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Seat allocation distributions and seat biases of stationary apportionment methods for proportional representation

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Abstract. In a proportional representation system, apportionment methods are used to round the vote proportion of a party to an integer number of seats in parliament. Assuming uniformly distributed vote proportions, we derive the seat allocation distributions for stationary divisor methods. An important characteristic of apportionment methods are seat biases, that is, expected differences between actual seat numbers and ideal shares of seats, when the parties are ordered from largest to smallest. We obtain seat bias formulas for the stationary divisor methods and for the quota method of greatest remainders.

Key words: Apportionment methods; rounding methods; Webster; Jefferson; Hamilton; Sainte-Laguë; d'Hondt; Hare.

1 Introduction

In a proportional representation system, apportionment methods translate the vote proportion of a party into an integer number of seats in parliament (Balinski and Young [1], Kopfermann [6]). This rounding process leaves an inevitable gap between the ideal seat allocation based on an essentially continuous fraction and the actual seat allocation based on the accuracy given by the size of the parliament. A "good" apportionment method should, on average, treat smaller and larger parties equally and not allow a systematic advantage in either direction.

Taking up original work by Pólya [7–9], Schuster *et al.* [10] introduce the notion of seat biases in order to quantify how much a given apportionment method favors smaller or larger parties. The shares of votes of a party are assumed to follow a uniform distribution, and the seat biases are defined as the expected differences between actual seat allocations and ideal shares of seats, under the condition that the parties are decreasingly ordered by their vote counts. For three-party systems, Schuster *et al.* [10] derive formulas for the seat biases of popular apportionment methods. For an arbitrary number of parties they provide numerical evidence about asymptotic seat biases, as the number of seats in parliament grows.

In this paper we first obtain the seat allocation distributions of stationary divisor methods, when the size of the parliament M is given (Section 2). Then we present a systematic method for calculating seat biases (Sections 3 and 4). Focusing on the divisor methods with standard rounding (Webster, Sainte-Laguë) and rounding down (Jefferson, d'Hondt), we confirm the conjecture of Schuster *et al.* [Appendix A.3,10] about seat biases in four-party systems (Section 5). In Appendix A, we give analogous results for the quota method of greatest remainders (Hamilton, Hare).

2 Seat allocation distributions of stationary divisor methods

Let the *probability simplex* S^ℓ be the set of all non-negative *weight vectors* $\mathbf{w} = (w_1, \dots, w_\ell)^t$ summing to one,

$$S^\ell = \left\{ \mathbf{w} \in [0,1]^\ell : \sum_{i=1}^{\ell} w_i = 1 \right\}.$$

We interpret w_i as the share of votes for party i, where $i = 1, \dots, \ell$. For a given *district magnitude* or *house size* M, that is the number of seats to be allocated among the parties, the possible *seat allocations* $\mathbf{m} = (m_1 \dots, m_\ell)^t$ form the grid set

$$G^\ell(M) = \left\{ \mathbf{m} \in \{0, \dots, M\}^\ell : \sum_{i=1}^{\ell} m_i = M \right\}.$$

An *apportionment method* A maps a weight vector $\mathbf{w}$ into a seat allocation vector $\mathbf{m}$,

$$A : S^\ell \to G^\ell(M).$$

Note that rounding the weights w_i individually does not generally result in a feasible apportionment method because the side condition $\sum_{i=1}^{\ell} m_i = M$ is not enforced automatically (Happacher [4, Section 1]).

The *stationary divisor methods* with parameter $q \in [0,1]$ are defined via the rounding function r_q which rounds down when the fractional part is less than q and up when it is greater than q. More formally, denote integer and fractional part of a nonnegative number $x \geq 0$ by $\lfloor x \rfloor = IntegerPart(x)$ and $x - \lfloor x \rfloor = FractionalPart(x)$, respectively. Then

$$r_q(x) = \begin{cases} \lceil x \rceil = \text{IntegerPart}(x) + 1 & \text{for FractionalPart}(x) > q, \\ \lfloor x \rfloor = \text{IntegerPart}(x) & \text{for FractionalPart}(x) < q. \end{cases}$$

A tie occurs when $FractionalPart(x) = q$; there the definition may stipulate $r_q(x) = \lfloor x \rfloor$ or $r_q(x) = \lceil x \rceil$. Because we consider random weights for which ties appear with probability zero, this ambiguity does not affect our results. The *q-stationary divisor method* maps the weight vector $\mathbf{w}$ into the integer vector

$$A_{q,\ell,M}(\mathbf{w}) \in G^\ell(M)$$

whose components m_i are such that there exists a divisor $D \in (0, \infty)$ with $m_i = r_q(w_i/D)$ for each i. The *divisor method with standard rounding* (Webster, Sainte-Laguë) has $q = 0.5$, while the *divisor method with rounding down* (Jefferson, d'Hondt) has $q = 1$.

