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ORIGINAL ARTICLE

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Constructions of Nash equilibria in stochastic games of resource extraction with additive transition structure

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Abstract A class of N-person stochastic games of resource extraction with discounted payoffs in discrete time is considered. It is assumed that transition probabilities have special additive structure. It is shown that the Nash equilibria and corresponding payoffs in finite horizon games converge as horizon goes to infinity. This implies existence of stationary Nash equilibria in the infinite horizon case. In addition the algorithm for finding Nash equilibria in infinite horizon games is discussed.

1 Introduction

This paper deals with stochastic games of a resource extraction, which belong to the class of N-person noncooperative discounted dynamic games with uncountable state space. In general the existence of Nash equilibria in this class of games is an open problem. In some special cases the existence of equilibria in these games has been proved – for example by Himmelberg et al. (1976), Parthasarathy and Sinha (1991) or Nowak (1985). There are some economic games of this type that have stationary Nash equilibria. For example, they have been studied by Curtat (1996) and Dutta and Sundaram (1992). For a good survey of the existing literature on Nash equilibria in stochastic games with infinitely many states the interested reader is refereed to Nowak (2003) and Amir (2003).

The special case of these games are games of resource extraction or capital accumulation. The pioneering work in this field is Levhari and Mirman (1980). Papers of Amir (1996) and Nowak (2003) also deal with such games and the model in this paper is somewhat similar to theirs. However, unlike in this paper, for proofs

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of existence of Nash equilibria they used fixed point theorems. In this paper we construct convergent sequences of equilibria and corresponding payoffs in finite horizon games.

The method for construction of equilibria in finite horizon games has been developed by Rieder (1979) in more general case. In this paper we can restrict ourselves to pure Markov strategies. The convergence of equilibria and corresponding payoffs in finite horizon games in general case is an open problem. In our model the assumption of special additive transition probabilities structure gives such convergence. This assumption is similar to one used in Basar and Olsder (1995) in other classes of dynamic games. The only other paper about such convergence is Balbus and Nowak (2004), but only symmetric games are studied there.

We present also some considerations on a best reply algorithm. Similar algorithms has also been studied by Gabay and Moulin (1980) for one step games.

This paper is organized in the following way: in the first section the model and basic assumptions are described. The main results are placed in the second section and in the third one there are some auxiliary lemmas needed in preceding proofs.

2 The model and basic assumptions

Definition 2.1 *An N-person nonzero-sum stochastic game of resource extraction is defined by the following objects*:

$$(S, D, \bar{u}, p),$$

where

(1) *S is an interval in $[0, \infty)$ containing zero called the set of all resource levels or the state space,*
(2) $D := \{(s, \bar{x}) : s \in S, \bar{x} = (x_i)_{i=1}^N, x_i \in A_i(s)\}$, *where* $A_i(s) := [0, a_i(s)]$ *represents the set of actions available to player i in state s. Here $a_i : S \to R$ are nonnegative Borel measurable functions such that*

$$\sum_{i=1}^{N} a_i(s) \le s \quad \textit{for every} \quad s \in S$$

This implies that $a_i(0) = 0$. The quantity $a_i(s)$ represent the consumption capacity of player i in state s.
(3) *$\bar{u} = (u_i)_{i=1}^N$ and $u_i : S \to R$ is a nonnegative bounded utility function for player i.*
(4) *p is a transition probability from D to S, called the law of motion among states.*

The game is played in discrete time. If in some stage the game is in a state s, then player i chooses his/her consumption level $x_i \in A_i(s) \subseteq R_+ = [0, +\infty)$ and obtains his/her payoff $u_i(x_i)$ (which depends on his/her own consumption only). The game changes then its state according to the transition probability $p(\cdot|s, \bar{x})$, where $\bar{x} = (x_1, \ldots, x_N)$.

Definition 2.2 *Let* $H^1 = S$ *and for* $n \geq 2$

$$H^n = D_1 \times \cdots \times D_{n-1} \times S,$$

where $D_i = D, i = 1, \ldots, n-1$.

H^n for $n = 1, 2, \ldots$ is called the space of all n-stage histories of the game.

Let $H^\infty = D \times D \times \cdots$

H^∞ is called the space of all infinite histories of the game.

Note

(a) Both spaces of infinite and finite histories of the game are endowed with the product σ-algebra.
(b) In this paper we deal only with nonrandomized (pure) strategies of the game. However Nash equilibria that we obtain in the class of pure strategies are also Nash equilibria in the class of randomized strategies. This follows from the theory of dynamic programming (see Blackwell 1965).
(c) Every player has full information about the past history of the game and the actions performed by the other players.

Definition 2.3 *A strategy of player i is a sequence*

$$\pi_i = (\pi_{i,1}, \pi_{i,2}, \ldots),$$

where $\pi_{i,n}$ is Borel measurable mapping from H^n into S such that for every

$$h^n = (s_0, a_0, \ldots, s_n) \in H^n,$$

we have

$$\pi_{i,n}(h^n) \in A_i(s_n).$$

The set of all strategies of player i is denoted by Π_i.

Definition 2.4 *Strategy $\pi_i \in \Pi_i$, $\pi_i = (\pi_{i,1}, \pi_{i,2}, \ldots)$ such that $\pi_{i,n}$ depends only on n and the n-th state of the game s_n is called Markov.*

Definition 2.5 *Borel measurable mapping form S into $A_i(s)$ (independent of the stage of the game) is called a stationary strategy of player i. By F_i we denote the set of stationary strategies of player i.*

Let us note that a stationary strategy is a special case of a Markov strategy.

For each strategy profile of the players $\bar{\pi} = (\pi_i)_{i=1}^N$ for any initial state of the game $s_1 = s \in S$, a probability $P_s^{\bar{\pi}}$ and stochastic process $\{(s_n, \bar{x}_n)\}$ (where $\bar{x}_n = (x_{i,n})_{i=1}^N$) is defined on H^∞ in the canonical way (see Chap 7 in Bertsekas and Shreve 1978), where the random variables s_n and $\bar{x}_n$ represent a state of the game and a vector of players' decisions on n-th stage of the game respectively.

Note

$\{(s_n, \bar{x}_n)\}$ and $P_s^{\bar{\pi}}$ exist by the Ionescu-Tulcea Theorem (see Proposition V.1.1 in Neveu 1965).

Definition 2.6 *Let* $\Pi = \Pi_1 \times \cdots \times \Pi_N$, $\bar{\pi} \in \Pi$ *and* β *be a discount factor* ($\beta \in (0, 1)$). *We define*:

(a) β*-discounted expected payoff of player* i *in the* m*-stage game as the following function* $\gamma_i^m(\bar{\pi}) : S \to R_+$:

$$\gamma_i^m(\bar{\pi})(s) := E_s^{\bar{\pi}} \left(\sum_{n=1}^{m} \beta^{n-1} u_i(x_{i,n}) \right) \tag{1}$$

where $E_s^{\bar{\pi}}$ *is expected value operator with respect to the probability measure* $P_s^{\bar{\pi}}$ *and* $x_{i,n}$ *is consumption of player* i *on stage* n.

