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Symmetric strong vector quasi-equilibrium problems

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Abstract In this paper, we introduce the symmetric strong vector quasi-equilibrium problem. We then demonstrate that the symmetric strong vector quasi-equilibrium problem is solvable under the suitable assumptions. As an application, we get an existence theorem of the strong saddle points of vector-valued functions. In addition, we give a characterization of vector-valued properly quasi-convex functions.

Keywords Symmetric strong vector quasi-equilibrium problems · Existence theorem · Strong saddle point · Vector-valued properly quasi-convex function

1 Introduction and preliminaries

Let X, Y be real locally convex Hausdorff topological vector spaces, let $E \subset X$, $D \subset Y$ be nonempty subsets. Let $S : E \times D \to 2^E$ and $T : E \times D \to 2^D$ be set-valued mappings and let $f, g : E \times D \to R$ be real functions. According to Noor and Oettli (1994), the symmetric quasi-equilibrium problems consists in finding $(\bar{x}, \bar{y}) \in E \times D$ such that $\bar{x} \in S(\bar{x}, \bar{y})$, $\bar{y} \in T(\bar{x}, \bar{y})$, and

$$f(\bar{x}, \bar{y}) \leqslant f(x, \bar{y}), \quad \text{for any } x \in S(\bar{x}, \bar{y}),$$
$$g(\bar{x}, \bar{y}) \leqslant g(\bar{x}, y), \quad \text{for any } y \in T(\bar{x}, \bar{y}).$$

The problem is a generalization of the equilibrium problem proposed by Blum and Oettli (1994). The special cases of the equilibrium problem include, for

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instance, optimization problems; problems of Nash equilibriums; fixed point problems; variational inequalities; and complementarity problems (see Blum and Oettli (1994)).

Fu (2003) introduced the symmetric weak vector quasi-equilibrium problems, for which he obtained an existence theorem.

Recently, Ansari et al. (1997) introduced the concept of the strong minimal solution in the vector equilibrium problem. In Chen and Hou (2000), stated that the existence of the solution for the (strong) vector variational inequalities was still an open problem. Fu (2000) discussed the existence of the solution for the strong generalized vector equilibrium problems. Tan (2004) discussed the existence of the solution for the strong vector quasi-variational inclusion problems.

In this paper, we will introduce the symmetric strong vector quasi-equilibrium problem, and give an existence theorem for the solution of this problem. As an application, we get an existence theorem of the strong saddle points of vector-valued functions.

Throughout this paper, let Z be a real topological vector space, let $C \subset Z$ be a closed convex pointed cone. Cone C induces a partially ordering in Z, defined by

$$z_1 \leqslant z_2 (\text{or } z_2 \geqslant z_1) \quad \text{if and only if } z_2 - z_1 \in C.$$

Let X, Y, E, D, S, T be as above. Let the vector-valued functions $f, g : E \times D \to Z$ be given.

The symmetric strong vector quasi-equilibrium problem (in short, SSVQEP) consists in finding $(\bar{x}, \bar{y}) \in E \times D$ such that $\bar{x} \in S(\bar{x}, \bar{y})$, $\bar{y} \in T(\bar{x}, \bar{y})$, and

$$f(\bar{x}, \bar{y}) \leqslant f(x, \bar{y}), \quad \text{for any } x \in S(\bar{x}, \bar{y}), \tag{1.1}$$

$$g(\bar{x}, \bar{y}) \leqslant g(\bar{x}, y), \quad \text{for any } y \in T(\bar{x}, \bar{y}). \tag{1.2}$$

We call this $(\bar{x}, \bar{y})$ the solution of SSVQEP.

If intC $\neq \emptyset$, and (1.1), (2.1) are replaced with

$$f(x, \bar{y}) - f(\bar{x}, \bar{y}) \notin -\operatorname{int} C, \quad \text{for any } x \in S(\bar{x}, \bar{y}),$$
$$g(\bar{x}, y) - g(\bar{x}, \bar{y}) \notin -\operatorname{int} C, \quad \text{for any } y \in T(\bar{x}, \bar{y}),$$

the problem then becomes the symmetric weak vector quasi-equilibrium problem (in short, SWVQEP); and we call this $(\bar{x}, \bar{y})$ the solution of SWVQEP (see Fu (2003)).

