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On approximate efficiency in multiobjective programming

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Abstract This paper is focused on approximate (ε-efficient) solutions of multiobjective mathematical programs. We introduce a new ε-efficiency concept which extends and unifies different notions of approximate solution defined in the literature. We characterize these ε-efficient solutions in convex multiobjective programs through approximate solutions of linear scalarizations, which allow us to obtain parametric representations of different ε-efficiency sets. Several classical ε-efficiency notions are considered in order to show the concepts introduced and the results obtained.

Keywords Multiobjective mathematical programming · ε-efficiency · Scalarization · Weighting method · Parametric representation

Mathematics Subject Classification (2000) 49J52 · 90C29 · 49M37

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1 Introduction

In the last years, researchers and practitioners have been interested on approximate solutions of optimization problems. They agree that a lot of usual resolution methods, as for example, the iterative and heuristic methods, give as solution feasible points near to the theoretical solution. This is the most important reason to study this kind of solutions.

In vector optimization, as a consequence to model the preferences of the decision maker by a partial order, there exist different approximate efficiency notions, called ε-efficiency concepts. The first concept was introduced by Kutateladze (1979) and has been used to establish vector variational principles, approximate Kuhn-Tucker type conditions, approximate duality theorems, resolution methods, etc. (see Dentcheva and Helbig 1996; Dutta and Vetrivel 2001; Gutiérrez 2004; Gutiérrez et al. 2005a,b; Idrissi et al. 1998; Isac 1996; Liu 1991, 1996; Liu and Yokoyama 1999; Loridan 1984, 1992; Ruhe and Fruhwirth 1990; Tammer 1992; Vályi 1987; White 1998).

The ε-efficiency set obtained according to the Kutateladze's definition is sometimes too big, which has some undesirable consequences. For example, it is possible to attain as a limit of ε-efficient solutions when ε tends to zero, a weak efficient solution a long way from the efficiency set. So, several authors have proposed other ε-efficiency concepts (see for example Helbig 1992; Németh 1986; Tanaka 1995; Vályi 1985; White 1986), in order to achieve better features.

In this work, we introduce a new concept of approximate solution for a multiobjective program that allows us to study various well-known ε-efficiency notions in a unified way. We characterize this new ε-efficiency notion in convex multiobjective mathematical programs via linear scalarization, i.e., by means of approximate solutions of associated scalar optimization problems. As a consequence of this characterization, we extend the classical Weighting Method to approximate efficiency sets obtained through different ε-efficiency notions and we deduce parametric representations of these ε-efficiency sets via a notion of parametric representation for ε-efficiency sets introduced by the authors in (Gutiérrez et al. 2006a,b).

The outline of the paper is as follows. In Sect. 2, the multiobjective mathematical program and the preference relation are fixed. Moreover, we describe some notations used in the sequel. In Sect. 3, we propose a new ε-efficiency concept and we prove some properties of this notion when ε tends to zero. In Sect. 4, it is shown that our concept extends and unifies several ε-efficiency notions introduced previously in the literature by Kutateladze (1979), Németh (1986), Helbig (1992) and Tanaka (1995). In Sect. 5, we characterize and give parametric representations of the ε-efficiency set in convex multiobjective mathematical programs through approximate solutions of scalar optimization problems. The scalarization process is based on the Weighting Method. In the last part of Sect. 5, the results attained previously are applied to obtain parametric representations of several ε-efficiency sets in a convex Paretian context. Finally, in Sect. 6, conclusions are presented that summarize this work.

2 Preliminaries

We denote by $\mathrm{int}(C)$, $\mathrm{cl}(C)$, $\mathrm{bd}(C)$ and C^c the interior, the closure, the boundary and the complement of a set $C \subset \mathbb{R}^p$, respectively. A cone is a set $K \subset \mathbb{R}^p$

such that $\alpha K \subset K, \forall \alpha > 0$. We do not require that $0 \in K$. Therefore, the cone generated by a set C is defined as

$$\text{cone}(C) := \bigcup_{\alpha > 0} \alpha C.$$

We say that a set C is proper (resp. solid) if $\emptyset \neq C \neq \mathbb{R}^p$ (resp. $\text{int}(C) \neq \emptyset$). We denote the nonnegative orthant in $\mathbb{R}^p$ by $\mathbb{R}^p_+$.

For a cone $K \subset \mathbb{R}^p$, its positive polar cone (resp. strict positive polar cone) is

$$K^+ = \{h \in \mathbb{R}^p : \langle h, d \rangle \geq 0, \forall d \in K\}$$

(resp. $K^{s+} = \{h \in \mathbb{R}^p : \langle h, d \rangle > 0, \forall d \in K \backslash \{0\}\}$).

In this paper, we consider the multiobjective mathematical program

$$\text{Min}\{f(x) : x \in S\}, \tag{P}$$

where $f : \mathbb{R}^n \to \mathbb{R}^p$ and $S \subset \mathbb{R}^n$, $S \neq \emptyset$. As usual, to solve (P) the following preference relation $\leq$ defined in $\mathbb{R}^p$ by a solid convex cone $D \subset \mathbb{R}^p$ is used, which models the preferences stated by the decision-maker:

$$y, z \in \mathbb{R}^p, \; y \leq z \iff y - z \in -D.$$

We suppose that D is a pointed cone, i.e., a cone such that $D \cap (-D) \subset \{0\}$.

Definition 2.1 *It is said that a feasible point $x_0 \in S$ is an efficient solution of* (P) *with respect to D (or an efficient solution for short) if*

$$(f(x_0) - D) \cap f(S) \subset \{f(x_0)\}. \tag{2.1}$$

Let us note that if $0 \in D$ (resp. $0 \notin D$) then (2.1) becomes

$$(f(x_0) - D) \cap f(S) = \{f(x_0)\}$$

(resp. $(f(x_0) - D) \cap f(S) = \emptyset$). We denote the set of efficient solutions of (P) with respect to D by $\text{E}(f, S, D)$ and with respect to $\text{int}(D)$ by $\text{WE}(f, S, D)$ (these last efficient solutions of (P) are called weakly-efficient solutions). It is obvious that $\text{E}(f, S, D) \subset \text{WE}(f, S, D)$.

3 ε-efficiency in multiobjective programming

It is clear that an approximate solution of (P) is a feasible point $x_0 \in S$ such that for all feasible point $x \in S$ whose image $f(x)$ is better than $f(x_0)$, the improvement $f(x_0) - f(x)$ is near to zero.

To make this idea a useful ε-efficiency notion for multiobjective mathematical programs (see Definition 3.2), we consider a solid pointed ($C \cap (-C) \subset \{0\}$) convex set $C \subset \mathbb{R}^p$ and we assume that C is co-radiant, i.e., a set such that $\alpha d \in C, \forall d \in C, \forall \alpha > 1$. Notice that a cone is a co-radiant set. Moreover, we denote $C(\varepsilon) := \varepsilon C, \forall \varepsilon > 0$ and

$$C(0) := \bigcup_{\varepsilon > 0} C(\varepsilon). \tag{3.1}$$

Lemma 3.1

(i) $C(\varepsilon)$ *is a solid pointed convex co-radiant set,* $\forall\varepsilon > 0$.
(ii) $C(\varepsilon_2) \subset C(\varepsilon_1)$, $\forall\varepsilon_1, \varepsilon_2 > 0, \varepsilon_1 < \varepsilon_2$.
(iii) $C + C(\alpha) \subset C, \forall\alpha > 0$.
(iv) $C(\varepsilon) + C(\delta) \subset C(\varepsilon)$, $\forall\varepsilon, \delta > 0$.
(v) $C(\varepsilon) + C(0) \subset C(\varepsilon)$, $\forall\varepsilon > 0$.
(vi) $C(0)$ *is a solid pointed convex cone.*

Proof Part (i). Consider $\varepsilon > 0$. It is obvious that $C(\varepsilon)$ is a solid pointed convex set, since C is a solid pointed convex set. Let $y \in C(\varepsilon)$ and $\alpha > 1$. There exists $d \in C$ such that $y = \varepsilon d$. As C is a co-radiant set, it follows that $\alpha y = \varepsilon(\alpha d) \in C(\varepsilon)$ and $C(\varepsilon)$ is a co-radiant set.

Part (ii). Let $\varepsilon_1, \varepsilon_2 > 0$, $\varepsilon_1 < \varepsilon_2$ and $y \in C(\varepsilon_2)$. There exists $d \in C$ such that $y = \varepsilon_2 d$. For

$$\alpha := 1 + (\varepsilon_2 - \varepsilon_1)/\varepsilon_1$$

we have that $y = \alpha(\varepsilon_1 d) \in C(\varepsilon_1)$, since $\alpha > 1$ and $C(\varepsilon_1)$ is a co-radiant set. Then, $C(\varepsilon_2) \subset C(\varepsilon_1)$.