We assume that the weight vector $\mathbf{w} = (w_1, \ldots, w_\ell)^t$ follows the uniform distribution on S^ℓ. In the following, we are interested in the distribution of the random variable $A_{q,\ell,M}$. To this end we introduce the set of 0-1 vectors in $\mathbb{R}^{\ell-1}$ having component sum $r-1$,

$$\left\{ \begin{matrix} \ell-1 \\ r-1 \end{matrix} \right\} := \left\{ \mathbf{t} \in \{0,1\}^{\ell-1} : \sum_{i=1}^{\ell-1} t_i = r-1 \right\}.$$

Its cardinality is given by $\#\left\{ \begin{smallmatrix} \ell-1 \\ r-1 \end{smallmatrix} \right\} = \binom{\ell-1}{r-1}$. In addition, for $\mathbf{t} \in \left\{ \begin{smallmatrix} \ell-1 \\ r-1 \end{smallmatrix} \right\}$, the initial section $(t_1, \ldots, t_j)^t$ contains

$$t_{(j)} := \sum_{i=1}^{j} t_j, \quad j \leq \ell - 1,$$

many ones and $j - t_{(j)}$ many zeros.

Theorem 1. *Suppose the weight vectors* $\mathbf{w}$ *are uniformly distributed on* S^ℓ*. Use the stationary divisor method with parameter* $q \in [0,1]$ *to apportion the house size* $M > \ell$.

Then the seat allocation vector $A_{q,\ell,M}$ *is a discrete random variable, with values in the finite grid set* $G^\ell(M)$*, attaining a grid point* $\mathbf{m} \in G^\ell(M)$ *with probability*

$$\begin{aligned} P(A_{q,\ell,M} = \mathbf{m}) &= \frac{q^{\ell-r}(\ell-1)!}{\binom{\ell-1}{r-1}} \sum_{\mathbf{t} \in \left\{ \begin{smallmatrix} \ell-1 \\ r-1 \end{smallmatrix} \right\}} \prod_{j=1}^{\ell-1} \frac{1}{M + (r + j - t_{(j)})q - (r - t_{(j)})} \\ &=: p_{q,\ell,M}(r), \end{aligned}$$

where r *denotes the number of positive components of* $\mathbf{m}$*, and where we set* $0^0 := 1$ *for* $q = 0$ *and* $r = \ell$.

Proof. Due to the uniform distribution assumption, probabilities are proportional to surface volumes, with the constant of proportionality $\mathrm{vol}_{\ell-1}(S^\ell) = \sqrt{\ell}/(\ell-1)!$. The result follows from Drton and Schwingenschlögl [3, Theorem 4.5]. ∎

Corollary 1. *For the divisor method with rounding down we have*

$$P(A_{1,\ell,M} = \mathbf{m}) = \binom{M+\ell-1}{\ell-1}^{-1} \quad \textit{for } \mathbf{m} \in G^\ell(M).$$

For the divisor method with rounding up we have

$$P(A_{0,\ell,M} = \mathbf{m}) = \begin{cases} \binom{M-1}{\ell-1}^{-1} & \textit{for } \mathbf{m} \in G^\ell(M) \textit{ with } r = \ell \\ 0 & \textit{for } \mathbf{m} \in G^\ell(M) \textit{ with } r < \ell. \end{cases}$$

Proof. For $q = 1$, Theorem 1 implies that $p_{q,\ell,M}(r)$ is constant in r and hence equals the inverse of the cardinality $\#G^\ell(M)$. For $q = 0$, $p_{q,\ell,M}(r) = 0$ when $r < \ell$, and it is the inverse of $\#\{\mathbf{m} \in G^\ell(M) : m_i \geq 1 \textit{ for } i = 1, \ldots, \ell\}$ when $r = \ell$. ∎

As already indicated by Pólya [9, p. 367], all seat allocations occur with the same probability under the divisor method with rounding down; see also [3, Remark 4.6].

3 Seat biases

Let the ℓ parties be decreasingly ordered by votes and give the largest party index 1 and the smallest party index ℓ. We now consider the weight vectors in the ordered subset of the probability simplex,

$$S^\ell_\geq = \{\mathbf{w} \in S^\ell : w_1 \geq w_2 \geq \cdots \geq w_\ell\}.$$

We define the grid set of ordered seat allocation vectors in $G^\ell(M)$,

$$G^\ell_\geq(M) = \{\mathbf{m} \in G^\ell(M) : m_1 \geq m_2 \geq \cdots \geq m_\ell\}.$$

The definition of stationary divisor methods entails

$$\mathbf{w} \in S^\ell_\geq \Longrightarrow A_{q,\ell,M} \in G^\ell_\geq(M).$$

Conditional on $w_1 \geq \cdots \geq w_\ell$, i.e. $\mathbf{w} \in S^\ell_\geq$, the weight vectors are uniformly distributed in $S^\ell_\geq$. Therefore, probabilities are still proportional to volumes, and the constant of proportionality becomes, by symmetry,

$$\mathrm{vol}_{\ell-1}(S^\ell_\geq) = \frac{\mathrm{vol}_{\ell-1}(S^\ell)}{\ell!} = \frac{\sqrt{\ell}}{\ell!(\ell-1)!}.$$

