Math. Meth. Oper. Res. (2006) 64: 521–540
DOI 10.1007/s00186-006-0079-z

ORIGINAL ARTICLE

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Lagrangian conditions for vector optimization in Banach spaces

Received: 15 July 2005 / Accepted: 17 December 2005 / Published online: 6 September 2006

Abstract We consider vector optimization problems on Banach spaces without convexity assumptions. Under the assumption that the objective function is locally Lipschitz we derive Lagrangian necessary conditions on the basis of Mordukhovich subdifferential and the approximate subdifferential by Ioffe using a non-convex scalarization scheme. Finally, we apply the results for deriving necessary conditions for weakly efficient solutions of non-convex location problems.

Keywords Non-convex vector optimization · Lagrangian conditions · Mordukhovich subdifferential · Ioffe subdifferential

1 Introduction

In this paper we will be mainly concerned with the following vector minimization problem (VP) given as

$$V - \min\ F(x), \quad \text{subject to} \quad x \in C,$$

where X and Y are Banach spaces, $K \subset Y$ a closed, convex and pointed cone which induces a partial order on Y, $F : X \to Y$ and $C \subseteq X$. In order to describe solution concepts for the vector optimization problem (VP) we use the following notations: Let us consider $A \subseteq Y$.

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- A point $y_0 \in A$ is said to be a *minimal point* of A, if there exists no other point $y \in A$ such that $y - y_0 \in -(K \setminus \{0\})$. We denote this by $y_0 \in \mathrm{Eff}(A, K)$, where $\mathrm{Eff}(A, K)$ denotes the set of minimal points of A with respect to the ordering cone K. A point $x_0 \in C$ is called an *efficient point* of (VP), if $F(x_0) \in \mathrm{Eff}(F(C), K)$.
- A point $y_0 \in A$ is said to be a *weakly minimal element*, if $\mathrm{int}\, K \neq \emptyset$ and there exists no $y \in A$ such that $y - y_0 \in -\mathrm{int}\, K$. This is denoted by $y_0 \in w - \mathrm{Eff}(A, K)$. A point $x_0 \in C$ is a *weakly efficient* point of (VP), if $F(x_0) \in w - \mathrm{Eff}(F(C), K)$.
- Consider a closed convex and pointed cone D with non-empty interior such that $K \setminus \{0\} \subset \mathrm{int}\, D$. A point $y_0 \in A \subseteq Y$ is said to be *properly minimal*, if $y_0 \in \mathrm{Eff}(A, D)$. A point $x_0 \in C$ is called a *properly efficient* point for (VP), if $F(x_0) \in \mathrm{Eff}(F(C), D)$.

It is however important to note that in the above definition of minimal points one may just assume K to be a non-empty set in Y rather than a cone. Such general definitions are provided in Gerth(Tammer) and Weidner (1990) and we refer the reader to Gerth(Tammer) and Weidner (1990) for more details on this issue.

There are many papers where Lagrangian multiplier rules are shown for vector optimization problems under differentiability or convexity (or convexity-like) assumptions (see Jahn 1986, 2004). Necessary conditions for weakly minimal elements in non-differentiable vector optimization where the objective function takes its values in finite dimensional spaces are derived by Clarke (1983),Minami (1983), Mordukhovich (1985), Craven (1989) and Miettinen (1999), Miettinen and Mäkelä (2000). Lagrangian multiplier rules for weakly minimal elements of vector optimization problems on Banach spaces are presented by El Abdouni and Thibault (1992) and by Amahroq and Taa (1997).

In this article we are interested in developing the Lagrangian necessary conditions for the vector program (VP) and also try to demonstrate how Lagrangian multipliers can be interpreted as subgradients of some convex scalarizing functions. In the Lagrangian theory of vector optimization one of the most important approaches is by scalarization where the vector optimization problem is converted into an equivalent scalar optimization problem and then the usual techniques of scalar optimization is applied. For example let us consider the case where the function F is a K-convex function and C a convex subset of X and let x_0 be a weakly efficient point for (VP). Then by using the standard separation theorem for convex sets it is easy to show that there exists $0 \neq y^* \in K^*$ such that x_0 also solves the following scalar optimization problem (SP)

$$\min \langle y^*, F(x) \rangle, \quad \text{subject to} \quad x \in C.$$

The converse of this fact is also true. But when the function F is not convex such a nice scalarization through bounded linear operators is not possible since the convex separation results are no longer available. However, if $Y = R^l$ and $K = R^l_+$ then a scalarization using max function can be developed for the case of a weakly efficient point while a Chankong and Haimes (1983) type scalarization can be used for an efficient point. But if Y is an infinite dimensional Banach space such scalarizations cannot be used anymore. However, it is interesting to note that Gerth(Tammer) and Weidner (1990) have developed some non-linear scalarization schemes by developing non-convex separation theorems in linear topological spaces. In this article we precisely aim to develop Lagrangian multiplier rules for (VP) using these non-convex scalarization schemes. Moreover, we will also demonstrate that the Lagrangian multipliers are in fact subgradients of certain convex functions that are generated by these non-convex scalarization schemes. Furthermore, the Lagrangian conditions that we develop here are indeed very general by their very nature.

Throughout this paper X and Y will denote Banach Spaces and X^* and Y^* denotes the respective dual spaces. Let $\|.\|_X$ and $\|.\|_Y$ denote the norms in X and Y, respectively. However, one can drop the subscript denoting the space if there is no confusion. Let us consider a closed, convex and pointed cone K in Y which induces a partial order on Y. Thus $x \leq_K y$ iff $y - x \in K$ for $x, y \in Y$. If K has a non-empty interior that is $\text{int}\, K \neq \emptyset$ then $x <_K y$ iff $y - x \in \text{int}\, K$. A functional $z : Y \to \mathbb{R}$ is said to be K-monotone (increasing) if $y - x \in K$ implies that $z(y) \geq z(x)$. The functional z is called strictly K-monotone (increasing) if $y - x \in K \backslash \{0\}$ implies that $z(y) > z(x)$. Let us mention in the beginning that throughout the article by the term K-monotone we will mean monotone in the increasing sense. We will also need the notion of the dual cone to K denoted as K^* and given by

$$K^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \forall y \in K\}.$$

2 Preliminaries

Throughout this article we will be mainly concerned with the case where the function F is strictly differentiable and also the case where F is locally Lipschitz (cf. Clarke 1983). When F is locally Lipschitz, we will consider the case when Y is finite dimensional and also the case when Y is infinite dimensional. When Y is finite dimensional our main tool to express the optimality conditions is the Clarke subdifferential and the subdifferential by Mordukhovich where as when Y is infinite dimensional we will see that our main tool will be the approximate subdifferential of Ioffe.

A comprehensive theory of basic/limiting normals and subgradients is presented in the book by Mordukhovich (2005).

In what follows we shall represent by $F'(x)$ the strict derivative of F at x. Given any linear map $T : X \to Y$ we denote by T^* its adjoint map. Moreover, for any set B we denote by $\text{co}B$ the convex hull of B and by $\text{cl}B$ we mean the closure of the set B. We denote by $\langle ., .\rangle$ the usual duality pairing between X and X^*. For any function $F : X \to Y$ and $y^* \in Y^*$ we evaluate the function $\langle y^*, F\rangle : X \to R$ as $\langle y^*, F\rangle(x) = \langle y^*, F(x)\rangle$. For any set B in X^* we denote by $\overline{\text{co}}^* B$ the weak-star closed convex hull of B. Moreover, for any set $C \subseteq X$ we denote the Bouligand tangent cone or the contingent cone to C at $x_0 \in C$ as $T(C, x_0)$. The Bouligand tangent cone is closed though not necessarily convex. For more details on Bouligand tangent cones see for example Rockafellar and Wets (1998).