Part (iii). For each $d_1, d_2 \in C$ and $\alpha > 0$ it follows that

$$d_1 + \alpha d_2 = (1+\alpha)\left(\left(1 - \frac{\alpha}{1+\alpha}\right)d_1 + \frac{\alpha}{1+\alpha}d_2\right)$$

and $d_1 + \alpha d_2 \in C$, since C is a co-radiant convex set.

Part (iv). For each $\varepsilon, \delta > 0$ it follows from part (iii) that

$$C(\varepsilon) + C(\delta) = \varepsilon(C + C(\delta/\varepsilon)) \subset \varepsilon C = C(\varepsilon)\,.$$

Part (v). By part (iv) we see that

$$C(\varepsilon) + C(0) = \bigcup_{\delta>0}(C(\varepsilon) + C(\delta)) \subset C(\varepsilon)\,, \quad \forall\varepsilon > 0.$$

Part (vi). It is clear that $C(0) = \mathrm{cone}(C)$, and so we have that $C(0)$ is a cone.

If $y \in C(0) \cap (-C(0))$ then there exist $\delta,\ \nu > 0$ such that $y \in C(\delta) \cap (-C(\nu))$. Consider $\beta = \min\{\delta, \nu\} > 0$. By parts ($i$)-($ii$) we see that $y \in C(\beta) \cap (-C(\beta)) \subset \{0\}$, and therefore, $C(0)$ is a pointed set. Moreover, by a similar reasoning, if $y_1, y_2 \in C(0)$ then there exists $\beta > 0$ such that $y_1, y_2 \in C(\beta)$ and it follows that $\lambda y_1 + (1-\lambda)y_2 \in C(\beta)$, $\forall\lambda \in (0, 1)$, since $C(\beta)$ is a convex set. Consequently, $C(0)$ is convex. □

Definition 3.2 *Let* $\varepsilon \geq 0$. *We say that a feasible point* $x_0 \in S$ *is an* ε*-efficient solution of* (P) *with respect to C (or an* ε*-efficient solution for short) if*

$$(f(x_0) - C(\varepsilon)) \cap f(S) \subset \{f(x_0)\}. \tag{3.2}$$

We denote by $\mathrm{AE}(f, S, C, \varepsilon)$ the set of ε-efficient solutions of (P) with respect to C. Let us observe that when $\varepsilon = 0$ we have $\mathrm{AE}(f, S, C, 0) = \mathrm{E}(f, S, C(0))$.

Taking $p = 1$ and $C = (1, \infty)$ in Definition 3.2 we obtain the classical concept of approximate solution in (scalar) mathematical programming. We recall this notion in the following definition.

Definition 3.3 *Consider* $p = 1$ *in program* (P) *and* $\varepsilon \geq 0$. *It is said that* $x_0 \in S$ *is an* ε*-solution of* (P) *if*

$$f(x_0) - \varepsilon \leq f(x), \quad \forall x \in S.$$

We denote the set of all ε-solutions of (P) when $p = 1$ by $\mathrm{AMin}(f, S, \varepsilon)$.

As C is a solid pointed convex co-radiant set, it follows that $\mathrm{int}(C)$ is a nonempty pointed convex co-radiant set and we can also consider the set of all ε-efficient solutions of (P) with respect to $\mathrm{int}(C)$ (or weakly ε-efficient solutions for short):

$$\{x \in S : (f(x) - \mathrm{int}(C)\,(\varepsilon)) \cap f(S) \subset \{f(x)\}\}.$$

We denote this set by $\mathrm{WAE}(f, S, C, \varepsilon)$. Notice that

$$\mathrm{int}(C)\,(0) = \bigcup_{\varepsilon>0} \varepsilon\, \mathrm{int}(C) = \bigcup_{\varepsilon>0} \mathrm{int}(C(\varepsilon)) \tag{3.3}$$

is an open cone. Moreover, as C is a solid convex set it follows that (see Jiménez and Novo 2003, Proposition 2.3(*ii*))

$$\mathrm{int}(C)\,(0) = \mathrm{cone}(\mathrm{int}(C)) = \mathrm{int}(\mathrm{cone}(C)) = \mathrm{int}(C(0)) \subset C(0)\,. \tag{3.4}$$

Therefore,

$$\mathrm{AE}(f, S, C, \varepsilon) \subset \mathrm{WAE}(f, S, C, \varepsilon)\,, \quad \forall \varepsilon \geq 0 \tag{3.5}$$

and

$$\mathrm{WAE}(f, S, C, 0) = \mathrm{E}(f, S, \mathrm{int}(C(0))) = \mathrm{WE}(f, S, C(0))\,.$$

To illustrate we give now an example (see Sect. 4 for more important notions).

Example 3.4 Let $h \in D^+\backslash\{0\}$ and define

$$C := \{y \in \mathbb{R}^p : \langle h, y\rangle > 1\}.$$

It is clear that C is a proper solid pointed convex co-radiant set and $C(\varepsilon) = \{y \in \mathbb{R}^p : \langle h, y\rangle > \varepsilon\}$. Then we have for each $\varepsilon \geq 0$ and $x \in S$

$$\begin{aligned} x \in \mathrm{AE}(f, S, C, \varepsilon) &\iff (f(x) - f(S)) \subset C(\varepsilon)^c \\ &\iff \forall z \in S, \quad \langle h, f(z)\rangle \geq \langle h, f(x)\rangle - \varepsilon. \end{aligned}$$

This means that x is an ε-efficient solution with respect to $\langle h, \cdot\rangle$ in the sense introduced by Vályi (1985).

Theorem 3.5 shows several properties of the family $\{\mathrm{AE}(f, S, C, \varepsilon)\}_{\varepsilon \geq 0}$.

Theorem 3.5

(i) $\mathrm{AE}(f, S, C, 0) \subset \mathrm{AE}(f, S, C, \varepsilon)$, $\forall \varepsilon > 0$.
(ii) $\mathrm{AE}(f, S, C, \varepsilon_1) \subset \mathrm{AE}(f, S, C, \varepsilon_2)$, $\forall \varepsilon_1, \varepsilon_2 > 0, \varepsilon_1 < \varepsilon_2$.
(iii) $\bigcap_{\varepsilon>0} \mathrm{AE}(f, S, C, \varepsilon) = \mathrm{AE}(f, S, C, 0)$.

(iv) *Let* $(x_n) \subset S$, $(\varepsilon_n) \subset \mathbb{R}_+$ *and* $y \in \mathbb{R}^p$ *such that* $x_n \in \mathrm{AE}(f, S, C, \varepsilon_n)$, $\varepsilon_n \downarrow 0$ *and* $f(x_n) \to y$. *Then* $f^{-1}(y) \cap S \subset \mathrm{WAE}(f, S, C, 0)$.

(v) *Let* $(x_n) \subset S$ *and* $(\varepsilon_n) \subset \mathbb{R}_+$ *such that* $x_n \in \mathrm{AE}(f, S, C, \varepsilon_n)$ *and* $\varepsilon_n \downarrow 0$. *Consider*

$$K := \bigcap_n (f(x_n) - C(\varepsilon_n)).$$

Then $f^{-1}(K) \cap S \subset \mathrm{AE}(f, S, C, 0)$.

Proof Part (*i*). Let $\varepsilon > 0$ and $x \in \mathrm{AE}(f, S, C, 0)$. It follows that

$$\begin{aligned}(f(x) - C(\varepsilon)) \cap f(S) &\subset \left(f(x) - \bigcup_{\delta>0} C(\delta) \right) \cap f(S) \\ &= (f(x) - C(0)) \cap f(S) \subset \{f(x)\}\end{aligned}$$

and $x \in \mathrm{AE}(f, S, C, \varepsilon)$.

Part (*ii*). Let $\varepsilon_1, \varepsilon_2 > 0$, $\varepsilon_1 < \varepsilon_2$ and $x \in \mathrm{AE}(f, S, C, \varepsilon_1)$. By Lemma 3.1(*ii*) we have that $C(\varepsilon_2) \subset C(\varepsilon_1)$ and we deduce that

$$(f(x) - C(\varepsilon_2)) \cap f(S) \subset (f(x) - C(\varepsilon_1)) \cap f(S) \subset \{f(x)\}.$$

Then, $x \in \mathrm{AE}(f, S, C, \varepsilon_2)$.

Part (*iii*). From part (*i*) it follows that

$$\mathrm{AE}(f, S, C, 0) \subset \bigcap_{\varepsilon>0} \mathrm{AE}(f, S, C, \varepsilon).$$

Conversely, if $x \in \bigcap_{\varepsilon>0} \mathrm{AE}(f, S, C, \varepsilon)$ then

$$(f(x) - C(\varepsilon)) \cap f(S) \subset \{f(x)\}, \quad \forall \varepsilon > 0.$$

Therefore, we have that

$$(f(x) - C(0)) \cap f(S) = \bigcup_{\varepsilon>0} \big((f(x) - C(\varepsilon)) \cap f(S)\big) \subset \{f(x)\},$$

and so $x \in \mathrm{AE}(f, S, C, 0)$.