(b) β*-discounted expected payoff of player* i *in the infinite horizon game as the following function* $\gamma_i(\bar{\pi}) : S \to R_+$:

$$\begin{aligned}\gamma_i(\bar{\pi})(s) &:= E_s^{\bar{\pi}} \left(\sum_{n=1}^{\infty} \beta^{n-1} u_i(x_{i,n}) \right) \\ &= \lim_{m\to\infty} \gamma_i^m(\bar{\pi})(s).\end{aligned}$$

Definition 2.7 $\bar{\pi}^* \in \Pi$ *is a Nash equilibrium in the discounted stochastic game with an infinite horizon if and only if, for each* $s \in S$, *we have*:

$$\gamma_i(\bar{\pi}^*)(s) \geq \gamma_i(\bar{\pi}^*_{-i}, \pi_i)(s)$$

for each $\pi_i \in \Pi_i$, *where*

$$\begin{aligned}\bar{\pi}^*_{-i} &= (\pi_1^*, \ldots, \pi_{i-1}^*, \pi_{i+1}^*, \ldots, \pi_N^*), \\ (\bar{\pi}^*_{-i}, \pi_i) &= (\pi_1^*, \ldots, \pi_{i-1}^*, \pi_i, \pi_{i+1}^*, \ldots, \pi_N^*).\end{aligned}$$

Nash equilibrium in the m-stage discounted game is defined in a similar way. We make additional assumptions about the model of the game described above.

Assumption 2.1 (a) $0 \in S$ *and* $p(\{0\}|0, 0, \ldots, 0) = 1$.

(b) *For any* $(s, \bar{x}) \in D$, *where* $s > 0$, *transition probability can be expressed in the following form*:

$$p(\cdot|s, \bar{x}) = q(\cdot|s, \bar{x}) + g_0(s, \bar{x})\delta_0(\cdot), \tag{2}$$

where

$$q(\cdot|s, \bar{x}) = \sum_{i=1}^{N} g_i(a_i(s) - x_i) H_i(\cdot|s), \tag{3}$$

(1) $g_i : S \to [0, 1]$ ($i = 1, \ldots, N$) *is increasing, strictly concave and twice differentiable,*

$$\sum_{i=1}^{N} g_i(t) \leq 1, \quad \forall t \in S$$

and $g_i(0) = 0$,

(2) $H_i(\cdot|s)$ *are transition probabilities from* S *into* S *with support* $S_+ := S\backslash\{0\}$,

(3) δ_0 *is Dirac delta at the point zero.*

(c) *For any $s \in S$, we have*

$$\sum_{i=1}^{N} a_i(s) = s.$$

Note

(a) The form of transition probability (2) implies that

$$g_0(s, \bar{x}) = 1 - \sum_{i=1}^{N} g_i(a_i(s) - x_i). \tag{4}$$

(b) From (4) it follows that if $a_i(s) = x_i$ for each player i, then $g_0(s, \bar{x}) = 1$.

(c) Assumption 2.1 (c) may be interpreted that there is a possibility that resource runs out. This assumption can be replaced with a condition that for all $s \in S$ we have

$$\sum_{i=1}^{N} a_i(s) < s,$$

which means that on every stage of the game some amount of resource is left. Then we should however omit also the assumption that $g_i(0) = 0$ and assume that for some i we have $g_i(0) > 0$. Otherwise, we obtain a model, which is difficult to justify. In this model, in situation that all players make maximal consumption ($x_i = a_i(s)$ for each i) – we have some amount of resource left as investment for the next stage and the state of the game moves to zero with probability 1 in spite of that.

(d) Assumption 2.1 (c) can be replaced with the condition that for $s \in \tilde{S} \subset S$, we have

$$\sum_{i=1}^{N} a_i(s) < s$$

and for other $s \in S \setminus \tilde{S}$

$$\sum_{i=1}^{N} a_i(s) = s.$$

Such assumption can be interpreted that for some states of the game (from the set $\tilde{S}$) complete consumption of resource is not possible, and for other states – it is. In this case we should make the following assumption on the functions g_i: they depend also on s, for $s \in \tilde{S}$ we have $g_i(s, 0) > 0$, and for $s \in S \setminus \tilde{S}$ we have $g_i(s, 0) = 0$.

(e) If in some stage of the game the state is zero, with probability 1 it stays there forever.

(f) Assumption (3) is typical in the theory of dynamic games, specially differential games (see Amir 2003 or Basar and Olsder 1995). It means that the influence of the players on evolution of the state process is – in some sense – additive. Similar assumption can be found in Himmelberg et al. (1976), but unlike in that paper here we restrict ourselves to nonrandomized equilibria and study some convergence problems, which have relevance to computing approximate equilibria.

Assumption 2.2 *Utility functions u_i are increasing, concave, bounded and twice differentiable. We also assume that $u_i(0) = 0$.*

Let us denote by $B_0(S)$ the space of bounded, Borel measurable, nonnegative functions $v : S \to R$ such that $v(0) = 0$. In the space $B_0(S)$ we consider the metric

$$\rho(v, w) := \sup_{s \in S} |v(s) - w(s)|,$$

where $v, w \in B_0(S)$.

The metric space $B_0(S)$ is complete. Let $f_i \in F_i$ and $v \in B_0(S)$. Let us introduce now the following notation:

$$(T^i_{\bar{f}_{-i}} v)(s) = \max_{x_i \in A_i(s)} [u_i(x_i) + \beta \int_S v(s') p(ds'|s, (\bar{f}_{-i}(s), x_i))]$$

and

$$(L^i_{\bar{f}} v)(s) = u_i(f_i(s)) + \beta \int_S v(s') p(ds'|s, \bar{f}(s)),$$

where $(\bar{f}_{-i}(s), x_i) = (f_1(s), \dots, f_{i-1}(s), x_i, f_{i+1}(s), \dots, f_N(s)), \bar{f}(s) = (f_1(s), \dots, f_N(s))$.

Put

$$v_{\bar{f}_{-i}}(s) = \sup_{\pi_i \in \Pi_i} \gamma_i(\bar{f}_{-i}, \pi_i)(s).$$

It is easy to see that for $\beta < 1$ the operator $L^i_{\bar{f}} : B_0(S) \to B_0(S)$ is a contraction. This implies that, for any $v \in B_0(S)$, we have

$$\begin{aligned} \lim_{n\to\infty} ((L^i_{\bar{f}})^n v)(s) &= \gamma_i(\bar{f})(s) \\ &= (L^i_{\bar{f}} \gamma_i(\bar{f}))(s) \end{aligned} \tag{5}$$

for every $s \in S$.