For a given locally Lipschitz function $f : X \to \mathbb{R}$ we denote the Clarke generalized directional derivative at the point x and in the direction v by $f^\circ(x, v)$ and the Clarke subdifferential (see Clarke 1983) of f at x by $\partial^\circ f(x)$, furthermore we denote by $T_c(C, x_0)$ and $N_c(C, x_0)$ the Clarke tangent cone and the Clarke normal cone to the set C at x_0, respectively (see Clarke 1983). It is important to note that the Clarke tangent cone is always a closed and convex set. If $f : X \to \mathbb{R}$ be a convex and continuous function then we will denote the subdifferential of f at x as $\partial f(x)$. An extended-valued function $f : X \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ is said to be proper if $f(x) > -\infty$ and the set $domf = \{x \in X : f(x) < +\infty\} \neq \emptyset$.

The main tool in the proofs is the application of the following non-convex scalarization scheme.

Lemma 2.1 (Gerth(Tammer) and Weidner 1990) *Let $K \subset Y$ be a closed convex cone with a non-empty interior and let A be a subset of Y with non-empty interior. We have $y_0 \in w - \text{Eff}(A, K)$ if and only if $y_0 \in A$ and there exists a continuous convex functional on Y which is strictly-$(\text{int } K)$-monotone with the range $(-\infty, +\infty)$ and*

(i) $z(y_0) = 0$.
(ii) $z(A) \geq 0$.
(iii) $z(\text{int } A) > 0$.
(iv) $z(y_0 - K) \leq 0$.
(v) $z(y_0 - \text{int } K) < 0$.
(vi) $z(y_0 - bdK) = 0$.

If $y_0 = 0$ then one may choose z to be sublinear.

For our results we mainly require (i) and (ii) in Lemma 2.1. However a close look at the proof of Lemma 2.1 in Gerth(Tammer) and Weidner (1990) will reveal that the assumption $\text{int } A \neq \emptyset$ is not essential to prove (i) and (ii) in Lemma 2.1. Therefore we summarize our requirement in the form of the following Lemma.

Lemma 2.2 *Let $K \subset Y$ be a closed convex cone with a non-empty interior and let A be subset of Y. We have $y_0 \in w - \text{Eff}(A, K)$ if and only if $y_0 \in A$ and there exists a continuous convex functional on Y which is strictly-$(\text{int } K)$-monotone with the range $(-\infty, +\infty)$ and*

(i) $z(y_0) = 0$.
(ii) $z(A) \geq 0$.

Remark 2.1 Corresponding results can be shown for properly minimal elements: Let K be a closed convex cone with a non-empty interior, $D \subset Y$ a closed convex and pointed cone with non-empty interior such that $K \setminus \{0\} \subset \text{int } D$ and let A be subset of Y. We have $y_0 \in \text{Eff}(A, D)$ if and only if $y_0 \in A$ and there exists a continuous convex functional on Y which is strictly-K-monotone with the range $(-\infty, +\infty)$ and

(i) $z(y_0) = 0$.
(ii) $z(A) \geq 0$.

Definition 2.1 *For a set-valued map $F : X \to X^*$ we denote by*

$$\limsup_{x \to x_0} F(x)$$

the sequential Kuratowski-Painlevé upper limit with respect to the norm topology on X and weak star topology on X^*, which is given as

$$\limsup_{x \to x_0} F(x) = \{x^* \in X^* : \exists \text{ sequences } x_k \to x_0, \text{ and } x_k^* \xrightarrow{w^*} x^*, \text{ with } x_k^* \in F(x_k), \forall k = 1, 2, \ldots\},$$

where $\xrightarrow{w^*}$ denotes convergence in the weak-star topology of X^*.

Definition 2.2 (Mordukhovich and Shao 1996) *Let S be a non-empty subset of X and let $\varepsilon \geq 0$. Given $x \in clS$ the non-empty set*

$$N_\varepsilon^F(S, x) = \left\{x^* \in X^* : \limsup_{y \to x, y \in S} \frac{\langle x^*, y - x\rangle}{\|y - x\|} \leq \varepsilon\right\}$$

is called the set of Fréchet ε-normals to S at x. When $\varepsilon = 0$, then the above set is a cone, called the set of Fréchet normals and denoted by $N^F(S, x)$.
Let $x_0 \in cl\ S$. The non-empty cone

$$N_L(S, x_0) = \limsup_{x \to x_0, \varepsilon \downarrow 0} N_\varepsilon^F(S, x)$$

is called the limiting normal cone or the Mordukhovich normal cone to S at x_0.

It is important to note that the set of Frechet ε-normals is a convex set for every $\varepsilon \geq 0$ but the limiting normal cone is in general non-convex. For more details on limiting normals see for example Mordukhovich and Shao (1996), for a treatment of limiting normals in finite dimensional spaces see for example Mordukhovich (1994) and Rockafellar and Wets (1998). When S is a convex set, then the limiting normal cone reduces to the standard normal cone of convex analysis which has been defined earlier. Moreover, if X is an Asplund space, then we have

$$N_L(S, x_0) = \limsup_{x \to x_0} N^F(S, x).$$

Further, in an Asplund space one also has

$$N_c(S, x_0) = \overline{\text{co}}^* N_L(S, x_0). \tag{1}$$

Definition 2.3 *Let $f : X \to \overline{\mathbb{R}}$ be a given proper function and $x_0 \in dom f$. The set*

$$\partial_L f(x_0) = \{x^* \in X^* : (x^*, -1) \in N_L(epi f, (x_0, f(x_0)))\}$$

is called the limiting subdifferential or the Mordukhovich subdifferential of f at x_0. If $x_0 \notin dom f$, then we set $\partial_L f(x_0) = \emptyset$.

Definition 2.4 *Let $f : X \to \overline{\mathbb{R}}$ be a given proper function and $x \in dom f$. The following set*

$$\partial_\varepsilon^F f(x) = \left\{x^* \in X^* : \liminf_{u \to x} \frac{f(u) - f(x) - \langle x^*, u - x\rangle}{\|u - x\|} \geq -\varepsilon\right\}$$

is called the Frechet ε-subdifferential of f at x. If $\varepsilon = 0$, then we denote the above set by $\partial^F(x)$ and is known as the Frechet subdifferential of f at x.

If f is lower-semicontinuous around x_0, then it has been shown in Mordukhovich and Shao (1996) that

$$\partial_L f(x_0) = \limsup_{x \xrightarrow{f} x_0, \varepsilon \downarrow 0} \partial_\varepsilon^F f(x),$$

where $x \xrightarrow{f} x_0$ means that $x \to x_0$ with $f(x) \to f(x_0)$.

If f is a continuous convex function, then the limiting subdifferential coincides with the usual subdifferential $\partial f(x)$ of a convex function. If X is an Asplund space, then one has

$$\partial_L f(x_0) = \limsup_{x \xrightarrow{f} x_0} \partial^F f(x).$$

Moreover, if X is an Asplund space and $f : X \to R$ be locally Lipschitz around x_0 then we have

$$\partial^\circ f(x_0) = \overline{\text{co}}^* \partial_L f(x_0). \tag{2}$$

We need the following calculus rules from Mordukhovich and Shao (1996) for proving one of our main results.