Part (*iv*). Let $x \in f^{-1}(y) \cap S$ and suppose that there exists $z \in S$ such that $f(z) \in f(x) - \mathrm{int}(C)(0)$. From (3.3) it follows that there exists $\varepsilon > 0$ verifying $f(z) \in f(x) - \mathrm{int}(C(\varepsilon))$. As $f(x_n) \to y$ we deduce that there exists $n_0 \in \mathbb{N}$ such that

$$f(z) + y - f(x_n) \in f(x) - C(\varepsilon), \forall n \geq n_0.$$

As $\varepsilon_n \downarrow 0$ it follows from Lemma 3.1(*ii*) that there exists $n_1 \geq n_0$ such that

$$f(z) \in f(x_n) - C(\varepsilon_n), \forall n \geq n_1$$

and therefore $f(z) = f(x_n)$, $\forall n \geq n_1$, since $x_n \in \mathrm{AE}(f, S, C, \varepsilon_n)$. Then, taking the limit, we have $f(z) = y = f(x)$,

$$(f(x) - \mathrm{int}(C)(0)) \cap f(S) \subset \{f(x)\}$$

and $x \in \mathrm{WAE}(f, S, C, 0)$.

Part (v). Consider $x \in f^{-1}(K) \cap S$. As $f(x) \in K$ and $x_n \in \mathrm{AE}(f, S, C, \varepsilon_n)$ we have that

$$f(x) \in (f(x_n) - C(\varepsilon_n)) \cap f(S) \subset \{f(x_n)\}, \quad \forall n$$

and we deduce that $f(x) = f(x_n), \forall n$. Therefore,

$$(f(x) - C(\varepsilon_n)) \cap f(S) = (f(x_n) - C(\varepsilon_n)) \cap f(S) \subset \{f(x_n)\} = \{f(x)\}, \quad \forall n.$$

Thus, by (3.1) we see that

$$(f(x) - C(0)) \cap f(S) \subset \{f(x)\}$$

and we conclude that $x \in \mathrm{AE}(f, S, C, 0)$. □

Remark 3.6 From Theorem 3.5(iv) it is clear that if f is continuous at $x_0 \in S$ and there exist $(x_n) \subset S$ and $(\varepsilon_n) \subset \mathbb{R}_+$ such that $x_n \in \mathrm{AE}(f, S, C, \varepsilon_n)$, $x_n \to x_0$ and $\varepsilon_n \downarrow 0$ then $x_0 \in \mathrm{WAE}(f, S, C, 0)$.

Remark 3.7 Theorem 3.5 holds if we change C by $\mathrm{int}(C)$ and $\mathrm{AE}(f, S, C, \varepsilon)$ by $\mathrm{WAE}(f, S, C, \varepsilon)$, since $\mathrm{int}(C)$ is also a nonempty solid pointed convex co-radiant set. In this case, let us observe that $\mathrm{WAE}(f, S, \mathrm{int}(C), 0) = \mathrm{WAE}(f, S, C, 0)$.

4 Relations with other ε-efficiency concepts

In this section, we suppose that $0 \in D$ and we prove that several well-known ε-efficiency concepts are particular cases of our ε-efficiency notion by choosing suitable sets C in Definition 3.2.

4.1 ε-efficiency in the senses of Kutateladze and Németh

Let $q \in D\backslash\{0\}$. It is clear that

$$C := q + D$$

is a solid pointed convex co-radiant set, since $\mathrm{int}(C) = q + \mathrm{int}(D)$, $C \subset D$ and

$$\alpha C = q + ((\alpha - 1)q + \alpha D) \subset q + D = C, \quad \forall \alpha > 1.$$

Moreover, $C(\varepsilon) = \varepsilon q + D, \forall \varepsilon > 0$. Then, we can consider the set of ε-efficient solutions of (P) with respect to C and for each $\varepsilon > 0$ we have

$$x \in \mathrm{AE}(f, S, C, \varepsilon) \iff x \in S,\ (f(x) - \varepsilon q - D) \cap f(S) \subset \{f(x)\}. \quad (4.1)$$

As D is a pointed cone, it follows that (4.1) is equivalent to

$$x \in \mathrm{AE}(f, S, C, \varepsilon) \iff x \in S,\ (f(x) - \varepsilon q - D) \cap f(S) = \emptyset. \quad (4.2)$$

This concept was introduced by Kutateladze (1979) and it is the most popular notion of ε-efficiency (see Helbig and Pateva 1994; Kutateladze 1979; Staib 1988;

White 1986; Yokoyama 1999, for more details about it). We denote the set of ε-efficient (resp. weak ε-efficient) solutions of (P) in this sense by $\mathrm{AEKu}(f, S, D, q, \varepsilon)$ (resp. $\mathrm{WAEKu}(f, S, D, q, \varepsilon)$).

Next, let us consider

$$C := H + D,$$

where $H \subset D\backslash\{0\}$ is a nonempty D-convex set, i.e., such that $H + D$ is a convex set. This set C becomes the previous one considering $H = \{q\}$. C is a pointed convex set, since $C \subset D$ and D is a pointed cone. Moreover, as

$$C = \bigcup_{q \in H} (q + D)$$

and $q + D$ is a solid co-radiant set, $\forall q \in H$, then C is a solid co-radiant set,

$$C(\varepsilon) = \bigcup_{q \in H} \varepsilon(q + D) = \bigcup_{q \in H} (\varepsilon q + D) = \varepsilon H + D, \quad \forall \varepsilon > 0 \tag{4.3}$$

and an ε-efficiency notion can be deduced from Definition 3.2 by taking $C = H + D$. With this notion, for each $\varepsilon > 0$ the following ε-efficiency set is obtained:

$$x \in \mathrm{AE}(f, S, C, \varepsilon) \iff x \in S, \ (f(x) - \varepsilon H - D) \cap f(S) \subset \{f(x)\}. \tag{4.4}$$

As D is a pointed cone, for each $\varepsilon > 0$ condition (4.4) becomes

$$x \in \mathrm{AE}(f, S, C, \varepsilon) \iff x \in S, \ (f(x) - \varepsilon H - D) \cap f(S) = \emptyset.$$

This notion was introduced by Németh (1986). We denote the set of ε-efficiency (resp. weak ε-efficiency) in the sense of Németh by $\mathrm{AENe}(f, S, D, H, \varepsilon)$ (resp. $\mathrm{WAENe}(f, S, D, H, \varepsilon)$), i.e., $\mathrm{AENe}(f, S, D, H, \varepsilon) = \mathrm{AE}(f, S, C, \varepsilon)$ (resp. $\mathrm{WAENe}(f, S, D, H, \varepsilon) = \mathrm{WAE}(f, S, C, \varepsilon)$), $\forall \varepsilon \geq 0$, with $C = H + D$.

Some properties of this kind of approximate solutions are collected in Proposition 4.3. The following lemma is necessary.

Lemma 4.1

(i) $\mathrm{int}(D) \subset C(0) \subset D\backslash\{0\}$.
(ii) $H \subset \mathrm{int}(D) \iff C(0) = \mathrm{int}(D)$.
(iii) $\mathrm{bd}(D) \cap (D\backslash\{0\}) \subset \mathrm{cone}(H) \Rightarrow C(0) = D\backslash\{0\}$.
(iv) $\mathrm{int}(C)(0) = \mathrm{int}(D)$.

Proof Part (*i*). Let $d \in \mathrm{int}(D)$. Taking a point $q \in H$ we deduce that there exists $\varepsilon > 0$ such that $d - \varepsilon q \in D$. Therefore, $d \in \varepsilon q + D \subset \varepsilon H + D$ and we see that $\mathrm{int}(D) \subset C(0)$. Moreover, for each $\varepsilon > 0$ it is clear that $\varepsilon H + D \subset D\backslash\{0\}$, since $H \subset D\backslash\{0\}$ and D is a pointed convex cone. Thus, by (3.1) and (4.3) we have that $C(0) \subset D\backslash\{0\}$.

Part (*ii*). Suppose that $H \subset \mathrm{int}(D)$. As D is a solid convex cone, it follows that

$$C(\varepsilon) = \varepsilon H + D \subset \varepsilon\, \mathrm{int}(D) + D \subset \mathrm{int}(D)\,, \quad \forall \varepsilon > 0,$$

and $C(0) \subset \mathrm{int}(D)$. By part (*i*) we have the converse inclusion and so $\mathrm{int}(D) = C(0)$.

Next, suppose $C(0) = \text{int}(D)$. Then, taking $\varepsilon = 1$ in (4.3) we deduce that $H \subset H + D \subset C(0) = \text{int}(D)$.

Part (*iii*). From part (*i*) we see that $\text{int}(D) \subset C(0)$ and by the hypothesis,

$$\text{bd}(D) \cap (D\backslash\{0\}) \subset \text{cone}(H) = \bigcup_{\alpha>0} \alpha H \subset C(0) \tag{4.5}$$

Thus, it follows that $D\backslash\{0\} = \text{int}(D) \cup (\text{bd}(D) \cap (D\backslash\{0\})) \subset C(0)$.