3 Main results

We first introduce two well known helpful lemmas. These are facts from the theory of discounted dynamic programming (see Dynkin and Yushkevich 1979 or Blackwell 1965).

Lemma 3.1 (Blackwell) $\gamma_i(\bar{f})(s) = v_{\bar{f}_{-i}}(s)$ *for every* $s \in S$ *if and only if*

$$v_{\bar{f}_{-i}}(s) = (L^i_{\bar{f}} v_{\bar{f}_{-i}})(s) = (T^i_{\bar{f}_{-i}} v_{\bar{f}_{-i}})(s) \tag{6}$$

for every $s \in S$.

In the following sections the question of existence of best replies of player i to strategies of other players is an important issue:

Lemma 3.2 *Each player* i *has stationary strategy* $f_i \in F_i$ *that is optimal if the other players choose stationary strategies* $\bar{f}_{-i} \in \prod_{j \neq i} F_j$.

For proof – see Blackwell (1965), Dynkin and Yushkevich (1979) or Bertsekas and Shreve (1978).

3.1 Convergence of Nash equilibria and payoffs in finite horizon games

For $\bar{v} = (v_1, v_2, \ldots, v_N)$, where $v_i \in B_0(S)$ and $s \in S$ let us define $\Gamma(\bar{v}, s)$ as the auxiliary game with the following payoff function of player i

$$h_i(\bar{v}, s, \bar{x}) = u_i(x_i) + \beta \int_S v_i(s') p(ds'|s, \bar{x}),$$

where $(s, \bar{x}) \in D$.

Note

(a) As $v_i(0) = 0$ $(i = 1, \ldots, N)$, we have $h_i(\bar{v}, s, \bar{x})$ in the following form

$$h_i(\bar{v}, s, \bar{x}) = u_i(x_i) + \beta \int_{S_+} v_i(s') q(ds'|s, \bar{x}),$$

where $S_+ := S \setminus \{0\}$.

(b) If in $\Gamma(\bar{v}, s)$ the state is $s = 0$, then $\bar{x} = (0, \ldots, 0)$ is unique vector of feasible actions. It is then also a trivial Nash equilibrium in $\Gamma(\bar{v}, 0)$.

Theorem 3.1 *In every* N*-person* m*-stage discounted stochastic game of resource extraction described in section* 2 *fulfilling Assumptions* 2.1 *and* 2.2 *exists unique nonrandomized Markov Nash equilibrium.*

Note

To prove this theorem we construct a Nash equilibrium by the algorithm of the backward induction. For similar construction – see Rieder (1979). However, here we obtain nonrandomized Nash equilibria.

Proof (a) Note that $\bar{f}^1(s) = (s_1(s), \dots, a_N(s))$ is the unique nonrandomized Nash equilibrium in the one stage game.

(b) Assume that in the m-stage game exist a unique nonrandomized Markov Nash equilibrium

$$\bar{f}^{*m} = (\bar{f}^m, \dots, \bar{f}^1),$$

where $\bar{f}^k = (f_1^k, \dots, f_N^k)$, $f_i^k \in F_i$.

Denote the vector of corresponding payoffs by

$$\bar{v}^m := (v_1^m, \dots, v_N^m),$$

where $v_i^m \in B_0(S)$.

Now, finding the equilibrium strategies in the first stage of the $(m+1)$-stage game is equivalent to finding the Nash equilibrium in $\Gamma(\bar{v}^m, s)$. We can rewrite $h_i(\bar{v}^m, s, \bar{x})$ in the following way:

$$h_i(\bar{v}^m, s, \bar{x}) = u_i(x_i) + \sum_{j=1}^{N} g_j(a_j(s) - x_j)\tilde{I}_j(s),$$

where $\tilde{I}_j(s) = \beta \int\limits_{S_+} v_j^m(\tilde{s}) H(d\tilde{s}|s)$.

It is easy to see that the strict concavity of $u_i(x_i) + \tilde{I}_i g_i(a_i(s) - x_i)$ implies the existence and the uniqueness of the nonrandomized Nash equilibrium

$$\bar{f}^{m+1} = (f_1^{m+1}, \dots, f_N^{m+1})$$

in $\Gamma(\bar{v}^m, s)$ (where $f_i^{m+1} \in F_i$).

Note that

$$f^{*m+1} = (\bar{f}^{m+1}, \bar{f}^m, \dots, \bar{f}^1)$$

is the unique nonrandomized Markov Nash equilibrium in the $(m+1)$-stage game.

Theorem 3.2 *Consider the N-person m-stage stochastic game of resource extraction described in section 2 and fulfilling Assumptions 2.1 and 2.2. Let $\bar{v}^k = (v_1^k, \dots, v_N^k)$ be the vector of payoffs in the Nash equilibrium in a k-stage game and*

$$\pi_{i,k} := (f_i^k, f_i^{k-1}, \dots, f_i^1)$$

be the strategy of player i from the Markov equilibrium $(\pi_{1,k}, \dots, \pi_{N,k})$. Then

$$v_i^m(s) \le v_i^{m+1}(s) \quad \forall i = 1, \dots, N, \quad m \ge 1, \quad s \in S.$$

Moreover

$$f_i^m(s) \ge f_i^{m+1}(s) \quad \forall i = 1, \dots, N, \quad m \ge 1$$

for all $s \in S$.

Proof For $s = 0$ we have $\bar{v}^m(s) = (0, \dots, 0)$ for any m.

Denote

$$v_i^1(s) := \max_{x \in A_i(s)} u_i(x), \quad s \in S.$$

Clearly $v_i^1 \in B_0(S)$.

Let us consider the game $\Gamma(\bar{v}^1, s)$, $s \in S_+$. For the payoff vector in the Nash equilibrium $\bar{v}^2$ in this game we have

$$\max_{x \in A_i(s)} u_i(x) = v_i^1(s) \le v_i^2(s) = \max_{x \in A_i(s)} [u_i(x) + \beta \int_S v_i^1(\tilde{s}) p(d\tilde{s}|s, \bar{x})]$$
$$= \max_{x \in A_i(s)} [u_i(x) + \beta \int_{S_+} v_i^1(\tilde{s}) q(d\tilde{s}|s, \bar{x})]$$

for all $i = 1, \dots, N$. This is implied by the special form of payoffs and transition probabilities. We also have following correspondence between strategies in the first and second-stage games: $f_i^1(s) \ge f_i^2(s)$. (This inequality is trivial, since in first-stage game players "take all". In second-stage game consumption in the first stage may be smaller, because players may want to invest some amount of the resource).