Lemma 2.3 *Let X be an Asplund space and let $x_0 \in X$. Let $f_i : X \to \bar{\mathbb{R}}$, $i = 1, 2$ be proper lower-semicontinuous functions and one of these is Lipschitz near x_0 (i.e., locally Lipschitz at x_0). Then one has*

$$\partial_L(f_1 + f_2)(x_0) \subseteq \partial_L f_1(x_0) + \partial_L f_2(x_0).$$

Lemma 2.4 *Let X be an Asplund space. Let $F : X \to \mathbb{R}^m$ be locally Lipschitz at x_0. Let $\phi : \mathbb{R}^m \to \mathbb{R}$ be Lipschitz around $F(x_0)$. Then one has*

$$\partial_L(\phi \circ F)(x_0) \subseteq \bigcup_{y^* \in \partial_L \phi(F(x_0))} \partial_L \langle y^*, F \rangle (x_0).$$

It is important to note that the notion of the limiting subdifferential was first introduced by Mordukhovich (1976) in context of optimal control. For detailed theory of the limiting subdifferential along with calculus rules and applications see the comprehensive two volume book of Mordukhovich (2005).

As we have observed above, the limiting subdifferential admits a very good calculus in Asplund spaces. In order to represent optimality conditions in arbitrary Banach spaces one needs to consider the approximate subdifferential by Ioffe (1986, 1989, 2000). For an arbitrary function $f : X \to \bar{\mathbb{R}}$, where X is a Banach space, such subdifferentials are constructed via the Dini-Hadamard subdifferential (for more details see for example Ioffe 1989). Since in this article we will need to work with locally Lipschitz function and lower semicontinuous function we will use the approach in Ioffe (2000) by first defining the approximate subdifferential for the locally Lipschitz case and then use this idea to define the approximate normal cone and then use the normal cone to define the approximate subdifferential for a lower semicontinuous function.

Let $f : X \to \mathbb{R}$ be a locally Lipschitz function. Then the *lower Dini directional derivative* at $x \in X$ and in the direction $h \in X$ is given as

$$f^-(x,h) = \liminf_{\lambda \downarrow 0} \frac{f(x+\lambda h) - f(x)}{\lambda}.$$

The *lower Dini subdifferential* of f at x is given as

$$\partial^- f(x) = \{x^* \in X^* : f^-(x,h) \geq \langle x^*, h\rangle, \forall h \in X\}.$$

Let $\mathcal{L}$ be a closed subspace of X. We set

$$\partial^-_{\mathcal{L}} f(x) = \{x^* \in X^* : f^-(x,h) \geq \langle x^*, h\rangle, \forall h \in \mathcal{L}\}.$$

Let $\mathcal{F}$ denote the collection of finite dimensional subspaces of X. Then the approximate subdifferential $\partial_a f(x)$ of a locally Lipschitz function f at $x \in X$ is given as

$$\partial_a f(x) = \bigcap_{\mathcal{L} \in \mathcal{F}} \limsup_{u \to x} \partial^-_{\mathcal{L}} f(u).$$

For a locally Lipschitz function it holds $\partial_a f(x) \neq \emptyset$ for all $x \in X$. For any $\alpha > 0$ it is thus clear that $\partial_a f(\alpha x) \subseteq \alpha \partial_a f(x)$. The normal cone to a closed set C at a point $x \in C$ is given as

$$N_a(C,x) = \bigcup_{\lambda \geq 0} \lambda \partial_a d_C(x). \tag{3}$$

Here d_C represents the distance function associated with the set C and it is well known that the distance function d_C is Lipschitz with rank one.
Let $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function. Then the approximate subdifferential $\partial_a f(x)$ of f at x is given as

$$\partial_a f(x) = \{x^* \in X^* : (x^*, -1) \in N_a(\text{epi} f, (x, f(x)))\}.$$

If f is locally Lipschitz, then the above definition of the approximate subdifferential coincides with the definition of approximate subdifferential for locally Lipschitz function given earlier. One can also show that if C is a closed set then

$$N_a(C,x) = \partial_a \delta_C(x), \tag{4}$$

where the δ_C represents the indicator function of the set C (see Ioffe 2000).
The asymptotic subdifferential associated with a proper lower semicontinuous function f at x is given as

$$\partial_a^\infty f(x) = \{x^* \in X^* : (x^*, 0) \in N_a(\text{epi} f, (x, f(x)))\}.$$

If f is a locally Lipschitz, then we have $\partial_a^\infty f(x) = \{0\}$. Further, if x_0 is a local minimum for a lower-semicontinuous function $f : X \to \mathbb{R}$, then $0 \in \partial_a f(x_0)$.
Let X and Y be Banach spaces and let $\Phi : X \to Y$ be a set-valued map. The set-valued map $D_a^*\Phi(x,y) : Y^* \to X^*$ is defined as the co-derivative of Φ at the point (x,y) and is given as

$$D_a^*\Phi(x,y)(y^*) = \{x^* \in X; (x^*, -y^*) \in N_a(\text{gph}\Phi, (x,y))\},$$

where gphΦ denote the graph of the set-valued map Φ. If Φ is single-valued then we write the co-derivative as $D_a^* \Phi(x)(y^*)$. We present the following calculus rules which we shall use in the sequel. For more details, see Ioffe (1989).

The approximate normal cone and the approximate subdifferential for a proper lower-semicontinuous function presented above is termed as the G-nucleus of the G-normal cone and the G-nucleus of the G-subdifferential in Ioffe (1989). For details on the G-normal cone and the G-subdifferential, see Ioffe (1989). Ioffe (1989) has also introduced the notion of an A-subdifferential for an arbitary function by slightly modifying the definition of the approximate subdifferential of a locally Lipschitz function given here. However, for a locally Lipschitz function the A-subdifferential, the G-subdifferential and G-nucleus all coincide. Further, if we consider X to be a weakly compactly generated (WCG) Asplund space, then the approximate subdifferential and the limiting subdifferential for a locally Lipschitz function coincide (see Theorem 9.2 in Mordukhovich and Shao 1996). Moreover, the limiting subdifferential and the approximate subdifferential coincide in finite dimensional spaces.

Now, using Theorem 7.4 in Ioffe (1989) we have the following result:

Lemma 2.5 *Let X be a Banach space and let $f, g : X \to \overline{\mathbb{R}}$. Let f be locally Lipschitz and g be a proper lower-semicontiuous function. Then*

$$\partial_a (f + g)(x) \subset \partial_a f(x) + \partial_a g(x).$$

In the following we present the chain rules for approximate subdifferentials which will lead us to derive optimality conditions for a vector minimization problem in Banach spaces. Now, we consider the function $f = g \circ F$ where g and F are locally Lipschitz. One needs certain conditions on F in order to get chain rules for the approximate subdifferential. One of the conditions is that F has a strict prederivative with norm compact values (see Ioffe 1989). The other one is that F is strongly compactly Lipschitzian (see Jourani and Thibault 1993). For simplicity in the representation we shall split the results in the form of two lemmas. In the first one we will show that without any additional assumption on F one can relate the approximate subdifferential of $g \circ F$ to the approximate coderivative of F. We now provide the definition of the prederivative of F from Ioffe (1989) and the definition of a strongly compactly Lipschitzian map from Jourani and Thibault (1993).

