(N)\cdot\Psi_i(c^0) = 0$ and thus $\Psi_i(c) = 0$. We can conclude that

$$\text{if } c(\{i\}) = 0 \text{ then } c_N(\{i\}) = 0 \text{ and so } \Psi_i(c) = 0. \tag{9}$$

Define the number $I(c)$ to be the number of players $i \in N$ with $c(\{i\}) > 0$. We show that $\Psi_i(c) = \mathcal{P}_i(c)$ for all $i \in N$ by induction on $I(c)$.

If $I(c) = 0$ then by (9), $\Psi_i(c) = 0$ for all $i \in N$.

If $I(c) = 1$ then there is a single player $k \in N$ with $c(\{k\}) > 0$. For all $i \in N\backslash\{k\}$, $c(\{i\}) = 0$, so by (9), $\Psi_i(c) = 0 = \mathcal{P}_i(c)$, since $c_N(\{i\}) = 0$. By efficiency it follows that $\Psi_k(c) = c(N) - \sum_{i \neq k} \Psi_i(c) = c(N) - \sum_{i \neq k} \mathcal{P}_i(c) = \mathcal{P}_k(c)$.

Assume now that $\Psi(c) = \mathcal{P}(c)$ for all generalized holding cost games (N, c) with $I(c) \leq I, I \leq n-1$. Consider a generalized holding cost game $(N, \overline{c})$ corresponding to $\langle N, \overline{a}, \{\overline{d_i}, \overline{h_i}, \overline{f_i}\}_{i \in N}\rangle$ with $I(\overline{c}) = I + 1$. Without loss of generality assume that $\overline{c}(\{i\}) > 0$ for the players $i = 1, 2, \ldots, I+1$. Define the game (N, c) to be corresponding to $\langle N, a, \{d_i, h_i, f_i\}_{i \in N}\rangle$ where $a = \overline{a}, d_i = \overline{d_i}, h_i = \overline{h_i}, f_i = \overline{f_i}$ for all $j \in N\backslash\{I+1\}$ and $d_{I+1} = 0$. Then $I(c) = I$ and $\Psi(c) = \mathcal{P}(c)$. Since $c_N(\{k\}) = \overline{c}_N(\{k\}) > 0$ for all $k = 1, 2, \ldots, I$ it follows by (8) that $\overline{c}(N) \cdot \Psi_k(\overline{c}) = c(N) \cdot \Psi_k(c) = c(N) \cdot \mathcal{P}_k(c)$. By (7),

$$\mathcal{P}_k(c) = \frac{2ad_k f_k(h_N)}{\sqrt{\sum_{j \in N} 2ad_j f_j(h_N)}} = \frac{c_N(\{k\})^2}{c(N)},$$

so using induction

$$\overline{c}(N) \cdot \Psi_k(\overline{c}) = c(N) \cdot \mathcal{P}_k(c) = c(N)\frac{c_N(\{k\})^2}{c(N)} = c_N(\{k\})^2 = \overline{c}_N(\{k\})^2.$$

From this it follows that $\Psi_k(\overline{c}) = \overline{c}_N(\{k\})^2/\overline{c}(N) = \mathcal{P}_k(\overline{c})$. We also have $\overline{c}(\{j\}) = 0$ for all $j = I+2, \ldots, n-1, n$ so by (9) $\Psi_j(\overline{c}) = 0 = \mathcal{P}_j(\overline{c})$. Finally, efficiency implies that

$$\Psi_{I+1}(\overline{c}) = \overline{c}(N) - \sum_{k \neq I+1} \Psi_k(\overline{c}) = \overline{c}(N) - \sum_{k \neq I+1} \mathcal{P}_k(\overline{c}) = \mathcal{P}_{I+1}(\overline{c}),$$

which concludes the proof. □

6 Concluding remarks

Holding cost games are introduced and studied in Meca et al. (2004). In the way of extending this study to the cost games corresponding to the n-person EPQ inventory situations with shortages, we took advantage of a common property underlying both models (EOQ and EPQ with shortages). Specifically we use the increasing character of the functions involved to introduce the more general framework of generalized holding cost games.

Since every generalized holding cost game is totally balanced but not concave, in general, we study a core-allocation family for it: the N-rational solution family. For generalized holding cost games corresponding to situations with positive constant demand and the same strictly increasing function for all players (for instance, holding cost games with positive constant demand), it turns out to be an interesting relation of inclusion between the N-rational solution family and the core: there are a fixed number of common extreme points for them—exactly all of those marginal vectors related to orders with an M-player first, enabling to order the rest of the players in any way. The above result is also true for the 3-player generalized holding cost games corresponding to situations with positive demands and strictly increasing functions; for instance, the 3-player generalized holding cost games corresponding to n-person EPQ inventory situations with shortages and positive demands.

A particular N-rational solution, the minimum square proportional rule, is proposed. Finally we prove that for generalized holding cost games that come from situations with the same function f for each player (holding cost games, among others), the minimum square proportional rule can be reached through a pmas.

It is a topic for further research to find out if the minimum square proportional rule can be reached through a pmas for every generalized holding cost game. Another direction for future research would be to complete the analysis of core structure for generalized holding cost games; for instance, to study the common extreme points for their core.

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