Let us assume that $v_i^{m-1}(s) \le v_i^m(s), i = 1, \dots, N$ for some $m \ge 2$ and $\bar{v}^m(s) = (v_1^m(s), \dots, v_N^m(s))$ is a vector of payoffs corresponding to the Nash equilibrium in the game $\Gamma(\bar{v}^{m-1}, s)$.

Fix $s > 0$. Let G_1 be the game $\Gamma(\bar{v}^{m-2}, s)$ and let G_2 be the game $\Gamma(\bar{v}^{m-1}, s)$. In G_1 we have the following payoff functions

$$k_i(\bar{x}) := h_i(\bar{v}^{m-1}, s, \bar{x}) = u_i(x_i) + \sum_{j=1}^N I_j g_j(c_j - x_j),$$

and in G_2:

$$l_i(\bar{x}) := h_i(\bar{v}^m, s, \bar{x}) = u_i(x_i) + \sum_{j=1}^N J_j g_j(c_j - x_j),$$

where

$$I_j = \beta \int_{S_+} v_j^{m-1}(s') H_j(ds'|s),$$
$$J_j = \beta \int_{S_+} v_j^m(s') H_j(ds'|s),$$
$$c_j = a_j(s).$$

Since $v_i^{m-1}(s) \le v_i^m(s)$, we obtain $I_i \le J_i \quad \forall_{i=1,\cdots,N}$. Let $f_i^{m+1}(s) := \operatorname{argmax}_{x \in A_i(s)}[u_i(x) + \beta \int_S v_i^m(\tilde{s}) p(d\tilde{s}|s, \bar{x})]$. From Lemma 4.2 we have $k_i(\bar{f}^m(s)) \le l_i(\bar{f}^{m+1}(s))$, and this implies that $v_i^m(s) \le v_i^{m+1}(s)$. We also have $f_i^m(s) \ge f_i^{m+1}(s)$. □

Note
From the above theorem we obtain not only existence of Nash equilibrium, but also possibility of calculation of ϵ-equilibria in the class of stationary strategies in the infinite horizon game. This is illustrated in example at the end of this section.

Corollary *The sequence $\{\bar{f}^m\}$ converges pointwise to a Nash equilibrium in the infinite horizon game as $m \to \infty$. Similarly the sequence $\{\bar{v}^m\}$ converges to the vector of equilibrium payoffs in infinite horizon game (this is implied by Bellman equation – see Lemma* 3.1*). So the outcome from Theorem* 3.2 *is*:

Theorem 3.3 *In every N-person game with infinite horizon game of resource extraction described in section* 2 *and fulfilling Assumptions* 2.1 *and* 2.2 *exists non-randomized stationary Nash equilibrium $(f_1^*, \dots, f_N^*)$, where $f_i^*(s) = \lim_{m\to\infty} f_i^m(s)$. Moreover for every $s \in S$ we have*

$$v_i^*(s) := \lim_{m\to\infty} v_i^m(s) = \gamma_i(f_1^*, \dots, f_N^*)(s) \qquad \forall i = 1, \dots, N$$

where $v_i^m(s)$ are total payoffs in Nash equilibrium in m-stage game.

Proof From the definition of f_i^m and v_i^m we conclude

$$v_i^{m+1}(s) = u_i(f_i^{m+1}(s)) + \beta \int_S v_i^m(\tilde{s}) p(d\tilde{s}|s, \bar{f}^{m+1}) \quad \forall s \in S \tag{7}$$

and $v_i^k(s)$ and $f_i^k(s)$ converge monotonically as $k \to \infty$ for every $i = 1, \dots, N$, $s \in S$.

If $m \to \infty$ right hand side of (7) converges to (this is implied by monotone convergence Lebesgue theorem)

$$\begin{aligned}(L^i_{\bar{f}^*} v_i^*)(s) &= u_i(f_i^*(s)) + \beta \int_S v_i^*(\tilde{s}) p(d\tilde{s}|s, \bar{f}^*) \\ &= (T^i_{\bar{f}^*_{-i}} v_i^*)(s).\end{aligned}$$

We obtain

$$v_i^*(s) = (L^i_{\bar{f}^*} v_i^*)(s) = (T^i_{\bar{f}^*_{-i}} v_i^*)(s),$$

which is the Bellman optimality equation [see Lemma (3.1)] in the game with infinite horizon. So we can conclude that $(f_1^*, \dots, f_N^*)$ is a Nash equilibrium in this game and

$$\gamma_i(f_1^*, \dots, f_N^*)(s) = v_i^*(s). \qquad \square$$

Corollary *If we assume that $a_i(s)$ and $\int_S v(s') H_i(ds'|s)$ are continuous in S, for any $v \in B_0(s)$, S is a compact interval, $v_i^m(s)$ are continuous, $v_i^*(s)$ is continuous (for $i = 1, \dots, N$) and also $f_i^m(s)$, $f_i^*(s)$ are continuous, then from the Dini Theorem convergence v_i^m to v_i^* and f_i^m to f_i^* is uniform.*

Theorem 3.4 *Assume that $H_i(\cdot|s)$ do not depend on s and $a_i(s)$ have Lipschitz property with constant* 1 *and are nondecreasing.*

Then $f_i^m(s)$ are nondecreasing and have Lipschitz property with constant 1.

Corollary *$f_i^*(s)$ is Lipschitz continuous for every i.*

Proof From (19) $f_i^m(s)$ are of the form

$$f_i^m(s) = \operatorname*{argmax}_{x \in [0, a_i(s)]} [u_i(x) + I_i g_i(a_i(s) - x)]$$

for some $I_i \geq 0$.

By Lemma 4.4 we can conclude that $u_i(x) + I_i g_i(a_i(s) - x)$ is supermodular with respect to (s, x). $a_i(s)$ is nondecreasing, so from Topkis' theorem (see Topkis 1978) we obtain that $f_i^m(s)$ is a nondecreasing function.

Consider

$$k(s) := s - f_i^m(s) = \operatorname*{argmax}_{z \in [s - a_i(s), a_i(s)]} [u_i(s - z) + I_i g_i(a_i(s) - (s - z))].$$

Again from Lemma 4.4 we can conclude that $u_i(s - z) + I_i g_i(a_i(s) - s + z)$ is supermodular with respect to (s, z). Both $a_i(s)$ and $s - a_i(s)$ are nondecreasing ($s - a_i(s)$ is nondecreasing because $a_i(s)$ has Lipschitz property with constant 1) and from Topkis' theorem $k(s)$ is nondecreasing. This implies that $f_i^m(s)$ has Lipschitz property with constant 1. □

Example Let us consider a second-person game described in section 2 with:

$$S = [0, 1], \quad A_i(s) = [0, \frac{s}{2}], \quad u_1(x) = 4(x - x^2), \quad u_2(y) = 3y - 2y^2,$$
$$\beta = 0.99, \quad q(\cdot|s, x, y) = [g(\frac{s}{2} - x) + g(\frac{s}{2} - y)]\mu(\cdot),$$

where $g(t) = \frac{2}{3}(2t - t^2)$, and μ is the probability on [0, 1] with the distribution function $F(s) = s^2$.