Definition 2.5 *Consider $F : X \to Y$, where X and Y are Banach spaces. Then the set-valued map $R : X \to Y$ is called a strict prederivative of F at x if R is positively homgeneous (i.e., $R(\lambda x) = \lambda R(x), \quad \lambda > 0$), $0 \in R(0)$ and*

$$F(u + h) - F(u) \in R(h) + r(u, h)\|h\| B_Y,$$

where $r(u, h) \to 0$ as $u \to x$ and $h \to 0$. Here B_Y denotes the unit ball around the origin in Y.

Definition 2.6 *A mapping $F : X \to Y$ is said to be strongly compactly Lipschitzian at $x_0 \in X$, if there exist a multifunction $R : X \to Comp(Y)$, where $Comp(Y)$ denotes the set of all norm compact subsets of Y, and a function $r : X \times X \to \mathbb{R}_+$ satisfying*

(i) $\lim_{x \to x_0, v \to 0} r(x, v) = 0$;

(ii) there exists $\alpha > 0$ *such that*

$$t^{-1}[F(x + tv) - F(x)] \in R(v) + ||v||\, r(x,t)\, \mathcal{B}_Y$$

for all $x \in x_0 + \alpha\mathcal{B}_X$, $v \in \alpha\, \mathcal{B}_X$ *and* $t \in (0, \alpha)$ *(here* $\mathcal{B}_Y$ *denotes the closed unit ball around the origin of* Y*);*

(iii) $R(0) = \{0\}$ *and* R *is upper semicontinuous.*

Remark 2.2 Any strongly compactly Lipschitzian mapping F at x_0 is locally Lipschitz near x_0.

If Y is finite dimensional, then F is strongly compactly Lipschitzian at x_0 if and only if it is locally Lipschitz near x_0 (see Jourani and Thibault 1993). Every locally Lipschitz function F which has a upper semicontinuous strict prederivative with norm compact values is also strongly compactly Lipschitzian, if we modify the definition of positive homogeneity in Definition 2.5 to include $\lambda = 0$.

Lemma 2.6 *Let* $F : X \to Y$ *be a locally Lipschitz function where* X *and* Y *are Banach spaces. Let us consider the function* $f(x) = (g \circ F)(x)$ *where* $g : Y \to \mathbb{R}$ *is locally Lipschitz. Then*

$$\partial_a f(x) \subset \bigcup_{y^* \in \partial_a g(F(x))} D_a^* F(x)(y^*).$$

Further, if F *has a strict prederivative with norm compact values then one has*

$$\partial_a f(x) \subset \bigcup_{y^* \in \partial_a g(F(x))} \partial_a \langle y^*, F\rangle(x).$$

Proof Since F and g are locally Lipschitz it is clear that $f = g \circ F$ is also locally Lipschitz. Since g is locally Lipschitz we have $\partial_a^\infty g(y) = \{0\}$. Thus the qualification conditions of Theorem 7.5 in Ioffe (1989) are satisfied automatically and thus using Theorem 7.5 in Ioffe (1989) we get

$$\partial_a f(x) \subset \bigcup_{y^* \in \partial_a g(F(x))} D_a^* F(x)(y^*).$$

Further, if F has strict prederivative with norm compact values, then we get

$$\partial_a f(x) \subset \bigcup_{y^* \in \partial_a g(F(x))} \partial_a \langle y^*, F\rangle(x)$$

using Corollary 7.8.1 in Ioffe (1989). Observe that since $\partial_a^\infty g(y) = \{0\}$ the qualification condition in Corollary 7.8.1 in Ioffe (1989) is automatically satisfied. □

Remark 2.3 In Theorem 7.5 in Ioffe (1989) one merely requires F to be continuous near x and g to be directionally Lipschitz at $F(x)$ and lower semicontinuous around $F(x)$ (see for example Ioffe 1989 for the definition of a directionally Lipschitz function). However, in our case both of these conditions are automatically satisfied since F and g are locally Lipschitz functions.

The following lemma is a mere restatement of Theorem 2.5 in Jourani and Thibault (1993):

Lemma 2.7 *Let $F : X \to Y$ be a locally Lipschitz function where X and Y are Banach spaces. Let us consider the function $f(x) = (g \circ F)(x)$ where $g : Y \to \mathbb{R}$ is locally Lipschitz. If F is strongly compactly Lipschitzian then one has*

$$\partial_a f(x) \subset \bigcup_{y^* \in \partial_a g(F(x))} \partial_a \langle y^*, F \rangle(x).$$

Remark 2.4 By using Proposition 2.4 in Jourani and Thibault (1993) one can show that

$$\partial_a \langle y^*, F \rangle(x) = D_a^* F(x)(y^*) \quad \forall y^* \in Y^*.$$

Thus Lemma 2.7 can be deduced from Lemma 2.6. However, Jourani and Thibault (1993) provides an elegant proof of Lemma 2.7 without using the machinery of the coderivative.

3 Main results

Theorem 3.1 *Let us consider the program (VP), where X and Y are Banach spaces and $F : X \to Y$ is a locally Lipschitz function and C is a closed subset of X. Moreover, assume that F is strictly differentiable. Let x_0 be a weakly efficient point of (VP). Then there exists $0 \neq v^* \in K^*$ such that*

$$0 \in (F'(x_0))^* v^* + N_c(C, x_0).$$

Proof Since x_0 is a weakly efficient point for (VP) by Lemma 2.2 there exists a continuous convex function $z : Y \to \mathbb{R}$ with range $(-\infty, +\infty)$, which is strictly int$K$-monotone such that x_0 solves the problem

$$\min z \circ F(x) \qquad \text{subject to} \quad x \in C.$$

Thus from Clarke (1983) we have

$$0 \in \partial^\circ (z \circ F)(x_0) + N_c(C, x_0).$$

Since $z : Y \to \mathbb{R}$ is a continuous convex function it is locally Lipschitz and thus by using Theorem 2.3.10 in Clarke (1983) we have

$$\partial^\circ (z \circ F)(x_0) \subseteq (F'(x_0))^* \partial z(F(x_0)).$$

Thus there exists $v^* \in \partial z(F(x_0))$ such that

$$0 \in (F'(x_0))^* v^* + N_c(C, x_0).$$

We will now show that for any $y \in Y$ one has $\partial z(y) \subseteq K^*$ using the fact that z is strictly int K-monotone. Let $e \in \text{int}\, K$. Thus we have $z(y) > z(y - e)$. Since z is a continuous convex function on the Banach Space Y one has $\partial z(y) \neq \emptyset$ for each $y \in Y$. Thus we have

$$z(y) > z(y - e) \geq z(y) + \langle v^*, -e \rangle \qquad \forall v^* \in \partial z(y).$$

This shows that $\langle v^*, e \rangle > 0$ for any $e \in \text{int}\, K$. This immediately yields that $v^* \in K^*$. More it also shows that $v^* \neq 0$. This completes the proof. □

Remark 3.1 From the proof of the above theorem it is interesting to note that the Lagrangian multiplier associated with the vector optimization problem (VP) is actually the subgradient of the convex scalarizing function at the point $F(x_0)$. Thus in a manner similar to the scalar optimization the above theorem provides the interpretation of the Lagrangian multiplier of a vector minimization problem as a subgradient.