Define the expected ideal share of seats to be

$$\mathbf{I}^\ell(M) = \mathrm{E}[\mathbf{w}M | w_1 \geq \cdots \geq w_\ell],$$

and the expected number of seats to be

$$\mathbf{E}^\ell(M) = \mathrm{E}[A(\mathbf{w}) | w_1 \geq \cdots \geq w_\ell].$$

The vector of seat biases, sorted from the largest to the smallest party, then becomes

$$\mathbf{B}^\ell(M) = \mathrm{E}[A(\mathbf{w}) - \mathbf{w}M | w_1 \geq \cdots \geq w_\ell] = \mathbf{E}^\ell(M) - \mathbf{I}^\ell(M). \tag{1}$$

The components $B^\ell_1(M), \ldots, B^\ell_\ell(M)$ of $\mathbf{B}^\ell(M)$ must sum to zero,

$$\sum_{i=1}^{\ell} B^\ell_i(M) = E\left[\sum_{i=1}^{\ell} A(\mathbf{w})_i - M \sum_{i=1}^{\ell} w_i \middle| w_1 \geq \cdots \geq w_\ell\right] = M - M = 0.$$

The expected ideal seat allocation $\mathbf{I}^\ell(M) = (I^\ell_1(M), \ldots, I^\ell_\ell(M))^t$ has the components

$$I^\ell_i(M) = \mathrm{E}[w_i M | w_1 \geq w_2 \geq \cdots \geq w_\ell] = \frac{M}{\ell} \sum_{j=i}^{\ell} \frac{1}{j},$$

as shown in Drton and Schwingenschlögl [3, Lemma 5.1]. The expected ideal seat proportions $\mathbf{I}^\ell(M)/M$ do not depend on the house size M, but only on the number ℓ of parties in the system: $\mathbf{I}^2(M)/M = (3/4, 1/4)^t$, $\mathbf{I}^3(M)/M = (11/18, 5/18, 2/18)^t$, and $\mathbf{I}^4(M)/M = (25/48, 13/48, 7/48, 3/48)^t$.

We denote by $\mathbf{e}_i \in \mathbb{R}^\ell$ the Euclidean unit vector with the i-th component one and the other components zero. Defining $\mathbf{v}_r := \frac{1}{r}\sum_{j=1}^{r} \mathbf{e}_j$, $r = 1, \ldots, \ell$, we have

$$\mathbf{I}^\ell(M) = \frac{M}{\ell}\sum_{r=1}^{\ell} \mathbf{v}_r. \tag{2}$$

It remains the task to calculate the expectated number of seats $\mathbf{E}^\ell(M) = (E_1^\ell(M), \ldots, E_\ell^\ell(M))^t$, for the various apportionment methods. We can state

$$\mathbf{E}^\ell(M) = \sum_{\mathbf{m}\in G_\geq^\ell(M)} \mathbf{m}\, P(A_{q,\ell,M} = \mathbf{m} \mid w_1 \geq \cdots \geq w_\ell). \tag{3}$$

Let $\mathscr{G}^\ell$ be the permutation group on $\{1, \ldots, \ell\}$, and define

$$\sigma(\mathbf{x}) = (x_{\sigma(1)}, \ldots, x_{\sigma(\ell)})^t, \quad \sigma \in \mathscr{G}^\ell, \mathbf{x} \in \mathbb{R}^\ell.$$

Then

$$b(\mathbf{m}) = \#\{\sigma \in \mathscr{G}^\ell : \sigma(\mathbf{m}) = \mathbf{m}\}$$

counts the number of permutations leaving the seat allocation $\mathbf{m}$ invariant. We call $b(\mathbf{m})$ the *boundary factor* for $\mathbf{m}$ since we have $b(\mathbf{m}) \neq 1$ only if the weight vector $\mathbf{m}/M$ is located on the boundary of $S_\geq^\ell$.

A stationary divisor method maps weight vectors with permuted entries to permuted seat allocations,

$$A_{\ell,q,M}(\mathbf{w}) = \mathbf{m} \Longrightarrow A_{\ell,q,M}(\sigma(\mathbf{w})) = \sigma(\mathbf{m}) \quad \text{for all } \sigma \in \mathscr{G}^\ell;$$

a property called "anonymity" by Balinski and Young [1]. By symmetry we obtain, for $\mathbf{m} \in G_\geq^\ell(M)$,

$$P(A_{q,\ell,M} = \mathbf{m} \mid w_1 \geq \cdots \geq w_\ell) = \frac{\ell!}{b(\mathbf{m})} P(A_{q,\ell,M} = \mathbf{m}), \tag{4}$$

where the unconditional probabilities $P(A_{q,\ell,M} = \mathbf{m})$ are given in Theorem 1. Note that $P(A_{q,\ell,M} = \mathbf{m} \mid w_1 \geq \cdots \geq w_\ell) = 0$ for $\mathbf{m} \notin G_\geq^\ell(M)$.