It is clear that if $\operatorname{int} C \neq \emptyset$, and $(\bar{x}, \bar{y})$ is a solution of SSVQEP, then $(\bar{x}, \bar{y})$ is a solution of SWVQEP.

Let F be a set-valued map from a Hausdorff topological space W to another topological space Q. We say that F is upper semicontinuous at $x_0 \in W$, if for any neighborhood $U(F(x_0))$ of $F(x_0)$, there exists a neighborhood $U(x_0)$ of x_0 such that

$$F(x) \subset U(F(x_0)), \quad \text{for all } x \in U(x_0).$$

We say that F is upper semicontinuous on W if F is upper semicontinuous at every point $x \in W$.

We say that F is lower semicontinuous at $x_0 \in W$, if for any $y_0 \in F(x_0)$ and any neighborhood $U(y_0)$ of y_0, there exists a neighborhood $U(x_0)$ of x_0 such that

$$F(x) \cap U(y_0) \neq \emptyset, \quad \text{for all } x \in U(x_0).$$

We say that F is lower semicontinuous on W if F is lower semicontinuous at every point $x \in W$.

From Aubin and Ekeland (1984), we can see that F is lower semicontinuous at $x_0 \in W$ if and only if for any $y_0 \in F(x_0)$, and any net $\{x_\alpha\}$ with $x_\alpha \to x_0$, there is a net $\{y_\alpha\}$ such that $y_\alpha \in F(x_\alpha)$ and $y_\alpha \to y_0$.

We say that F is continuous on W if F is both upper semicontinuous and lower semicontinuous on W.

The set

$$\text{graph}(F) = \{(x, y) \in W \times Q : y \in F(x)\}$$

is said to be the graph of F. We say that F is closed if graph (F) is closed.

2 The vector-valued properly quasi-convex function

In this section, we give a characterization of vector-valued properly quasi-convex functions.

Definition 2.1 (Ferro (1989)) *Let $A \subset X$ be a convex subset, $h : A \to Z$ be a vector-valued function. The function h is called properly quasi-convex if for every $x_1, x_2 \in A$ and for any $t \in [0, 1]$, we have*

either $h(tx_1 + (1-t)x_2) \leqslant h(x_1)$ or $h(tx_1 + (1-t)x_2) \leqslant h(x_2)$.

The function h is called properly quasi-concave if $-h$ is properly quasi-convex.

The concept of vector-valued properly quasi-convex function is of great importance in the study of the minimax theorem for the vector-valued function, the strong generalized vector equilibrium problems, and the strong vector quasi-variational inclusion problems (see Fu 2000; Tan 2004; Ferro 1989).

Definition 2.2 *Let $A \subset X$ be a nonempty convex set, $h : A \to Z$ be a vector-valued function. The function h is called quasi-convex if for any $z \in Z$, the set $L(z) =: \{x \in A : h(x) \leqslant z\}$ is convex. The function h is called lower semicontinuous if for any $z \in Z$, the set $L(z) =: \{x \in A : h(x) \leqslant z\}$ is closed.*

Lemma 2.1 *Assume that A is a nonempty convex set, and the vector-valued function $h : A \to Z$ is lower semicontinuous. Then h is properly quasi-convex if and only if*

(i) *for any $x_1, x_2 \in A$, there exists $t_0 \in [0, 1]$ such that*

$$h(t_0x_1 + (1-t_0)x_2) \leqslant h(x_1) \text{ and } h(t_0x_1 + (1-t_0)x_2) \leqslant h(x_2);$$

(ii) *h is quasi-convex.*

Proof If h is properly quasi-convex, then for every $x_1, x_2 \in A$, we have

$$[0, 1] = \{t \in [0, 1] : h(x(t)) \leqslant h(x_1)\} \cup \{t \in [0, 1] : h(x(t)) \leqslant h(x_2)\} \quad (2.1)$$

where $x(t) = tx_1 + (1-t)x_2$. Since h is lower semicontinuous and $[0, 1]$ is connected, the two sets in the right-hand side of (2.1) are nonempty and closed. It follows that there exists $t_0 \in [0, 1]$ such that

$$h(t_0x_1 + (1-t_0)x_2) \leqslant h(x_1) \quad \text{and} \quad h(t_0x_1 + (1-t_0)x_2) \leqslant h(x_2);$$

It is easy to see that h is quasi-convex.