Let us now assume that $f : X \to Y$ is locally Lipschitz and Y is finite dimensional, say $Y = \mathbb{R}^l$ and let $K = \mathbb{R}^l_+$. Then if $x_0 \in C$ is a weakly efficient point for (VP) one has

$$(f_1^+(x_0, v), \ldots, f_l^+(x_0, v)) \notin -\text{int}\mathbb{R}^l_+ \qquad \forall v \in T(C, x_0),$$

where $f^+(x, v)$ denotes the upper-Dini directional derivative of a locally Lipschitz function $f : X \to \mathbb{R}$ at the point x and in the direction given as

$$f^+(x, v) = \limsup_{\lambda \downarrow 0} \frac{f(x + \lambda v) - f(x)}{\lambda}.$$

Since $T_c(C, x_0) \subseteq T(C, x_0)$ we have

$$(f_1^+(x_0, v), \ldots, f_l^+(x_0, v)) \notin -\text{int}\mathbb{R}^l_+, \qquad \forall v \in T_c(C, x_0).$$

This implies the following system

$$f_1^+(x_0, v) < 0, \ldots, f_l^+(x_0, v) < 0, \qquad v \in T_c(C, x_0),$$

has no solutions. Since for a locally Lipschitz function $f : X \to \mathbb{R}$ one has $f^+(x_0, v) \leq f^\circ(x_0, v)$ we immediately get that the following system

$$f_1^\circ(x_0, v) < 0, \ldots, f_l^\circ(x_0, v) < 0, \qquad v \in T_c(C, x_0)$$

has no solutions. Thus by applying the Gordan's theorem of the alternative and observing that $T_c(C, x_0)$ is a convex set we conclude that there exists $\tau \in \mathbb{R}^l_+ \backslash \{0\}$, such that

$$\sum_{j=1}^{l} \tau_j f_j^\circ(x_0, v) \geq 0 \qquad \forall v \in T_c(C, x_0).$$

By using the calculus of support functions we immediately have

$$0 \in \sum_{j=1}^{l} \tau_j \partial^\circ f_j(x_0) + N_c(C, x_0).$$

However, we can often have the situation where $K \neq \mathbb{R}^l_+$. For example, if one wants to talk about (say) Benson properly efficient points then one has to consider the cone K having a non-empty interior and also satisfying $\mathbb{R}^l_+ \backslash \{0\} \subset \text{int}K$ and seek for an efficient point with respect to such a cone K. Efficient points obtained with respect to such a cone K are called Benson properly efficient points. Observe that we have a similar definition in Sect. 1 in a general setting which we call as properly efficient point. In this situation we cannot take the advantage of componentwise description. Thus in such a case we need to consider the following approach to derive a necessary optimality condition in terms of the Clarke subdifferential. Assume that $x_0 \in C$ is a weakly efficient point of (VP), then by Lemma 2.2 there exists a continuous convex function $z : Y \to \mathbb{R}$ with range $(-\infty, +\infty)$, which is strictly $\text{int}K$-monotone such that x_0 solves the problem

$$\min\ z \circ F(x) \qquad \text{subject to} \quad x \in C.$$

Thus from Clarke (1983) we immediately get that

$$0 \in \partial^\circ(z \circ F)(x_0) + N_c(C, x_0).$$

Now, by applying Theorem 2.4.5 in Clarke et al. (1998) we immediately get that

$$0 \in \overline{\text{co}}^* \left\{ \bigcup_{v^* \in \partial z(F(x_0))} \partial^\circ \langle v^*, F\rangle(x_0) \right\} + N_c(C, x_0).$$

However, from the above expression it is not possible to deduce the existence of $v^* \in \partial z(F(x_0))$ such that

$$0 \in \partial^\circ \langle v^*, F\rangle(x_0) + N_c(C, x_0).$$

Thus we see that the Clarke's subdifferential calculus is not the appropriate tool to develop the necessary optimality conditions for weak minimization of locally Lipschitz vector functions in terms of Clarke subdifferential and the Clarke normal cone when X is an arbitrary Banach Space. However, we shall show that when Y is finite dimensional and X is an Asplund space the limiting subdifferential of Mordukhovich can be used to develop a sharp necessary optimality condition for weak minimization of a vector optimization problem with a locally Lipschitz objective function. Then using the condition thus developed one can then pass on to a representation with Clarke subdifferential and Clarke normal cone using the relations (2) and (1).

Theorem 3.2 *Let us consider the program (VP). Consider that X is an Asplund Space and $Y = \mathbb{R}^l$ and $F : X \to Y$ be locally Lipschitzian. Additionally assume that C is a closed subset of X. Let x_0 be a weakly efficient point of (VP). Then there exists $0 \neq v^* \in K^*$ such that*

$$0 \in \partial_L \langle v^*, F\rangle(x_0) + N_L(C, x_0). \tag{5}$$

Further, if F is strictly differentiable then one has

$$0 \in (F'(x_0))^* v^* + N_L(C, x_0).$$

Proof Since x_0 is a weakly efficient point for (VP) by Lemma 2.2 there exists a continuous convex function $z : Y \to \mathbb{R}$ with range $(-\infty, +\infty)$, which is strictly $\text{int}K$-monotone such that x_0 solves the problem

$$\min\ z \circ F(x) \qquad \text{subject to} \quad x \in C.$$

Hence x_0 is also a solution of the problem

$$\min z \circ F(x) + \delta_C(x),$$

where δ_C denotes the indicator function of the set C. Thus from Mordukhovich and Shao (1996) we have

$$0 \in \partial_L(z \circ F + \delta_C)(x_0).$$

Since F is locally Lipschitz and z is a continuous convex function and hence locally Lipschitz, it is clear that $z \circ F$ is also locally Lipschitz. Moreover, since C is a closed subset of X we have δ_C to be a proper lower-semicontinuous function and thus by Lemma 2.3 we have

$$0 \in \partial_L(z \circ F)(x_0) + N_L(C, x_0),$$

where $\partial_L \delta_C(x_0) = N_L(C, x_0)$ (see Mordukhovich and Shao 1996). Now, by applying Lemma 2.4 we can claim the existence of a $v^* \in \partial_L z(F(x_0))$ such that

$$0 \in \partial_L \langle v^*, F\rangle(x_0) + N_L(C, x_0).$$

Moreover, analogously to the proof of Theorem 3.1 we can conclude that $0 \neq v^* \in K^*$.

The last assertion follows by a direct application of Theorem 5.2 in Mordukhovich and Shao (1996). □

Remark 3.2 It is clear that, if X is an Asplund space and x_0 is a weakly efficient point of (VP), we see using (5) that there exists $v^* \in K \backslash \{0\}$ such that

$$0 \in \overline{\text{co}}^*(\partial_L \langle v^*, F\rangle(x_0) + N_L(C, x_0)).$$

Now, using (2) and (1) we have

$$0 \in \partial^\circ \langle v^*, F\rangle(x_0) + N_c(C, x_0) \tag{6}$$

taking into account the weak*-compactness of Clarke's subdifferential (see Proposition 1.5 in Clarke et al. 1998). Thus we now have an representation in terms of the Clarke's subdifferential and Clarke's normal cone. Further, let us note that (2) and (1) shows that optimality conditions given by (5) is sharper than the ones given by (6).

Remark 3.3 We get corresponding results like in Theorems 3.1, 3.2 for properly efficient elements x_0 (with v^* belongs to the quasi-interior of K^*) taking into account Remark 2.1.