Finally, part (*iv*) is a direct consequence of part (*i*) and (3.4). □

Example 4.2 In $\mathbb{R}^3$, consider $H = \{(\alpha, 1 - \alpha, 0) : 0 \leq \alpha \leq 1\}$.

(a) If $D = \mathbb{R}^3_+\backslash\{(0, 0, y_3) : y_3 > 0\}$ then these data show that the converse of Lemma 4.1(*iii*) is false.

(b) If $D = \mathbb{R}^3_+$, then these data show that the inclusions in Lemma 4.1(*i*) can be strict.

Proposition 4.3

(i) *If* $H \subset \text{int}(D)$ *then*

$$\bigcap_{\varepsilon>0} \text{AENe}(f, S, D, H, \varepsilon) = \text{WE}(f, S, D)\,,$$

if $\text{bd}(D) \cap (D\backslash\{0\}) \subset \text{cone}(H)$ *then*

$$\bigcap_{\varepsilon>0} \text{AENe}(f, S, D, H, \varepsilon) = \text{E}(f, S, D)\,,$$

otherwise

$$\text{E}(f, S, D) \subset \bigcap_{\varepsilon>0} \text{AENe}(f, S, D, H, \varepsilon) \subset \text{WE}(f, S, D)\,.$$

(ii) *Let* $(x_n) \subset S$, $(\varepsilon_n) \subset \mathbb{R}_+$ *and* $y \in \mathbb{R}^p$ *such that* $x_n \in \text{AENe}(f, S, D, H, \varepsilon_n)$, $\varepsilon_n \downarrow 0$ *and* $f(x_n) \to y$. *Then* $f^{-1}(y) \cap S \subset \text{WE}(f, S, D)$.

(iii) *Let* $(x_n) \subset S$ *and* $(\varepsilon_n) \subset \mathbb{R}_+$ *such that* $x_n \in \text{AENe}(f, S, D, H, \varepsilon_n)$ *and* $\varepsilon_n \downarrow 0$. *Consider*

$$K := \bigcap_n (f(x_n) - \varepsilon_n H - D).$$

Then $f^{-1}(K) \cap S \subset \text{WE}(f, S, D)$ *and if* $\text{bd}(D) \cap (D\backslash\{0\}) \subset \text{cone}(H)$ *then* $f^{-1}(K) \cap S \subset \text{E}(f, S, D)$.

Proof If $H \subset \text{int}(D)$, then by Lemma 4.1(*ii*) we deduce that $C(0) = \text{int}(D)$, and so $\text{AENe}(f, S, D, H, 0) = \text{WE}(f, S, D)$. From Lemma 4.1(*iv*), it follows that $\text{int}(C)(0) = \text{int}(D)$ and we have $\text{WAENe}(f, S, D, H, 0) = \text{WE}(f, S, D)$. If $\text{bd}(D) \cap (D\backslash\{0\}) \subset \text{cone}(H)$, we deduce from Lemma 4.1(*iii*) that $C(0) = D\backslash\{0\}$ and $\text{AENe}(f, S, D, H, 0) = \text{E}(f, S, D)$. Then properties (*i*)-(*iii*) follow from Theorem 3.5(*iii*)-(*v*) taking into account that $\text{E}(f, S, D) \subset \text{AENe}(f, S, D, H, 0) \subset \text{WE}(f, S, D)$ by Lemma 4.1(*i*). □

In Proposition 4.3 we have extended several properties proved in the literature for the ε-efficiency set in the sense of Kutateladze (see for example Helbig and Pateva 1994, Lemma 3.3 and Theorem 3.4) to the approximate solutions in the sense of Németh. We can deduce these properties by considering $H = \{q\}$ in Proposition 4.3.

4.2 ε-efficiency in the sense of Helbig

Let $h \in D^+\backslash\{0\}$. For each $\alpha \in \mathbb{R}$ we denote

$$[h > \alpha] = \{y \in \mathbb{R}^p : \langle h, y\rangle > \alpha\}.$$

Consider the following set:

$$C := D \cap [h > 1]. \tag{4.6}$$

Lemma 4.4

(i) *C is a solid pointed convex co-radiant set.*
(ii) $C(\varepsilon) = D \cap [h > \varepsilon]$, $\forall \varepsilon \geq 0$.
(iii) $\mathrm{int}(D) = \mathrm{int}(C)\,(0) \subset C(0) \subset D\backslash\{0\}$.
(iv) *If $h \in D^{s+}$ then $C(0) = D\backslash\{0\}$.*

Proof Part (*i*). It is obvious that C is a pointed convex co-radiant set and we prove just that C is solid. Indeed, as D is a proper solid cone, there exists $d \in \mathrm{int}(D)$ such that $\langle h, d\rangle = \alpha > 0$. Then $(2/\alpha)d \in \mathrm{int}(D) \cap [h > 1] = \mathrm{int}(C)$ and C is solid.

Part (*ii*). For $\varepsilon > 0$ it follows easily since D is a cone and $\langle h, \cdot\rangle$ is a linear function. For $\varepsilon = 0$ it is clear.

Part (*iii*). Let $d \in \mathrm{int}(D)$. As $h \in D^+\backslash\{0\}$ we see that $\langle h, d\rangle > 0$. Then, there exists $\varepsilon > 0$ such that $d \in [h > \varepsilon]$ and we deduce that $d \in C(\varepsilon)$. Thus, it follows that $\mathrm{int}(D) \subset C(0)$.

By part (*ii*) it is obvious that $C(0) \subset D\backslash\{0\}$. Hence we deduce that $\mathrm{int}(D) = \mathrm{int}(C(0))$ and by (3.4) we see that $\mathrm{int}(C(0)) = \mathrm{int}(C)\,(0)$. Therefore it follows that $\mathrm{int}(D) = \mathrm{int}(C)\,(0) \subset C(0)$.

Part (*iv*). If $h \in D^{s+}$ then $\langle h, d\rangle > 0$, $\forall d \in D\backslash\{0\}$ and we see that $D\backslash\{0\} \subset C(0)$. By part (*iii*) we have the converse inclusion and so $C(0) = D\backslash\{0\}$. □

By Definition 3.2, for each $\varepsilon \geq 0$ we obtain the following ε-efficiency set:

$$\begin{aligned} x \in \mathrm{AE}(f, S, C, \varepsilon) &\iff x \in S,\ (f(x) - (D \cap [h > \varepsilon])) \cap f(S) = \emptyset \\ &\iff x \in S,\ (f(x) - f(S)) \cap (D \cap [h > \varepsilon]) = \emptyset \\ \iff x \in S,\ \langle h, f(x)\rangle - \varepsilon &\leq \langle h, f(z)\rangle, \quad \forall z \in S,\ f(z) \in f(x) - D. \end{aligned}$$

This notion was introduced by Helbig (1992). We denote the set of ε-efficient (resp. weak ε-efficient) solutions in this sense by $\mathrm{AEHe}(f, S, D, h, \varepsilon)$ (resp. $\mathrm{WAEHe}(f, S, D, h, \varepsilon)$). Their elements satisfy the following properties.

Proposition 4.5

(i) *If $h \in D^+\backslash\{0\}$ then*

$$\mathrm{E}(f, S, D) \subset \bigcap_{\varepsilon>0} \mathrm{AEHe}(f, S, D, h, \varepsilon) \subset \mathrm{WE}(f, S, D)\,,$$

and if $h \in D^{s+}$ then

$$\bigcap_{\varepsilon>0} \mathrm{AEHe}(f, S, D, h, \varepsilon) = \mathrm{E}(f, S, D)\,.$$

(ii) *Let* $(x_n) \subset S$, $(\varepsilon_n) \subset \mathbb{R}_+$ *and* $y \in \mathbb{R}^p$ *such that* $x_n \in \mathrm{AEHe}(f, S, D, h, \varepsilon_n)$, $\varepsilon_n \downarrow 0$ *and* $f(x_n) \to y$. *Then* $f^{-1}(y) \cap S \subset \mathrm{WE}(f, S, D)$.

(iii) *Let* $(x_n) \subset S$ *and* $(\varepsilon_n) \subset \mathbb{R}_+$ *such that* $x_n \in \mathrm{AEHe}(f, S, D, h, \varepsilon_n)$ *and* $\varepsilon_n \downarrow 0$. *Consider*

$$K := \bigcap_n (f(x_n) - (D \cap [h > \varepsilon_n])).$$

If $h \in D^+\backslash\{0\}$ *then* $f^{-1}(K) \cap S \subset \mathrm{WE}(f, S, D)$, *and if* $h \in D^{s+}$ *then* $f^{-1}(K) \cap S \subset \mathrm{E}(f, S, D)$.