Now that the conditions (i) and (ii) are satisfied, we show that h is properly quasi-convex.

By condition (i), for any $x_1, x_2 \in A$, there exists $t_0 \in [0, 1]$ such that

$$h(t_0x_1 + (1 - t_0)x_2) \leqslant h(x_1) \quad \text{and} \quad h(t_0x_1 + (1 - t_0)x_2) \leqslant h(x_2). \tag{2.2}$$

Let $x_{t_0} = t_0x_1 + (1 - t_0)x_2$. If t_0= 1, by (2.2), we have $h(x_1) \leqslant h(x_2)$.Hence $x_1, x_2 \in L(h(x_2))$. By the quasi-convexity of h, for any $t \in [0, 1]$, we have

$$tx_1 + (1 - t)x_2 \in L(h(x_2)),$$

that is

$$h(tx_1 + (1 - t)x_2) \leqslant h(x_2) \quad \text{for any } t \in [0, 1]. \tag{2.3}$$

If t_0= 0, by (2.2), we have $h(x_2) \leqslant h(x_1)$. By the quasi-convexity of h, we can get

$$h(tx_1 + (1 - t)x_2) \leqslant h(x_1), \quad \text{for any } t \in [0, 1]. \tag{2.4}$$

If $t_0 \in (0, 1)$, For any $t \in [0, 1]$, if $t \geqslant t_0$, take $\alpha = (t - t_0)/(1 - t_0) \in [0, 1]$. By $h(x_{t_0}) \leqslant h(x_1)$, $h(x_1) \leqslant h(x_1)$ and the quasi-convexity of h, we have

$$h(\alpha x_1 + (1 - \alpha)x_{t_0}) \leqslant h(x_1). \tag{2.5}$$

It is easy to see that $\alpha x_1 + (1 - \alpha)x_{t_0} = tx_1 + (1 - t)x_2$. By (2.5), we have

$$h(tx_1 + (1 - t)x_2) \leqslant h(x_1). \tag{2.6}$$

If $t \leqslant t_0$, take $\alpha = (t_0 - t)/t_0 \in [0, 1]$, by $h(x_{t_0}) \leqslant h(x_2)$, $h(x_2) \leqslant h(x_2)$ and the quasi-convexity of h, we have

$$h(\alpha x_2 + (1 - \alpha)x_{t_0}) \leqslant h(x_2). \tag{2.7}$$

It is easy to see that $\alpha x_2 + (1 - \alpha)x_{t_0} = tx_1 + (1 - t)x_2$. By (2.7), we have

$$h(tx_1 + (1 - t)x_2) \leqslant h(x_2). \tag{2.8}$$

By (2.3), (2.4), (2.6), and (2.8), for every $x_1, x_2 \in A$ and for any $t \in [0, 1]$, we have either $h(tx_1 + (1 - t)x_2) \leqslant h(x_1)$ or $h(tx_1 + (1 - t)x_2) \leqslant h(x_2)$. Thus h is properly quasi-convex.

3 Main result

In this section, we give an existence theorem for the solution of the symmetric strong vector quasi-equilibrium problem.

Theorem 3.1 *Assume that*

(i) $E \subset X, D \subset Y$ *are nonempty convex compact subsets,*$S : E \times D \to 2^E$ *and* $T : E \times D \to 2^D$ *are continuous; and for each* $(x, y) \in E \times D$, $S(x, y), T(x, y)$ *are nonempty closed convex subsets;*

(ii) $f, g : E \times D \to Z$ *are continuous;*

(iii) *For any* $y \in D$, $f(x, y)$ *is properly quasi-convex in x; for any* $x \in E$, $g(x, y)$ *is properly quasi-convex in y.*

Then SSVQEP has a solution.