We decompose the grid set $G_\geq^\ell(M)$ into disjoint subsets, for $r = 1, \ldots, \ell$,

$$K_r(M) = \left\{\mathbf{m} \in G_\geq^\ell(M) : m_r > 0 = m_{r+1}\right\}.$$

Therefore, $\mathbf{m} \in K_r(M)$ has the first r components positive and the last $\ell - r$ components zero. Furthermore, for $r = 1, 2, \ldots$ the subset $K_r(M)$ comprises the grid points in the polytope which is generated by the vertices $\mathbf{v}_1, \ldots, \mathbf{v}_r$. For $\mathbf{m} \in K_r(M)$, the probability $p_{q,\ell,M}(r) = P(A_{q,\ell,M} = \mathbf{m})$ is constant (Theorem 1), and the boundary factor $b(\mathbf{m})$ decomposes according to

$$b(\mathbf{m}) = (\ell - r)!\, b_r(\mathbf{m}),$$

where $b_r(\mathbf{m}) = b\big((m_1, \ldots, m_r)^t\big)$. From (3) and (4) we obtain

$$\mathbf{E}^\ell(M) = \sum_{r=1}^{\ell} \frac{\ell!}{(\ell - r)!} p_{q,\ell,M}(r) \sum_{\mathbf{m}\in K_r(M)} \frac{1}{b_r(\mathbf{m})}\, \mathbf{m}. \tag{5}$$

This leaves us with the task of determining

$$\mathbf{S}^r(M) = \left(S_1^r(M), \ldots, S_\ell^r(M)\right)^t = \sum_{\mathbf{m}\in K_r(M)} \frac{1}{b_r(\mathbf{m})}\mathbf{m}. \tag{6}$$

Because the components $S_i^r(M)$ are polynomials in M, we call them *apportionment polynomials*.

The definition of $K_r(M)$ implies that the last $\ell - r$ components are zero,

$$\mathbf{S}^r(M) = \left(S_1^r(M), \ldots, S_r^r(M), 0, \ldots, 0\right)^t.$$

The polynomials $S_i^r(M)$ for $i \le r$ reflect the combinatorial-geometric structure of the boundary classes $K_r(M)$ of the ordered grid set $G_>^\ell(M)$; they do not depend on the size ℓ of the system, nor on the particular apportionment method.

Using the apportionment polynomials, seat biases can be represented as follows.

Theorem 2. *Under the assumptions of Theorem 1 the seat biases satisfy*

$$\mathbf{B}^\ell(M) = \left(\sum_{r=1}^{\ell} \frac{\ell!}{(\ell-r)!} p_{q,\ell,M}(r)\mathbf{S}^r(M)\right) - \frac{M}{\ell}\sum_{r=1}^{\ell} \mathbf{v}_r.$$

Proof. The Theorem is a consequence of (1), (2), (5), and (6). ∎

4 Apportionment polynomials

In view of Theorem 2 there remains the task to efficiently handle the polynomials $S_i^r(M)$ for $r = 1, 2, \ldots$ and $i \le r$. Because we frequently will have to distinguish different cases according to the divisibility of M it is convenient to introduce the notation

$$[y_1, y_2, \ldots, y_k]_k^M = \begin{cases} y_1 & \text{for } \frac{M}{k} \in \mathbb{N} \\ y_2 & \text{for } \frac{M-1}{k} \in \mathbb{N} \\ \vdots & \vdots \\ y_k & \text{for } \frac{M-(k-1)}{k} \in \mathbb{N} \end{cases}$$

For $r = 1$, the class $K_1(M)$ contains only $\mathbf{m} = (M, 0, \ldots, 0)^t$. From (6), we immediately see that

$$S_1^1(M) = M.$$

For $r = 2$ we find

$$S_1^2(M) = \left(\sum_{m_2=1}^{\left\lfloor\frac{M-1}{2}\right\rfloor} (M - m_2)\right) + \left[\frac{M}{4}, 0\right]_2^M,$$

$$S_2^2(M) = \left(\sum_{m_2=1}^{\left\lfloor\frac{M-1}{2}\right\rfloor} m_2\right) + \left[\frac{M}{4}, 0\right]_2^M.$$

Note that the polynomial $S_1^2(M)$ sums over the seats $m_1 = M - m_2$ of the largest party, whereas $S_2^2(M)$ sums over the seats m_2 of the second-largest party. When M is even, the tied seat allocation $\mathbf{m}^* = (M/2, M/2, 0, \ldots, 0)^t$ generates the additional term $M/4$.