Proof From Lemma 4.4(*iii*) we deduce that

$$\mathrm{E}(f, S, D) \subset \mathrm{AEHe}(f, S, D, h, 0) \subset \mathrm{WE}(f, S, D), \quad \forall h \in D^+\backslash\{0\}.$$

Moreover, by Lemma 4.4(*iii*)–(*iv*), we have that $\mathrm{AEHe}(f, S, D, h, 0) = \mathrm{E}(f, S, D)$, $\forall h \in D^{s+}$ and for each $h \in D^+\backslash\{0\}$ we see that $\mathrm{WAEHe}(f, S, D, h, 0) = \mathrm{WE}(f, S, D)$. Then, parts (*i*)-(*iii*) follow easily from Theorem 3.5(*iii*)–(*v*). □

Remark 4.6 As an example of Definition 3.2, let us introduce a new ε-efficiency concept, which extends the previous one due to Helbig. Consider $h_1, h_2, \ldots, h_m \in D^+\backslash\{0\}$ and let $h : \mathbb{R}^p \to \mathbb{R}$ be the function $h(y) := \min\{h_i(y) : i = 1, 2, \ldots, m\}$. It is clear that $C := D \cap [h > 1]$ is a solid pointed convex co-radiant set and

$$C(\varepsilon) = D \cap [h > \varepsilon], \quad \forall \varepsilon \geq 0.$$

Then, from Definition 3.2 the following ε-efficiency concept is obtained: $x \in S$ is an ε-efficient solution of (P) with respect to C if and only if for each $z \in S$ such that $f(z) \in f(x) - D$ there exists $i \in \{1, 2, \ldots, m\}$ such that $h_i(f(x)) - \varepsilon \leq h_i(f(z))$.

Lemma 4.4 and Proposition 4.5 can be easily generalized in order to obtain several properties of these ε-efficient solutions.

4.3 ε-efficiency in the sense of Tanaka

Next, we suppose that $D \subset \mathbb{R}^p_+$ and we define the set

$$C := D \cap \mathrm{B}^c,$$

where B denotes the unit open ball in $\mathbb{R}^p$,

$$\mathrm{B} = \{y \in \mathbb{R}^p : \|y\|_1 < 1\},$$

and $\|\cdot\|_1$ is the l_1 norm in $\mathbb{R}^p$.

Lemma 4.7

(i) *C is a solid pointed convex co-radiant set.*

(ii) $C(\varepsilon) = D \cap (\varepsilon \mathrm{B})^c$, $\forall \varepsilon > 0$.

(iii) $C(0) = D\backslash\{0\}$ *and* $\mathrm{int}(C)(0) = \mathrm{int}(D)$.

Proof Part (*i*). It is obvious that C is convex. Moreover, C is a pointed set, since $C \subset D$ and D is a pointed cone. Let $y \in C$ and $\alpha > 1$. As D is a cone, we have that $\alpha y \in D$. Moreover, $\|\alpha y\|_1 = \alpha \|y\|_1 > 1$ since $\alpha > 1$ and $y \notin \mathrm{B}$. Then $\alpha y \in C$ and it follows that C is a co-radiant set.

Next, we prove that C is a solid set. Indeed, there exists a point $q \in \mathrm{int}(D)$, $q \neq 0$, since D is a solid cone. Then, $2q/\|q\|_1 \in \mathrm{int}(D) \cap \mathrm{int}(\mathrm{B}^c) = \mathrm{int}(C)$ and C is a solid set.

Part (*ii*). Let $y \in C$ and $\varepsilon > 0$. It is clear that $\varepsilon y \in D$ and $\|\varepsilon y\|_1 = \varepsilon \|y\|_1 \geq \varepsilon$ since $y \in \mathrm{B}^c$. It follows that $\varepsilon y \in D \cap (\varepsilon \mathrm{B})^c$ and $C(\varepsilon) \subset D \cap (\varepsilon \mathrm{B})^c$. Similarly, if $y \in D \cap (\varepsilon \mathrm{B})^c$ then $y/\varepsilon \in D \cap \mathrm{B}^c = C$. Thus, $y \in \varepsilon C$ and $C(\varepsilon) = D \cap (\varepsilon \mathrm{B})^c$.

Part (*iii*). By part (*ii*) it is clear that

$$C(0) = \bigcup_{\varepsilon>0} D \cap (\varepsilon \mathrm{B})^c = D \cap \left(\bigcap_{\varepsilon>0} \varepsilon \mathrm{B} \right)^c = D \backslash \{0\}.$$

Analogously,

$$\begin{aligned} \mathrm{int}(C)\,(0) &= \bigcup_{\varepsilon>0} \mathrm{int}\big(D \cap (\varepsilon \mathrm{B})^c\big) = \bigcup_{\varepsilon>0} \mathrm{int}(D) \cap \mathrm{int}\big((\varepsilon \mathrm{B})^c\big) \\ &= \mathrm{int}(D) \cap \left(\bigcap_{\varepsilon>0} \mathrm{cl}(\varepsilon \mathrm{B}) \right)^c = \mathrm{int}(D)\,. \end{aligned}$$

□

The set of all approximate solutions (resp. weak approximate solutions) with respect to this set C is denoted by $\mathrm{AETa}(f, S, D, \varepsilon)$ (resp. $\mathrm{WAETa}(f, S, D, \varepsilon)$). It follows that

$$\begin{aligned} x \in \mathrm{AETa}(f, S, D, \varepsilon) &\iff x \in S,\ (f(x) - (D \cap (\varepsilon \mathrm{B})^c)) \cap f(S) \subset \{f(x)\} \\ &\iff x \in S,\ (f(x) - D) \cap f(S) \subset f(x) + \varepsilon \mathrm{B}. \end{aligned}$$

This ε-efficiency concept was introduced by Tanaka (1995) and it is equivalent to other previous one due to White (1986) when the open ball B is given by the l_∞ norm (see Yokoyama 1996, Proposition 3.2, for more details).

Next, we give several properties of this notion, which extend others previously proved by Tanaka 1995, Proposition 3.3.

Proposition 4.8

(i) $\bigcap_{\varepsilon>0} \mathrm{AETa}(f, S, D, \varepsilon) = \mathrm{E}(f, S, D)$.
(ii) *Let* $(x_n) \subset S$, $(\varepsilon_n) \subset \mathbb{R}_+$ *and* $y \in \mathbb{R}^p$ *such that* $x_n \in \mathrm{AETa}(f, S, D, \varepsilon_n)$, $\varepsilon_n \downarrow 0$ *and* $f(x_n) \to y$. *Then* $f^{-1}(y) \cap S \subset \mathrm{WE}(f, S, D)$.
(iii) *Let* $(x_n) \subset S$ *and* $(\varepsilon_n) \subset \mathbb{R}_+$ *such that* $x_n \in \mathrm{AETa}(f, S, D, \varepsilon_n)$, $\varepsilon_n \downarrow 0$ *and*

$$K := \bigcap_n (f(x_n) - (D \cap (\varepsilon_n \mathrm{B})^c)).$$

Then $f^{-1}(K) \cap S \subset \mathrm{E}(f, S, D)$.

Proof By Lemma 4.7(*iii*) we deduce that $\mathrm{AETa}(f, S, D, 0) = \mathrm{E}(f, S, D)$ and $\mathrm{WAETa}(f, S, D, 0) = \mathrm{WE}(f, S, D)$. Then, the proposition is a consequence of Theorem 3.5(*iii*)–(*v*). □

5 Linear scalarization, ε-efficiency and parametric representation in convex multiobjective programs

In the literature, approximate solutions of (P) are usually studied in convex problems via the Kutateladze's definition (see for example White 1986, Lemma 3.2; Deng 1997, Theorem 2.1; Liu and Yokoyama 1999, Theorems 1 and 2; Liu 1999, Lemma 2.1; Dutta and Vetrivel 2001, Theorem 2.1). However, results about ε-efficiency notions different to the Kutateladze's concept are very limited (see Yokoyama 1996, Lemmas 4.1 and 4.2; Gutiérrez et al. 2005b, Lemma 3.1).

Our objective in this section is to characterize the ε-efficient solutions with respect to a set C, in order to obtain additional results about ε-efficiency concepts different to the Kutateladze's notion by applying this characterization to suitable sets C.

5.1 Necessary conditions

Next, necessary conditions for the approximate solutions of (P) with respect to a set C are obtained via approximate solutions of linear scalarizations, i.e., by the Weighting Method. With this aim, we suppose that the objective function of (P) satisfies the following generalized convexity condition.

Definition 5.1 (Frenk and Kassay 1999, Definition 2.4) *It is said that $f : S \subset \mathbb{R}^n \to \mathbb{R}^p$ is a subconvexlike function on S with respect to a solid convex cone $K \subset \mathbb{R}^p$ if the set $f(S) + \operatorname{int}(K)$ is convex.*

Let us observe that if $M \subset \mathbb{R}^p$ is a solid convex cone such that $\operatorname{int}(M) = \operatorname{int}(K)$ and f is subconvexlike on S with respect to K, then f is also subconvexlike on S with respect to M.

It is well-known that if $f : \mathbb{R}^n \to \mathbb{R}^p$ is K-convex then f is subconvexlike on S with respect to K for each convex set $S \subset \mathbb{R}^n$ (see Frenk and Kassay 1999).