Proof Define $A : E \times D \to 2^E$ and $B : E \times D \to 2^D$ by

$$A(x, y) = \{v \in S(x, y) : f(v, y) \leqslant f(u, y) \text{ for all } u \in S(x, y)\}, \quad \text{for all } (x, y) \in E \times D,$$
$$B(x, y) = \{w \in T(x, y) : g(x, w) \leqslant g(x, d) \text{ for all } d \in T(x, y)\}, \quad \text{for all } (x, y) \in E \times D.$$

(I) For any fixed $(x, y) \in E \times D$, $A(x, y)$ is nonempty.
Indeed, for every $u \in S(x, y)$, we set

$$H(u) = \{v \in S(x, y) : f(v, y) \leqslant f(u, y)\}.$$

We have $u \in H(u)$, $H(u) \neq \emptyset$. Now by induction, we show that the family $\{H(u) : u \in S(x, y)\}$ has the finite intersection property. Let $u_1, u_2 \in S(x, y)$. By assumptions (ii) and (iii), and Lemma 2.1, there exists some $t \in [0, 1]$ such that

$$f(tu_1 + (1-t)u_2, y) \leqslant f(u_1, y) \quad \text{and} \quad f(tu_1 + (1-t)u_2, y) \leqslant f(u_2, y).$$

Since $S(x, y)$ is a convex set, $v =: tu_1 + (1-t)u_2 \in S(x, y)$. Hence $v \in H(u_1) \cap H(u_2)$. Let $u_1, \ldots, u_n \in S(x, y)$, and $\bigcap_{i=1}^{n} H(u_i) \neq \emptyset$. Then there exists $v \in S(x, y)$ such that

$$f(v, y) \leqslant f(u_i, y), \quad i = 1, \ldots, n. \tag{3.1}$$

Let $u_{n+1} \in S(x, y)$. By assumptions (ii) and (iii), and Lemma 2.1, there exists some $t \in [0, 1]$ such that

$$f(tv + (1-t)u_{n+1}, y) \leqslant f(u_{n+1}, y) \quad \text{and} \quad f(tv + (1-t)u_{n+1}, y) \leqslant f(v, y). \tag{3.2}$$

From (3.1) and (3.2), we can get

$$f(tv + (1-t)u_{n+1}, y) \leqslant f(u_i, y), \quad i = 1, \ldots, n+1.$$

Since $S(x, y)$ is a convex set, $tv + (1-t)u_{n+1} \in S(x, y)$. Thus $tv + (1-t)u_{n+1} \in \bigcap_{i=1}^{n+1} H(u_i)$. Since $S(x, y)$ is a closed subset of *E*, and *E* is a compact set, $S(x, y)$ is a compact subset of *E*. By the continuity of *f*, we can see that *H*(*u*) is closed for every $u \in S(x, y)$.This follows that

$$\cap\{H(u) : u \in S(x, y)\} \neq \emptyset.$$

Hence there exists $v \in \cap\{H(u) : u \in S(x, y)\}$. This means that $v \in S(x, y)$ and

$$f(v, y) \leqslant f(u, y), \quad \text{for all} \quad u \in S(x, y).$$

Thus $A(x, y) \neq \emptyset$.

(II) For any fixed $(x, y) \in E \times D$, $A(x, y)$ is a closed subset of E.

In fact, let a net $\{v_\alpha : \alpha \in I\} \subset A(x, y)$, $v_\alpha \to v \in E$. Since $v_\alpha \in S(x, y)$ and $S(x, y)$ is closed, $v \in S(x, y)$. It follows from $f(v_\alpha, y) \leqslant f(u, y)$, for all $u \in S(x, y)$ and the continuity of f that

$$f(v, y) \leqslant f(u, y), \quad \text{for all} \quad u \in S(x, y).$$

Thus $v \in A(x, y)$.

(III) For any fixed $(x, y) \in E \times D$, $A(x, y)$ is a convex subset of E.