We start our algorithm with $f_1^1(s) = f_2^1(s) = s/2$ for both players (that is we assume that players consume all available resource).

In such a game functions $f_i^n(s)$ (actions of player i on the first stage of the n-stage game) are of the following form:

$$f_i^n(s) = \begin{cases} \frac{s}{2} & \text{for } 0 \leq s \leq s_i^n \\ a_i^n s + b_i^n & \text{for } s_i^n < s \leq 1 \end{cases}$$

where a_i^n, b_i^n, s_i^n are positive constants (see Table 1).

The functions $v_i^n(s)$ – total payoffs of player i in the n-stage game – are piecewise square functions on the intervals $[0, \min\{s_1^n, s_2^n\}]$, $[\min\{s_1^n, s_2^n\}, \max\{s_1^n, s_2^n\}]$, $[\max\{s_1^n, s_2^n\}, 1]$. The supremum norm of the differences $v_i^{n+1} - v_i^n$ decreases as n increases (see Table 2).

Table 1 Constants a_i^n, b_i^n, s_i^n (approximate values) in n-stage games with n from 2 to 5

n	a_1^n	b_1^n	s_1^n	a_2^n	b_2^n	s_2^n
2	0.06043956	0.31868132	0.725	0.09919840	0.40280561	1
3	0.06116209	0.31651372	0.72125428	0.10221392	0.39225129	0.98608600
4	0.06119600	0.31641200	0.72107820	0.10240218	0.39159238	0.98489568
5	0.06119773	0.31640680	0.72106921	0.10241307	0.39155426	0.98482679

Table 2 The supremum norm of differences between payoffs in successive iterations (approximate values)

i	$\sup_{s\in[0,1]} \lvert v_1^{i+1}(s) - v_1^i(s)\rvert$	$\sup_{s\in[0,1]} \lvert v_2^{i+1}(s) - v_2^i(s)\rvert$
1	0.06648352	0.11243751
2	0.00786529	0.00567793
3	0.00060686	0.00034841
4	0.00003454	0.00001979

Note

The pair $(f_1^n, f_2^n) := f^n$ is a stationary ϵ-equilibrium in the infinite horizon game that is, for some $\epsilon > 0$ we have:

$$\forall i \in \{1,2\} \quad \forall \pi_i \in \Pi_i \quad \gamma_i(f^n)(s) \geq \gamma_i(f_{-i}^n, \pi_i)(s) - \epsilon.$$

First we note that for any n we have $v_i^{n+1}(s) \geq v_i^n(s)$ for $s \in S$ and $i = 1, 2$, where $v_i^{n+1}(s) = (L^i_{(f_1^{n+1}, f_2^{n+1})} v_n)(s)$. Iterating $L^i_{(f_1^{n+1}, f_2^{n+1})}$ we obtain:

$$\begin{aligned} v_i^n(s) &\leq (L^i_{(f_1^{n+1}, f_2^{n+1})} v_n)(s) \leq ((L^i_{(f_1^{n+1}, f_2^{n+1})})^2 v_n)(s) \leq \cdots \\ &\cdots \leq ((L^i_{(f_1^{n+1}, f_2^{n+1})})^k v_n)(s). \end{aligned}$$

If $k \to \infty$ we get

$$v_i^n(s) \leq \lim_{k\to\infty} ((L^i_{(f_1^{n+1}, f_2^{n+1})})^k v_n)(s) = \gamma_i(f_1^{n+1}, f_2^{n+1})(s). \tag{8}$$

f_1^{n+1} and f_2^{n+1} are stationary strategies of players 1 and 2 in the infinite horizon game. Denote $\delta_1 := 0.00003455$ and $\delta_2 := 0.0000198$. From Table 2 we can see that $v_i^{n+1}(s) \leq v_i^n(s) + \delta_i$ for all $s \in S$ with $n = 4$.

We have

$$v_1^n(s) + \delta_1 \geq v_1^{n+1}(s) = (T^1_{f_2^{n+1}} v_1^n)(s),$$

hence

$$(T^1_{f_2^{n+1}} v_1^n)(s) + \beta\delta_1 \geq (T^1_{f_2^{n+1}}(v_1^n + \delta_1))(s) \geq ((T^1_{f_2^{n+1}})^2 v_1^n)(s).$$

Iterating above inequality we obtain

$$\begin{aligned} ((T^1_{f_2^{n+1}})^k v_1^n)(s) &\leq ((T^1_{f_2^{n+1}})^{k-1}(v_1^n + \delta_1))(s) \\ &\leq ((T^1_{f_2^{n+1}})^{k-2}((T^1_{f_2^{n+1}} v_1^n) + \beta\delta_1))(s) \leq \cdots \\ &\leq ((T^1_{f_2^{n+1}})^{k-1} v_1^n)(s) + \beta^{k-1}\delta_1 \\ &\leq \cdots \leq v_1^n(s) + \delta_1(1 + \beta + \beta^2 + \cdots + \beta^{k-1}). \end{aligned}$$

If $k \to \infty$, from above inequality we conclude that

$$v_1^n(s) + \frac{\delta_1}{1-\beta} \geq \sup_{\pi \in \Pi_1} \gamma_1(\pi, f_2^{n+1})(s). \tag{9}$$

[Recall (5).] We can also show that

$$v_2^n(s) + \frac{\delta_2}{1-\beta} \geq \sup_{\pi \in \Pi_2} \gamma_2(f_1^{n+1}, \pi)(s). \tag{10}$$

From (8) and (9), it follows that

$$\gamma_1(f_1^{n+1}, f_2^{n+2})(s) + \frac{\delta_1}{1-\beta} \geq v_1^n(s) + \frac{\delta_1}{1-\beta} \geq \sup_{\pi \in \Pi_1} \gamma_1(\pi, f_2^{n+1})(s).$$

From (8) and (10) we can obtain also similar result for player 2:

$$\gamma_2(f_1^{n+1}, f_2^{n+2})(s) + \frac{\delta_2}{1-\beta} \geq v_2^n(s) + \frac{\delta_2}{1-\beta} \geq \sup_{\pi \in \Pi_2} \gamma_2(f_1^{n+1}, \pi)(s).$$

This means that (f_1^{n+1}, f_2^{n+1}) is an ϵ-equilibrium in the infinite horizon game with

$$\epsilon = \max\left\{\frac{\delta_1}{1-\beta}, \frac{\delta_2}{1-\beta}\right\}.$$

For $n = 4$ we have $\epsilon = \max\{\frac{0.00003455}{1-0.99}, \frac{0.0000198}{1-0.99}\} = 0.003455$.