It is difficult to determine $S_1^r(M), \ldots, S_r^r(M)$ individually, whereas their sum $\sum_{i=1}^r S_i^r(M)$ has a simple form not depending on the divisibility of the house size M.

Lemma 1. *The apportionment polynomials* $S_1^r(M), \ldots, S_r^r(M)$ *fulfill, for all* $r = 1, 2, \ldots$,

$$\sum_{i=1}^{r} S_i^r(M) = \frac{M}{r!}\binom{M-1}{r-1} = \frac{1}{(r-1)!}\binom{M}{r}.$$

Proof. Choose $\ell = r$. Recall $E_i^\ell(M) = \mathrm{E}[m_i | w_1 \geq w_2 \geq \cdots \geq w_\ell]$. Generally, we have for every $q \in [0, 1]$

$$M = \sum_{i=1}^{\ell} E_i^\ell(M) \overset{(5)}{=} \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} \frac{\ell!}{(\ell - j)!} p_{q,\ell,M}(j) \sum_{\mathbf{m} \in K_j(M)} \frac{m_i}{b_j(\mathbf{m})}.$$

Specifically, selecting $q = 0$, Corollary 2 leads to $p_{0,\ell,M}(j) = 0$ for $j < \ell = r$, and

$$\begin{aligned} M &= \sum_{i=1}^{r} r! p_{0,r,M}(r) \sum_{\mathbf{m} \in K_r(M)} \frac{m_i}{b_r(\mathbf{m})} = \sum_{i=1}^{r} r! p_{0,r,M}(r) S_i^r(M) \\ &= r! \binom{M-1}{r-1}^{-1} \sum_{i=1}^{r} S_i^r(M). \end{aligned}$$

From this $\sum_{i=1}^r S_i^r(M)$ follows. ∎

In general, we obtain $\mathbf{S}^r(M)$ from recursions on M and r. According to our previous definition, the first r components of the vectors $r\mathbf{v}_r \in \left\{ {\ell \atop r} \right\}$ are equal to one, the last $\ell - r$ components are equal to zero.

Theorem 3. *Starting with* $\mathbf{S}^1(M) = (M, 0, \ldots, 0)^t$ *the vectors* $\mathbf{S}^r(M)$ *of the apportionment polynomials for* $r = 2, 3, \ldots$ *obey the recursive scheme*

$$\begin{aligned} \mathbf{S}^r(M) = &\left(\sum_{h=1}^{r-1} \frac{1}{(r-h)!} \sum_{k=1}^{\left\lfloor \frac{M-1}{r} \right\rfloor} \left(\mathbf{S}^h(M - kr) + \frac{\binom{M-kr-1}{h-1} kr}{h!} \mathbf{v}_r \right) \right) \\ &+ \left[\frac{M}{r!}, 0, \ldots, 0 \right]_r^M \mathbf{v}_r. \end{aligned}$$

Proof. The case $r = 1$ has been considered previously. For $r \geq 2$, we split the sum over $\mathbf{m} \in K_r(M)$ into an iterated sum; first over the last non-vanishing component m_r and then over $m_1, \ldots, m_{r-1}$. In addition, we decompose the ordered seat allocation $\mathbf{m} \in G_{\geq}^\ell(M)$ as

$$\mathbf{m} = (\mathbf{m} - m_r r\mathbf{v}_r) + m_r r\mathbf{v}_r.$$

From $\mathbf{m} \in K_r(M)$ we conclude that $(\mathbf{m} - m_r r \mathbf{v}_r) \in K_h(M - rm_r)$, for some $h \leq r-1$, and obtain

$$\mathbf{S}^r(M) = \left(\sum_{m_r=1}^{\left\lfloor \frac{M-1}{r} \right\rfloor} \sum_{h=1}^{r-1} \sum_{\substack{\mathbf{m} \in \\ K_h(M-rm_r)}} \frac{m_r r \mathbf{v}_r + \mathbf{m}}{(r-h)! b_h(\mathbf{m})} \right) + \left[\frac{M}{r!}, 0, \ldots, 0 \right]_r^M \mathbf{v}_r.$$

Applying (6) and

$$\sum_{\substack{\mathbf{m} \in \\ K_h(M-rm_r)}} \frac{1}{b_h(\mathbf{m})} = \frac{1}{M - rm_r} \sum_{\substack{\mathbf{m} \in \\ K_h(M-rm_r)}} \sum_{i=1}^{h} \frac{m_i}{b_h(\mathbf{m})} = \sum_{i=1}^{h} S_i^h(M - rm_r),$$