Indeed, let $v_1, v_2 \in A(x, y)$. Then $v_1, v_2 \in S(x, y)$ and

$$f(v_i, y) \leqslant f(u, y), \quad \text{for all } u \in S(x, y), \ i = 1, 2.$$

Since $f(\cdot, y)$ is quasi-convex, $L(f(u, y))$ is a convex set. Thus for any $u \in S(x, y)$,

$$tv_1 + (1-t)v_2 \in L(f(u, y)) = \{v \in E : f(v, y) \leqslant f(u, y)\} \quad \text{for all } t \in [0, 1].$$

Hence

$$f(tv_1 + (1-t)v_2, y) \leqslant f(u, y), \quad \text{for all } u \in S(x, y).$$

Since $S(x, y)$ is a convex set,

$$tv_1 + (1-t)v_2 \in S(x, y), \quad \text{for all } t \in [0, 1].$$

We have $tv_1 + (1-t)v_2 \in A(x, y)$. Hence $A(x, y)$ is convex.

(IV) $A(x, y)$ is upper semicontinuous on $E \times D$,.

Since E is a compact set, we need only to show that A is a closed mapping (see Aubin and Ekeland (1984)). Let a net $\{(x_\alpha, y_\alpha) : \alpha \in I\} \subset E \times D$ converging to $(x, y) \in E \times D$. Let $v_\alpha \in A(x_\alpha, y_\alpha)$ and $v_\alpha \to v$. We will show that $v \in A(x, y)$. Since S is an upper semicontinuous set-valued map and for each $(x, y) \in E \times D$, $S(x, y)$ is a closed set, S is a closed set-valued map (see Aubin and Ekeland (1984)). It follows from $(x_\alpha, y_\alpha) \to (x, y)$, $v_\alpha \in S(x_\alpha, y_\alpha)$, and $v_\alpha \to v$ that $v \in S(x, y)$. Since S is lower semicontinuous on $E \times D$, for any $u \in S(x, y)$, there exists a net $\{u_\alpha\}$ with $u_\alpha \in S(x_\alpha, y_\alpha)$ such that $u_\alpha \to u$. Since $v_\alpha \in A(x_\alpha, y_\alpha)$, we have

$$f(v_\alpha, y_\alpha) \leqslant f(u_\alpha, y_\alpha).$$

It follows from continuity of f that

$$f(v, y) \leqslant f(u, y).$$

Thus

$$f(v, y) \leqslant f(u, y), \quad \text{for all } u \in S(x, y),$$

hence $v \in A(x, y)$.

(V) Similarly, for each $(x, y) \in E \times D$, $B(x, y)$ is a nonempty convex closed subset of D, and B is upper semicontinuous on $E \times D$.

(VI) Define $F : E \times D \to 2^{E\times D}$ by

$$F(x, y) = (A(x, y), B(x, y)), \quad \text{for all } (x, y) \in E \times D.$$

Then for each $(x, y) \in E \times D$, $F(x, y)$ is a nonempty convex closed subset of $E \times D$, and F is upper semicontinuous on $E \times D$. By Kakutani-Fan-Glicksberg fixed point theorem (see Holmes 1975, p. 186), there is a point $(\bar{x}, \bar{y}) \in E \times D$ such that $(\bar{x}, \bar{y}) \in F(\bar{x}, \bar{y})$, i.e. $\bar{x} \in A(\bar{x}, \bar{y})$, $\bar{y} \in B(\bar{x}, \bar{y})$. By the definition of A and B, we have $\bar{x} \in S(\bar{x}, \bar{y})$, $\bar{y} \in T(\bar{x}, \bar{y})$, and

$$f(\bar{x}, \bar{y}) \leqslant f(x, \bar{y}), \quad \text{for any } x \in S(\bar{x}, \bar{y}),$$
$$g(\bar{x}, \bar{y}) \leqslant g(\bar{x}, y), \quad \text{for any } y \in T(\bar{x}, \bar{y}).$$

Thus $(\bar{x}, \bar{y})$ is a solution of SSVQEP. The proof is completed.