In the above theorem we consider Y to be a finite dimensional space. The natural question is why we are not considering Y to be an arbitrary Asplund space. In most infinite dimensional Asplund spaces we will not be able to find an ordering cone with a non-empty interior. For example, the natural ordering cone of any l_p space with $p \in [1, +\infty)$ has an empty interior. However, all these l_p spaces are Asplund spaces. Let Ω be a compact Hausdorff space and let $C(\Omega)$ denote the space of all continuous real-valued functions defined on Ω. The natural ordering cone of $C(\Omega)$ has a non-empty interior though $C(\Omega)$ is not an Asplund space. Thus it is not of much interest to talk about weak minimum for the case when Y is an infinite dimensional Asplund space. For more details regarding infinite dimensional spaces which are useful in the study of vector optimization see for example Jahn Jahn (2004).

Now, we shall consider the case where Y will be an arbitrary Banach space and $F : X \to Y$ is a locally Lipschitz function. The optimality conditions as we shall see will be represented via the co-derivative associated with the approximate normal cone. Further, under additional assumptions on F we shall get a Lagrangian multiplier rule for weak minimization in arbitrary Banach spaces:

Theorem 3.3 *Let X and Y be arbitrary Banach spaces and $F : X \to Y$ be a locally Lipschitz function. Let $x_0 \in C$ be a weakly efficient point of F over C, where C is a closed set. Then there exists $y^* \in K^*\backslash\{0\}$ such that*

$$0 \in D_a^* F(x_0)(y^*) + N_a(C, x_0).$$

Further, if F has a strict prederivative with norm compact values or F is strongly compactly Lipschitzian, then one has that there exists $y^* \in K^*\backslash\{0\}$ such that

$$0 \in \partial_a \langle y^*, F\rangle(x_0) + N_a(C, x_0).$$

Proof Since x_0 is a weakly efficient point of (VP) by Lemma 2.2 there exists a continuous convex function $z : Y \to \mathbb{R}$ with range $(-\infty, +\infty)$ and *strictly* $-$ (int K)-monotone, such that x_0 solves the problem

$$\min \; z \circ F(x), \qquad \text{subject to} \quad x \in C.$$

Then x_0 also solves the following unconstrained problem

$$\min (z \circ F)(x) + \delta_C(x).$$

Since $(z \circ F)$ is locally Lipschitz and δ_C is a proper lower-semicontinuous function since C is a closed set, by using Lemma 2.5 we have

$$0 \in \partial_a (z \circ F)(x_0) + \partial_a \delta_C(x_0).$$

From (4) we have

$$0 \in \partial_a (z \circ F)(x_0) + N_a(C, x_0).$$

By Lemma 2.6 we have

$$0 \in \bigcup_{y^* \in \partial z(F(x_0))} D_a^* F(x_0)(y^*) + N_a(C, x_0).$$

Thus there exists $y^* \in \partial z(F(x_0))$ such that

$$0 \in D_a^* F(x_0)(y^*) + N_a(C, x_0).$$

Since z is a continuous convex function on Y which is also *strictly* $-$ (int K) $-$ *monotone*, it is clear that $\partial z(F(x_0)) \subset K^*$. Furthermore, similar to the proof of Theorem 3.1 we get $y^* \in K^*\backslash\{0\}$.

Now, if we additionally assume that F has a strict prederivative with norm compact values or F is strongly compactly Lipschitzian then using Lemma 2.6 or Lemma 2.7 we can easily show that

$$0 \in \partial_a \langle y^*, F\rangle(x_0) + N_a(C, x_0).$$

This completes the proof. □

Remark 3.4 A nearly similar result was shown by El Abdouni and Thibault (1992) in terms of the approximate subdifferential by assuming the objective and constraints to be strictly compactly Lipschitzian. However, using the non-convex scalarization scheme our proof is shorter and simpler.

We will now discuss the case when the set C is defined by inequality constraints, i.e.,

$$C = \{x \in X : g_i(x) \leq 0, i = 1, \dots, m\} \tag{7}$$

where each $g_i : X \to R$.

Theorem 3.4 *Let us consider the program (VP), where $F : X \to \mathbb{R}^l$ is locally Lipschitz, where X is an Asplund space and the set C is described by (7). Further each g_i is locally Lipschitz. Let x_0 be a weakly efficient point for (VP). Assume that the active index set $I(x_0) = \{i : g_i(x_0) = 0\} \neq \emptyset$ and $0 \notin \operatorname{co} \bigcup_{i \in I(x_0)} \partial g_i(x_0)$. Then there exist $0 \neq v \in K^*$ and scalars $\lambda_i \geq 0$, $i \in I(x_0)$, such that*

$$0 \in \partial_L \langle v^*, F\rangle(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial_L g_i(x_0).$$

Proof Since x_0 is a weakly efficient point of (VP) by using Theorem 3.2 we immediately conclude the existence of $0 \neq v^* \in K^*$ such that

$$0 \in \partial_L \langle v^*, F\rangle(x_0) + N_L(C, x_0).$$

Since $0 \notin \operatorname{co} \bigcup_{i \in I(x_0)} \partial g_i(x_0)$ we have from Theorem 4.4 in Mordukhovich (2001)

$$N_L(C, x_0) \subset \left\{ \sum_{i \in I(x_0)} \lambda_i \partial_L g_i(x_0) : \lambda_i \geq 0, i \in I(x_0) \right\}.$$

This immediately yields the result. □

Let us now consider the case where C is defined by equality constraints, i.e.,

$$C = \{x \in X : h_j(x) = 0, j = 1, \dots, k\}, \tag{8}$$

where $h_j : X \to R$.

We have the following result which can be proved by an application of Theorem 4.4 in Mordukhovich (2001):

Theorem 3.5 *Let X be an Asplund Space. Let $F : X \to \mathbb{R}^l$ be locally Lipschitz and h_j be a locally Lipschitz function for each j. Consider the problem (VP), where C is defined as in (8). Let x_0 be a weakly efficient point for (VP). Assume further that $0 \notin \text{co} \bigcup_j (\partial_L h_j(x_0) \cup \partial_L(-h_j(x_0)))$. Then there exist $0 \neq v^* \in K^*$ and scalars $\mu_j \geq 0$, $j = 1, \ldots, k$ such that*

$$0 \in \partial_L \langle v^*, F\rangle(x_0) + \sum_{j=1}^{k} \mu_j(\partial_L h_j(x_0) \cup \partial_L h_j(-x_0)).$$

We would now study the case where Y is finite dimensional and $K = \mathbb{R}^l_+$. Instead of a weakly efficient point we will now characterize an efficient point or a Pareto minimum. To keep the exposition simple we will consider the case where the feasible set is described by inequality constraints. It is well known from Chankong and Haimes (1983) that a point $x_0 \in C$ is an efficient point for (VP) if and only if x_0 solves the problem P(k, x_0) for all $k = 1, \ldots, l$, where P(k, x_0) is given as

$$\min \ f_k(x), \qquad \text{subject to} \quad x \in F_k,$$

where $F_k = \{f_j(x) \leq f_j(x_0), \quad j \neq k, j = 1, \ldots, l; \quad x \in C\}$.