Lemma 1 results in the expression claimed in the assertion. ■

The recursion of Theorem 3 allows us to write the apportionment polynomials as iterated sums. These sums may be simplified via the well known formulas for $\sum_{k=1}^{m} k^s$, $s \in \mathbb{N}_0$, see e.g. Burrows and Talbot [2], Edwards [5], leading to polynomial expressions for the vectors $\mathbf{S}^r(M)$. For $r = 2$ we have

$$\begin{aligned} S_1^2(M) &= \frac{3}{8}M^2 - \frac{1}{2}M + \left[0, \frac{+1}{8}\right]_2^M, \\ S_2^2(M) &= \frac{1}{8}M^2 \qquad + \left[0, \frac{-1}{8}\right]_2^M. \end{aligned}$$

For $r = 3$ we obtain

$$\begin{aligned} S_1^3(M) &= \frac{11}{216}M^3 - \frac{3}{16}M^2 + \frac{13}{72}M + \left[0, \frac{+1}{54}, \frac{-1}{54}\right]_3^M + \left[0, \frac{-1}{16}\right]_2^M, \\ S_2^3(M) &= \frac{5}{216}M^3 - \frac{1}{16}M^2 + \frac{1}{72}M + \left[0, \frac{-2}{54}, \frac{+2}{54}\right]_3^M + \left[0, \frac{+1}{16}\right]_2^M, \\ S_3^3(M) &= \frac{2}{216}M^3 \qquad - \frac{2}{72}M + \left[0, \frac{+1}{54}, \frac{-1}{54}\right]_3^M. \end{aligned}$$

For $r = 4$ we find

$$\begin{aligned} S_1^4(M) &= \frac{25}{6912}M^4 - \frac{11}{432}M^3 + \frac{203}{3456}M^2 - \frac{7}{144}M + \left[0, \frac{+3}{256}, \frac{-1}{96}, \frac{+3}{256}\right]_4^M + \left[0, 0, \frac{+1}{54}\right]_3^M, \\ S_2^4(M) &= \frac{13}{6912}M^4 - \frac{5}{432}M^3 + \frac{71}{3456}M^2 - \frac{1}{144}M + \left[0, \frac{-1}{256}, \frac{+3}{96}, \frac{-1}{256}\right]_4^M + \left[0, 0, \frac{-2}{54}\right]_3^M, \\ S_3^4(M) &= \frac{7}{6912}M^4 - \frac{2}{432}M^3 + \frac{5}{3456}M^2 + \frac{2}{144}M + \left[0, \frac{-3}{256}, \frac{-3}{96}, \frac{-3}{256}\right]_4^M + \left[0, 0, \frac{+1}{54}\right]_3^M, \\ S_4^4(M) &= \frac{3}{6912}M^4 \qquad - \frac{15}{3456}M^2 \qquad + \left[0, \frac{+1}{256}, \frac{+1}{96}, \frac{+1}{256}\right]_4^M. \end{aligned}$$

The divisibility of M affects only the constant terms of the individual polynomials $S_i^r(M)$. According to Lemma 1, the sum of the polynomials has no constant term, which is readily verified for $r \leq 4$.

5 Seat bias formulas for traditional apportionment methods

With the apportionment polynomials $\mathbf{S}^r(M)$, $r \leq 4$, we are able to calculate the vector of seat biases $\mathbf{B}^\ell(M)$ for $\ell \leq 4$ by Theorem 3. For $l \leq 3$, Schuster *et al.* [10] give seat bias formulas for both the divisor method with standard rounding and the divisor method with rounding down, which are confirmed via our approach.

We now derive seat bias formulas for these methods in the case of $\ell = 4$ parties. We state our results up to the highest order in $1/M$ not depending explicitly on the divisibility of M. For general q-stationary divisor methods, Theorems 1 and 2 yield ($\ell = 4$)

$$\begin{aligned}\mathbf{B}^4(M) =& \frac{24q^3}{(M+4q-1)(M+3q-1)(M+2q-1)}\mathbf{S}^1(M)\\ &+\frac{72(M^2+M+2)q^2}{(M+4q-1)(M+4q-2)(M+3q-1)(M+3q-2)(M+2q-1)}\mathbf{S}^2(M)\\ &+\frac{48(3M^2+21qM-12M+37q^2-42q+11)q}{(M+4q-1)(M+4q-2)(M+4q-3)(M+3q-1)(M+3q-2)}\mathbf{S}^3(M)\\ &+\frac{144}{(M+4q-1)(M+4q-2)(M+4q-3)}\mathbf{S}^4(M)-\frac{M}{\ell}\sum_{r=1}^{\ell}\mathbf{v}_r\\ =& \begin{pmatrix} +\frac{13}{12}(q-\frac{1}{2})+\left(\frac{11}{3}(q-q^2)-\frac{25}{48}\right)\frac{1}{M}\\ +\frac{1}{12}(q-\frac{1}{2})+\left(\frac{5}{3}(q-q^2)-\frac{13}{48}\right)\frac{1}{M}\\ -\frac{5}{12}(q-\frac{1}{2})+\left(\frac{2}{3}(q-q^2)-\frac{7}{48}\right)\frac{1}{M}\\ -\frac{9}{12}(q-\frac{1}{2})-\left(\frac{18}{3}(q-q^2)-\frac{45}{48}\right)\frac{1}{M}\end{pmatrix}+\mathcal{O}\left(\frac{1}{M^2}\right).\end{aligned} \tag{7}$$