Corollary 3.1 *Assume that*

(i) *$E \subset X, D \subset Y$ are nonempty convex compact subsets,$S : D \to 2^E$ and $T : E \to 2^D$ are continuous; and for each $y \in D, x \in E$, $S(y)$ and $T(x)$ are nonempty closed convex subsets;*
(ii) *$f, g : E \times D \to Z$ are continuous;*
(iii) *For any fixed $y \in D$, $f(x, y)$ is properly quasiconvex in x; for any fixed $x \in E$, $g(x, y)$ is properly quasiconvex in y.*

Then there exists a point $(\bar{x}, \bar{y}) \in E \times D$ such that $\bar{x} \in S(\bar{y}), \bar{y} \in T(\bar{x})$, and

$$f(\bar{x}, \bar{y}) \leqslant f(x, \bar{y}), \quad \textit{for any } x \in S(\bar{y}),$$
$$g(\bar{x}, \bar{y}) \leqslant g(\bar{x}, y), \quad \textit{for any } y \in T(\bar{x}).$$

If int $C \neq \emptyset$*, we can get the following corollary.*

Corollary 3.2 (Fu 2003) *Assume that*

(i) *$E \subset X, D \subset Y$ are nonempty convex compact subsets,$S : E \times D \to 2^E$ and $T : E \times D \to 2^D$ are continuous; and for each $(x, y) \in E \times D$, $S(x, y), T(x, y)$ are nonempty closed convex subsets;*
(ii) *$f, g : E \times D \to Z$ are continuous;*
(iii) *For any $y \in D$, $f(x, y)$ is properly quasiconvex in x; for any $x \in E$, $g(x, y)$ is properly quasiconvex in y.*
Then SWVQEP has a solution.

By comparing Theorem 3.1 and Corollary 3.2, we can see that under the same conditions, our result is better than the main result in Fu (2003).

4 Application

In this section, we will apply Theorem 3.1 to get the existence theorem of strong saddle points of the vector-valued functions.

Definition 4.1 *Let $E \subset X$, $D \subset Y$ be nonempty subset. Let $f : E \times D \to Z$ be a vector-valued function. A point $(\bar{x}, \bar{y}) \in E \times D$ is called a strong saddle point of f in $E \times D$ if*

$$f(\bar{x}, y) \leqslant f(\bar{x}, \bar{y}) \leqslant f(x, \bar{y}), \quad \textit{for all } (x, y) \in E \times D.$$

Definition 4.2 *Let $E \subset X, D \subset Y$ be nonempty subset. Let $S : E \times D \to 2^E, T : E \times D \to 2^D$, and $f : E \times D \to Z$ be a vector-valued function. A point $(\bar{x}, \bar{y}) \in E \times D$ is called a strong saddle point of f in $E \times D$ with constraints if $\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$ and*

$$f(\bar{x}, y) \leqslant f(\bar{x}, \bar{y}) \leqslant f(x, \bar{y}), \quad \text{for any } x \in S(\bar{x}, \bar{y}) \text{ and } y \in T(\bar{x}, \bar{y}).$$

Using the concepts of weakly minimal (weakly maximal) points and minimal (maximal) points in multiobjective optimization, some authors obtained the existence theorems of the saddle point for vector-valued functions (see Tanaka 1988, 1994; Shi and Ling 1995; Luc and Vargas 1992; Tan et al. 1996). It is clear that a strong saddle point of f is an ideal saddle point; it is better than other saddle points. Up to now, no paper deals with strong saddle point problem.

Theorem 4.1 *Assume that*

(i) *$E \subset X, D \subset Y$ are nonempty convex compact subsets, $S : E \times D \to 2^E$ and $T : E \times D \to 2^D$ are continuous; and for each $(x, y) \in E \times D$, $S(x, y), T(x, y)$ are nonempty closed convex subsets;*
(ii) *$f : E \times D \to Z$ is continuous;*
(iii) *For any $y \in D$, $f(x, y)$ is properly quasi-convex in x;*
(iv) *For any $x \in E$, $f(x, y)$ is properly quasi-concave in y.*

Then there exists a point $(\bar{x}, \bar{y}) \in E \times D$ such that $\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$, and

$$f(\bar{x}, y) \leqslant f(\bar{x}, \bar{y}) \leqslant f(x, \bar{y}) \quad \text{for any } x \in S(\bar{x}, \bar{y}) \text{ and } y \in T(\bar{x}, \bar{y}).$$