3.2 An algorithm based on best responses in infinite horizon games

In previous section we have shown convergence of strategies from the algorithm based on finding Nash equilibria in finite horizon games. In this section we study a natural algorithm based on best responses in an infinite horizon game. We point out that such an approach does not lead to a Nash equilibrium in general – without concavity assumptions as made in this paper.

In the first step of the algorithm ($n = 0$), we assume that in an infinite horizon game all players consume all of available resource that is, $f_i^0(s) := a_i(s)$. Then $v_i^0(s) = u_i(a_i(s))$, because from second stage onwards the game is in state $s = 0$ with probability 1.

In the second step the first player ($i = 1$) chooses his or her strategy as the best response $f_1^1 \in F_1$ to the strategies of the other players $(f_2^0, \ldots, f_N^0)$. Let

$$v_1^1(s) := \gamma_1(f_1^1, f_2^0, \ldots, f_N^0)(s).$$

In the third step the second player ($i = 2$) chooses the best response $f_2^1 \in F_2$ to the strategies $(f_1^1, f_3^0, \ldots, f_N^0)$. Put

$$v_2^1(s) := \gamma_2(f_1^1, f_2^1, f_3^0, \ldots, f_N^0)(s).$$

Generally – in every step some player i chooses the best response $f_i^n \in F_i$ to the strategies $(f_1^n, \dots, f_{i-1}^n, f_{i+1}^{n-1}, \dots, f_N^{n-1})$. Define

$$v_i^n(s) := \gamma_i(f_1^n, \dots, f_i^n, f_{i+1}^{n-1}, \dots, f_N^{n-1})(s).$$

The existence of such stationary best responses follows from Blackwell (1965) paper – see Lemma 3.2.

Let us now consider the sequence of strategies $\{(f_i^n(s))_{i=1}^N\}_{n=1}^\infty$ constructed above.

Theorem 3.5 *When $n \to \infty$ the sequence $\{(f_i^n(s))_{i=1}^N\}_{n=1}^\infty$ converges to a Nash equilibrium in the infinite horizon game.*

Proof From Lemma 4.3 we can conclude that for any i both $\{f_i^n(s)\}_{n=1}^\infty$ and $\{v_i^n(s)\}_{n=1}^\infty$ are monotone: $(\{f_i^n(s)\}_{n=1}^\infty$ is nonincreasing, $\{v_i^n(s)\}_{n=1}^\infty$ is nondecreasing) and these sequences are bounded. This implies their convergence that is,

$$f_i^*(s) := \lim_{n\to\infty} f_i^n(s)$$

and

$$v_i^*(s) := \lim_{n\to\infty} v_i^n(s)$$

exist.

Argumentation based on the Bellman optimality equations allows to conclude that $\{f_i^*(s)\}_{i=1}^N$ is a Nash equilibrium in the infinite horizon game. □

4 Auxiliary lemmas

Lemma 4.1 *Let $x \in [0, c], c > 0, I_k > 0$, for $k = 1, 2$, and functions $\Phi_k : [0, c] \longrightarrow R$ be defined as follows*:

$$\Phi_k(x) = u(x) + I_k g(c - x),$$

where u and g are increasing, nonnegative, concave, twice differentiable and at least one of them is strictly concave.

Denote:

$$x_k^* := \operatorname*{argmax}_{x\in[0,c]} \Phi_k(x),$$
$$m_k^* := \max_{x\in[0,c]} \Phi_k(x).$$

If

$$I_1 \le I_2,$$

then

$$x_1^* \ge x_2^* \tag{11}$$

and

$$m_1^* \le m_2^*. \tag{12}$$

Proof For $I_1 = I_2$ inequality (11) is obvious. Assume that

$$0 < I_1 < I_2 \tag{13}$$

and consider three following cases:
(a) $x_1^* = 0$,
(b) $x_1^* = c$,
(c) $x_1^* \in (0, c)$.

The function $\Phi_k(x)$ $(k = 1, 2)$ is strictly concave as a sum of concave and strictly concave function.

So ${\Phi_k}'(x) = u'(x) - I_k g'(c - x)$ is decreasing in x.

(a) If $x_1^* = 0$, then ${\Phi_1}'(x_1^{*+}) \leq 0$, because ${\Phi_1}'(x)$ is decreasing. We obtain

$$u'(x) - I_1 g'(c - x) < 0 \qquad \forall x \in (0, c].$$

So $u'(x) < I_1 g'(c - x)$, and from (13) we obtain

$$u'(x) < I_2 g'(c - x).$$

From this we get

$${\Phi_2}'(x) < 0 \quad \forall x \in (0, c].$$

This means that $\Phi_2(x)$ is decreasing in x and attains its maximum at $x_2^* = 0$.

In case (b) relation in (11) is obvious, because c is maximal value for both x_1^* and x_2^*.

In case (c) we can conclude that

$${\Phi_1}'(x_1^*) = 0.$$

Let

$$f_1(x) = u'(x) - I_1 g'(c - x),$$
$$f_2(x) = u'(x) - I_2 g'(c - x).$$

Note that since g is increasing and concave and since (13) holds we have

$$f_1(x) > f_2(x) \quad \forall x \in (0, c]. \tag{14}$$

Both functions are decreasing, so

$$f_1'(x) = {\Phi_1}''(x) \leq 0 \tag{15}$$

and

$$f_2'(x) = {\Phi_2}''(x) \leq 0.$$

Let us assume that

$$x_2^* > x_1^*.$$

From $x_1^* \in (0, c)$ we can conclude that $x_2^* \in (0, c]$, so – from the definition of f_2 we obtain

$$f_2(x_2^*) \geq 0,$$

and from (14) we have

$$f_1(x_2^*) > 0.$$

By Lagrange's theorem

$$\exists \xi \in (x_1^*, x_2^*) : \quad f_1'(\xi) = \frac{f_1(x_2^*) - f_1(x_1^*)}{x_2^* - x_1^*} > 0,$$

which is a contradiction to (15). Hence (11) follows.

Now we will prove inequality (12). Form $I_1 \leq I_2$ and the fact that g are nonnegative we obtain

$$u(x) + I_1 g(c - x) \leq u(x) + I_2 g(c - x) \quad \forall x \in [0, c],$$

and consequently

$$\begin{aligned} m_1^* &= \max_{x \in [0,c]} \Phi_1(x) = u(x_1^*) + I_1 g(c - x_1^*) \\ &\leq u(x_1^*) + I_2 g(c - x_1^*) \\ &\leq \max_{x \in [0,c]} [u(x) + I_2 g(c - x)] = m_2^*. \end{aligned} \tag{16}$$

□

Let us now consider two auxiliary one-stage games.