From (7), we obtain the vector of seat biases for the *divisor method with standard rounding* ($q = 1/2$) in the case of $\ell = 4$ parties,

$$\mathbf{B}^4(M) = \begin{pmatrix} +\frac{19}{48}\frac{1}{M}\\ +\frac{7}{48}\frac{1}{M}\\ +\frac{1}{48}\frac{1}{M}\\ -\frac{27}{48}\frac{1}{M}\end{pmatrix}+\mathcal{O}\left(\frac{1}{M^3}\right).$$

The leading terms are of the order of magnitude $1/M$, and there is no term proportional to $1/M^2$. As visible from Figure 1(a), the divisor method with standard rounding is asymptotically unbiased when the number of available seats M increases.

For the *divisor method with rounding down* ($q = 1$) and for $\ell = 4$ parties we find from (7)

$$\mathbf{B}^4(M) = \begin{pmatrix} +\frac{13}{24}-\frac{25}{48}\frac{1}{M}+\frac{25}{24}\frac{1}{M^2}\\ +\frac{1}{24}-\frac{13}{48}\frac{1}{M}+\frac{13}{24}\frac{1}{M^2}\\ -\frac{5}{24}-\frac{7}{48}\frac{1}{M}+\frac{7}{24}\frac{1}{M^2}\\ -\frac{9}{24}+\frac{45}{48}\frac{1}{M}-\frac{45}{24}\frac{1}{M^2}\end{pmatrix}+\mathcal{O}\left(\frac{1}{M^3}\right).$$

We now have noticeable seat biases in favor of the larger parties. When the number of available seats M tends to infinity the asymptotic seat biases are

$$\lim_{M\to\infty}\mathbf{B}^4(M) = \left(\frac{13}{24},\frac{1}{24},-\frac{5}{24},-\frac{9}{24}\right)^t.$$

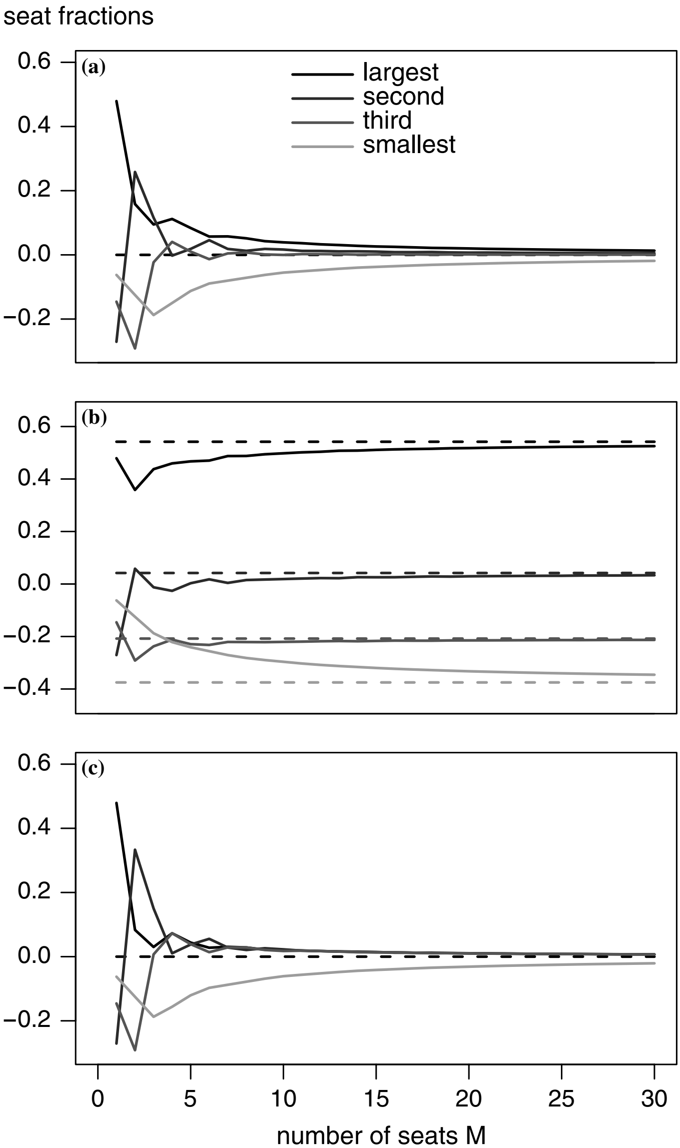

Fig. 1. Seat biases $B_1^4(M), \ldots, B_4^4(M)$ for $\ell = 4$ parties, as functions of the house size $M \leq 30$. (a) Divisor method with standard rounding (Webster, Sainte-Laguë). (b) Divisor method with rounding down (Jefferson, d'Hondt). (c) Quota method of greatest remainders (Hamilton, Hare). All seat biases of the Webster and Hamilton methods are tiny and quickly converge to zero, whence the methods legitimately are termed "practically unbiased". In contrast, the Jefferson method comes with strong seat biases, favoring the two larger parties at the expense of the two smaller parties. The dashed lines indicate the asymptotic behaviour of the various seat bias curves.