For each $K \subset \mathbb{R}^p$ and $y \in \mathbb{R}^p$ we denote $d(y, K) = \inf\{\|y - z\| : z \in K\}$, where $\|\cdot\|$ is the Euclidean norm.

Theorem 5.2 *Consider program* (P) *and suppose that f is subconvexlike on S with respect to $C(0)$ and $d(0, C) \leq \delta$. Then, $\forall \varepsilon \geq 0$,*

$$\operatorname{WAE}(f, S, C, \varepsilon) \subset \bigcup_{l \in C(0)^+,\, \|l\|=1} \operatorname{AMin}(l \circ f, S, \varepsilon\delta)\,,$$

where $l \circ f : \mathbb{R}^n \to \mathbb{R}$ is the function $\langle l, f(\cdot)\rangle$.

Proof Let $\varepsilon > 0$ and $x_0 \in \operatorname{WAE}(f, S, C, \varepsilon)$. Then $x_0 \in S$ and

$$(f(x_0) - \operatorname{int}(C)(\varepsilon)) \cap f(S) = \emptyset,$$

since $0 \notin \operatorname{int}(C)(\varepsilon)$. By Lemma 3.1($v$) it follows that

$$(f(x_0) - \operatorname{int}(C)(\varepsilon)) \cap (f(S) + \operatorname{int}(C)(0)) = \emptyset,$$

and by Definition 5.1 we see that $f(S) + \operatorname{int}(C)(0)$ is a convex set, since from (3.4) we have that $\operatorname{int}(C)(0) = \operatorname{int}(C(0))$ and f is a subconvexlike function on

S with respect to $C(0)$. Then, by the Separation Theorem (see, for example, Jahn 2004, Theorem 3.14) we deduce that there exists $l \in \mathbb{R}^p\backslash\{0\}$ such that

$$\langle l, f(x_0)-\varepsilon d_1\rangle \leq \langle l, f(x)+d_2\rangle, \quad \forall d_1 \in C, \quad \forall d_2 \in \text{int}(C)\,(0), \quad \forall x \in S. \quad (5.1)$$

We can suppose that $\|l\| = 1$ since $l \neq 0$. Moreover, by continuity, (5.1) holds $\forall d_2 \in \text{cl}(C(0))$ since $C(0)$ is convex. As $C(0)$ is a cone we deduce that $l \in C(0)^+$ and by (3.4) we see that

$$\text{cl}(\text{int}(C)\,(0)) = \text{cl}(\text{int}(C(0))) = \text{cl}(C(0))\,.$$

Taking $d_2 = 0 \in \text{cl}(C(0))$ in (5.1) we obtain

$$\langle l, f(x_0)\rangle - \varepsilon\langle l, d_1\rangle \leq \langle l, f(x)\rangle, \quad \forall d_1 \in C, \quad \forall x \in S,$$

and by the Cauchy–Schwartz inequality we have that

$$\langle l, f(x_0)\rangle - \varepsilon\|d_1\| \leq \langle l, f(x)\rangle, \quad \forall d_1 \in C, \quad \forall x \in S.$$

By assumption and by the definition of $d(0, C)$ we derive that

$$\langle l, f(x_0)\rangle - \varepsilon\delta \leq \langle l, f(x_0)\rangle - \varepsilon d(0, C) \leq \langle l, f(x)\rangle, \quad \forall x \in S,$$

and it follows that $x_0 \in \text{AMin}(l \circ f, S, \varepsilon\delta)$.

If $\varepsilon = 0$ and $x_0 \in \text{AE}(f, S, C, 0)$ then, repeating the same reasoning we deduce that

$$\langle l, f(x_0)\rangle \leq \langle l, f(x)\rangle, \quad \forall x \in S,$$

since (5.1) is true for all $\varepsilon > 0$. Therefore, it follows that $x_0 \in \text{AMin}(l \circ f, S, 0)$ with $l \in C(0)^+$ and $\|l\| = 1$. □

5.2 Sufficient conditions

Next, we obtain sufficient conditions for ε-efficient solutions of (P) with respect to a set C. For this we need the following result (see Bolintineanu 2001, Lemma 2.7 for more detail).

Lemma 5.3 *Let $K \subset \mathbb{R}^p$ be a convex cone such that K^+ is solid. If $l \in \text{int}(K^+)$ then*

$$d(l, \mathbb{R}^p\backslash K^+) \leq \inf\{\langle l, y\rangle : y \in K, \|y\| = 1\}.$$

Theorem 5.4 *Suppose that $0 \notin \text{cl}(C)$, $C(0)^+$ is solid and consider $l \in \text{int}(C(0)^+)$ and $\delta \geq 0$. Then*

$$\text{AMin}(l \circ f, S, \delta) \subset \text{AE}(f, S, C, \varepsilon)\,, \quad \forall \varepsilon > \delta/c,$$
$$\text{AMin}(l \circ f, S, \delta) \subset \text{WAE}(f, S, C, \varepsilon)\,, \quad \forall \varepsilon \geq \delta/c,$$

where $c = d(0, C) \cdot d(l, \mathbb{R}^p\backslash C(0)^+)$.

Proof Let $x_0 \in \text{AMin}(l \circ f, S, \delta)$ and $\varepsilon > \delta/c$. Reasoning "ad absurdum", let us suppose that $x_0 \notin \text{AE}(f, S, C, \varepsilon)$. Then there exist $x \in S$ and $d \in C(\varepsilon)$, $d \neq 0$, such that $f(x_0) - d = f(x)$. As l is a linear function and $x_0 \in \text{AMin}(l \circ f, S, \delta)$ we deduce that

$$\langle l, f(x)\rangle = \langle l, f(x_0)\rangle - \langle l, d\rangle \leq \langle l, f(x)\rangle + \delta - \langle l, d\rangle$$

and it follows that $\langle l, d\rangle \leq \delta$. By Lemma 5.3 we have that

$$\langle l, d\rangle \geq \|d\| d(l, \mathbb{R}^p \backslash C(0)^+) \geq \varepsilon d(0, C) d(l, \mathbb{R}^p \backslash C(0)^+) = \varepsilon c > \delta,$$

which is a contradiction. Therefore, $x_0 \in \text{AE}(f, S, C, \varepsilon)$.

By the same reasoning we see that if $\varepsilon \geq \delta/c$ and $x_0 \notin \text{WAE}(f, S, C, \varepsilon)$ then there exists $d \in \text{int}(C)\,(\varepsilon)$ such that $\langle l, d\rangle \leq \delta$. If $\varepsilon > 0$ then $\|d\| > \varepsilon d(0, C)$ since $(1/\varepsilon)d \in \text{int}(C)$, and by Lemma 5.3 it follows that

$$\langle l, d\rangle \geq \|d\| d(l, \mathbb{R}^p \backslash C(0)^+) > \varepsilon d(0, C) d(l, \mathbb{R}^p \backslash C(0)^+) = \varepsilon c \geq \delta,$$

which is a contradiction. If $\varepsilon = 0$ then $\delta = 0$ and so $\langle l, d\rangle = 0$, which is contrary to

$$\langle l, d\rangle \geq \|d\| d(l, \mathbb{R}^p \backslash C(0)^+) > 0.$$

Therefore, we conclude that $x_0 \in \text{WAE}(f, S, C, \varepsilon)$. □

There are some particular ε-efficiency notions for which it is possible to obtain other results similar to Theorem 5.4, because for a fixed $l \in C(0)^+$ there exists a specific positive lower bound for the images $\langle l, d\rangle$ when $d \in C$. The following theorem shows one of them.

Theorem 5.5 *Let* $H \subset \text{int}(D)$ *be a nonempty compact* D*-convex set. Consider* $0 \in D, l \in D^+ \backslash \{0\}$ *and* $\delta \geq 0$. *Then*

$$\text{AMin}(l \circ f, S, \delta) \subset \text{AENe}(f, S, D, H, \varepsilon)\,, \quad \forall \varepsilon > \delta/m_l,$$
$$\text{AMin}(l \circ f, S, \delta) \subset \text{WAENe}(f, S, D, H, \varepsilon)\,, \quad \forall \varepsilon \geq \delta/m_l,$$

where $m_l = \min\{\langle l, d\rangle : d \in H\}$.

Proof Firstly, notice that $\text{AENe}(f, S, D, H, \varepsilon)$ is the ε-efficiency set with respect to $C = H + D$. Let $x_0 \in \text{AMin}(l \circ f, S, \delta)$, $\varepsilon > \delta/m_l$ and suppose that $x_0 \notin \text{AENe}(f, S, D, H, \varepsilon)$. As in the proof of Theorem 5.4, we see that there exists $d \in C(\varepsilon)$ such that $\langle l, d\rangle \leq \delta$. By (4.3) we have that $C(\varepsilon) = \varepsilon H + D$ and we deduce that there exists $q \in H$ such that $d \in \varepsilon q + D$. As $l \in D^+$ it follows that

$$\langle l, d\rangle \geq \varepsilon \langle l, q\rangle \geq \varepsilon\, m_l > \delta$$

which is a contradiction.