Theorem 3.6 *Let X be an Asplund space, $Y = \mathbb{R}^l$ and $K = \mathbb{R}^l_+$. Consider the problem (VP) where $F(x) = (f_1(x), \ldots, f_l(x))$ is locally Lipschitz and C is given as*

$$C = \{x \in X : g_i(x) \leq 0, i = 1, \ldots, m\}.$$

Let x_0 be an efficient point for (VP). Let there exists an index k for which the following qualification condition holds

$$0 \notin \text{co} \left\{ \left(\bigcup_{j \neq k} \partial_L f_j(x_0) \right) \bigcup \left(\bigcup_i \partial_L g_i(x_0) \right) \right\}. \tag{9}$$

Then there exist scalars $\tau_j \geq 0$, $j = 1, \ldots, l$ and $\lambda_i \geq 0$, $i = 1, \ldots, m$, such that

(i) $0 \in \sum_{j=1}^{l} \tau_j \partial_L f_j(x_0) + \sum_{i=1}^{m} \lambda_i \partial_L g_i(x_0)$, (ii) $\lambda_i g_i(x_0) = 0$,
(iii) $\tau = (\tau_1, \ldots, \tau_l) \neq 0$.

Proof Since x_0 is an efficient point, by the Chankong and Haimes (1983) criteria x_0 solves P(k, x_0) for all $k = 1, \ldots, l$. Hence for the particular k for which the qualification condition (9) holds we have

$$0 \in \partial_L f_k(x_0) + N_{F_k}(x_0).$$

Then using Theorem 4.4 in Mordukhovich (2001) we are immediately led to the fact that there exists $\tau_j \geq 0$ and $\lambda_i \geq 0$, $i \in I(x_0)$, such that

$$0 \in \partial_L f_k(x_0) + \sum_{j \neq k} \tau_j \partial_L f_j(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial_L g_i(x_0).$$

By setting $\tau_k = 1$ and $\lambda_i = 0$ if $g_i(x_0) < 0$ the result is immediately established. □

Remark 3.5 It is important to understand why one needs such an optimality condition in terms of the limiting subdifferential rather than the Clarke subdifferential. In fact in terms of the Clarke subdifferential the result can be stated in an arbitary Banach Space. However, in Asplund spaces the above optimality conditions are more robust and sharper than the corresponding ones given in terms of the Clarke subdifferential. To the best of our knowledge the qualification condition (9) in terms of the limiting subdifferential to be new for the inequality constrained vector optimization problem (VP). For a study of qualification conditions in nonsmooth vector optimization using the Clarke subdifferential see for example Li (2000) or Chandra et al. (2004).

We will now turn our attention to the case where X and Y are both finite dimensional, say $X = \mathbb{R}^n$ and $Y = \mathbb{R}^l$ but $K \neq \mathbb{R}^l_+$. If F is locally Lipschitz then one would like to express the optimality conditions for a weakly efficient point of (VP) in terms of the Clarke generalized Jacobian of F (see Clarke 1983 for details on the Clarke generalized Jacobian). However a scalarization is possible if the function F is additionally K-convex. To the best of our knowledge the nonlinear scalarization scheme of Gerth(Tammer) and Weidner (1990) has not being used to derive optimality conditions in the finite dimensional case. However we will show that even if we use the nonlinear scalarization scheme of Gerth(Tammer) and Weidner (1990) one will not get a robust optimality condition in terms of the Clarke generalized Jacobian. Let us denote the Clarke generalized Jacobian of F at x_0 by $\partial_C F(x_0)$. Let x_0 be a weakly efficient point for (VP). Then using Lemma 2.2 we have that there exists an increasing convex function $z : \mathbb{R}^l \to \mathbb{R}$ such that

$$0 \in \partial^\circ(z \circ F)(x_0) + N_c(C, x_0).$$

Now by using for example Theorem 4.1 in Demyanov and Rubinov (1995) we have

$$0 \in \text{co}\{(\partial_C F(x_0))^{\mathrm{T}} v : v \in \partial z(F(x_0))\} + N_c(C, x_0),$$

where T denotes the transpose of a matrix. Since $\partial z(F(x_0)) \in K^*$ the above expression also implies that

$$0 \in \text{co}\{(\partial_C F(x_0))^{\mathrm{T}} v : v \in K^*\} + N_c(C, x_0).$$

However, it is not immediate from the above expression whether there exits $0 \neq v \in K^*$ such that

$$0 \in (\partial_C F(x_0))^{\mathrm{T}} v + N_c(C, x_0).$$

Thus we conclude that for locally Lipschitz non-convex vector optimization problems the Clarke subdifferential and the Clarke generalized Jacobians are not useful tools to develop robust optimality conditions. In this direction it seems the limiting subdifferential of Mordukhovich is very well suited for the purpose.

4 Applications to approximation problems

We consider a class of vector-valued control approximation problems in which each objective function is a sum of two terms, a locally Lipschitz function and a power of a norm of a linear vector function and the feasible set is not necessary convex. An important special case of such problems are (non-convex) vector-valued location problems with forbidden regions. Necessary conditions for solutions of these problems were derived using the results from Sect. 3.

In our proofs it is possible to use the special structure of the subdifferential of a power of the norm and the Theorem 3.2.

We will introduce a general vector-valued control approximation problem.

We suppose that

(A) $(X, \|\cdot\|_X)$ is an Asplund space and $(Y_i, \|\cdot\|_i)$ $(i = 1, \dots, n)$ are real Banach spaces, $x \in X$, $a^i \in Y_i$, $\alpha_i \geq 0$, $\beta_i \geq 1$, $A_i \in L(X, Y_i)$, $(i = 1, \dots, n)$, $f_1 : X \to \mathbb{R}^n)$ is a locally Lipschitz cost function,
(B) $C \subseteq X$ is a nonempty closed set,
(C) $K \subset \mathbb{R}^n$ is a pointed closed convex cone with nonempty interior and $K + \mathbb{R}^n_+ \subseteq K$.

Moreover, $\|\cdot\|_*$ denotes the dual norm to $\|\cdot\|_X$, and $\|\cdot\|_{i*}$ the dual norm to $\|\cdot\|_i$. Let us recall that the dual norm $\|\cdot\|_*$ to $\|\cdot\|_X$ is defined by

$$\|p\|_* := \sup_{\|x\|_X = 1} |\ p(x)\ | \ .$$

Now we consider the following (non-convex) *vector control approximation problem*

(P_{app}) Compute the set $\text{Eff}(F(C), K)$,

where

$$F(x) := f_1(x) + \begin{pmatrix} \alpha_1 \|A_1(x) - a^1\|_1^{\beta_1} \\ \cdots \\ \alpha_n \|A_n(x) - a^n\|_n^{\beta_n} \end{pmatrix}$$

is the objective vector function. We will derive necessary conditions for weakly efficient elements using Theorem 3.2.

In order to prove this assertion we need the following assertion (cf. Zălinescu 2002, Corollary 2.4.16) concerning the subdifferential of norm terms (for the usual subdifferential $\partial\|x\|$ of a convex function):

Lemma 4.1 *If X is a Banach space then we have*

$$\partial\|x\|_X = \begin{cases} \{p \in L(X, \mathbb{R}) \mid p(x) = \|x\|_X,\ \|p\|_* = 1\} & \textit{if } x \neq 0, \\ \{p \in L(X, \mathbb{R}) \mid \|p\|_* \leq 1\} & \textit{if } x = 0, \end{cases}$$

and for $\beta > 1$,

$$\partial\left(\frac{1}{\beta}\|\cdot\|_X^\beta\right)(x) = \{p \in L(X, \mathbb{R}) \mid \|p\|_* = \|x\|_X^{\beta-1},\ p(x) = \|x\|_X^\beta\}.$$

In the next theorem we will derive necessary conditions for weakly efficient solutions of (P_{app}).