Proof In Theorem 3.1, let $g(x, y) = -f(x, y)$, $(x, y) \in E \times D$. For any $x \in E$, since $f(x, y)$ is properly quasi-concave in y, $g(x, y) = -f(x, y)$ is properly quasi-convex in y. In view of Theorem 3.1, there exists a point $(\bar{x}, \bar{y}) \in E \times D$ such that $\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$, and

$$f(\bar{x}, \bar{y}) \leqslant f(x, \bar{y}), \quad \text{for any } x \in S(\bar{x}, \bar{y}),$$

and

$$g(\bar{x}, \bar{y}) \leqslant g(\bar{x}, y), \quad \text{for any } y \in T(\bar{x}, \bar{y}).$$

It follows that

$$f(\bar{x}, y) \leqslant f(\bar{x}, \bar{y}) \leqslant f(x, \bar{y}), \quad \text{for any } x \in S(\bar{x}, \bar{y}) \text{ and } y \in T(\bar{x}, \bar{y}).$$

Corollary 4.1 *Assume that*

(i) *$E \subset X, D \subset Y$ are nonempty convex compact subsets.*
(ii) *$f : E \times D \to Z$ is continuous;*
(iii) *For any $y \in D$, $f(x, y)$ is properly quasi-convex in x;*
(iv) *For any $x \in E$, $f(x, y)$ is properly quasi-concave in y.*

Then there exists a point $(\bar{x}, \bar{y}) \in E \times D$ such that

$$f(\bar{x}, y) \leqslant f(\bar{x}, \bar{y}) \leqslant f(x, \bar{y}), \quad \text{for any } (x, y) \in E \times D.$$

Proof In Theorem 4.1, let $S(x, y) = E$ for all $(x, y) \in E \times D$, and $T(x, y) = D$ for all $(x, y) \in E \times D$. It is clear that $S : E \times D \to 2^E$ and $T : E \times D \to 2^D$ are continuous; and for each $(x, y) \in E \times D$, $S(x, y)$, $T(x, y)$ are nonempty closed convex subsets. Then Theorem 4.1 yields the conclusion. □

Example 4.1 Let $X = Y = R$, $E = D = [-1, 1] \subset R$, $Z = R^2$,
$C = R^2_+ = \{(x, y) \in R^2 : x \geqslant 0, y \geqslant 0\}$. Define $f : [-1, 1] \times [-1, 1] \to R^2$ by

$$f(x, y) = (y - x, (y - x)^3), (x, y) \in [-1, 1] \times [-1, 1].$$

Then (i) f is continuous on $[-1, 1] \times [-1, 1]$.

(ii) For any $y \in [-1, 1]$, $f(x, y)$ is properly quasi-convex in x. In fact, for every $x_1, x_2 \in [-1, 1]$ and for any $t \in [0, 1]$, let $x_1 < x_2$. We have

$$\begin{aligned} f(tx_1 + (1-t)x_2, y) &= (y - (tx_1 + (1-t)x_2), (y - \big(tx_1 + (1-t)x_2)^3\big) \\ &\leqslant (y - x_1, (y - x_1)^3) = f(x_1, y) \end{aligned}$$

since $h(t) = t^3$ is a monotone increasing function on $(-\infty, +\infty)$.

(iii) For any $x \in [-1, 1]$, $f(x, y)$ is properly quasi-concave in y. In fact, for every $y_1, y_2 \in [-1, 1]$ and for any $t \in [0, 1]$, let $y_1 < y_2$. We have

$$\begin{aligned} f(x, ty_1 + (1-t)y_2) &= \big(ty_1 + (1-t)y_2 - x, (ty_1 + (1-t)y_2 - x)^3\big) \\ &\geqslant \big(y_1 - x, (y_1 - x)^3\big) = f(x, y_1). \end{aligned}$$

By Corollary 4.1, there exists a point $(\bar{x}, \bar{y}) \in [-1, 1] \times [-1, 1]$ such that

$$f(\bar{x}, y) \leqslant f(\bar{x}, \bar{y}) \leqslant f(x, \bar{y}), \quad \text{for any } (x, y) \in [-1, 1] \times [-1, 1].$$

We know that point (1,1)$\in [-1, 1] \times [-1, 1]$ is a strong saddle point of f in $[-1, 1] \times [-1, 1]$.

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