By the same reasoning we have that if $x_0 \notin \text{WAENe}(f, S, D, H, \varepsilon)$, $\varepsilon \geq \delta/m_l$ and $\varepsilon > 0$ then there exists $d \in D \backslash \{0\}$ such that $\langle l, d\rangle \leq \delta$ and $(1/\varepsilon)d \in \text{int}(H + D)$. As D is a solid convex cone, by Frenk and Kassay (1999, Theorem 3.2) it follows that $\text{int}(H + D) = \text{int}(H + \text{int}(D))$, and since $H + \text{int}(D) =$

$\cup_{q\in H}(q+\text{int}(D))$ is an open set, we obtain that $\text{int}(H+D)=H+\text{int}(D)$. Hence there exists $q\in H$ such that $d\in\varepsilon q+\text{int}(D)$ and

$$\langle l,d\rangle>\varepsilon\langle l,q\rangle\geq\varepsilon\, m_l\geq\delta$$

which is a contradiction. If $\varepsilon=0$ then $\delta=0$ and $d\in\text{int}(C)(0)$. From Lemma 4.1(*iv*) we see that $\text{int}(C)(0)=\text{int}(D)$ and so $\langle l,d\rangle>0$, which is contrary to $\langle l,d\rangle\leq\delta=0$. □

Corollary 5.6 *Let $H\subset\text{int}(D)$ be a nonempty compact D-convex set, $0\in D$ and $\delta\geq 0$. Then*

$$\bigcup_{l\in D^+,\ \|l\|=1}\text{AMin}(l\circ f,S,\delta)\subset\text{AENe}(f,S,D,H,\varepsilon),\quad\forall\varepsilon>\delta/m,$$

$$\bigcup_{l\in D^+,\ \|l\|=1}\text{AMin}(l\circ f,S,\delta)\subset\text{WAENe}(f,S,D,H,\varepsilon),\quad\forall\varepsilon\geq\delta/m,$$

where $m=\min\{\langle l,d\rangle: l\in D^+,\ \|l\|=1, d\in H\}$.

Proof By Theorem 5.5 and Theorem 3.5(*ii*) we deduce that

$$\bigcup_{l\in D^+,\ \|l\|=1}\text{AMin}(l\circ f,S,\delta)\subset\text{AENe}(f,S,D,H,\varepsilon),\quad\forall\varepsilon>k,\tag{5.2}$$

$$\bigcup_{l\in D^+,\ \|l\|=1}\text{AMin}(l\circ f,S,\delta)\subset\text{WAENe}(f,S,D,H,\varepsilon),\quad\forall\varepsilon\geq k,\tag{5.3}$$

where $k=\delta/\inf\{m_l: l\in D^+,\ \|l\|=1\}$. As $\{(l,d)\in\mathbb{R}^p\times\mathbb{R}^p: l\in D^+,\|l\|=1, d\in H\}$ is a compact set, by the Weierstrass' Theorem we deduce that

$$m=\min\{\langle l,d\rangle: l\in D^+,\|l\|=1,d\in H\}=\inf\{m_l\in\mathbb{R}_+: l\in D^+,\|l\|=1\}>0,$$

and so $k=\delta/m$ and we obtain the conclusion from (5.2) and (5.3). □

5.3 Parametric representations

In the literature, necessary and sufficient conditions on the efficient solutions of multiobjective mathematical programs obtained via scalarization are frequently used to attain parametric representations of the efficiency set (see for example Wierzbicki 1986, Sect. 2). Next, following this idea, we provide parametric representations of the ε-efficiency sets via linear scalarizations and Theorems 5.2 and 5.4 in multiobjective mathematical programs whose objective functions are subconvexlike on the feasible set.

Previously, we recall a definition of parametric representation introduced by the authors in (Gutiérrez et al. 2006a,b), which extends the notion of parametric representation of efficiency sets to ε-efficiency sets.

Let $C\subset\mathbb{R}^p$ be a solid pointed convex co-radiant set and let $\{\varphi_\alpha\}_{\alpha\in\mathcal{P}}$ be a family of scalar functions $\varphi_\alpha:\mathbb{R}^p\to\mathbb{R}$, where $\mathcal{P}$ is a parametric index set. Consider the following two properties:

(P1) There exist a subset $\mathcal{P}_s$ of $\mathcal{P}$ and $c_s > 0$ such that

$$\bigcup_{\alpha\in\mathcal{P}_s} \text{AMin}(\varphi_\alpha \circ f, S, \delta) \subset \text{AE}(f, S, C, c_s\delta)\,, \quad \forall\delta \geq 0. \tag{5.4}$$

(P2) There exist a subset $\mathcal{P}_n$ of $\mathcal{P}$ and $c_n > 0$ such that

$$\text{AE}(f, S, C, \varepsilon) \subset \bigcup_{\alpha\in\mathcal{P}_n} \text{AMin}(\varphi_\alpha \circ f, S, c_n\varepsilon)\,, \quad \forall\varepsilon \geq 0. \tag{5.5}$$

Definition 5.7 *We say that* $\{\varphi_\alpha\}_{\alpha\in\mathcal{P}}$ *gives a parametric representation of* $\text{AE}(f, S, C, \varepsilon)$ *if properties* (P1) *and* (P2) *hold. We say that a parametric representation of* $\text{AE}(f, S, C, \varepsilon)$ *is complete when* $\mathcal{P}_s = \mathcal{P}_n = \mathcal{P}$.

Theorem 5.8 *Suppose that* $0 \notin \text{cl}(C)$ *and* $C(0)^+$ *is solid. Consider*

$$\mathcal{P} = \{l \in C(0)^+ : \|l\| = 1\},$$

and a nonempty compact set $\mathcal{F} \subset \{l \in \text{int}\big(C(0)^+\big) : \|l\| = 1\}$. *If* f *is a subconvexlike function on* S *with respect to* $C(0)$*, then the family* $\{\langle l, \cdot\rangle\}_{l\in\mathcal{P}}$ *gives a parametric representation of* $\text{WAE}(f, S, C, \varepsilon)$ *and* $\text{AE}(f, S, C, \varepsilon)$ *with* $\mathcal{P}_n = \mathcal{P}$, $c_n = d(0, C)$, $\mathcal{P}_s = \mathcal{F}$, $c_s = k_s := 1/(m\, d(0, C))$ *and* $\mathcal{P}_n = \mathcal{P}$, $c_n = d(0, C)$, $\mathcal{P}_s = \mathcal{F}$, $c_s > k_s$*, respectively, where* $m := \min\{d(l, \mathbb{R}^p\backslash C(0)^+) : l \in \mathcal{F}\}$.

Proof By Theorem 5.2 we see that

$$\text{AE}(f, S, C, \varepsilon) \subset \text{WAE}(f, S, C, \varepsilon) \subset \bigcup_{l\in\mathcal{P}} \text{AMin}(l \circ f, S, \varepsilon d(0, C))\,, \quad \forall\varepsilon \geq 0$$

and condition (5.5) holds for the ε-efficiency and weak ε-efficiency sets taking $\mathcal{P}_n = \mathcal{P}$ and $c_n = d(0, C)$.

As $\mathcal{F}$ is a compact set it follows that $m > 0$ and by Theorems 5.4 and 3.5(*ii*) we deduce that

$$\bigcup_{l\in\mathcal{F}} \text{AMin}(l \circ f, S, \delta) \subset \text{AE}(f, S, C, \varepsilon)\,, \quad \forall\varepsilon > \delta/(m\, d(0, C)),$$

$$\bigcup_{l\in\mathcal{F}} \text{AMin}(l \circ f, S, \delta) \subset \text{WAE}(f, S, C, \varepsilon)\,, \quad \forall\varepsilon \geq \delta/(m\, d(0, C)).$$

Therefore, condition (5.4) holds for the weak ε-efficiency set taking $\mathcal{P}_s = \mathcal{F}$ and $c_s = k_s$ and for the ε-efficiency set taking $\mathcal{P}_s = \mathcal{F}$ and $c_s > k_s$. □

For the Németh's ε-efficiency notion we have the following complete parametric representation.

Theorem 5.9 *Assume that* $0 \in D$ *and* $H \subset \text{int}(D)$ *is a nonempty compact* D*-convex set. If* f *is a subconvexlike function on* S *with respect to* D *then the family* $\{\langle l, \cdot\rangle\}_{l\in\mathcal{P}}$*, where*

$$\mathcal{P} = \{l \in D^+ : \|l\| = 1\},$$

gives a complete parametric representation of $\text{WAENe}(f, S, D, H, \varepsilon)$ *and* $\text{AENe}(f, S, D, H, \varepsilon)$ *with* $c_n = d(0, H + D)$, $c_s = 1/m$ *and* $c_n = d(0, H + D)$, $c_s > 1/m$*, respectively, where* $m := \min\{\langle l, d\rangle : l \in D^+, \|l\| = 1, d \in H\}$.