Theorem 4.1 *Consider a weakly efficient element x_0 of the approximation problem (P_{app}). Then there exists a functional $v^* \in K^*\backslash\{0\}$, such that*

$$0 \in \Big\{ \partial_L \langle v^*, f_1 \rangle + \sum_{i=1}^{n} \alpha_i \beta_i A_i^* v_i^* M_{0i} \mid M_{0i} \in L(Y_i, \mathbb{R}),\ M_{0i}(A_i(x_0) - a^i) = \|A_i(x_0) - a^i\|_i^{\beta_i},\ \|M_{0i}\|_{i^*} \leq 1 \text{ if } \beta_i = 1 \text{ and } A_i(x_0) = a^i,\ \|M_{0i}\|_{i^*} = \|A_i(x_0) - a^i\|_i^{\beta_i - 1} \text{otherwise } (\forall\, i,\ 1 \leq i \leq n) \Big\} + N_L(C, x_0).$$

Proof Assume x_0 is a weakly efficient element of (P_{app}). Then from Theorem 3.2 we get the existence of $v^* \in K^*\backslash\{0\}$ and

$$0 \in \partial_L \langle v^*, F \rangle (x_0) + N_L(C, x_0). \tag{10}$$

Furthermore, we have

$$\partial_L \langle v^*, F \rangle (x_0) = \partial_L \langle v^*, f_1(.) + \begin{pmatrix} \alpha_1 \|A_1(.) - a^1\|_1^{\beta_1} \\ \cdots \\ \alpha_n \|A_n(.) - a^n\|_n^{\beta_n} \end{pmatrix} \rangle (x_0).$$

The rule of sums for Mordukhovich subdifferentials (Lemma 2.3) yields the relation

$$\partial_L \langle v^*, F \rangle (x_0) \subseteq \partial_L \langle v^*, f_1 \rangle (x_0) + \sum_{i=1}^{n} \alpha_i v_i^* \partial \|A_i(\cdot) - a^i\|_i^{\beta_i}(x_0). \tag{11}$$

Applying Lemma 4.1, relation (11), implies

$$\begin{aligned} \partial \langle v^*, F \rangle (x_0) &\subseteq \partial_L \langle v^*, f_1 \rangle (x_0) + \sum_{i=1}^{n} \alpha_i A_i^* v_i^* \partial (\|u\|_i^{\beta_i}) \mid_{u = A_i(x_0) - a^i} \\ &= \Big\{ \partial_L \langle v^*, f_1 \rangle (x_0) + \sum_{i=1}^{n} \alpha_i \beta_i A_i^* v_i^* M_{0i} \mid M_{0i} \in L(Y_i, \mathbb{R}),\ M_{0i}(A_i(x_0) - a^i) \\ &\quad = \|A_i(x_0) - a^i\|_i^{\beta_i},\ \|M_{0i}\|_{i^*} \leq 1 \text{ if } \beta_i = 1 \text{ and } A_i(x_0) = a^i, \\ &\quad \|M_{0i}\|_{i^*} = \|A_i(x_0) - a^i\|_i^{\beta_i - 1} \text{otherwise } (\forall\, i,\ 1 \leq i \leq n) \Big\}. \end{aligned}$$

Then we get together with (10) the desired relation. □

Acknowledgments The authors wish to express their thanks to Professor B. S. Mordukhovich and to Professor A. D. Ioffe for valuable comments and suggestions. Furthermore, we like to thank the anonymous reviewers.

References

Amahroq T, Taa A (1997) On Lagrange–Kuhn–Tucker multipliers for multiobjective optimization problems. Optimization 41:159–172
Chankong V, Haimes YY (1983) Multiobjective decision making : theory and methodology. North-Holland, Amsterdam
Clarke FH (1983) Optimization and nonsmooth analysis. Wiley, New York
Clarke FH, Ledyaev YuS, Stern RJ, Wolenski PR (1998) Nonsmooth analysis and control theory. Springer, Berlin Heidelberg New York
Chandra S, Dutta J, Lalitha CS (2004) Regularity cconditions and optimality in nonsmooth vector optimization. Numer Funct Anal Appl 25:479–501
Craven BD (1989) Nonsmooth multiobjective programming. Numer Funct Anal Optim 10(1–2): 65–76
Demyanov VF, Rubinov A (1995) Constructive nonsmooth analysis. Peter-Verlag, Frankfurt
El Abdouni B, Thibault L (1992) Lagrange multipliers for Pareto nonsmooth programming problems in Banach spaces. Optimization 26:277–285
Gerth(Tammer) C, Weidner PW (1990) Nonconvex separation theorems and some applications in vector optimization. J Optim Theory Appl 67:297–320
Ioffe AD (1986) Approximate subdifferentials and applications II. Mathematika 33:111–128
Ioffe AD (1989) Approximate subdifferentials and applications III. The metric theory. Mathematika 36:1–38
Ioffe AD (2000) Metric regularity and subdifferential calculus. Russ Math Surveys 55:501–558
Jahn J (1986) Mathematical vector optimization in partially ordered spaces. Peter Lang, Frankfurt Bern New York
Jahn J (2004) Vector optimization: theory applications and extensions. Springer, Berlin Heidelberg New York
Jourani A, Thibault L (1993) The approximate subdifferential of composite functions. Bull Aust Math Soc 47:443–455
Li XF (2000) Constraint qualification in nonsmooth multiobjective optimization. J Optimi Theory Appl 106:373–398
Miettinen K (1999) Nonlinear multiobjective optimization. Kluwer, Boston
Miettinen K, Mäkelä MM (2000) Tangent and normal cones in nonconvex multiobjective optimization. In: Haimes YY, Steuer RE (eds) Research and practice in multiple criteria decision making. Lecture notes in economils and mathematical systems, vol 487. Springer, Berlin Heidelberg New York, pp 114–124
Minami M (1983) Weak Pareto-optimal necessary conditions in a nondifferentiable multiobjective program on a Banach space. J Optim Theory Appl 41:451–461
Mordukhovich BS (1976) Maximum principle in problems of time optimal control with nonsmooth constraints. J Appl Math Mech 40:960–969
Mordukhovich BS (1985) On necessary conditions for an extremum in non-smooth optimization. Sov Math Dokl 32:215–220
Mordukhovich BS (1994) Generalized differential calculus for nonsmooth and set-valued mappings. J Math Anal Appl 183:250–288
Mordukhovich BS (2001) The extremal principle and its application to optimization and economics. In: Optimization and related topic, Ballarat 1999, applied optimization vol 47. Kluwer, Dodrecht
Mordukhovich BS (2005) Variational analysis and generalized differentiation, I: basic theory, II: applications (series: fundamental principles of mathematics), vol 330 and 331. Springer, Berlin Heidelberg New York
Mordukhovich BS, Shao Y (1996) Nonsmooth sequential analysis in Asplund spaces. Transa Am Math Soc 348:1235–1280
Rockafellar RT, Wets RJB (1998) Variational analysis. Springer, Berlin Heidelberg New York
Zălinescu C (2002) Convex analysis in general vector spaces. World Scientific, New Jersey