Therefore, the largest party may expect an extra seat, beyond its ideal share of seats, every other election. Non-zero asymptotes stand out in Figure 1(b).

For additional results on $\ell = 5$ parties see [11]. Our results on $l = 4$ parties confirm, in particular, the conjecture of Schuster *et al.* [10] on asymptotic seat biases for $M \to \infty$. A general proof of these formulas is still missing however.

A The quota method of greatest remainders

The *quota method of greatest remainders* (Hamilton, Hare) operates in two stages. First, the proportion $w_i M$ is rounded down to its integer part, $\tilde{m}_i = \lfloor w_i M \rfloor$. In the case that all $w_i M$ are integers we get $M - \sum_{i=1}^{\ell} \tilde{m}_i = 0$ and set $m_i = \tilde{m}_i$. Otherwise, there is a positive discrepancy $\delta = M - \sum_i \tilde{m}_i \geq 1$.

Then the fractional parts $\delta_i = w_i M - \tilde{m}_i$ are ranked to obtain $\delta_{(1)} \geq \delta_{(2)} \geq \cdots \geq \delta_{(\ell)}$ (ties are broken arbitrarily). The seat allocation vector

$$A_{\ell,M}(\mathbf{w}) \in G^{\ell}(M)$$

has the components $m_{(i)} = \tilde{m}_{(i)} + 1$ if $i \leq \delta$ and $m_{(i)} = \tilde{m}_{(i)}$ if $i > \delta$. Thus, the δ largest remainders are rounded up, the $\ell - \delta$ smallest remainders are rounded down.

The following Theorem 6 gives an analog to Theorem 1.

Theorem 4. *Suppose the weight vectors* $\mathbf{w}$ *are uniformly distributed on* S^{ℓ}*. Use the quota method of greatest remainders to apportion the house size* $M > \ell$*.*

Then the seat allocation vector $A_{\ell,M}$ *is a discrete random variable, with values in the finite grid set* $G^{\ell}(M)$*, attaining a grid point* $\mathbf{m} \in G^{\ell}(M)$ *with probability*

$$P(A_{\ell,M} = \mathbf{m}) = \frac{(\ell-1)!}{\binom{\ell}{r} M^{\ell-1}} \sum_{\mathbf{t} \in \left\{ {\ell-1 \atop r-1} \right\}} \prod_{j=1}^{\ell-2} \left(1 - \frac{j}{r + j - t_{(j)}} \right)^{1 - t_{j+1}}$$

$$=: p_{\ell,M}(r),$$

where r denotes the number of positive components of $\mathbf{m}$.

Proof. See Drton and Schwingenschlögl [3, Theorem 3.4]. ∎

As $P(A_{\ell,M} = \mathbf{m})$ depends on $\mathbf{m}$ only through r it is possible to calculate seat biases in analogy to the calculations for q-stationary divisor methods. Theorems 2 and 4 yield for $\ell = 4$ parties the following seat biases, which are depicted in Figure 1(c),

$$\mathbf{B}^4(M) = \frac{1}{M^3} \left(\mathbf{S}^1(M) + 14\mathbf{S}^2(M) + 72\mathbf{S}^3(M) + 144\mathbf{S}^4(M) \right) - \frac{M}{\ell} \sum_{r=1}^{\ell} \mathbf{v}_r$$

$$= \begin{pmatrix} +\frac{5}{24}\frac{1}{M} + \frac{1}{M^3}\left[0, \frac{-17}{16}, \frac{-24}{16}, \frac{-17}{16}\right] + \frac{1}{M^3}\left[0, \frac{+4}{3}, \frac{+4}{3}\right] \\ +\frac{5}{24}\frac{1}{M} + \frac{1}{M^3}\left[0, \frac{+35}{16}, \frac{+72}{16}, \frac{+35}{16}\right] + \frac{1}{M^3}\left[0, \frac{-8}{3}, \frac{-8}{3}\right] \\ +\frac{5}{24}\frac{1}{M} + \frac{1}{M^3}\left[0, \frac{-27}{16}, \frac{-72}{16}, \frac{-27}{16}\right] + \frac{1}{M^3}\left[0, \frac{+4}{3}, \frac{+4}{3}\right] \\ -\frac{15}{24}\frac{1}{M} + \frac{1}{M^3}\left[0, \frac{+9}{16}, \frac{+24}{16}, \frac{+9}{16}\right] \end{pmatrix}.$$

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