Definition 4.1 *Let G_1 and G_2 be games, in which the sets of actions are*

$$X_i := [0, c_i], i = 1, \ldots, N,$$

where $c_i > 0$.

The payoff function of player i in the game G_1 is

$$h_i(\bar{x}) := u_i(x_i) + \sum_{j=1}^{N} g_j(c_j - x_j) I_j$$

and

$$l_i(\bar{x}) := u_i(x_i) + \sum_{j=1}^{N} g_j(c_j - x_j) J_j$$

is the payoff function of player i in the game G_2 and

(a) I_i, J_i are constants such that

$$0 \leq I_i \leq J_i;$$

(b) the functions g_i are increasing, nonnegative, concave and twice differentiable;

(c) the functions u_i are increasing, nonnegative, concave and twice differentiable;
(d) for each i the function u_i or the function g_i is strictly concave.

Lemma 4.2 *Let $\bar{x}^*$ and $\bar{y}^*$ be Nash equilibria in G_1 and G_2, respectively. Then we have*

$$h_i(\bar{x}^*) \leq l_i(\bar{y}^*) \tag{17}$$

and

$$x_i^* \geq y_i^* \tag{18}$$

for every $i = 1, \ldots, N$.

Moreover, in both considered games equilibria are unique.

Proof Note that $\bar{x}^* = (x_i^*)_{i=1}^N$ is Nash equilibrium in game G_1 if and only if

$$x_i^* = \operatorname*{argmax}_{x \in [0,c_i]}[u_i(x) + I_i g_i(c_i - x)] \tag{19}$$

for every player i. (For $\bar{y}^*$ – Nash equilibrium in G_2 – analogously.)

For any player i we can rewrite his or her payoffs in the following form:

$$h_i(\bar{x}) := u_i(x_i) + I_i g_i(c_i - x_i) + \Theta(\bar{x}_{-i}),$$
$$l_i(\bar{x}) := u_i(x_i) + J_i g_i(c_i - x_i) + \Xi(\bar{x}_{-1}),$$

in games G_1 and G_2, respectively, where $\Theta(\bar{x}_{-i})$ and $\Xi(\bar{x}_{-i})$ are defined as

$$\Theta(\bar{x}_{-i}) = \sum_{j \in \{1,\cdots,N\}\setminus\{i\}} g_j(c_j - x_j) I_j$$

and

$$\Xi(\bar{x}_{-1}) = \sum_{j \in \{1,\cdots,N\}\setminus\{i\}} g_j(c_j - x_j) J_j.$$

Observe that both $\Theta(\bar{x}_{-i})$ and $\Xi(\bar{x}_{-i})$ do not depend on x_i (In connection with strict concavity of h_i and l_i for $I_i > 0$ and the fact that u_i in increasing this implies uniqueness of Nash equilibria).

If $I_i > 0$, from Lemma 4.1 we obtain (18). If $I_i = 0$, then (18) is obvious since u_i is increasing.

Since $g_i(\cdot)$ are increasing and we have $0 \leq I_i \leq J_i (i = 1, \ldots, N)$, so $\Theta(\bar{x}^*_{-i}) \leq \Xi(\bar{y}^*_{-i})$, which together with Lemma 4.1 implies (17). □

Our next lemma is needed in the proof of Theorem 3.5.

Fix player i and consider two profiles π_{-i}, σ_{-i} of stationary strategies of the other players. Let $\pi_{-i} = (f_n^1)_{n \neq i}$ and $\sigma_{-i} = (f_n^2)_{n \neq i}$, $f_n^j \in F_n$, $j = 1, 2$.

Let

$$v_i^1(s) := \sup_{\pi_i \in \Pi_i} \gamma_i(\pi_{-i}, \pi_i)(s),$$
$$v_i^2(s) := \sup_{\pi_i \in \Pi_i} \gamma_i(\sigma_{-i}, \pi_i)(s).$$

We will show that under our assumptions about transition probabilities the fact that if the profile of stationary strategies π_{-i} means greater consumption of players $n \neq i$ (comparing to the profile σ_{-i}), then player i consumes more, when he or she plays against π_{-i}, than in the situation where he or she plays against σ_{-i}. In the first case player i have worse "perspectives", because other players "care less" for the future resource renewal, which partly explains player i's behaviour.

Lemma 4.3 *Assume that π_{-i} and σ_{-i} are such that*

$$f_n^1(s) \geq f_n^2(s) \quad \forall_{n \neq i, s \in S}. \tag{20}$$

Let $\phi_i \in F_i$ and $\xi_i \in F_i$ be the best response of player i to strategy profile π_{-i} and σ_{-i} respectively.

Then

$$\phi_i(s) \geq \xi_i(s) \quad \forall s \in S$$

and

$$v_i^1(s) \leq v_i^2(s) \quad \forall s \in S.$$

Proof We define

$$v_{i,m}^1 := \sup_{\pi_i \in \Pi_i} \gamma_i^m(\pi_{-i}, \pi_i)(s)$$

and

$$v_{i,m}^2 := \sup_{\pi_i \in \Pi_i} \gamma_i^m(\pi_{-i}, \sigma_i)(s)$$

for the m-stage game.

These are the optimal payoffs of player i in the m-stage games if player i plays against π_{-i} and σ_{-i}, respectively. In the one stage game we have $v_{i,1}^j = \max_{x \in A_i(s)} u_i(x)$ for $j = 1, 2$, and optimal strategies are in both cases identical that is, player i chooses $f_{i,1}^j = a_i(s)$, $s \in S$.

For $m \geq 1$, let

$$\pi_{i,m}^j = (f_{i,m}^j, f_{i,m-1}^j, \ldots, f_{i,1}^j) \tag{21}$$

be optimal Markov response of the player i to strategy profile $(f_n^j)_{n \neq i}$ of the other players (in the m-stage game). $f_{i,1}^j(s) = a_i(s)$, $s \in S$. We additionally assume that for some $m \geq 1$, it holds

$$f_{i,m}^j(s) \leq f_{i,m-1}^j(s) \leq \cdots \leq f_{i,1}^j(s) \tag{22}$$

and

$$v_{i,m}^j(s) \geq v_{i,m-1}^j(s) \geq \cdots \geq v_{i,1}^j(s) \tag{23}$$

for each $s \in S$. Moreover, we have $f_{i,k}^1(s) \geq f_{i,k}^2(s)$ and $v_{i,k}^1(s) \leq v_{i,k}^2(s)$ for $k \leq m$, $s \in S$ and $v_{i,k}^j \in B_0(S)$. (This is our inductional hypothesis.)