Proof Approximate solutions of (P) in the sense of Németh are ε-efficient solutions with respect to $C = H + D$. By Lemma 4.1(*ii*) we see that $C(0) = \text{int}(D)$. Then $C(0)^+ = D^+$ and from Theorem 5.2 and (3.5) we deduce that (5.5) holds for the weak ε-efficiency and ε-efficiency sets taking $\mathcal{P}_n = \mathcal{P}$ and $c_n = d(0, H + D)$.

By Corollary 5.6 we see that (5.4) holds for the weak ε-efficiency and ε-efficiency sets taking $\mathcal{P}_s = \mathcal{P}$, $c_s = 1/m$ and $\mathcal{P}_s = \mathcal{P}$, $c_s > 1/m$, respectively.

Thus, the family $\{\langle l, \cdot\rangle\}_{l\in\mathcal{P}}$ gives a complete parametric representation of $\text{WAENe}(f, S, D, H, \varepsilon)$ and $\text{AENe}(f, S, D, H, \varepsilon)$. □

Remark 5.10 If $\|\cdot\|$ is monotone on D whit respect to the preference relation defined by D, i.e. $\|v + d\| \geq \|v\|$, $\forall v, d \in D$, then $d(0, H + D) = d(0, H)$ and we can replace $d(0, H + D)$ by $d(0, H)$ in the previous theorem.

5.4 Application to convex Pareto programs

In this subsection we consider that (P) is a Pareto multiobjective mathematical program (i.e., $D = \mathbb{R}^p_+$) and we assume that the objective function $f : S \subset \mathbb{R}^n \to \mathbb{R}^p$ is subconvexlike on S with respect to $\mathbb{R}^p_+$. Under these hypotheses we apply Theorems 5.8 and 5.9 to obtain parametric representations of the ε-efficiency sets in the senses of Németh, Helbig and Tanaka via linear scalarizations.

We denote the components of a vector $y \in \mathbb{R}^p$ by y_i, $i = 1, 2, \ldots, p$, and we define the following solid pointed convex co-radiant sets:

$$C_{\text{Ne}} = H + \mathbb{R}^p_+, \quad C_{\text{He}} = \mathbb{R}^p_+ \cap [\langle h, \cdot\rangle > 1], \quad C_{\text{Ta}} = \mathbb{R}^p_+ \cap \mathrm{B}^c,$$

where $H \subset \mathbb{R}^p_+\backslash\{0\}$ is a nonempty compact $\mathbb{R}^p_+$-convex set and $h \in \mathbb{R}^p_+\backslash\{0\}$. Notice that these sets define the Németh, Helbig and Tanaka's ε-efficiency notions, respectively.

Theorem 5.11 *Consider* $\mathcal{P} = \{l \in \mathbb{R}^p_+ : \|l\| = 1\}$, *a nonempty compact set* $\mathcal{F} \subset \{l \in \text{int}(\mathbb{R}^p_+) : \|l\| = 1\}$ *and* $m = \min_{1\leq i\leq p}\{l_i : l \in \mathcal{F}\}$. *The family* $\{\langle l, \cdot\rangle\}_{l\in\mathcal{P}}$ *gives a parametric representation of*

- (i) $\text{WAENe}(f, S, \mathbb{R}^p_+, H, \varepsilon)$ *and* $\text{AENe}(f, S, \mathbb{R}^p_+, H, \varepsilon)$ *with* $\mathcal{P}_n = \mathcal{P}, c_n = d(0, H), \mathcal{P}_s = \mathcal{F}, c_s = 1/(m\, d(0, H))$ *and* $\mathcal{P}_n = \mathcal{P}, c_n = d(0, H), \mathcal{P}_s = \mathcal{F}, c_s > 1/(m\, d(0, H))$, *respectively.*
- (ii) $\text{WAEHe}(f, S, \mathbb{R}^p_+, h, \varepsilon)$ *and* $\text{AEHe}(f, S, \mathbb{R}^p_+, h, \varepsilon)$ *with* $\mathcal{P}_n = \mathcal{P}, c_n = 1/\|h\|, \mathcal{P}_s = \mathcal{F}, c_s = \|h\|/m$ *and* $\mathcal{P}_n = \mathcal{P}, c_n = 1/\|h\|, \mathcal{P}_s = \mathcal{F}, c_s > \|h\|/m$, *respectively.*
- (iii) $\text{WAETa}(f, S, \mathbb{R}^p_+, \varepsilon)$ *and* $\text{AETa}(f, S, \mathbb{R}^p_+, \varepsilon)$ *with* $\mathcal{P}_n = \mathcal{P}, c_n = 1/\sqrt{p}, \mathcal{P}_s = \mathcal{F}, c_s = \sqrt{p}/m$ *and* $\mathcal{P}_n = \mathcal{P}, c_n = 1/\sqrt{p}, \mathcal{P}_s = \mathcal{F}, c_s > \sqrt{p}/m$, *respectively.*

Proof By (3.4) and Lemma 4.1(*iv*), Lemma 4.4(*iii*) and Lemma 4.7(*iii*) we see that

$$\text{int}(C_{\text{Ne}}(0)) = \text{int}(C_{\text{He}}(0)) = \text{int}(C_{\text{Ta}}(0)) = \text{int}(\mathbb{R}^p_+) \tag{5.6}$$

and so

$$C_{\text{Ne}}(0)^+ = C_{\text{He}}(0)^+ = C_{\text{Ta}}(0)^+ = \mathbb{R}^p_+. \tag{5.7}$$

Moreover, easy calculations give

$$d(0, C_{\mathrm{Ne}}) = d(0, H), \tag{5.8}$$
$$d(0, C_{\mathrm{He}}) = 1/\|h\|, \tag{5.9}$$
$$d(0, C_{\mathrm{Ta}}) = 1/\sqrt{p}, \tag{5.10}$$
$$m = \min\{d(l, \mathbb{R}^p \backslash \mathbb{R}^p_+) : l \in \mathcal{F}\} = \min_{1 \le i \le p} \{l_i : l \in \mathcal{F}\}. \tag{5.11}$$

By (5.6) we see that the hypotheses of Theorem 5.8 hold. Then, parts (*i*)–(*iii*) follow from Theorem 5.8 and Eqs. (5.8)-(5.11). □

In (Gutiérrez et al. 2006b, Theorem 3.4(b)) we have obtained a complete parametric representation of $\mathrm{AEKu}\big(f, S, \mathbb{R}^p_+, q, \varepsilon\big)$ and $\mathrm{WAEKu}\big(f, S, \mathbb{R}^p_+, q, \varepsilon\big)$ in Pareto multiobjective programs whose feasible set and objective function are convex. Next, we extend this result to the Németh's ε-efficiency set in Pareto programs whose objective function is subconvexlike on S with respect to $\mathbb{R}^p_+$.

Theorem 5.12 *Let* $H \subset \mathrm{int}\big(\mathbb{R}^p_+\big)$ *be a nonempty compact* $\mathbb{R}^p_+$*-convex set and* $\mathcal{P} = \{l \in \mathbb{R}^p_+ : \|l\| = 1\}$. *The family* $\{\langle l, \cdot\rangle\}_{l \in \mathcal{P}}$ *gives a complete parametric representation of* $\mathrm{WAENe}\big(f, S, \mathbb{R}^p_+, H, \varepsilon\big)$ *and* $\mathrm{AENe}\big(f, S, \mathbb{R}^p_+, H, \varepsilon\big)$ *with* $c_n = d(0, H), c_s = 1/m$ *and* $c_n = d(0, H), c_s > 1/m$, *respectively, where* $m := \min\{\langle l, d\rangle : l \in \mathbb{R}^p_+, \|l\| = 1, d \in H\}$.

Proof By (5.6)–(5.8) we have that $\mathrm{int}(C_{\mathrm{Ne}}(0)) = \mathrm{int}\big(\mathbb{R}^p_+\big)$, $C_{\mathrm{He}}(0)^+ = \mathbb{R}^p_+$ and $d(0, C_{\mathrm{Ne}}) = d(0, H)$. Now, the theorem follows easily from Theorem 5.9. □

6 Conclusions

In this work, a new ε-efficiency notion in multiobjective mathematical programming has been defined. We have proved that several well-known ε-efficiency concepts can be studied in a unified way via this new notion. Actually, we have seen under mild assumptions that our concept unifies the Németh, Helbig and Tanaka's ε-efficiency notions.

We have proved some properties of this new concept and we have characterized it in a convex framework via approximate solutions of associated scalar optimization problems. We show that approximate solutions of (P) obtained through the Weighting Method are ε-efficiency solutions in the senses of Németh, Helbig and Tanaka. In other words, these ε-efficient solutions and approximate solutions of linear scalarizations whose error is less than or equal to δ are related just via their precisions ε and δ respectively.

As final conclusion, we think that several results of this work will be useful in order to improve the actual resolution techniques and develop new methods to solve multiobjective mathematical programs.

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