For the $(m+1)$-stage game define:

$$f^j_{i,m+1}(s) := \operatorname*{argmax}_{x\in[0,a_i(s)]} [u_i(x) + \beta \int\limits_S v^j_{i,m}(s')p(ds'|s, (f^j_n(s))_{n\neq i}, x)]$$
$$= \operatorname*{argmax}_{x\in[0,a_i(s)]} [u_i(x) + \beta \int\limits_{S_+} v^j_{i,m}(s')q(ds'|s, (f^j_n(s))_{n\neq i}, x)]$$

(second equation follows from $v^j_{i,m} \in B_0(S)$ that is, $v^j_{i,m}(0) = 0$). Put

$$v^j_{i,m+1}(s) := (T^i_{(f^j_n)_{n\neq i}} v^j_{i,m})(s).$$

The function $f^j_{i,m+1}$ defined above can be rewritten in the following form:

$$f^j_{i,m+1}(s) = \operatorname*{argmax}_{x\in[0,a_i(s)]} [u_i(x) + I^j_{i,m} g_i(a_i(s) - x) + \sum_{n\in\{1,\cdots,N\}\setminus\{i\}} I^j_{n,m}(s) g_n(a_n(s) - f^j_n(s))]$$
$$= \operatorname*{argmax}_{x\in[0,a_i(s)]} [u_i(x) + I^j_{i,m} g_i(a_i(s) - x)] \tag{24}$$

where $I^j_{n,m}(s) := \beta \int\limits_S v^j_{i,m}(s') H_n(ds'|s)$.

The payoff function $v^j_{i,m+1}(s)$ can be rewritten in the manner.

It is easy to see that $I^j_{n,k}(s) \le I^j_{n,k+1}(s)$ and $I^1_{n,k}(s) \le I^2_{n,k}(s)$ for $k < m$. So, from Lemma 4.1 we conclude that

$$v^1_{i,m+1}(s) \le v^2_{i,m+1}(s),$$
$$v^j_{i,m}(s) \le v^j_{i,m+1}(s),$$
$$f^1_{i,m+1}(s) \ge f^2_{i,m+1}(s)$$

and

$$f^j_{i,m}(s) \ge f^j_{i,m+1}(s).$$

By induction we obtained sequences that satisfy (22) and (23) for any m. The sequences $\{f^1_{i,m}(s)\}$ and $\{f^2_{i,m}(s)\}$ are respectively converging to $\phi_i(s)$ and $\xi_i(s)$ and $\phi_i(s) \ge \xi_i(s)$.

Define

$$v^{j*}_i(s) := \lim_{m\to\infty} v^j_{i,m}(s).$$

It is easy to see (using the Bellman equation for the m-stage games and the infinite horizon game) that

$$v^{1*}_i(s) = v^1_i(s) = \sup_{\pi_i\in\Pi_i} \gamma_i(\pi_{-i}, \pi_i)(s)$$

and

$$v_i^{2*}(s) = v_i^2(s) = \sup_{\pi_i \in \Pi_i} \gamma_i(\sigma_{-i}, \pi_i)(s).$$

Moreover, we have $v_i^1(s) = v_i^{1*}(s) \le v_i^{2*}(s) = v_i^2(s)$, $s \in S$. □

The last lemma refers to the problem of monotonicity and Lipschitz property of equilibrium strategies. This is a light modification of Lemma 0.2 from Amir (1996) paper.

In the below calculus we will treat the set $S \times S$ as poset with natural partial order relation defined as follows:

$$(x_1, y_1) \succ (x_2, y_2) \Leftrightarrow x_1 \ge x_2 \quad \text{and} \quad y_1 \ge y_2.$$

Lemma 4.4 *Let the function $U : S \to R$ be concave and let the function $h : S \to S$ be nondecreasing and $h(0) = 0$. Then the set*

$$\Delta = \{(s, x) : s \in S, x \in [0, h(s)]\}$$

is a lattice and $U(-x + h(s))$ is supermodular on Δ that is $\Psi(s, x) := U(-x + h(s))$ holds for any $(s_1, x_1), (s_2, x_2) \in \Delta$

$$\Psi((s_1, x_1) \vee (s_2, x_2)) + \Psi((s_1, x_1) \wedge (s_2, x_2)) \ge \Psi(s_1, x_1) + \Psi(s_2, x_2),$$

where $(s_1, x_1) \vee (s_2, x_2)$ and $(s_1, x_1) \wedge (s_2, x_2)$ denote the least upper bound and the greatest lower bound of the pair $((s_1, x_1), (s_2, x_2))$ in relation "$\succ$".

Proof Note that fact that Δ is a lattice comes straight from the fact that $h(s)$ is nondecreasing.

Now we will prove that $U(-x+h(s))$ has nondecreasing differences in $(s, x) \in \Delta$, which is equivalent to supermodularity in Δ.

Let $(s_1, x_1), (s_2, x_2) \in \Delta$ be such that $s_1 \le s_2$ and $x_1 \ge x_2$. Note that (s_1, x_2) and (s_2, x_1) are also in the lattice Δ (as the greatest lower bound and the least upper bound of the pair $((s_1, x_1), (s_2, x_2))$).

For $i = 1, 2$,

$$-x_2 + h(s_2) \ge -x_{3-i} + h(s_i) \ge -x_1 + h(s_1)$$

holds and the sum of the "inner" terms is equal to the sum of the "outer" terms. So, there exists $\alpha \in [0, 1]$ such that

$$-x_1 + h(s_2) = \alpha(-x_2 + h(s_2)) + (1 - \alpha)(-x_1 + h(s_1))$$

and

$$-x_2 + h(s_1) = (1 - \alpha)(-x_2 + h(s_2)) + \alpha(-x_1 + h(s_1)).$$

Using concavity of U we conclude

$$\begin{aligned}
&U(-x_1 + h(s_2)) + U(-x_2 + h(s_1)) \\
&\quad \ge \alpha U(-x_2 + h(s_2)) + (1 - \alpha)U(-x_1 + h(s_1)) \\
&\qquad +(1 - \alpha)U(-x_2 + h(s_2)) + \alpha U(-x_1 + h(s_1)) \\
&\quad = U(-x_1 + h(s_1)) + U(-x_2 + h(s_2)).
\end{aligned}$$

This inequality can be rewritten in the following form:

$$U(-x_1 + h(s_1)) - U(-x_2 + h(s_1)) \leq U(-x_1 + h(s_2)) \\ -U(-x_2 + h(s_2)), \quad (25)$$

which means that $U(-x + h(s))$ has nondecreasing differences. □

Note

(a) If we consider $U(-x + h(s)) + f(x)$ instead of $U(-x + h(s))$ where f is any real function on S, inequality (25) still holds.
(b) The sum of two supermodular functions is a supermodular function.

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